Discussion #1

Math 164 - Optimization

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OH: Monday -
Thursday -

This week, Friday 3-4 PM

Simple Function:

$$f(x) = x^2 - 4x + 3$$

Minimize f! (31A)

1. Set $f' = 0$

$$f'(x) = 2x - 4 = 0$$
$$\Rightarrow x = 2$$

2. Check $f''(x) > 0$

$$f''(x) = 2 > 0$$

1. First Order Necessary Condition (FONC)

If $f'(x) \neq 0$, x is not a minimizer of f.

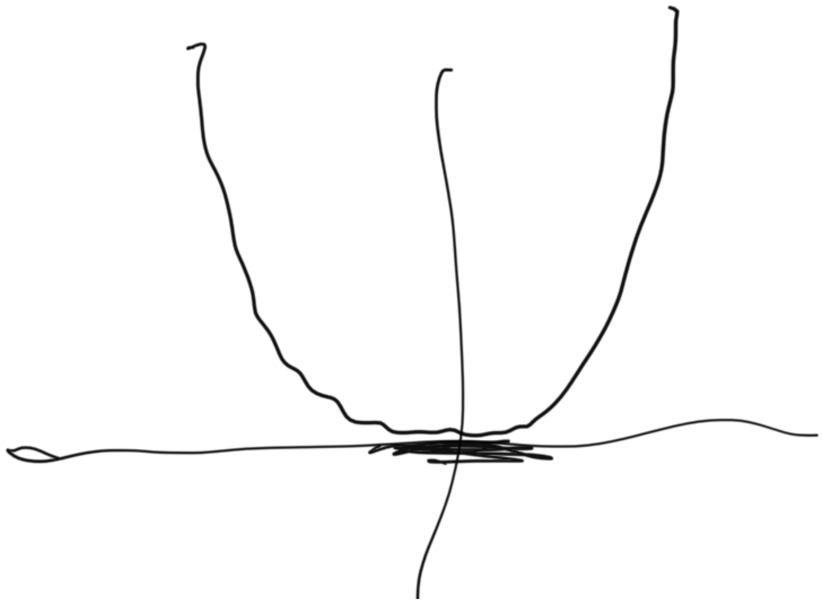
2. Second Order Sufficient Condition (SOSC)

If $f'(x) = 0$, and $f''(x) > 0$, then x is a strict local minimizer of f.

It is not necessary for SOSC to hold for x to be minimizer,

Can have $f''(x) = 0$, and x is a minimizer.

$f(x) = x^4$



Generalize to $f: \mathbb{R}^n \to \mathbb{R}$

1. FONC: $\nabla f(x) = 0$

2. SOSC: $\boxed{\underbrace{\nabla^2 f(x)}_{\text{Hessian}} > 0}$

$f(x,y)$,

$$\nabla f(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

$$\nabla^2 f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$\underline{\nabla^2 f(x) > 0}$ means this matrix is positive definite.

We say a symmetric matrix ... definite if

A is positive definite if any of the following conditions hold:

$$\approx ax^2$$

1. $\forall x \neq 0, \ \underbrace{x^T A x} > 0$
2. A has only positive eigenvalues
3. The $\boxed{\text{principal minors}}$ of A are all positive.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

1st minor: $\det [1] = 1$

2nd minor: $\det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 3-4 = -1$

3rd minor: $\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

$f(x,y) = x^2 - 4xy + y^2$

FONC: $\nabla f(\ldots) = \ldots (2x - 4y)$

1. FONC: $\nabla f(x,y) = \begin{pmatrix} 2x - 4y \\ 2y - 4x \end{pmatrix}$

2. SOSC: $\nabla^2 f(x,y)$

$$= \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 \\ -4 & 2 \end{pmatrix}$$

1. $\nabla f = 0$, $\begin{cases} 2x - 4y = 0 \\ 2y - 4x = 0 \end{cases}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. $\det [2] = 2 > 0$

$\det \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix} = 4 - (16)$

$= -12 < 0$

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ does not satisfy

SOSC

SOSC

not necessarily a minimizer

Special Class of Functions:

Convex Functions:

Def: $f \in C^2$

2x cont. diff.,

We say f is convex if

$\nabla^2 f(x) \geq 0$ (semipositive definite)

1. $x^T \nabla^2 f x \geq 0$
2. nonnegative eigenvalues
3. principal minors nonnegative

