

Discussion 10: Duality

- Constrained Optimization

Motivation:

Suppose we have LP

$$\min \quad 5x_1 + 8x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 = 4$$

$$3x_1 + 4x_2 = 5$$

$$\underline{x_1, x_2 \geq 0}$$

We want to find lower bound on the optimal value.

Idea: Add equations

$$x_1 + 2x_2 = 4$$

$$+ 3x_1 + 4x_2 = 5$$

$$f(x) = 5x_1 + 8x_2 \geq 4x_1 + 6x_2 = 9$$

So a lower bound on our LP is 9.

We can do better,

multiply one equation by λ_1 , another by λ_2
and add

$$\lambda_1 \cdot (x_1 + 2x_2 = 4)$$

$$\lambda_2 \cdot (3x_1 + 4x_2 = 5)$$

$$f(x) = 5x_1 + 8x_2 \geq (\lambda_1 + 3\lambda_2)x_1 + (2\lambda_1 + 4\lambda_2)x_2 = 4\lambda_1 + 5\lambda_2$$

When does this inequality hold?

When

$$\lambda_1 + 3\lambda_2 \leq 5$$

$$2\lambda_1 + 4\lambda_2 \leq 8$$

$$\lambda_1, \lambda_2 \in \mathbb{R}$$

To get best lower bound

$$\max 4\lambda_1 + 5\lambda_2$$

If we change the equality constraints to inequality, what happens if λ is negative?

$$\begin{aligned} & \lambda_1 (x_1 + 2x_2 \geq 4) \Rightarrow \lambda_1 x_1 + 2\lambda_1 x_2 \leq 4\lambda_1 \\ + & \lambda_2 (3x_1 + 4x_2 \geq 5) \Rightarrow 3\lambda_2 x_1 + 4\lambda_2 x_2 \leq 5\lambda_2 \end{aligned}$$

↑
Not a valid lower bound

Hence if we have inequality constraints

$$\min 5x_1 + 8x_2$$

$$x_1 + 2x_2 \geq 4$$

$$3x_1 + 4x_2 \geq 5$$

$$x_1, x_2 \geq 0$$

The dual requires the associated dual variables to be nonnegative

$$\max 4\lambda_1 + 5\lambda_2$$

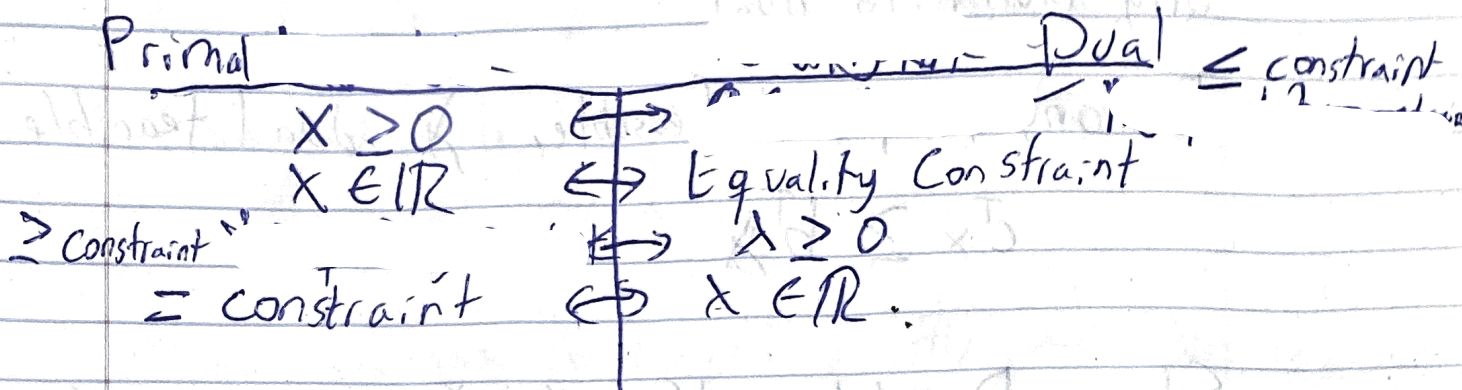
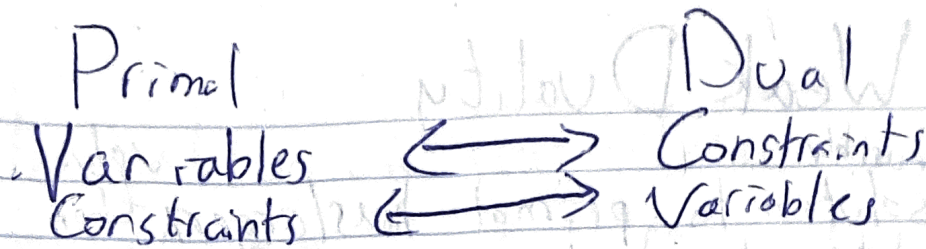
$$\text{s.t. } \lambda_1 + 3\lambda_2 \leq 5$$

