

Weighted ℓ_1 -Minimization for Sparse Recovery under Arbitrary Prior Information

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Abstract

Weighted ℓ_1 -minimization has been studied as a technique for the reconstruction of a sparse signal from compressively sampled measurements when prior information about the signal, in the form of a support estimate, is available. In this work, we study the recovery conditions and the associated recovery guarantees of weighted ℓ_1 -minimization when arbitrarily many distinct weights are permitted. For example, such a setup might be used when one has multiple estimates for the support of a signal, and these estimates have varying degrees of accuracy. Our analysis yields an extension to existing works that assume only a single constant weight is used. We include numerical experiments, with both synthetic signals and real video data, that demonstrate the benefits of allowing non-uniform weights in the reconstruction procedure.

Index Terms

Compressed sensing, weighted ℓ_1 -minimization, restricted isometry property

1 Introduction

Compressed sensing is a recently developed paradigm for the effective acquisition of sparse signals via few nonadaptive, linear measurements (e.g., see [19, 10, 9]). Specifically, we acquire

$$y = Ax + z, \quad (1)$$

where $x \in \mathbb{R}^n$ is an unknown s -sparse signal (that is, $\|x\|_0 = |\{i : x_i \neq 0\}| \leq s$ such that $s \ll n$), A is a known $m \times n$ measurement matrix, and $z \in \mathbb{R}^m$ is measurement noise where $\|z\|_2 \leq \epsilon$. Given knowledge of the noisy observations $y \in \mathbb{R}^m$ and the matrix A , we are interested in obtaining an estimate of x in the case where m is smaller than n . A common approach to this task is solving the convex optimization program

$$\min_{\tilde{x}} \|\tilde{x}\|_1 \text{ subject to } \|A\tilde{x} - y\|_2 \leq \epsilon, \quad (2)$$

where $\|\tilde{x}\|_1 = \sum_{i=1}^n |x_i|$ in the objective promotes sparsity and the constraint ensures fidelity to the model. The problem (2) can be recast as a linear program and can thus be solved efficiently

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in polynomial-time complexity [5]. It was shown in [10] that if A satisfies a certain restricted isometry property (which holds for certain random matrices with high probability whenever $m \gtrsim s \log(n/s)$) then the ℓ_1 -minimization in (2) can stably and robustly recover x from the potentially noisy measurements y .

In using (2) for reconstructing x , all the indices $i \in \{1, \dots, n\}$ are treated equally. That is, (2) does not make use of any information or assumptions on the support of x . In many applications, however, additional prior knowledge about the signal support may be available. For instance, in the acquisition of video or audio signals there is often high correlation from one frame to the next, suggesting that the information learned in the previous frames should be exploited in the acquisition of subsequent frames. In applications dealing with natural images, the signal may often be expected to follow a certain structured sparsity model which can be utilized by the recovery algorithm (e.g., [2]). A *weighted* version of the ℓ_1 -minimization in (2) has been studied to exploit prior information during the signal reconstruction (see Section 1.1 for a discussion of prior work). More precisely, suppose $\tilde{T} \subset \{1, \dots, n\}$ is a prior support estimate. Then the weighted ℓ_1 -minimization problem can be formulated as

$$\min_{\tilde{x}} \|\tilde{x}\|_{1, \vec{w}} \text{ subject to } \|A\tilde{x} - y\|_2 \leq \epsilon, \text{ where } \|x\|_{1, \vec{w}} = \sum_{i=1}^n \mathbf{w}_i |x_i| \text{ and } \mathbf{w}_i = \begin{cases} w \in [0, 1] & i \in \tilde{T} \\ 1 & i \in \tilde{T}^c \end{cases}. \quad (3)$$

Here, $\vec{w} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n]$ is the vector of weights (with the only distinct weights being either w or 1). Of course, (3) reduces to (2) when $w = 1$. The intuition behind this formulation is that selecting a weight w less than 1 will encourage nonzero entries on \tilde{T} in the minimizer of (3). If \tilde{T} is an accurate estimate of the true support of x , this weighted procedure has been shown to outperform (2), see e.g. [29] and references therein.

A limitation of (3) and much of the corresponding theory is that the weight w is the same on the entire set \tilde{T} . It is plausible that a practitioner may not have the same level of confidence on the entire set \tilde{T} depending on the type of prior information available, which suggests that allowing distinct weights on different pieces of \tilde{T} may be desirable. In other settings, a statistical prior on the signal may be known, providing probabilities on each entry being in the signal support; this information should also be leveraged by using non-uniform weights. Although the formulation and implementation of (3) can be easily modified to capture this feature, the modification to the theoretical analysis is less straight-forward. In this work, we provide a generalized theory studying the recovery conditions and error guarantees associated with weighted ℓ_1 -minimization, when *arbitrary* weight assignments are permitted.

1.1 Prior Work

Compressed sensing in the presence of prior information has previously been studied under various models. For example, the paper [16], which we believe to be the first of its kind, considers prior information in the form of a similar signal known beforehand. The authors propose solving a minimization problem similar to (2), but where the function being minimized includes two terms: one for measuring the sparsity of the recovered signal and the other for measuring the sparsity of the difference between the recovered signal and the prior known signal. Along the same line, the authors of [33] study a modification of the ℓ_1 -minimization (2) by minimizing either $\|\tilde{x}\|_1 + \beta \|\tilde{x} - x'\|_1$ or $\|\tilde{x}\|_1 + \frac{\beta}{2} \|\tilde{x} - x'\|_2^2$, where x' denotes the similar, previously known signal, and $\beta > 0$ is a parameter that establishes the tradeoff between the signal sparsity and the fidelity to the prior signal x' . Similarly, in the context of longitudinal Magnetic Resonance Imaging (MRI) [40] assumes knowledge of a previous MRI scan. Their proposed minimization is similar to that studied in [16] and [33], but

the authors also introduce weights in both terms of the objective function. Their final proposed scheme (which is not studied analytically) solves the optimization iteratively, with the possibility of adaptively selecting measurements at each iteration.

Another common type of prior information studied is in the form of a support estimate. The paper [39] assumes a support estimate \tilde{T} , and then minimizes $\|\tilde{x}_{\tilde{T}^c}\|_1$, where $\tilde{x}_{\tilde{T}^c}$ denotes the signal \tilde{x} restricted to the indices on the complement of \tilde{T} with zeros elsewhere, which is equivalent to (3) with $w = 0$. In the noise-free setting, the authors show that when \tilde{T} is a reasonable estimate of the true support, the recovery conditions are milder than those needed for classical compressed sensing. The work [4] considers prior information on the support of the discrete Fourier transform of the signal. Using a similar method as in [39] with noise-free measurements, the authors show empirically that a reduced number of measurements is required for recovery. Assuming partially known support information, the authors of [28] also propose a modification to ℓ_1 -minimization similar to [39] and provide experimental results in the application of dynamic MRI. The authors of [23] analyze the weighted ℓ_1 -minimization problem (3) under a *restricted isometry property* on the sensing matrix (see Section 2), thus generalizing the results of [10] to the weighted case. The authors show that if at least 50% of the estimated support \tilde{T} is correct, then weighted ℓ_1 -minimization is stable and robust under weaker sufficient conditions than those for standard ℓ_1 -minimization. As an extension to [23], an analysis of weighted ℓ_1 -minimization with multiple support estimate sets with distinct weights, which is the focus of this paper, is provided in [30]; however, we will demonstrate in Section 3 that the main result of [30] only provides a sub-optimal generalization to [23], which we remedy here. The analysis in [29] studies the noise-free weighted ℓ_1 -minimization problem in the presence of a support estimate \tilde{T} , but, in contrast to [23], the sensing matrix is assumed to possess a *null space property* (see [29] for details). Exact recovery conditions of the noise-free version of (3) under a null space property and the restricted isometry property are also studied in [41]. In a separate but related direction, modifications of *greedy* algorithms for compressed sensing have also been studied under the assumption of a partially known support. The work [14] proposes a modification of the IHT [3] algorithm to incorporate partially known support information, with a theoretical bound on the reconstruction error provided. Similarly, [13] proposes a modification of the OMP [38], CoSaMP [34], and re-weighted least squares [15] algorithms, and [35] proposes a modification of the BIHT [24] algorithm for one-bit compressed sensing to incorporate partially known support information.

