Discussion #3

Agenda:

1. Discuss Homework
2. 1D Optimization methods.

Comments on 1st HW

Note there is a difference between convexity and strict convexity

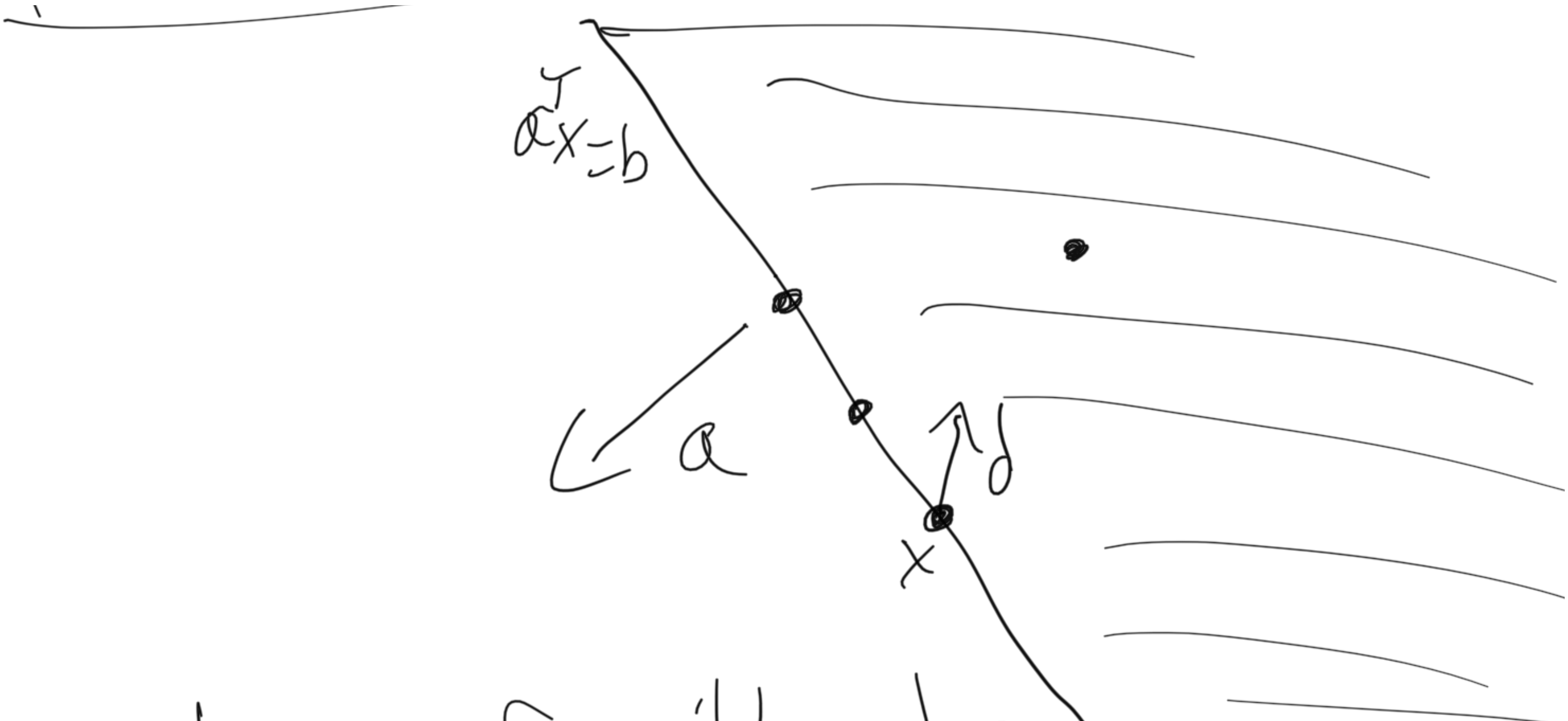
$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

For convexity, use $\leq$

For strict convexity, use $<$

Strict convexity $\Rightarrow$ unique global minimum

$$\boxed{S = \{ \underline{a^T x \leq b} \}}$$



What are feasible directions for $x \in S$.

