Discussion #2

Agenda: More on FONC, SONC
SOSC
Optimality Conditions

HW due today, On Gradescope
at 11:59
LA time

OH: M 1-2, Th 4-5

$$f(y) = \frac{1}{2} y^T A y, \quad A > 0$$

Want to show that
f is strictly convex

f is strictly convex if

$$f(tx + (1-t)y) < t f(x) + (1-t) f(y)$$

$\forall x, y \in \mathcal{X}, \ t \in [0,1]$

If $f \in C^2$, f is strictly convex if $\nabla^2 f \succ 0$.

$$g(x,y) = \frac{1}{2} x^T y$$

$$y = h(x) = Ax$$

$$f(x) = g(x, h(x)) = \frac{1}{2} x^T A x$$

$$\begin{tikzcd} g \arrow[d, "\frac{\partial g}{\partial x}"'] \arrow[r, "\frac{\partial g}{\partial y}"] & h \arrow[dl, "\frac{\partial h}{\partial x}"] \\ x & \end{tikzcd}$$

$$\nabla f(x) = \frac{\partial g}{\partial x} + \underbrace{\left(\frac{\partial h}{\partial x}\right)^T \frac{\partial g}{\partial y}}$$

$$\nabla f(x) = \nabla\left(\frac{1}{2} x^T A x\right)$$

$$= \frac{1}{2} A x + A^T \left(\frac{1}{2} x\right)$$

$$= \frac{1}{2}(A + A^T)x$$

$$= Ax$$

$\underline{\nabla^2 f = A}$, Since $A > 0$ we have f strictly convex.

Optimality Conditions

$f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$

We want to minimize f.

FONC: If x^* is a minimizer then $\nabla f(x^*) = 0$

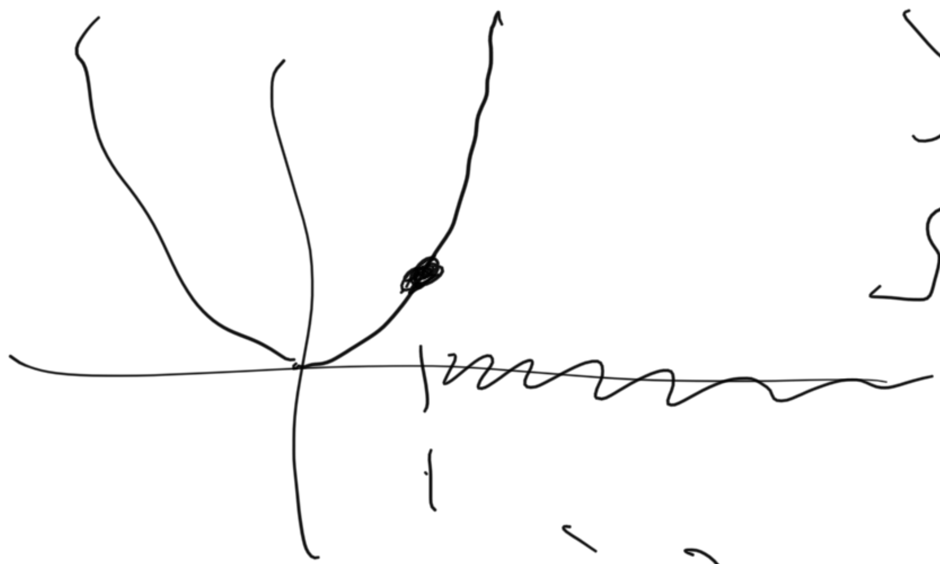
SONC: If x^* is a minimizer, then $\nabla^2 f(x^*) \geq 0$

SOSC: If $\nabla f(x^*) = 0$, and $\nabla^2 f(x^*) > 0$, then

x^* ...

x^* is a minimizer:

$x^* \in$ interior of Ω



$y = x^2$

$\Omega = \{x \geq 1\}$

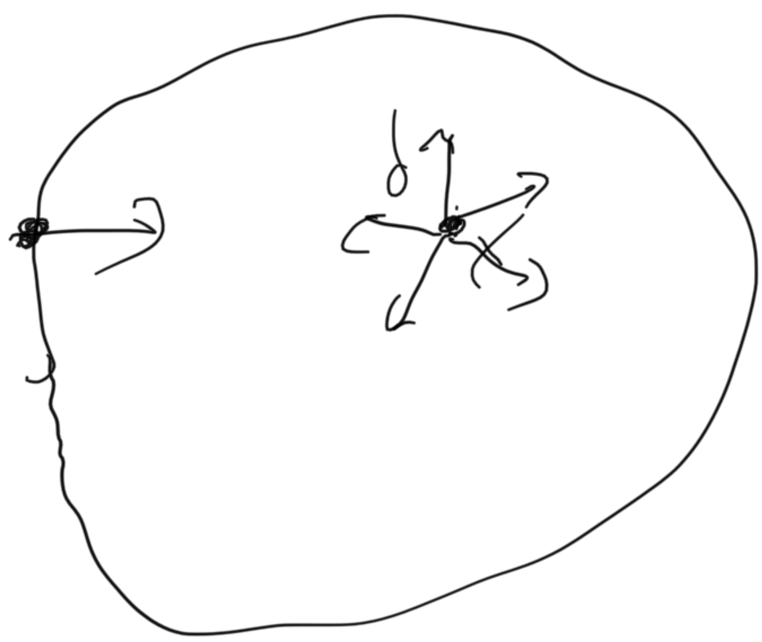
$f'(1) = 2$

Given $\Omega \subset \mathbb{R}^n$, $x \in \Omega$,

We say $d \neq 0 \in \mathbb{R}^n$ is feasible if

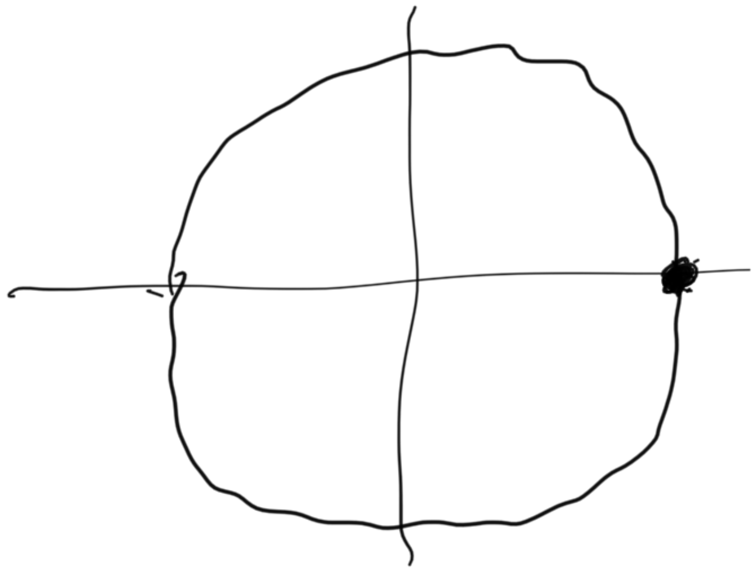
$\exists \alpha_0 > 0$, such that

$x + \alpha d \in \Omega$, $\forall 0 < \alpha < \alpha_0$



$$\Omega = \{(x,y) : x^2+y^2=1\}$$

$$f(x,y) = x$$



$x^*(1,0)$

There are no feasible directions at x^*.

FONC (boundary):

If x^* is a minimizer, for all feasible directions d at x^*,

$$\underbrace{d^T \nabla f(x^*)}_{\text{directional derivative in direction } d} \geq 0$$

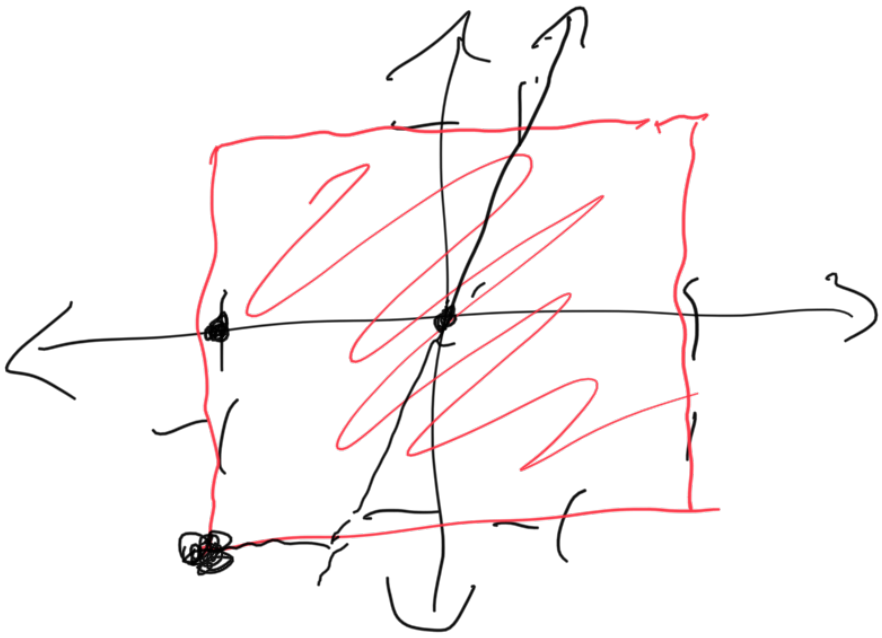
SONC. If x^* is a minimizer

for all feasible directions d at x^*

$$d^T \nabla^2 f(x^*)\, d \geq 0$$

$$f(x,y) = x + 2y$$

s.t. $(x,y) \in \Omega$, $\Omega = \left\{ \begin{array}{l} -1 \leq x \leq 1 \\ -1 \leq y \leq 1 \end{array} \right\}$



