Subleties in Optimality conditions

Warmup Exercise:

Find example 1D functions $f : \mathbb{R} \to \mathbb{R}$ ~~such that~~ and points $x \in \mathbb{R}$, such that

1. x satisfies FONC ($f'(x) = 0$) but is not local minimizer
2. x satisfies SONC ($f'(x) = 0, f''(x) \geq 0$) but is not a local minimizer
3. x is a local minimizer but does not satisfy SOSC ($f'(x) = 0, f''(x) > 0$)

Hint: For all these, think polynomials

1. $f(x) = -x^2$, $x = 0$
2. $f(x) = x^3$, $x = 0$
3. $f(x) = x^4$, $x = 0$

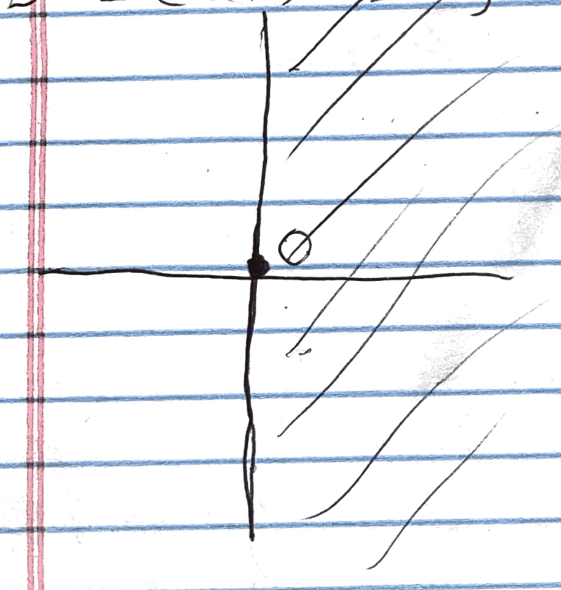
~~Boundary Case~~

~~1. Satisfies FONC but not optimizer~~

Reminder

	Interior	Boundary (General)
FONC	$\nabla f(x) = 0$	$d^T \nabla f(x) \geq 0$ for feasible d
SONC	$\nabla f(x) = 0$, $\nabla^2 f(x) \geq 0$	$d^T \nabla f(x) \geq 0$ for feasible d, if $= 0$, $d^T \nabla^2 f(x) d \geq 0$
SOSC	$\nabla f(x) = 0$, $\nabla^2 f(x) > 0$	see problem 6.32

$\Omega = \{(x,y) : x \geq 0\}$ $f(x,y) = x - y^2$



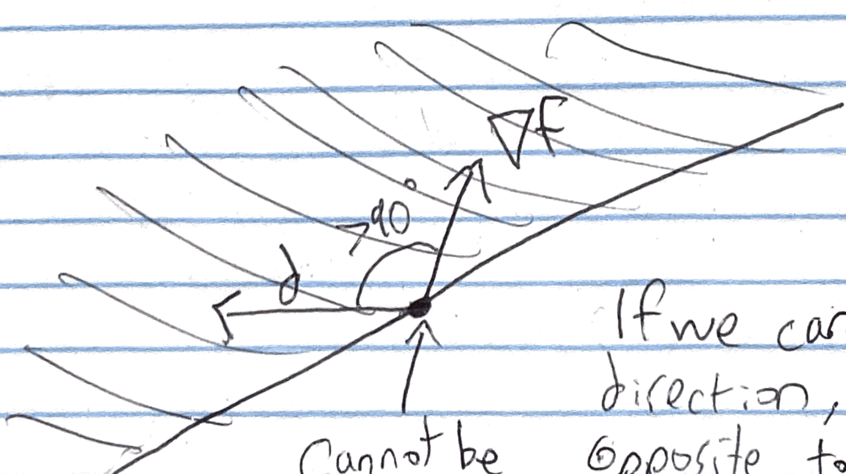
$$\nabla f = \begin{pmatrix} 1 \\ -2y \end{pmatrix} \qquad \nabla f(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\nabla^2 f = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= -2 < 0$$

FONC in other words:



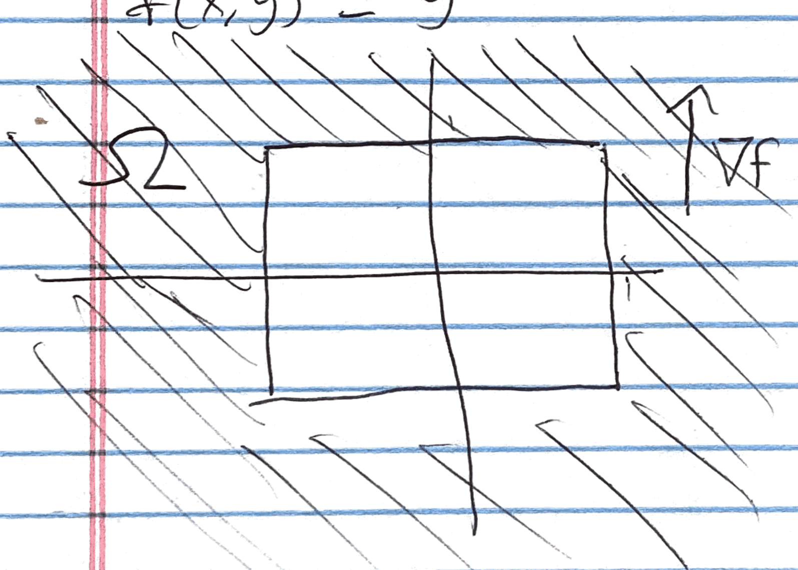
Cannot be a Minimizer

If we can find a feasible direction, pointing opposite to gradient, it is not a local minimizer

Problem

$\Omega = \{(x,y) \in \mathbb{R}^2 : x \leq -1$ or $x \geq 1$
$y \leq -1$ ~~and~~ or $y \geq 1\}$

$f(x,y) = y$



What points satisfy FONC?

$$\nabla f(x,y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Cases: 1) Interior point - does not satisfy FONC since $\nabla f \neq 0$ everywhere

2) Left, bottom, right boundaries

$d = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is a feasible direction and $d^T \nabla f(x) = -1 < 0$ so does not satisfy FONC

3) top side (minus corners) - only feasible directions are $d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ with $d_2 \geq 0$ so

$d^T \nabla f(x) = d_2 \geq 0$. So it satisfies FONC

These points <u>are</u> local minimizers since all neighbors are above or at same y-value.

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \geq 1\} \quad f(x,y) = y$$

Only point satisfying FONC is $(0,1)$.

but it is not a minimizer because every neighborhood contains points below it