As an alternative to prior support information, the papers [25, 26, 27] assume a non-uniform sparsity model and analyze the noise-free weighted ℓ_1 -minimization while allowing for non-uniform weights. Specifically, the authors consider a model where the entries of the unknown signal fall into two (or more) sets, each with a different probability of being nonzero. Indices within the same set would then get assigned the same weight. The study of this type of model is further generalized in [31]. The prior information studied in [37] is also in the form of probabilities that each entry of the signal is nonzero (i.e., a prior distribution). They study information-theoretic limits on the number of (noisy) measurements needed to recover the support set exactly, and show that significantly fewer measurements can be used if the prior distribution is sufficiently non-uniform.

Other relevant works include [26, 31, 18], which propose methods for determining optimal weights. We also mention that perhaps the first study of a weighted ℓ_1 -minimization approach was in [12]; the algorithm proposed there does not assume any prior information, but consists of solving a sequence of weighted ℓ_1 -minimization problems where the weights for the next iteration are determined from the solution of the previous iteration. Precise recovery guarantees for this scheme have remained elusive, but it is possible the analysis we present here may lend further insight into this related approach.

1.2 Contribution

We study the weighted ℓ_1 -minimization problem (3) in its full generality. That is, unlike in many of the prior works mentioned, we allow completely arbitrary and non-uniform (non-negative) weights rather than a constant weight w . We derive stability and robustness guarantees for weighted ℓ_1 -minimization that generalize the results of [23], further generalize the results of [10], and improve upon the results of [30], thus providing an extension to the existing literature. Our main technical result is Theorem 2, and in the ensuing discussion in Section 3, we compare the theoretical results associated with using a single weight to the general ones derived in this paper. We highlight scenarios under which the sufficient conditions associated with our “multi-weight” scenario are weaker than those associated with applying a single weight to the support estimate in (3), suggesting a practical benefit to the use of multiple weights. Indeed, we demonstrate using extensive numerical experiments that allowing arbitrary weights can often outperform weighted ℓ_1 -minimization with a constant weight.

1.3 Organization

The remainder of this paper is organized as follows. In Section 2 we recall the results on weighted ℓ_1 -minimization (3) when a constant weight w is used for signal reconstruction in the presence of a support estimate \tilde{T} , which reduce to the existing results for compressed sensing when ℓ_1 -minimization (2) is used for signal reconstruction without any prior information. In Section 3 we present our main theoretical results along with their proofs, thus providing a generalized and improved theory of weighted ℓ_1 -minimization when non-uniform weights are permitted. Numerical experiments involving synthetic and real signals are provided in Section 4. We conclude with a brief discussion in Section 5.

2 Existing Results for ℓ_1 -Minimization and Weighted ℓ_1 -Minimization with a Single Weight

As mentioned earlier, in [10] it was shown that the ℓ_1 -minimization problem (2), which utilizes no prior information about the signal, can stably and robustly recover x from the noisy measurements $y = Ax + z$ as long as A satisfies a particular property. The now well-known condition required on A is termed the *restricted isometry property* (RIP), and is defined in Definition 1 below.

Definition 1. An $m \times n$ matrix A is said to possess the RIP with s -restricted isometry constant $\delta_s < 1$ if δ_s is the smallest positive number such that

$$(1 - \delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s)\|x\|_2^2 \quad (4)$$

holds for all s -sparse vectors x .

Matrices constructed with independent and identically distributed standard Gaussian entries or rows subsampled from the discrete Fourier transform matrix are examples of matrices known to satisfy the RIP when $m \gtrsim s \log^a(n/s)$, for some constant $a \geq 1$ [11, 36].

The main result of [23] generalizes the recovery condition from [10] to the weighted ℓ_1 -minimization problem (3) where the constant weight w is applied on all of \tilde{T} . Theorem 1 below states the main result of [23], which reduces to the result from [10] when $w = 1$, or when \tilde{T} is empty (see [23] for details).

Theorem 1. (Friedlander et al. [23]) *Let $x \in \mathbb{R}^n$ and let x_s denote its best s -term approximation, supported on T_0 . Let $\tilde{T} \subset \{1, \dots, n\}$ be an arbitrary set and define ρ and α such that $|\tilde{T}| = \rho s$ and $\frac{|\tilde{T} \cap T_0|}{|\tilde{T}|} = \alpha$. Suppose that there exists an $a \in \frac{1}{s}\mathbb{Z}$ with $a \geq (1 - \alpha)\rho$, $a > 1$, and the measurement matrix A has RIP with*

$$\delta_{as} + \frac{a}{b^2} \delta_{(a+1)s} < \frac{a}{b^2} - 1 \quad (5)$$

where

$$b = b(w, \rho, \alpha) := w + (1 - w)\sqrt{1 + \rho - 2\alpha\rho} \quad (6)$$

for some given $0 \leq w \leq 1$. Then the solution \hat{x} to (3) obeys

$$\|\hat{x} - x\|_2 \leq C_0 \epsilon + C_1 s^{-1/2} \left(w \|x - x_s\|_1 + (1 - w) \|x_{\tilde{T}^c \cap T_0^c}\|_1 \right), \quad (7)$$

where C_0 and C_1 are well-behaved constants that depend on the measurement matrix A , the weight w , and the parameters ρ and α .

Remark 1. The constants C_0 and C_1 are explicitly given by

$$C_0 = \frac{2(1 + \frac{b}{\sqrt{a}})}{\sqrt{1 - \delta_{(a+1)s}} - \frac{b}{\sqrt{a}} \sqrt{1 + \delta_{as}}}, \quad C_1 = \frac{2a^{-1/2}(\sqrt{1 - \delta_{(a+1)s}} + \sqrt{1 + \delta_{as}})}{\sqrt{1 - \delta_{(a+1)s}} - \frac{b}{\sqrt{a}} \sqrt{1 + \delta_{as}}}. \quad (8)$$

Remark 2. Note that the classical result of [10] for un-weighted ℓ_1 -minimization (with $w = 1$ and $b = 1$) is proved using the condition (5) with $a = 3$, yielding the requirement $\delta_{3s} + 3\delta_{4s} < 2$. Thus, Theorem 1 requires a weaker RIP assumption than the classical un-weighted case for accurate support estimates (e.g. if $\rho = 1$ and $\alpha > 1/2$). Note also that the classical condition on the RIP constant has been improved several times [8, 1, 6, 7, 21, 22, 32]; although a version of Theorem 1 and the main results of this paper can likely be extended to the theory of these works, we do not pursue such refinements here.

Remark 3. Since $\delta_{as} < \delta_{(a+1)s}$, a sufficient condition for (5) to hold is

$$\delta_{(a+1)s} < \frac{a - b^2}{a + b^2} := \delta^b. \quad (9)$$

3 Weighted ℓ_1 -Minimization with Non-uniform Weights

In this section, we present our main results for generalizing the weighted ℓ_1 -minimization theory of [23], and improving the theory of [30], to allow for arbitrary weight assignments. Our main theorem is provided in Section 3.1, and Section 3.2 details the proof of this result.

