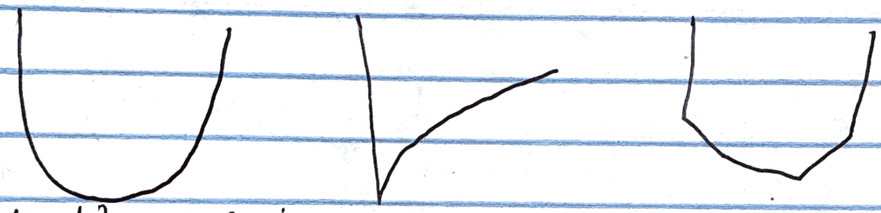
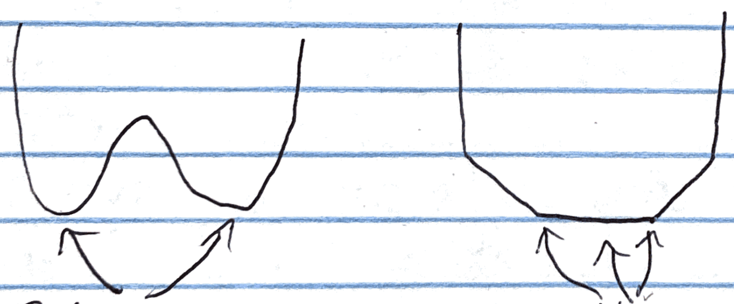
Golden Section Search

Basic idea: We have a function $f: \mathbb{R} \to \mathbb{R}$ which is unimodal, which means there is one local minimum

Example Unimodal Functions



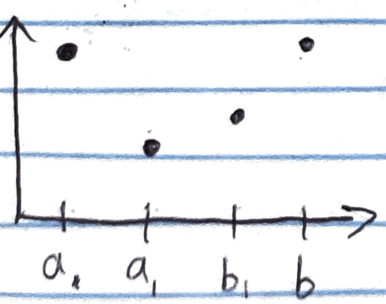
Not Unimodal



2 local mins

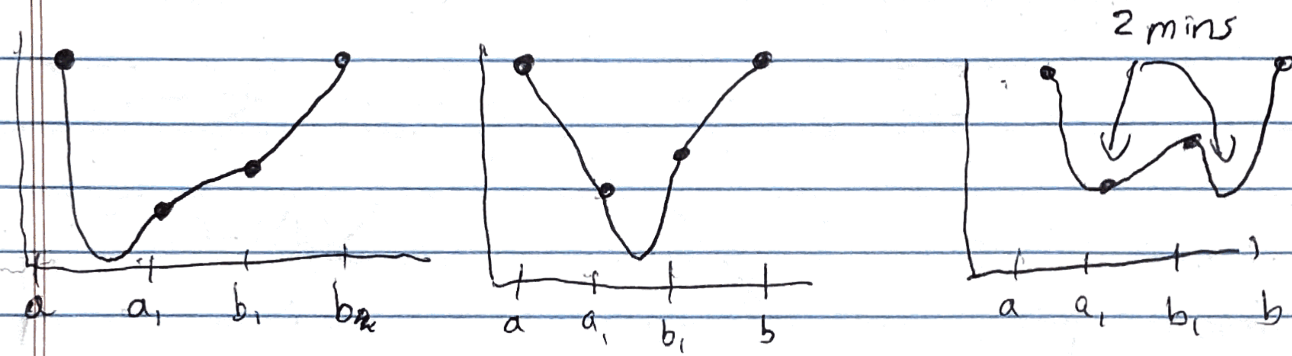
All points are local mins

Suppose we have interval $[a,b]$ containing the minimum. We evaluate at 2 additional points within $[a,b]$, a_1, b_1.



If $f(a_1) < f(b_1)$, we can rule out one of the intervals $[a,a_1]$, $[a_1,b_1]$, $[b_1,b]$, as not containing the minimum.