1) $x \in \operatorname{int} S$
   - All directions feasible

2) $x \in \partial S$

Recall, direction d is feasible if $\forall \alpha < \alpha_0$, $\alpha > 0$, we have $\underline{x + \alpha d} \in S$

$x \in \partial S \iff \underline{a^T x = b}$

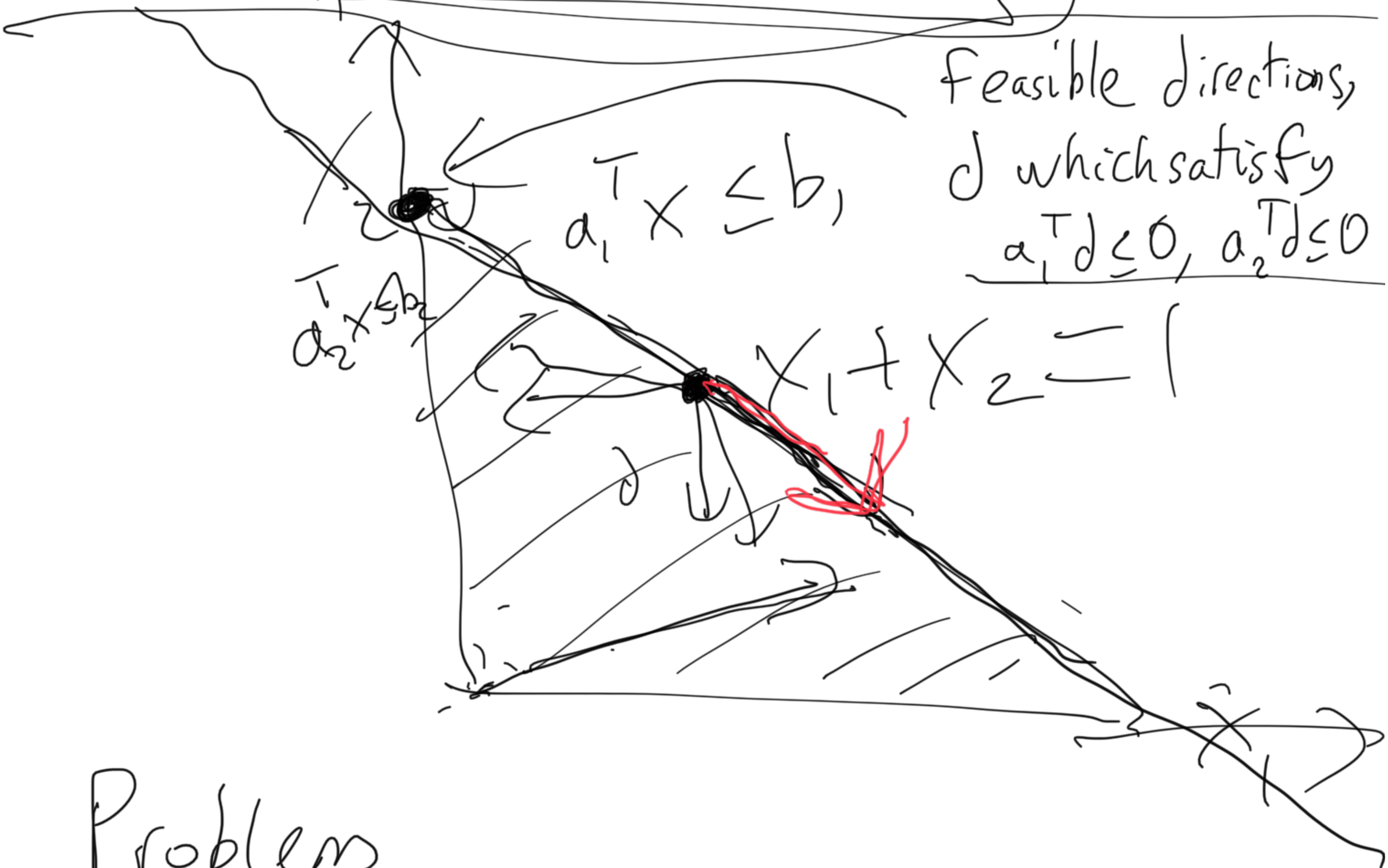
Want $x + \alpha d \in S$

$$\Leftrightarrow a^T(x+\alpha d) \le b$$

$$\| $$

$$b + \alpha a^T d \le b$$

$$\Leftrightarrow \boxed{a^T d \le 0}$$



Problem

$$\Omega = \left\{ \begin{array}{l} x_1 \ge 0 \\ x_2 \ge 0 \\ x_1 + x_2 \le 1 \end{array} \right.$$

$$a = \begin{pmatrix} \\ 1 \end{pmatrix} \qquad b = 1$$

$$a^T x \le b$$

$\boxed{d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}}$ such that

$$a^T d = \underline{d_1 + d_2} \le 0$$

FONC (for maximizer):

$d^T \nabla f(x) \le 0$ for all feasible d

$$f(x) = c_1 x_1 + c_2 x_2$$

$$\boxed{c_1, c_2 \ge 0, \quad c_1 > c_2}$$

$$\nabla f = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\boxed{d^T \nabla f = c_1 d_1 + c_2 d_2 > 0}$$

7. $f(x) = \underline{\frac{1}{2} x^T Q x - x^T b}$

where $Q = Q^T > 0$,

Show x^* minimizes $f \iff x^*$ satisfies FONC.