3.1 Weighted ℓ_1 -Minimization with N Distinct Weights

We consider weighted ℓ_1 -minimization with N distinct weights, where $1 \leq N \leq n$. To that end, suppose we have N disjoint support estimates $\tilde{T}_i \subset \{1, \dots, n\}$, $i = 1, \dots, N$, where $|\tilde{T}_i| = \rho_i s$. Define the accuracy of the support estimates to be $\alpha_i = \frac{|\tilde{T}_i \cap T_0|}{|\tilde{T}_i|}$. Also define $\tilde{T} = \bigcup_{i=1}^N \tilde{T}_i \subset \{1, \dots, n\}$. Then the general weighted ℓ_1 -minimization is formulated as

$$\min_{\tilde{x}} \|\tilde{x}\|_{1, \tilde{\mathbf{w}}} \text{ subject to } \|A\tilde{x} - y\|_2 \leq \epsilon, \text{ where } \mathbf{w}_k = \begin{cases} w_i \in [0, 1] & k \in \tilde{T}_i \\ 1 & k \in \tilde{T}^c \end{cases}. \quad (10)$$

Our main result provides recovery guarantees for (10). As we will discuss below, Theorem 2 recovers the classical un-weighted and weighted results for the single weight case. More importantly, in the arbitrary weight case, we show that the RIP requirements instated here are strictly weaker than those in the classical settings when sufficiently accurate prior information is available.

Theorem 2. *Let $x \in \mathbb{R}^n$, let x_s denote its best s -sparse approximation, and denote the support of x_s by T_0 . Let $\tilde{T}_i \subseteq \{1, \dots, n\}$ for $i = 1, \dots, N$, where $1 \leq N \leq n$, be arbitrary disjoint sets and denote $\tilde{T} = \bigcup_{i=1}^N \tilde{T}_i$. Without loss of generality, assume that the weights in (10) are ordered so that $1 \geq w_1 \geq w_2 \geq \dots \geq w_N \geq 0$. For each i , define the relative size ρ_i and accuracy α_i via $|\tilde{T}_i| = \rho_i s$ and $\frac{|\tilde{T}_i \cap T_0|}{|\tilde{T}_i|} = \alpha_i$. Suppose that there exists $a > 1$, $a \in \frac{1}{s}\mathbb{Z}$ with $\sum_{i=1}^N \rho_i(1 - \alpha_i) \leq a$, and that the measurement matrix A has the RIP with*

$$\delta_{as} + \frac{a}{K_N^2} \delta_{(a+1)s} < \frac{a}{K_N^2} - 1, \quad (11)$$

where

$$K_N = K_N(w_1, \dots, w_N, \rho_1, \dots, \rho_N, \alpha_1, \dots, \alpha_N) \\ := w_N + (1 - w_1) \sqrt{1 + \sum_{i=1}^N (\rho_i - 2\alpha_i \rho_i)} + \sum_{j=2}^N \left((w_{j-1} - w_j) \sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha_i \rho_i)} \right). \quad (12)$$

Then the minimizer \hat{x} to (10) obeys

$$\|\hat{x} - x\|_2 \leq C'_0 \epsilon + C'_1 s^{-1/2} \left(\|x - x_s\|_1 \sum_{i=1}^N w_i + (1 - \sum_{i=1}^N w_i) \|x_{\tilde{T}^c \cap T_0^c}\|_1 - \sum_{i=1}^N \sum_{j=1, j \neq i}^N w_j \|x_{\tilde{T}_i \cap T_0^c}\|_1 \right) \quad (13)$$

where C'_0 and C'_1 are well-behaved constants that depend on the measurement matrix A , the weights w_i , and the parameters ρ_i and α_i for $i = 1, \dots, N$.

Remark 4. The constants C'_0 and C'_1 are explicitly given by

$$C'_0 = \frac{2(1 + \frac{K_N}{\sqrt{a}})}{\sqrt{1 - \delta_{(a+1)s}} - \frac{K_N}{\sqrt{a}} \sqrt{1 + \delta_{as}}}, \quad C'_1 = \frac{2a^{-1/2}(\sqrt{1 - \delta_{(a+1)s}} + \sqrt{1 + \delta_{as}})}{\sqrt{1 - \delta_{(a+1)s}} - \frac{K_N}{\sqrt{a}} \sqrt{1 + \delta_{as}}}. \quad (14)$$

Note that C'_0 and C'_1 are identical to C_0 and C_1 from Theorem 1, respectively, except that b is replaced by K_N . Therefore, $C'_0 \leq C_0$ and $C'_1 \leq C_1$ whenever $K_N \leq b$.

Remark 5. In Theorem 3.3 of [30] the authors define instead of our constant K_N the quantity

$$\gamma = \sum_{i=1}^N w_i - (N - 1) + \sum_{i=1}^N (1 - w_i) \sqrt{1 + \rho_i - 2\alpha_i \rho_i}. \quad (15)$$

Thus when $N = 1$ or when $w_i = 1$ for all i , the result of [30] indeed reduces to that in [23] and [10], respectively. However, consider the simple case when $N = 2$, $\alpha_1 = \alpha_2 = 1$, and $w_1 = w_2 = w$. Then we would expect γ to reduce to b for any ρ_1 and ρ_2 such that $\rho_1 + \rho_2 = \rho$ and $\alpha = 1$. However, in this setting, γ only reduces to b when $\rho_1 \rho_2 = 0$. Thus, the single weight result of [23] is not recovered as expected. This sub-optimal behavior is further illustrated in Figure 1 below.

Remark 6. If $w_i = 1$ for all i , then $K_N = 1$ and the result reduces to the ℓ_1 -minimization result without weights (see Theorem 2 in [23], which is from [10]). If $w_i = w$ and $\alpha_i = \alpha$ for all i , and $\sum_{i=1}^N \rho_i = \rho$, then the single weight result of [23] is recovered. Theorem 2 thus recovers classical results in the un-weighted and single weight cases.

Remark 7. To build intuition about the term K_N , we can consider the expression (12) for small N . When $N = 2$, we obtain,

$$K_2 = w_2 + (1 - w_1)\sqrt{1 + \rho_1 - 2\alpha_1\rho_1 + \rho_2 - 2\alpha_2\rho_2} + (w_1 - w_2)\sqrt{1 + \rho_2 - 2\alpha_2\rho_2}.$$

When $N = 3$, we obtain,

$$K_3 = w_3 + (1 - w_1)\sqrt{1 + \rho_1 - 2\alpha_1\rho_1 + \rho_2 - 2\alpha_2\rho_2 + \rho_3 - 2\alpha_3\rho_3} \\ + (w_1 - w_2)\sqrt{1 + \rho_2 - 2\alpha_2\rho_2 + \rho_3 - 2\alpha_3\rho_3} + (w_2 - w_3)\sqrt{1 + \rho_3 - 2\alpha_3\rho_3}.$$

If either $w_1 = w_2$ or $w_2 = w_3$, then K_3 reduces to K_2 , as desired.

We will see that small values of K_N relax the requirement on the RIP constant. Suppose that $N = 2$ and $\alpha_1 = \alpha_2 = 1$. Then

$$K_2 = w_2 + (1 - w_1)\sqrt{1 - \rho_1 - \rho_2} + (w_1 - w_2)\sqrt{1 - \rho_2}.$$

Suppose $1 \geq \rho_1 \gg \rho_2 \geq 0$. Then we would want to choose w_1 as small as possible (or as close to w_2 as possible) in order to minimize the dominating term $(w_1 - w_2)\sqrt{1 - \rho_2}$. Similarly, if $1 \geq \rho_2 \gg \rho_1 \geq 0$, then the dominating term is w_2 . In order to minimize K_2 , we would want to choose w_2 as small as possible. Thus we see that when the support estimates are accurate, larger values of ρ_1 or ρ_2 encourage smaller corresponding weights. We also see that smaller values of α_1 and α_2 would encourage larger weights (as well as $w_1 \approx w_2$). This also agrees with our intuition, in that we would want to select larger weights on inaccurate support estimates.

Remark 8. Since $\delta_{as} < \delta_{(a+1)s}$, a sufficient condition for (11) to hold is

$$\delta_{(a+1)s} < \frac{a - K_N^2}{a + K_N^2} := \delta^{K_N}. \quad (16)$$

Note that this is the same sufficient condition as seen in (9), except with b replaced by K_N .

