Discussion 10: Duality

- Constrained Optimization

Motivation:

Suppose we have LP

$$\min \quad 5x_1 + 8x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 = 4$$
$$3x_1 + 4x_2 = 5$$
$$x_1, x_2 \geq 0$$

We want to find lower bound on the optimal value.

Idea: Add equations

$$x_1 + 2x_2 = 4$$
$$+\ 3x_1 + 4x_2 = 5$$

$$f(x) = 5x_1 + 8x_2 \geq 4x_1 + 6x_2 = 9$$

So a lower bound on our LP is 9.

We can do better,

multiply one equation by λ_1, another by λ_2 and add

$$\lambda_1 \cdot (x_1 + 2x_2 = 4)$$
$$\lambda_2 \cdot (3x_1 + 4x_2 = 5)$$

$$f(x) = 5x_1 + 8x_2 \geq (\lambda_1 + 3\lambda_2)x_1 + (2\lambda_1 + 4\lambda_2)x_2 = 4\lambda_1 + 5\lambda_2$$

When does this inequality hold?

When

$$\lambda_1 + 3\lambda_2 \le 5$$
$$2\lambda_1 + 4\lambda_2 \le 8$$
$$\lambda_1, \lambda_2 \in \mathbb{R}$$

To get best lower bound

$$\max \; 4\lambda_1 + 5\lambda_2$$

If we change the equality constraints to inequality,
What happens if λ is negative?

$$\lambda_1(x_1 + 2x_2 \ge 4) \Rightarrow \lambda_1 x + 2\lambda_1 x_2 \le 4\lambda_1$$
$$+ \quad \lambda_2(3x_1 + 4x_2 \ge 5)$$

not a valid lower bound

Hence if we have inequality constraints

$$\min \; 5x_1 + 8x_2$$
$$x_1 + 2x_2 \ge 4$$
$$3x_1 + 4x_2 \ge 5$$
$$x_1, x_2 \ge 0$$

The dual requires the associated dual variables to be nonnegative

$$\max \; 4\lambda_1 + 5\lambda_2$$
$$\text{s.t.} \quad \lambda_1 + 3\lambda_2 \le 5$$
$$2\lambda_1 + 4\lambda_2 \le 8$$
$$\lambda_1, \lambda_2 \ge 0$$

In general, very important

Each variable in primal is associated with constraint in dual. Each constraint is associated with variable

Primal $\longleftrightarrow$ Dual

Variables $\longleftrightarrow$ Constraints

Constraints $\longleftrightarrow$ Variables

Primal	Dual
$x \geq 0$	$\leq$ constraint
$x \in \mathbb{R}$	Equality Constraint
$\geq$ Constraint	$\lambda \geq 0$
$=$ constraint	$\lambda \in \mathbb{R}$.

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } Ax \geq b \\ &\quad\ x \geq 0 \end{aligned} \longrightarrow \begin{aligned} &\max\ b^T\lambda \\ &A^T\lambda \leq c \quad (\text{or } \lambda^T A \leq c^T) \\ &\lambda \geq 0 \end{aligned}$$

Primal

$$\begin{aligned} \min\ & 2x_1 + 3x_2 \\ \text{s.t. } & 4x_1 + 5x_2 \geq 1 \quad \lambda_1 \\ & 2x_1 + 2x_2 = 2 \quad \lambda_2 \\ & 3x_1 = 3 \quad \lambda_3 \\ & x_1 \in \mathbb{R},\ x_2 \geq 0 \end{aligned}$$

Dual

Q: How many constraints in dual?

$$\begin{aligned} \max\ & 1\lambda_1 + 2\lambda_2 + 3\lambda_3 \\ & 4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 2 \\ & 5\lambda_1 + 2\lambda_2 + 0\lambda_3 \leq 3 \\ & \lambda_1 \geq 0,\ \lambda_2 \in \mathbb{R},\ \lambda_3 \in \mathbb{R} \end{aligned}$$

Weak Duality

Any feasible solution to primal has larger value then any solution to dual

For any x, primal feasible, λ dual feasible

$$c^T x \geq b^T \lambda$$

Strong Duality; If x^*, λ^* are optimal

$$c^T x^* = b^T \lambda^*$$

~~Questions:~~

They meet at optimal value.

Possible Dual obj. values	Possible primal obj. values
$b^T\lambda$	$c^T x$

Questions: If dual problem is unbounded, what can we say about primal?

Analogous statement for primal.

4 possible cases:

1) Primal and dual both feasible, same optimal value
2) primal unbounded, dual infeasible
3) primal infeasible, dual unbounded
4) primal infeasible, dual infeasible.

Try to think of an example of this

Interpretation of Dual values

$f(x^*) = c^T x^* = b^T \lambda^* = b_1 \lambda_1^* + \dots + b_m \lambda_m^*$

If we adust b_1 our objective value changes by λ^*

$$\frac{\partial F}{\partial b_1} \approx \lambda_1^*$$

If we relax ~~our~~ constraint 1 a little, we can increase value of our function by λ_1^*