1. Is any point in the interior a minimizer?

$$\nabla f = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

No, since $\nabla f \neq 0$ anywhere

2) Check the boundary

Check $x^* = (-1, -1)$

Observe $d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ is feasible

if $d_1, d_2 \geq 0$

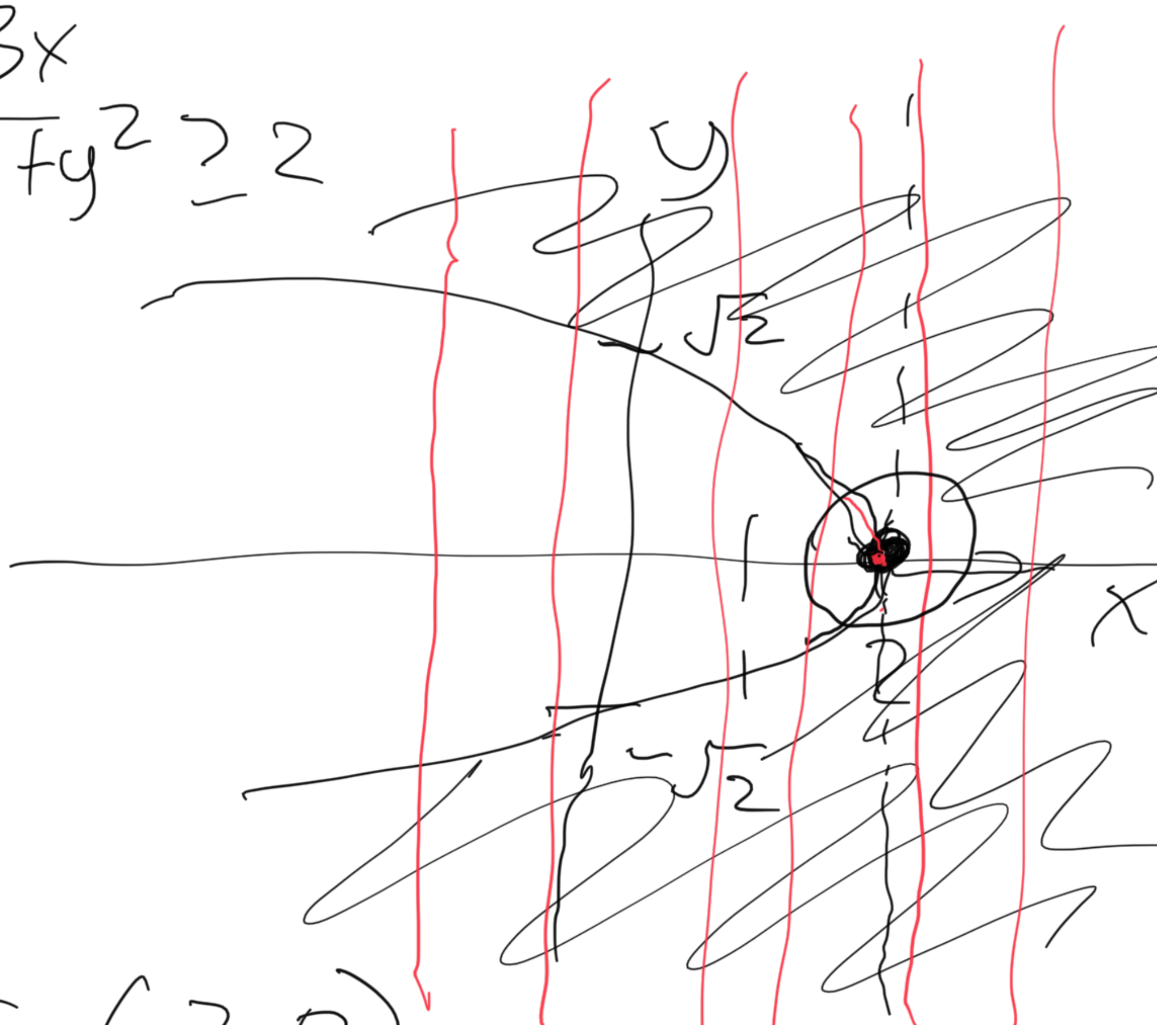
$d^T \nabla f(x^*) = d_1 + 2d_2$

≥ 0

Therefore x^* satisfies FONC

$\min 3x$

s.t. $x + y^2 \geq 2$



Consider (2, 0)

Consider $(2,0)$

at this point, d is feasible

if it points to the right,

$$d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad d_1 \ge 0$$

FONC: $\nabla f = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$$d^T \nabla f(2,0) = 3d_1 \ge 0$$

x^* satisfies FONC,

SONC: $\nabla^2 f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$d^T \nabla^2 f \, d = 0 \ge 0$$

so x^* satisfies SONC

x^* is not a local minimizer.