To formalize the above remarks, we consider the simple case when all accuracies α_i are the same value, and show that as long as the accuracy level is high enough (greater than $1/2$ to be precise) and the weights are smaller than that used in the single weight case, that the RIP requirements of Theorem 2 are strictly weaker than previous results for the single weight case. The following Proposition shows that the smallest weight is most beneficial in relaxing the sufficient RIP condition, the largest weight is least beneficial, and a combination of weights in between produces an intermediate RIP condition. This matches intuition, since if the support estimate is accurate one of course should use aggressive (small) weights on that set to encourage non-zero entries in the recovery. On the other hand, if one is only confident about portions of the support, this proposition shows that by using non-uniform weights, one can do much better than simply selecting a single conservative weight.

Proposition 1. Define δ^b and δ^{K_N} as in (9) and (16), respectively. Let $w_1 \geq w_2 \geq \dots \geq w_N$, $\sum_{i=1}^N \rho_i = \rho$ and $\alpha_1 = \alpha_2 = \dots = \alpha_N = \alpha$. For fixed a , α , ρ , and ρ_i for $i = 1, \dots, N$, $\delta^b = \delta^b(w)$ and $\delta^{K_N} = \delta^{K_N}(w_1, \dots, w_N)$. Then $\delta^b(w_1) \leq \delta^{K_N}(w_1, \dots, w_N) \leq \delta^b(w_N)$ if and only if $\alpha \geq \frac{1}{2}$.

Proof. Define all terms as stated in the Proposition. Then, for fixed a , α , ρ , and ρ_i for $i = 1, \dots, N$,

$$b(w, \rho, \alpha) = b(w) \quad \text{and} \quad K_N(w_1, \dots, w_N, \rho_1, \dots, \rho_N, \alpha_1, \dots, \alpha_N) = K_N(w_1, \dots, w_N).$$

It is sufficient to show $b(w_N) \leq K(w_1, \dots, w_N) \leq b(w_1)$ if and only if $\alpha \geq \frac{1}{2}$. Since $w_1 \geq w_N$, it is quickly seen that $b(w_N) \leq b(w_1)$ if and only if $\alpha \geq \frac{1}{2}$. Next, observe that $K(w_1, \dots, w_N) \leq b(w_1)$ holds if and only if

$$w_N + (1 - w_1)\sqrt{1 + \rho - 2\alpha\rho} + \sum_{j=2}^N \left((w_{j-1} - w_j) \sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha\rho_i)} \right) \leq w_1 + (1 - w_1)\sqrt{1 + \rho - 2\alpha\rho}$$

which is equivalent to

$$\sum_{j=2}^N \left((w_{j-1} - w_j) \sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha\rho_i)} \right) \leq (w_1 - w_2) + (w_2 - w_3) + \dots + (w_{N-1} - w_N)$$

and to

$$(w_1 - w_2) \left(\sqrt{1 + \sum_{i=2}^N (\rho_i - 2\alpha\rho_i)} - 1 \right) + \dots + (w_{N-1} - w_N) \left(\sqrt{1 + \sum_{i=N}^N (\rho_i - 2\alpha\rho_i)} - 1 \right) \leq 0.$$

For the above inequality to hold, we must have $\sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha\rho_i)} - 1 \leq 0$ for some $j \in \{2, \dots, N\}$. However, this requirement for any j means $\sum_{i=j}^N (\rho_i - 2\alpha\rho_i) \leq 0$. For *any* terms in this sum to be zero or less, we must have that $\alpha \geq \frac{1}{2}$. Similarly, the inequality $b(w_N) \leq K(w_1, \dots, w_N)$ holds if and only if

$$w_N + (1 - w_N)\sqrt{1 + \rho - 2\alpha\rho} \leq w_N + (1 - w_1)\sqrt{1 + \rho - 2\alpha\rho} + \sum_{j=2}^N \left((w_{j-1} - w_j) \sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha\rho_i)} \right)$$

which is equivalent to

$$(w_1 - w_2 + w_2 - w_3 + \dots + w_{N-1} - w_N)\sqrt{1 + \rho - 2\alpha\rho} \leq \sum_{j=2}^N \left((w_{j-1} - w_j) \sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha\rho_i)} \right)$$

and to

$$\begin{aligned} 0 \leq (w_1 - w_2) & \left(\sqrt{1 + \sum_{i=2}^N (\rho_i - 2\alpha\rho_i)} - \sqrt{1 + \rho - 2\alpha\rho} \right) + \dots \\ & + (w_{N-1} - w_N) \left(\sqrt{1 + \rho_N - 2\alpha\rho_N} - \sqrt{1 + \rho - 2\alpha\rho} \right). \end{aligned}$$

For the above inequality to hold, we must have $\sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha\rho_i)} - \sqrt{1 + \rho - 2\alpha\rho} \geq 0$ for some $j \in \{2, \dots, N\}$. However, this requirement for any j means $1 + \sum_{i=j}^N (\rho_i - 2\alpha\rho_i) \geq 1 + \rho - 2\alpha\rho$ and thus $(\sum_{i=j}^N \rho_i - \rho)(1 - 2\alpha) \geq 0$. Since $\sum_{i=j}^N \rho_i \leq \rho$ for all $j = 2, \dots, N$, we must have $\alpha \geq \frac{1}{2}$. Note that these results are tight since $K(w_N, \dots, w_N) = b(w_N)$ and $K(w_1, \dots, w_1) = b(w_1)$. \square

Figure 1 compares the value of δ^b defined in (9) when a single weight is used versus δ^{K_N} defined in (16) when two or three distinct weights are used as a function of the support estimate sizes. We set $\alpha_i = \alpha = 1$ for $i = 1, \dots, N$ and $\sum_{i=1}^N \rho_i = \rho = 1$, where $N = 2$ for the plot on the left and $N = 3$ for the plot on the right. When $N = 2$, the horizontal lines indicate δ^b when the weight $w = 0.5$ or $w = 0.25$ is used on the entire support estimate; in between, we see the transition of δ^{K_2} as ρ_1 varies with $w_1 = 0.5$ and $w_2 = 0.25$. Note that although the horizontal axis only shows the value of ρ_1 , this determines ρ_2 since we take $\rho_1 + \rho_2 = 1$. For comparison, we also include the value of

$$\delta^\gamma := \frac{a - \gamma^2}{a + \gamma^2} \quad (17)$$

from [30], where γ is defined in (15). Indeed, δ^γ behaves as expected at the endpoints $\rho_1 = 0$ and $\rho_1 = 1$, but falls below δ^b with $w = 0.5$ for many values of $\rho_1 \in (0, 1)$. This again highlights the sub-optimality of the prior result [30] and the improvement offered by Theorem 2. When $N = 3$, we again see the transition in δ^{K_3} as ρ_1 and ρ_2 are varied (which again determines ρ_3) with $w_1 = 0.5$, $w_2 = 0.4$, and $w_3 = 0.25$. The value of δ^b when a single weight is used is included for comparison. In both cases, we see that the smallest weight results in the best (largest) RIP condition and the largest weight results in the worst (smallest) RIP condition since the accuracy is assumed to be perfect, and intermediate behavior is seen in between. This illustrates that our main result recovers the classical results in the single weight case, and that the case of multiple weights interpolates as expected.