We try to draw a function with a local min in each interval. $[a, a_1]$ $[a_1, b_1]$ $[b_1, b]$



We cannot have a local min in $[b_1, b]$ without ~~creating~~ having at least 2 local minimums.
Hence we can rule out the interval $[b_1, b]$ as not containing the minimum.
~~Thus~~ Similarly if $f(a_1) > f(b_1)$, $[a, a_1]$ does not contain the min.
Then we have an algorithm starting with function f, points a, b.

Golden Section

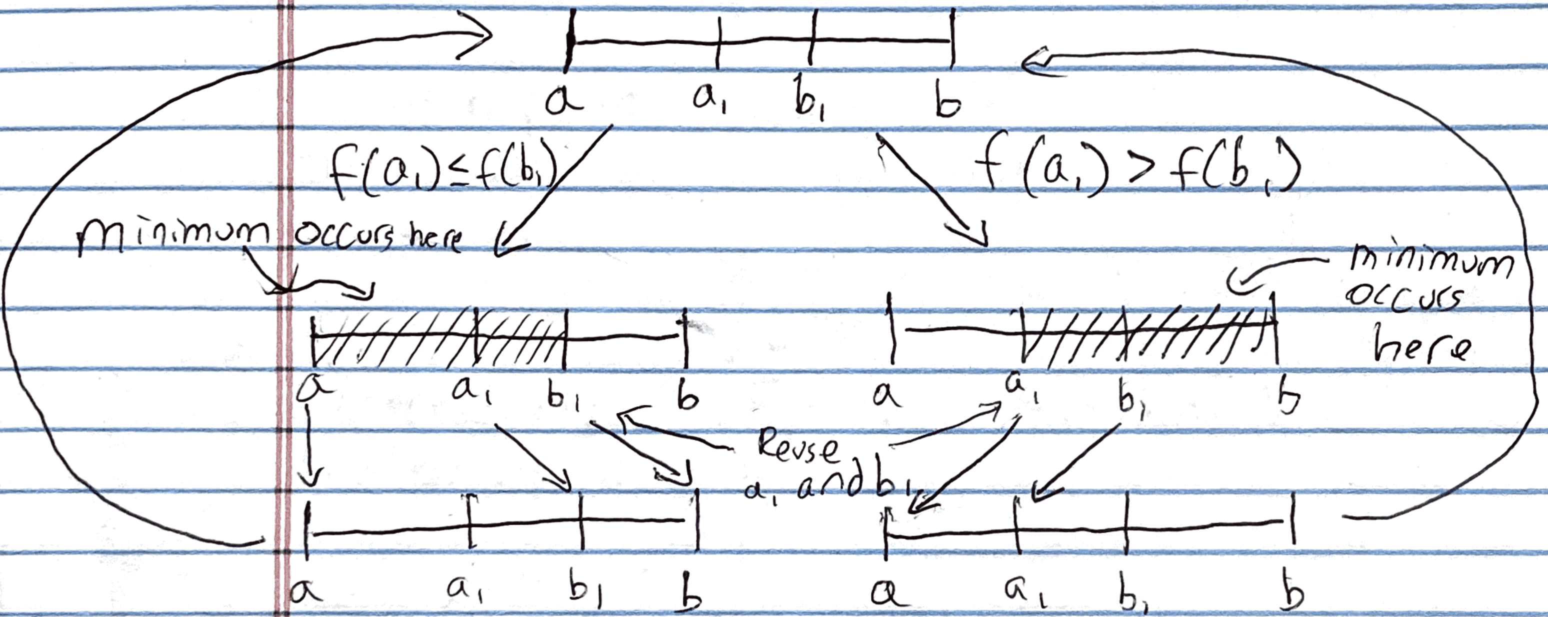
1. Pick points a_1, b_1
2. if $f(a_1) \le f(b_1)$
   - Go to 1 with new interval $[a, b_1]$

   else if $f(a_1) > f(b_1)$

   Go to 1 with new interval $[a_1, b]$

How to pick points a_1, b_1.

We would like to reuse points to reduce # function evals.



