Constrained Optimization

$$\min f(x)$$
$$\text{s.t.} \quad h(x) = 0$$
$$g(x) \le 0$$

Lagrangian

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x).$$

KKT Conditions:

(Under certain conditions), At an optimizer x^*, there exist λ^*, μ^*, such that the following conditions hold:

$$\left.\begin{array}{ll}(1) & h(x) = 0 \\ (2) & g(x) \le 0\end{array}\right\} \text{Primal Feasibility}$$

$$(3) \quad \nabla_x L(\cancel{x}, \lambda, \mu) = \nabla_x f(x^*) + \lambda^{*T} D_x h(x^*) + \mu^{*T} D_x g(x^*) = 0$$

$$(4) \quad \mu^* \ge 0 \; \} \text{ Dual Feasibility}$$

$$\left.\begin{array}{ll}(5) & \mu_1^* g(x_1^*) = 0 \\ & \vdots \\ & \mu_p^* g(x_p^*) = 0\end{array}\right\} \text{Complementary Slackness}$$

$$\min \; x_1 x_2^2$$

$$\text{s.t.} \quad x_1 - x_2 = 0 \quad \lambda \quad (1)$$

$$x_1 \geq 0 \quad \mu \quad (2)$$

$$L(x_1, x_2, \lambda, \mu)$$

$$= x_1 x_2^2 + \lambda(x_1 - x_2) - \mu x_1$$

KKT: (3) $\nabla_x L = \begin{pmatrix} x_2^2 + \lambda - \mu \\ 2x_1 x_2 - \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(4) $\mu \geq 0$

(5) $\mu x_1 = 0$

$\Rightarrow \lambda = 2x_1 x_2$

$\Rightarrow \lambda = 2x_2^2$

$x_2^2 + 2x_2^2 - \mu = 0$

If $x_1 > 0$, $\Rightarrow \mu = 0 \Rightarrow x_2^2 + 2x_2^2 = 0$

$3x_2^2 = 0$

$\Rightarrow x_2 = 0$

~~If $x_1 = 0$~~ Contradicts $x_1 = x_2$

If $x_1 = 0$, then $x_2 = 0$, $\lambda, \mu = 0$

So only point satisfying KKT conditions is $x_1 = 0, x_2 = 0$.

$$\min \ 2x_1 + 3x_2 - 4$$
$$\text{s.t. } \ x_1 x_2 = 6$$

$$L(x_1, x_2, \lambda) = 2x_1 + 3x_2 - 4 + \lambda(x_1 x_2 - 6)$$

$$\nabla_x L = \begin{pmatrix} 2 + \lambda x_2 \\ 3 + \lambda x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\cancel{2} \ \lambda = -\frac{2}{x_2}$$

$$\cancel{3+} \ 3 - 2\frac{x_1}{x_2} = 0$$

$$\Leftrightarrow \cancel{3} \ 3x_1 = 2x_2 \Rightarrow 3x_1 x_2 = 2x_2^2$$

$$\Rightarrow 3 \cdot 6 = 2x_2^2 \Rightarrow x_2 = \pm\sqrt{9} = \pm 3$$

$$x_1 = \pm 2$$

$$\min \ \tfrac{1}{2}\|x\|^2$$
$$\text{s.t. } \ Ax = b$$

$$L(x, \lambda) = \tfrac{1}{2}\|x\|^2 + \lambda(Ax - b)$$

$$\nabla_x L = x + A^T\lambda = 0$$

$$x = -A^T\lambda = A^T(AA^T)^{-1}b$$

$$= Ax + AA^T\lambda = 0$$

$$b + AA^T\lambda \Rightarrow \lambda = -(AA^T)^{-1}b$$