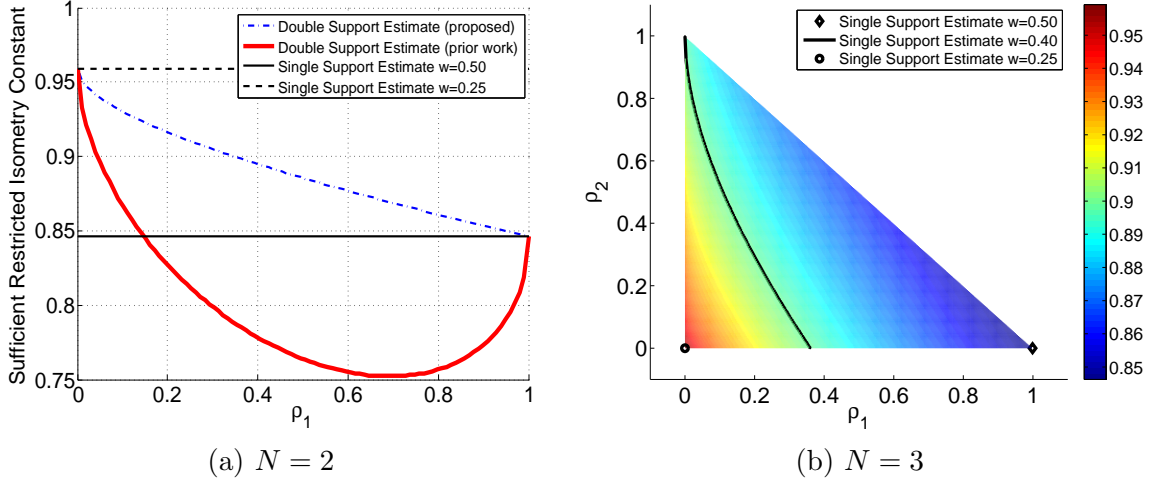


Figure 1: Comparison of δ^b (defined in (9)) and δ^{K_N} (defined in (16)) with $a = 3$. (a) We set $\alpha_1 = \alpha_2 = \alpha = 1$, $\rho_1 + \rho_2 = \rho = 1$, $w_1 = 0.5$, and $w_2 = 0.25$. The value of δ^{K_2} (dash-dotted line) is compared to δ^b in the single weight case with $w = 0.5$ (solid horizontal line) and $w = 0.25$ (dashed horizontal line). The value of δ^γ (from [30] and defined in (17)) is included for comparison (thick solid line). (b) We set $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 1$, $\rho_1 + \rho_2 + \rho_3 = \rho = 1$, $w_1 = 0.5$, $w_2 = 0.4$, and $w_3 = 0.25$. The value of δ^{K_3} (indicated by the color) is compared to δ^b in the single weight case with $w = 0.5$ (diamond marker), $w = 0.4$ (black line), and $w = 0.25$ (circle marker).

3.2 Proof of Main Result

We now present the proof of Theorem 2, which is inspired by that in [23] and [10]. Let $\hat{x} = x + h$ be the minimizer of (10), and let T_0 denote the set of the largest s coefficients of x in magnitude.

Our goal is to bound the norm of the error h .

Proof Roadmap. We will proceed with a sequence of lemmas, and then combine the results of the lemmas to obtain the final error bound. Briefly, the proof is organized as follows:

- *Lemma 1 - Cone Constraint:* The main challenge of the proof lies here, where we provide a cone constraint on $\|h_{T_0^c}\|_1$. This constraint is critical because it ultimately determines the parameter K_N given in (12).
- *Lemma 2 - Bounding the Tail:* Sorting the coefficients of $h_{T_0^c}$ together with Lemma 1 allows us to bound the tail of h .
- *Lemma 3 - Consequence of the RIP:* Due to the RIP assumption on A , along with the previous lemmas, we are able to define K_N and bound the largest portion of h .
- Combining Lemma 2 and Lemma 3, we obtain the final bound on the error $\|h\|_2$.

Proof Notation. We instate the notation of Theorem 2; let $|\tilde{T}_i| = \rho_i s$ for $i = 1, \dots, N$, where $0 \leq \rho_i \leq a_i$ and $\sum_{i=1}^N a_i > 1$. Define the accuracy of the support estimates $\alpha_i = \frac{|\tilde{T}_i \cap T_0|}{|\tilde{T}_i|}$ for $i = 1, \dots, N$. Set $\tilde{T} = \bigcup_{i=1}^N \tilde{T}_i$. For ease of notation, let us also define

$$\omega = \sum_{i=1}^N w_i$$

and

$$D = \omega \|x_{T_0^c}\|_1 + (1 - \omega) \|x_{\tilde{T}^c \cap T_0^c}\|_1 - \sum_{i=1}^N (\omega - w_i) \|x_{\tilde{T}_i \cap T_0^c}\|_1.$$

Proceeding as in [23], we will sort the coefficients of $h_{T_0^c}$ by partitioning T_0^c into disjoint sets T_j , $j \in \{1, 2, \dots\}$ each of size as , where $a \geq \sum_{i=1}^N a_i > 1$ and $a \in \frac{1}{s}\mathbb{Z}$ (to ensure the cardinality of each T_j is an integer). That is, T_1 indexes the as largest in magnitude coefficients of $h_{T_0^c}$, T_2 indexes the second as largest in magnitude coefficients of $h_{T_0^c}$, and so on. Define $T_{01} = T_0 \cup T_1$.

We now prove each of the above mentioned lemmas in sequence, and their combination will complete the proof.

Lemma 1 (Cone Constraint). *The vector h obeys the following cone constraint,*

$$\|h_{T_0^c}\|_1 \leq w_N \|h_{T_0}\|_1 + (1 - w_1) \|h_{T_0 \cup \bigcup_{i=1}^N \tilde{T}_i \setminus \bigcup_{i=1}^N (\tilde{T}_i \cap T_0)}\|_1 + \sum_{j=2}^N (w_{j-1} - w_j) \|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_1 + 2D. \quad (18)$$

Proof. Since $\hat{x} = x + h$ is a minimizer of (10), then $\|x + h\|_{1, \vec{w}} \leq \|x\|_{1, \vec{w}}$. By the choice of weights, we have

$$\sum_{i=1}^N w_i \|x_{\tilde{T}_i} + h_{\tilde{T}_i}\|_1 + \|x_{\tilde{T}^c} + h_{\tilde{T}^c}\|_1 \leq \sum_{i=1}^N w_i \|x_{\tilde{T}_i}\|_1 + \|x_{\tilde{T}^c}\|_1.$$

Furthermore, we have

$$\sum_{i=1}^N (w_i \|x_{\tilde{T}_i \cap T_0} + h_{\tilde{T}_i \cap T_0}\|_1 + w_i \|x_{\tilde{T}_i \cap T_0^c} + h_{\tilde{T}_i \cap T_0^c}\|_1) + \|x_{\tilde{T}^c \cap T_0} + h_{\tilde{T}^c \cap T_0}\|_1 + \|x_{\tilde{T}^c \cap T_0^c} + h_{\tilde{T}^c \cap T_0^c}\|_1$$

$$\leq \sum_{i=1}^N (w_i \|x_{\tilde{T}_i \cap T_0}\|_1 + w_i \|x_{\tilde{T}_i \cap T_0^c}\|_1) + \|x_{\tilde{T}^c \cap T_0}\|_1 + \|x_{\tilde{T}^c \cap T_0^c}\|_1.$$

Next, we use the reverse triangle inequality to get

$$\sum_{i=1}^N w_i \|h_{\tilde{T}_i \cap T_0^c}\|_1 + \|h_{\tilde{T}^c \cap T_0^c}\|_1 \leq \sum_{i=1}^N w_i \|h_{\tilde{T}_i \cap T_0}\|_1 + \|h_{\tilde{T}^c \cap T_0}\|_1 + 2 \left(\sum_{i=1}^N w_i \|x_{\tilde{T}_i \cap T_0^c}\|_1 + \|x_{\tilde{T}^c \cap T_0^c}\|_1 \right). \quad (19)$$