$$2\lambda_1 + 4\lambda_2 \leq 8$$

$$\lambda_1, \lambda_2 \geq 0$$

In general, very important

Each variable in primal is associated with constraint in dual. Each constraint is associated with variable



$$\begin{array}{l} \min c^T x \\ \text{s.t. } Ax \geq b \\ x \geq 0 \end{array} \rightarrow \begin{array}{l} \max b^T \lambda \\ A^T \lambda \leq c \quad (\text{or } A^T \lambda \leq c^T) \\ \lambda \geq 0 \end{array}$$

Primal		Possible Dual constraints
min $2x_1 + 3x_2$		
s.t. $4x_1 + 5x_2 \geq 1$	λ_1	
$2x_1 + 2x_2 = 2$		
$3x_1 = 3$	λ_3	
$x_1 \in \mathbb{R}, x_2 \geq 0$		

Dual Q: How many constraints in dual?

$$\begin{array}{l} \max 1\lambda_1 + 2\lambda_2 + 3\lambda_3 \\ 4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 2 \\ 5\lambda_1 + 2\lambda_2 + 0\lambda_3 \leq 3 \end{array}$$

$$x_1 \geq 0, \lambda_2 \in \mathbb{R}, \lambda_3 \in \mathbb{R}$$

Weak Duality

Any ^{feasible} solution to primal has ^{value} larger or equal than any solution to dual

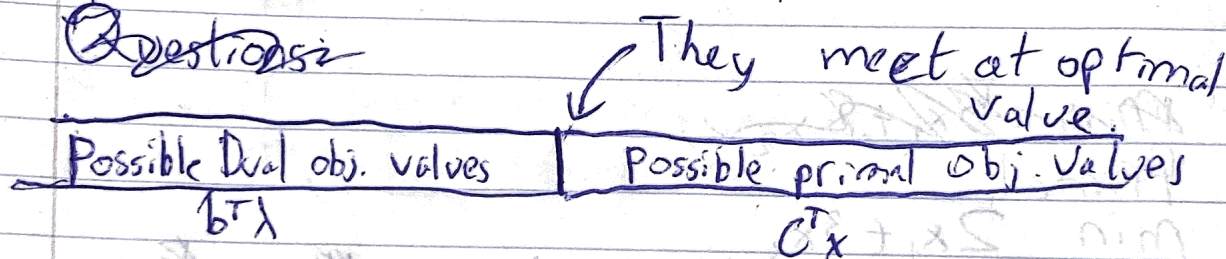
For any x , primal feasible, & dual feasible

$$c^T x \geq b^T \lambda$$

Strong Duality; If x^* , λ^* are optimal

$$c^T x^* = b^T \lambda^*$$

Questions:



Questions: If dual problem is unbounded, what can we say about primal?

Analogous statement for primal.

4 possible cases:

- 1) Primal and dual both feasible, same optimal value
- 2) Primal unbounded, dual infeasible
- 3) primal infeasible, dual unbounded
- 4) primal infeasible, dual infeasible

Try to think of an example of this

Interpretation of Dual values

$$f(x^*) = c^T x^* = b^T \lambda^* = b_1 \lambda_1^* + \dots + b_m \lambda_m^*$$

If we adjust b_i , our objective value changes by λ_i^*

$$\frac{\partial F}{\partial b_i} \approx \lambda_i^*$$

If we relax our constraint i a little, we can increase value of our function by λ_i^*