($\Rightarrow$) Immediate from FONC

($\Leftarrow$) Check SOSC

$\nabla^2 f = Q$

ID Search Methods (Line Search)

$f : \mathbb{R} \to \mathbb{R}$

minimize f

Q) Why we care?

A) Line Search is a building

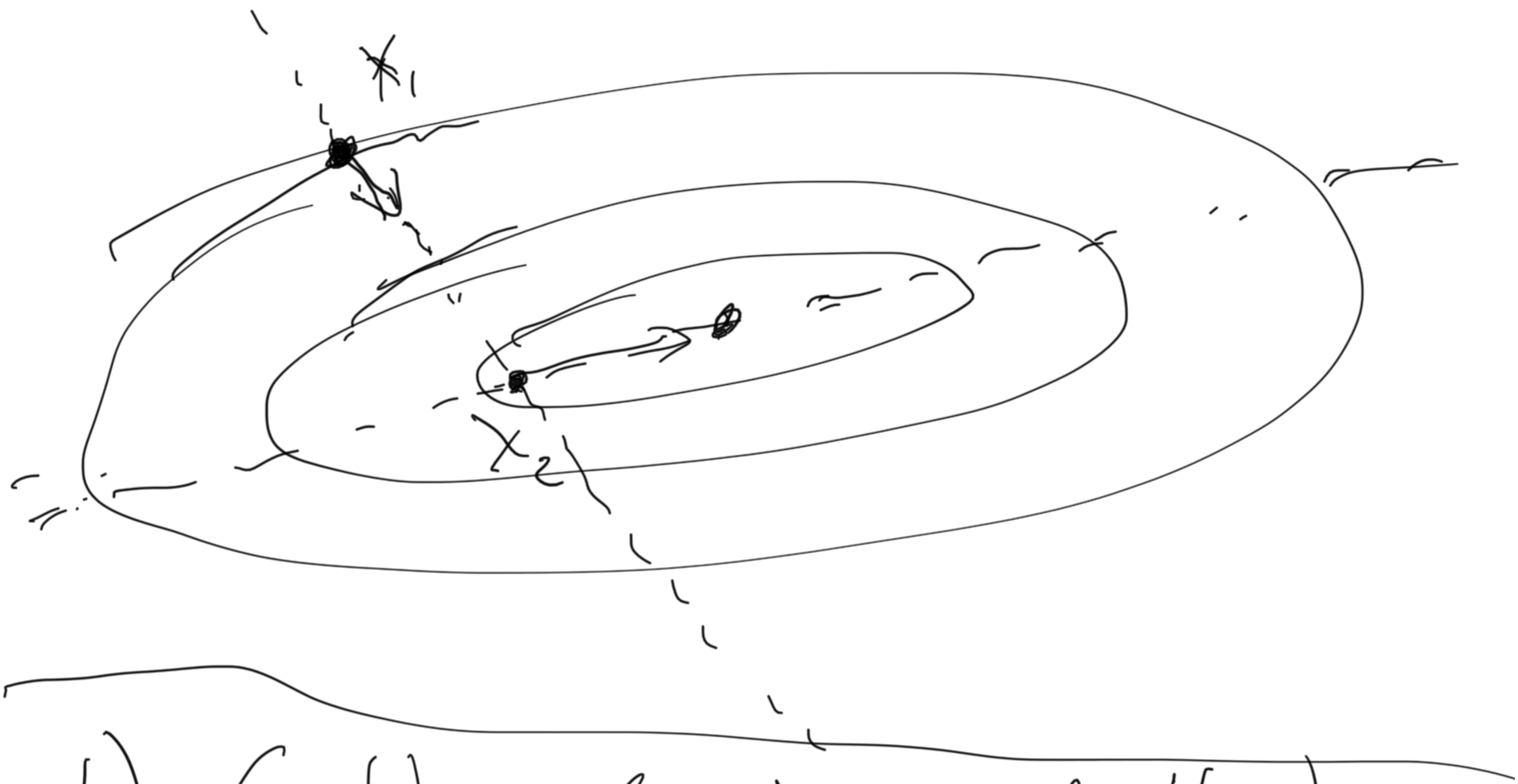
1) Line

block for multidimensional search.