Now, we can write $\|h_{T_0^c}\|_1 = \sum_{i=1}^N \|h_{\tilde{T}_i \cap T_0^c}\|_1 + \|h_{\tilde{T}^c \cap T_0^c}\|_1$. Let us add and subtract $w_i \|h_{\tilde{T}_i \cap T_0^c}\|_1$ for all pairs of i and j such that $i, j = 1, \dots, N$ and $i \neq j$, and $w_i \|h_{\tilde{T}^c \cap T_0^c}\|_1$ for $i = 1, \dots, N$ to the left side of (19). Then the left side of (19) becomes

$$\omega \|h_{T_0^c}\|_1 + (1 - \omega) \|h_{\tilde{T}^c \cap T_0^c}\|_1 - \sum_{i=1}^N (\omega - w_i) \|h_{\tilde{T}_i \cap T_0^c}\|_1.$$

Similarly, we can write $\|h_{T_0}\|_1 = \sum_{i=1}^N \|h_{\tilde{T}_i \cap T_0}\|_1 + \|h_{\tilde{T}^c \cap T_0}\|_1$. Let us add and subtract $w_i \|h_{\tilde{T}_i \cap T_0}\|_1$ for all pairs of i and j such that $i, j = 1, \dots, N$ and $i \neq j$, and $w_i \|h_{\tilde{T}^c \cap T_0}\|_1$ for $i = 1, \dots, N$ to the right side of (19), as well as $w_i \|x_{\tilde{T}_j \cap T_0^c}\|_1$ for all pairs of i and j such that $i, j = 1, \dots, N$ and $i \neq j$, and $w_i \|x_{\tilde{T}^c \cap T_0^c}\|_1$ for $i = 1, \dots, N$. Then the right side of (19) becomes

$$\begin{aligned} & \omega \|h_{T_0}\|_1 + (1 - \omega) \|h_{\tilde{T}^c \cap T_0}\|_1 - \sum_{i=1}^N (\omega - w_i) \|h_{\tilde{T}_i \cap T_0}\|_1 \\ & + 2(\omega \|x_{T_0^c}\|_1 + (1 - \omega) \|x_{\tilde{T}^c \cap T_0^c}\|_1 - \sum_{i=1}^N (\omega - w_i) \|x_{\tilde{T}_i \cap T_0^c}\|_1). \end{aligned}$$

Putting these together, and using our definition of D , we have

$$\begin{aligned} & \omega \|h_{T_0^c}\|_1 + (1 - \omega) \|h_{\tilde{T}^c \cap T_0^c}\|_1 - \sum_{i=1}^N (\omega - w_i) \|h_{\tilde{T}_i \cap T_0^c}\|_1 \\ & \leq \omega \|h_{T_0}\|_1 + (1 - \omega) \|h_{\tilde{T}^c \cap T_0}\|_1 - \sum_{i=1}^N (\omega - w_i) \|h_{\tilde{T}_i \cap T_0}\|_1 + 2D. \end{aligned} \quad (20)$$

But, we can also write $\|h_{T_0^c}\|_1$ as

$$\|h_{T_0^c}\|_1 = \omega \|h_{T_0^c}\|_1 + \sum_{i=1}^N \left((1 - \omega) \|h_{\tilde{T}_i \cap T_0^c}\|_1 \right) + (1 - \omega) \|h_{\tilde{T}^c \cap T_0^c}\|_1.$$

Solving for $\omega \|h_{T_0^c}\|_1$ and substituting into (20) gives

$$\begin{aligned} & \|h_{T_0^c}\|_1 - \sum_{i=1}^N \left((1 - \omega) \|h_{\tilde{T}_i \cap T_0^c}\|_1 \right) - (1 - \omega) \|h_{\tilde{T}^c \cap T_0^c}\|_1 + (1 - \omega) \|h_{\tilde{T}^c \cap T_0}\|_1 - \sum_{i=1}^N (\omega - w_i) \|h_{\tilde{T}_i \cap T_0^c}\|_1 \\ & \leq \omega \|h_{T_0}\|_1 + (1 - \omega) \|h_{\tilde{T}^c \cap T_0}\|_1 - \sum_{i=1}^N (\omega - w_i) \|h_{\tilde{T}_i \cap T_0}\|_1 + 2D. \end{aligned}$$

Simplifying, we get

$$\begin{aligned} \|h_{T_0^c}\|_1 &\leq \sum_{i=1}^N \left((1-\omega) \|h_{\tilde{T}_i \cap T_0^c}\|_1 \right) + \sum_{i=1}^N (\omega - w_i) \|h_{\tilde{T}_i \cap T_0^c}\|_1 \\ &\quad + \omega \|h_{T_0}\|_1 + (1-\omega) \|h_{\tilde{T}^c \cap T_0}\|_1 - \sum_{i=1}^N (\omega - w_i) \|h_{\tilde{T}_i \cap T_0}\|_1 + 2D \end{aligned} \quad (21)$$

$$\begin{aligned} &= \sum_{i=1}^N (1 - w_i) \|h_{\tilde{T}_i \cap T_0^c}\|_1 + \omega \|h_{T_0}\|_1 + \|h_{\tilde{T}^c \cap T_0}\|_1 \\ &\quad - \sum_{i=1}^N w_i \left(\|h_{\tilde{T}^c \cap T_0}\|_1 + \sum_{j=1, j \neq i}^N \|h_{\tilde{T}_j \cap T_0}\|_1 \right) + 2D \\ &= \sum_{i=1}^N (1 - w_i) \|h_{\tilde{T}_i \cap T_0^c}\|_1 + \omega \|h_{T_0}\|_1 + \|h_{\tilde{T}^c \cap T_0}\|_1 - \sum_{i=1}^N w_i \|h_{\tilde{T}_i \cap T_0}\|_1 \\ &\quad + \sum_{i=1}^N \|h_{\tilde{T}_i^c \cap T_0}\|_1 - \sum_{i=1}^N \|h_{\tilde{T}_i^c \cap T_0}\|_1 + 2D \end{aligned} \quad (22)$$

$$\begin{aligned} &= \omega \|h_{T_0}\|_1 + \|h_{\tilde{T}^c \cap T_0}\|_1 - \sum_{i=1}^N \|h_{\tilde{T}_i^c \cap T_0}\|_1 + \sum_{i=1}^N (1 - w_i) \left(\|h_{\tilde{T}_i \cap T_0^c}\|_1 + \|h_{\tilde{T}_i^c \cap T_0}\|_1 \right) + 2D \\ &= (\omega - (N - 1)) \|h_{T_0}\|_1 + \sum_{i=1}^N (1 - w_i) \left(\|h_{\tilde{T}_i \cap T_0}\|_1 + \|h_{\tilde{T}_i \cap T_0^c}\|_1 \right) + 2D, \end{aligned} \quad (23)$$

where in (22) we have added zero and observed that $\|h_{\tilde{T}^c \cap T_0}\|_1 + \sum_{j=1, j \neq i}^N \|h_{\tilde{T}_j \cap T_0}\|_1 = \|h_{\tilde{T}_i^c \cap T_0}\|_1$, and in (23) we have observed that $\sum_{i=1}^N \|h_{\tilde{T}_i^c \cap T_0}\|_1 = (N - 1) \|h_{T_0}\|_1 + \|h_{\tilde{T}^c \cap T_0}\|_1$. Then assuming, without loss of generality, $w_1 \geq w_2 \geq \dots \geq w_N$, and writing $1 - w_i = 1 - w_1 + w_1 - w_i$ for $i > 1$, we have

$$\begin{aligned} \|h_{T_0^c}\|_1 &\leq (\omega - (N - 1)) \|h_{T_0}\|_1 + (1 - w_1) \sum_{i=1}^N [\|h_{\tilde{T}_i^c \cap T_0}\|_1 + \|h_{\tilde{T}_i \cap T_0^c}\|_1] \\ &\quad + \sum_{i=2}^N (w_1 - w_i) [\|h_{\tilde{T}_i^c \cap T_0}\|_1 + \|h_{\tilde{T}_i \cap T_0^c}\|_1] + 2D. \end{aligned} \quad (24)$$

