Discussion #2

Agenda: More on FONC, SONC, SOSC

Optimality Conditions

HW due today, on Gradescope at 11:59
LA time

OH: M 1-2, Th 4-5

\[ f(y) = \frac{1}{2} y^T A y, \quad A > 0 \]

Want to show that

f is strictly convex

f is strictly convex if

\[ f(tx + (1-t)y) < t f(x) + (1-t) f(y) \]
\( \forall x, y \in \mathbb{R}, t \in \mathbb{R}_0^+ \)

If \( f \in \mathcal{C}^2 \), \( f \) is strictly convex if \( \nabla^2 f > 0 \).

\[
g(x, y) = \frac{1}{2} x^Ty
\]

\[
y = h(x) = Ax
\]

\[
f(x) = g(x, h(x)) = \frac{1}{2} x^T Ax
\]

\[
\nabla f(x) = \frac{\partial g}{\partial x} + \left( \frac{\partial h}{\partial x} \right)^T \frac{\partial g}{\partial y}
\]

\[
\nabla f(x) = \frac{\partial g}{\partial x} + A^T \left( \frac{1}{2} x \right)
\]
\[ \nabla^2 f = A, \quad \text{Since } A > 0 \quad \text{we have } f \text{ strictly convex.} \]

**Optimality Conditions**

\[ f: \Omega \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^n \]

We want to minimize \( f \).

**FONC:** If \( x^\ast \) is a minimizer, then \( \nabla f(x^\ast) = 0 \)

**SONC:** If \( x^\ast \) is a minimizer, then \( \nabla^2 f(x^\ast) \geq 0 \)

**SOSC:** If \( \nabla f(x^\ast) = 0 \) \text{ and } \nabla^2 f(x^\ast) \geq 0, \text{ then } x^\ast \text{ is a strict minimizer.} \]
\( x \) is a minimizer:

\[ x^* \in \text{interior of } \Omega \]

\[ y = x^2 \]

\[ \Omega = \{ x \geq 1 \} \]

\[ f'(1) = 2 \]

Given \( \Omega \subseteq \mathbb{R}^n \), \( x \in \Omega \), we say \( d \in \mathbb{R}^n \) is feasible if \( \exists \alpha_0 > 0 \) such that

\[ x + \alpha d \in \Omega, \ \forall 0 < \alpha \leq \alpha_0 \]
\[ S2 : z(x, y) = x^2 + y^2 = \frac{3}{2} \]

\[ f(x, y) = x \]

\[ x^* (1, 0) \]

There are no feasible directions at \( x^* \).

\[ \text{FONC (boundary):} \]

If \( x^* \) is a minimizer, for all feasible directions \( d \) at \( x^* \),

\[ d^T \nabla f(x^*) \geq 0 \]

directional derivative in direction \( d \).
SONC. If \( x^* \) is a minimizer, for all feasible directions \( d \) at \( x^* \):

\[
\nabla^T \nabla^2 f(x^*) d \geq 0
\]

\[
f(x, y) = x + 2y
\]

s.t. \((x) \in \mathbb{R})

\[
\begin{cases}
3 \leq x \leq 7 \\
-2 \leq y \leq 1
\end{cases}
\]

1. Is any point in the interior a minimizer?

\[
\nabla f = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

No, since \( \nabla f \neq 0 \) anywhere.
2) Check the boundary

Check \( x^* = (-1, -1) \)

Observe \( d = (d_1, d_2) \) is feasible if \( d_1, d_2 \geq 0 \)

\[
\nabla f(x^*) = d_1 + 2d_2 \\
\geq 0
\]

Therefore \( x^* \) satisfies FONC

\[
\min 3x \\
\text{s.t. } x + ty^2 \geq 2
\]
Consider $C(x)$ at this point, $d$ is feasible if it points to the right.

\[ d = (d_1, d_2), \quad d_1 \geq 0 \]

FONC: \( \forall f = (3) \)

\[ d^T \nabla f (2, 0) = 3d_1 \]

\( x^* \) satisfies FONC, \( \geq 0 \)

SONC: \( \nabla^2 f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \)

\[ d^T \nabla^2 f d = 0 \geq 0 \]

so \( x^* \) satisfies SONC

\( x^* \) is not a local minimizer.