Repeat

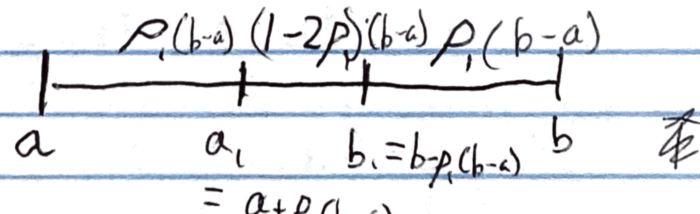
Let ρ_1 be proportion of interval to add to a to find a_1:

$$a_1 = a + \rho_1 (b-a)$$

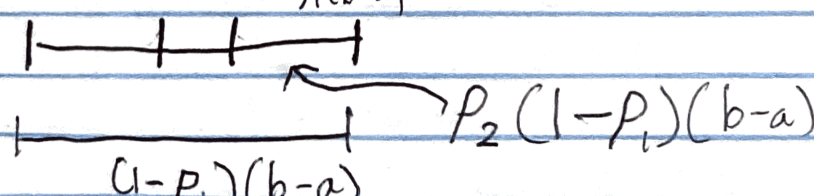
Similarly, we want b_1 to be symmetrically placed:

$$b_1 = b - \rho_1 (b-a)$$

First Division:



Second Division:



To have intervals line up, need $\rho_1(b-a) = \rho_2(1-\rho_1)(b-a)$

$\Rightarrow p_1 = p_2(1-p_1) \Rightarrow p_2 = 1 - \frac{p_1}{1-p_1}$

More generally, $p_{K+1} = 1 - \frac{p_K}{1-p_K}$

If we require $p_{K+1} = p_K$, $\forall K$,
then solving $p = 1 - \frac{p}{1-p}$ gives

$p = \frac{3-\sqrt{5}}{2} = 2-\varphi$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Alternatively, the Fibonacci method with N steps is given by

$$p_1 = 1 - \frac{F_N}{F_{N+1}}$$

$$p_2 = 1 - \frac{F_{N-1}}{F_N}$$

$$\vdots$$

$$p_N = 1 - \frac{F_1}{F_2} - \varepsilon \quad \text{for } \varepsilon > 0.$$

So our Final algorithm is

Start with f, $[a,b]$

1. Set $a_1 = a + p(b-a)$
   $b_1 = b - p(b-a)$
   with p given by

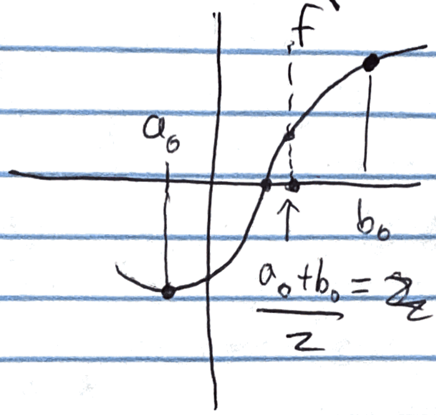
one of these

2. If $f(a) \leq f(b)$, repeat with new interval $[a, b_1]$
   else if $f(a) > f(b)$, repeat with new interval $[a_1, b]$

Root Finding methods of f'

~~Find $f'(x)$~~ Find x s.t. $f'(x) = 0$

Bisection Method



Suppose we have

$f'(a) < 0 < f'(b)$

Then by IVT, $\exists c$ btwn a and b with $f'(c) = 0$

Consider $c = \frac{a_0 + b_0}{2}$,

if $f'(c) > 0$, new interval is $[a_0, c]$
if $f'(c) < 0$, new interval is $[c, b_0]$
if $f'(c) = 0$, we're done.

Reduction Factor $\left(\frac{1}{2}\right)^N$

Golden Section Search	$\approx 0.618^N$
Fibonacci	$\frac{1+2\varepsilon}{F_{N+1}}$
Bisection	$\left(\frac{1}{2}\right)^N$

Why use these

$f(x) = (e^x - 2)^2$ on $[0, 5]$

How many iterations under each method to get an interval of size 0.001