Next, write $w_1 - w_i = w_1 - w_2 + w_2 - w_i$ for $i > 2$. Then we have

$$\begin{aligned} \|h_{T_0^c}\|_1 &\leq (\omega - (N - 1)) \|h_{T_0}\|_1 + (1 - w_1) \sum_{i=1}^N [\|h_{\tilde{T}_i^c \cap T_0}\|_1 + \|h_{\tilde{T}_i \cap T_0^c}\|_1] \\ &\quad + (w_1 - w_2) \sum_{i=2}^N [\|h_{\tilde{T}_i^c \cap T_0}\|_1 + \|h_{\tilde{T}_i \cap T_0^c}\|_1] + \sum_{i=3}^N (w_2 - w_i) [\|h_{\tilde{T}_i^c \cap T_0}\|_1 + \|h_{\tilde{T}_i \cap T_0^c}\|_1] + 2D. \end{aligned} \quad (25)$$

Continuing in this manner gives us

$$\|h_{T_0^c}\|_1 \leq (\omega - (N - 1)) \|h_{T_0}\|_1 + (1 - w_1) \sum_{i=1}^N [\|h_{\tilde{T}_i^c \cap T_0}\|_1 + \|h_{\tilde{T}_i \cap T_0^c}\|_1]$$

$$+ \sum_{j=2}^N (w_{j-1} - w_j) \sum_{i=j}^N [\|h_{\tilde{T}_i^c \cap T_0}\|_1 + \|h_{\tilde{T}_i \cap T_0^c}\|_1] + 2D. \quad (26)$$

Noting that $\|h_{\tilde{T}_i^c \cap T_0}\|_1 = \sum_{j=1, j \neq i}^N \|h_{\tilde{T}_j \cap T_0}\|_1 + \|h_{T_0 \cap \bigcap_{j=1}^N \tilde{T}_j^c}\|_1$ and $\|h_{T_0}\|_1 = \sum_{i=1}^N \|h_{\tilde{T}_i \cap T_0}\|_1 + \|h_{T_0 \cap \bigcap_{j=1}^N \tilde{T}_j^c}\|_1$, and that we can write $\sum_{i=j}^N \|h_{\tilde{T}_i \cap T_0^c}\|_1 + \|h_{T_0 \cap \bigcap_{i=j}^N \tilde{T}_i^c}\|_1 = \|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_1$ for any $j = 1, \dots, N$, the above can also be expressed as

$$\begin{aligned} \|h_{T_0^c}\|_1 &\leq (\omega - (N-1)) \|h_{T_0}\|_1 + (1 - w_1)[(N-1)\|h_{T_0}\|_1 + \|h_{T_0 \cup \bigcup_{i=1}^N \tilde{T}_i \setminus \bigcup_{i=1}^N (\tilde{T}_i \cap T_0)}\|_1] \\ &\quad + \sum_{j=2}^N (w_{j-1} - w_j)[(N-j)\|h_{T_0}\|_1 + \|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_1] + 2D. \end{aligned} \quad (27)$$

Combining all coefficients of $\|h_{T_0}\|_1$, we have

$$\begin{aligned} &\sum_{i=1}^N w_i - (N-1) + (1 - w_1)(N-1) + \sum_{j=2}^N (w_{j-1} - w_j)(N-j) \\ &= \sum_{i=1}^N w_i - (N-1)w_1 + (N-2)w_1 + \sum_{j=2}^{N-1} (N - (j+1))w_j - \sum_{j=2}^{N-1} (N-j)w_j \\ &= \sum_{i=2}^N w_i - \sum_{j=2}^{N-1} w_j \\ &= w_N. \end{aligned} \quad (28)$$

Finally, combining (27) with (28) we arrive at (18). \square

Lemma 2 (Bounding the Tail). *We have the following bound on the tail of the error h ,*

$$\begin{aligned} \|h_{T_{01}^c}\|_2 &\leq (as)^{-1/2} \left[w_N \|h_{T_0}\|_1 + (1 - w_1) \|h_{T_0 \cup \bigcup_{i=1}^N \tilde{T}_i \setminus \bigcup_{i=1}^N (\tilde{T}_i \cap T_0)}\|_1 \right. \\ &\quad \left. + \sum_{j=2}^N (w_{j-1} - w_j) \|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_1 + 2D \right]. \end{aligned} \quad (29)$$

Proof. As in [10], note that

$$\|h_{T_j}\|_2 \leq \sqrt{as} \|h_{T_j}\|_\infty \leq (as)^{-1/2} \|h_{T_{j-1}}\|_1, \quad (30)$$

where we have observed that (by construction) the average of the terms (in magnitude) on $h_{T_{j-1}}$ must be at least as large as $\|h_{T_j}\|_\infty$. Then noting that $h_{T_0^c} = \sum_{j \geq 1} h_{T_j}$ and $h_{T_{01}^c} = \sum_{j \geq 2} h_{T_j}$ and using the triangle inequality along with (30), we have

$$\begin{aligned} \|h_{T_{01}^c}\|_2 &\leq \sum_{j \geq 2} \|h_{T_j}\|_2 \\ &\leq (as)^{-1/2} \sum_{j \geq 1} \|h_{T_j}\|_1 \\ &\leq (as)^{-1/2} \|h_{T_0^c}\|_1. \end{aligned} \quad (31)$$

Combining (31) with (18) gives (29). \square

Lemma 3 (Consequence of the RIP). *Define*

$$K_N = w_N + (1 - w_1) \sqrt{1 + \sum_{i=1}^N (\rho_i - 2\alpha_i \rho_i)} + \sum_{j=2}^N \left((w_{j-1} - w_j) \sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha_i \rho_i)} \right). \quad (32)$$

Then the following inequality holds,

$$\|h_{T_{01}}\|_2 \leq \frac{2\epsilon + \frac{2\sqrt{1+\delta_{as}}}{\sqrt{as}} D}{\sqrt{1 - \delta_{(a+1)s}} - \frac{\sqrt{1+\delta_{as}}}{\sqrt{a}} K_N}, \quad (33)$$

when the denominator is positive.

Proof. Again, following [10], and noting that since A satisfies the RIP and $\|Ah\|_2 \leq 2\epsilon$ due to the feasibility of both x and \hat{x} , we have

$$\begin{aligned} \sqrt{1 - \delta_{(a+1)s}} \|h_{T_{01}}\|_2 &\leq \|Ah_{T_{01}}\|_2 \\ &= \|Ah_{T_{01}} + Ah_{T_{01}^c} - Ah_{T_{01}^c}\|_2 \\ &\leq \|Ah\|_2 + \|Ah_{T_{01}^c}\|_2 \\ &= 2\epsilon + \left\| \sum_{j \geq 2} Ah_{T_j} \right\|_2 \\ &\leq 2\epsilon + \sum_{j \geq 2} \|Ah_{T_j}\|_2 \\ &\leq 2\epsilon + \sqrt{1 + \delta_{as}} \sum_{j \geq 2} \|h_{T_j}\|_2 \\ &\leq 2\epsilon + \frac{\sqrt{1 + \delta_{as}}}{\sqrt{as}} \|h_{T_0^c}\|_1, \end{aligned}$$

where the last inequality follows from (31). Then applying (18), we get

$$\begin{aligned} \sqrt{1 - \delta_{(a+1)s}} \|h_{T_{01}}\|_2 &\leq 2\epsilon + 2 \frac{\sqrt{1 + \delta_{as}}}{\sqrt{as}} D \\ &\quad + \frac{\sqrt{1 + \delta_{as}}}{\sqrt{as}} \left[w_N \|h_{T_0}\|_1 + (1 - w_1) \|h_{T_0 \cup \bigcup_{i=1}^N \tilde{T}_i \setminus \bigcup_{i=1}^N (\tilde{T}_i \cap T_0)}\|_1 + \sum_{j=2}^N (w_{j-1} - w_j) \|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_1 \right]. \end{aligned}$$