f is strictly convex if

$\nabla^2 f > 0$

For strictly convex functions,

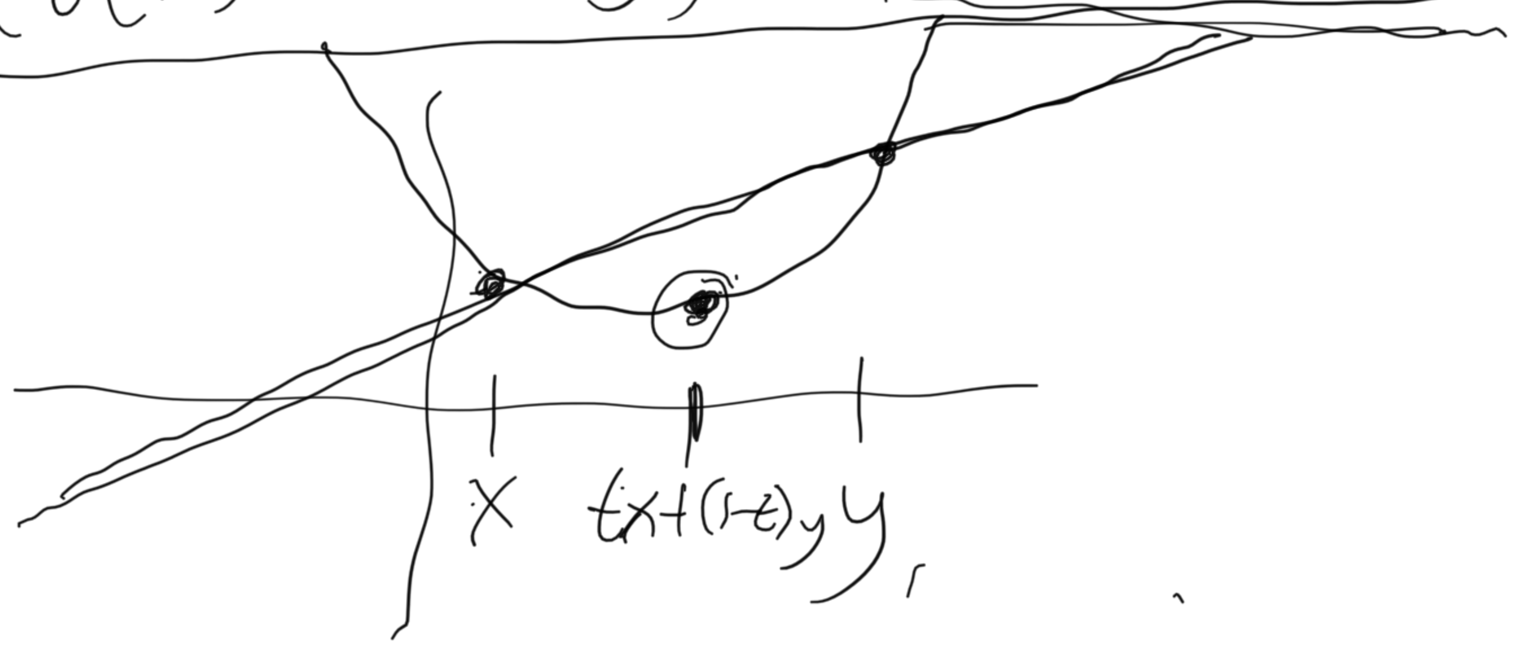
Only need to check FONC, because SOSC are automatically satisfied.

For arbitrary $f: \Omega \to \mathbb{R}$,

f is convex if

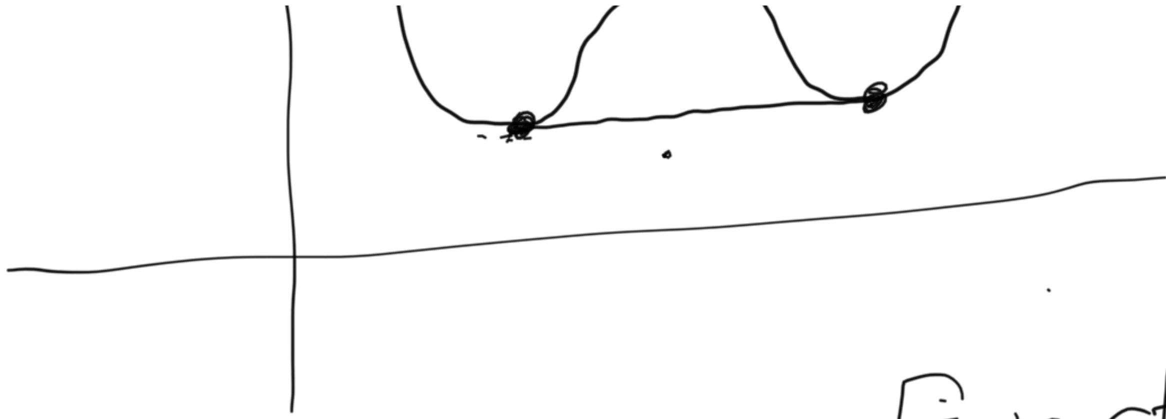
for all $x, y \in \Omega$, $t \in (0,1)$,

$$f(t(x) + (1-t)y) \le t f(x) + (1-t) f(y)$$



f is strictly convex if the $\le$ is a $<$

<u>Thm</u>: A strictly convex function has at most 1 minimizer



Strictly convex Functions have unique minimizer

$\Rightarrow$ If we minimize a strictly convex function, we are guaranteed to find the unique global minimizer

If x is a local minimizer of a convex function, it is a global minimizer.

Prove $f(x,y) = e^{x+y}$ is convex

Show $\nabla^2 f \geq 0$

$$\nabla f = \begin{pmatrix} e^{x+y} \\ e^{x+y} \end{pmatrix}$$

$$\nabla^2 f(x,y) = \begin{pmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix}$$

$$\det\left[e^{x+y}\right] = e^{x+y} > 0$$

$$\det\begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{bmatrix} = \underline{0} \geq 0$$

$\Rightarrow \underline{\underline{e^{x+y}}}$ is convex.