Many Optimization methods take the form:

1) Pick feasible direction d

2) Minimize $g(\alpha) = f(x + \alpha d)$



1) Golden Section Method

- For unimodal f (functions with 1 local min)

- Only requires evaluating

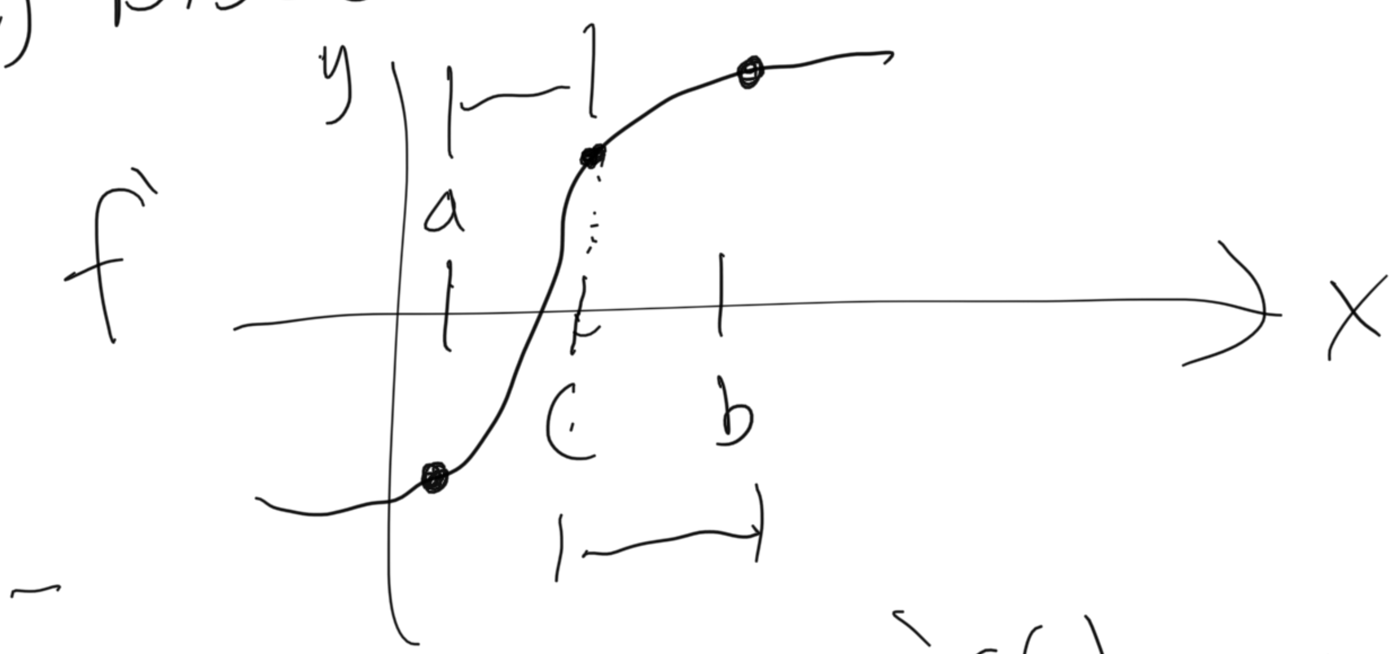
f

2) Zero-finding Methods

Find zeros of f'.

These might be minimizers

a) Bisection Method



Know $f'(a) < 0$, $f'(b) > 0$

By IVT, know a zero occurs in between.

Check $c = \frac{a+b}{2}$

If $f'(c) > 0$,
Next interval is $[a, c]$

If $f'(c) = 0$,
Done

If $f'(c) < 0$,
Next interval is $[c, b]$

Note: This method requires evaluating derivative

Newton's Method

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

At $(x_k, f'(x_k))$,
Using Taylor's Theorem, expanding at a minimizer x^* (zero of f')

$$0 = f'(x^*) = f'(x_k) + f''(x_k)(x^* - x_k) + \frac{1}{2} f'''(\xi)(x^* - x_k)^2$$

Solve for x^*,

$$\boxed{x^* = x_k - \frac{f'(x_k)}{f''(x_k)}} \boxed{- \frac{1}{2}\frac{f'''(\xi)}{f''(x_k)}(x^* - x_k)^2}$$

Comments:

If 1) $|f'''(\xi)|$ is large

2) $f''(x_k)$ close to zero

3) $(x^* - x_k)$ is large

This is not a good approximation.