Note that $|T_0 \cup \bigcup_{i=j}^N \tilde{T}_j \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)| = s + \sum_{i=j}^N (\rho_i s - 2\alpha_i \rho_i s)$ for any $j = 1, \dots, N$. Thus, we have

$$\|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_1 \leq \sqrt{s + s \sum_{i=j}^N (\rho_i - 2\alpha_i \rho_i)} \|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_2$$

for any $j = 1, \dots, N$. Now, since T_1 contains the largest as coefficients of $h_{T_0^c}$ with $a > 1$, and $|\bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)| = \sum_{i=j}^N (\rho_i s - \alpha_i \rho_i s) = s \sum_{i=1}^N \rho_i (1 - \alpha_i)$, then $\|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_2 \leq \|h_{T_{01}}\|_2$ for any $j = 1, \dots, N$ as long as we require $\sum_{i=1}^N \rho_i (1 - \alpha_i) \leq a$. Note that since $\rho_i \leq a_i$ for

$i = 1, \dots, N$, $\sum_{i=1}^N \rho_i(1 - \alpha_i) \leq \sum_{i=1}^N a_i(1 - \alpha_i) \leq \sum_{i=1}^N a_i \leq a$ as long as $\alpha_i \geq 0$ for $i = 1, \dots, N$. Consequently, we can bound

$$\|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_2 \leq \|h_{T_{01}}\|_2$$

for any $j = 1, \dots, N$. Also, $\|h_{T_0}\|_1 \leq \sqrt{s}\|h_{T_0}\|_2 \leq \sqrt{s}\|h_{T_{01}}\|_2$. Therefore, we have

$$\begin{aligned} \sqrt{1 - \delta_{(a+1)s}}\|h_{T_{01}}\|_2 &\leq 2\epsilon + 2\frac{\sqrt{1 + \delta_{as}}}{\sqrt{as}}D \\ &+ \frac{\sqrt{1 + \delta_{as}}}{\sqrt{a}}\|h_{T_{01}}\|_2 \left[w_N + (1 - w_1) \sqrt{1 + \sum_{i=1}^N (\rho_i - 2\alpha_i \rho_i)} + \sum_{j=2}^N \left((w_{j-1} - w_j) \sqrt{1 + \sum_{i=j}^N (\rho_i - 2\alpha_i \rho_i)} \right) \right]. \end{aligned} \quad (34)$$

Defining K_N as in (32) and solving (34) for $\|h_{T_{01}}\|_2$ gives (33), where we assume the denominator is positive. \square

Putting it all together. Finally, using that $\|h\|_2 = \|h_{T_{01}} + h_{T_{01}^c}\|_2 \leq \|h_{T_{01}}\|_2 + \|h_{T_{01}^c}\|_2$, along with (29), (33) and the arguments leading to (34), we get

$$\|h\|_2 \leq$$

$$\begin{aligned} &\|h_{T_{01}}\|_2 + \frac{1}{\sqrt{as}} \left[w_N \|h_{T_0}\|_1 + (1 - w_1) \|h_{T_0 \cup \bigcup_{i=1}^N \tilde{T}_i \setminus \bigcup_{i=1}^N (\tilde{T}_i \cap T_0)}\|_1 + \sum_{j=2}^N (w_{j-1} - w_j) \|h_{T_0 \cup \bigcup_{i=j}^N \tilde{T}_i \setminus \bigcup_{i=j}^N (\tilde{T}_i \cap T_0)}\|_1 + 2D \right] \\ &\leq \|h_{T_{01}}\|_2 + \frac{1}{\sqrt{as}} [\sqrt{s}K_N \|h_{T_{01}}\|_2 + 2D] \\ &= \left(1 + \frac{K_N}{\sqrt{a}} \right) \|h_{T_{01}}\|_2 + \frac{2}{\sqrt{as}}D \\ &\leq \left(1 + \frac{K_N}{\sqrt{a}} \right) \left(\frac{2\epsilon + \frac{2\sqrt{1 + \delta_{as}}}{\sqrt{as}}D}{\sqrt{1 - \delta_{(a+1)s}} - \frac{\sqrt{1 + \delta_{as}}}{\sqrt{a}}K_N} \right) + \frac{2}{\sqrt{as}}D \\ &= \frac{(1 + \frac{K_N}{\sqrt{a}})2\epsilon + \frac{2}{\sqrt{as}}(\sqrt{1 + \delta_{as}} + \sqrt{1 - \delta_{(a+1)s}})D}{\sqrt{1 - \delta_{(a+1)s}} - \frac{\sqrt{1 + \delta_{as}}}{\sqrt{a}}K_N}, \end{aligned} \quad (35)$$

with the condition that the denominator is positive. That is, we require

$$\sqrt{1 - \delta_{(a+1)s}} - \frac{\sqrt{1 + \delta_{as}}}{\sqrt{a}}K_N > 0,$$

which is equivalent to

$$\delta_{as} + \frac{a}{K_N^2} \delta_{(a+1)s} < \frac{a}{K_N^2} - 1.$$

4 Numerical Experiments

In this section, we include numerical experiments of weighted ℓ_1 -minimization and demonstrate that non-uniform weights can be preferable to a uniform weight. We stress that the purpose of these

experiments is to illustrate the potential benefits of the multi-weight setup, and not to investigate the associated application areas. We first present experiments for synthetically generated signals, and then present an example where we use weighed ℓ_1 minimization to recover a compressively sampled video signal.

4.1 Synthetic Experiments

For our first set of experiments, the signal x is synthetically generated and the recovery error when using weighted ℓ_1 -minimization with non-uniform weights is compared to that when using a single constant weight. Here, the signal is of dimension $n = 256$ and sparsity $s = |T_0| = |\text{supp}(x)| = 16$ with standard Gaussian nonzero values on T_0 , the measurement matrix is Gaussian, the measurement noise z is i.i.d. Gaussian with mean zero and standard deviation 0.01, and 500 trials are performed at each measurement level tested. We compare the recovery error $\|x - \hat{x}\|_2$ when using either a single support estimate set \tilde{T} or two disjoint support estimate sets \tilde{T}_1 and \tilde{T}_2 , such that $\tilde{T} = \tilde{T}_1 \cup \tilde{T}_2$ (this implies $\rho = \rho_1 + \rho_2$).

First, we set $\alpha = \alpha_1 = \alpha_2 = 1$, $\rho = 1$, and vary the sizes of ρ_1 and ρ_2 while maintaining that $\rho_1 + \rho_2 = 1$. We set $w_1 = 0.5$ (applied on \tilde{T}_1) and $w_2 = 0.25$ (applied on \tilde{T}_2) in the two support estimate setting, and compare the recovery error to the single support estimate case when $w = 0.5$ or $w = 0.25$ (applied on \tilde{T}). Figure 2 displays the mean recovery error of these results as a function of the number of measurements acquired. As expected, since all support estimates are completely accurate (that is, all elements of \tilde{T} , \tilde{T}_1 , and \tilde{T}_2 are elements of T_0), setting all weights to the smallest value of $w = 0.25$ performs the best while setting all weights to the largest value of $w = 0.5$ performs the worst; using a combination of these two weights as ρ_1 and ρ_2 are varied produces intermediate performance. This behavior reflects empirically the theoretical result of Figure 1.

Next, we set $\alpha = 0.5$, $\rho = 1$, $\rho_1 = \rho_2 = 0.5$, and vary the sizes of α_1 and α_2 while maintaining that $\rho_1\alpha_1 + \rho_2\alpha_2 = \rho\alpha$ (with our choice of parameters, this means $\alpha_1 + \alpha_2 = 1$). We again set $w_1 = 0.5$ and $w_2 = 0.25$ in the two support estimate setting, and compare the recovery error to the single support estimate case when $w = 0.5$ or $w = 0.25$. In Figure 2, we see that the best recovery is achieved when $\alpha_1 = 0$ and $\alpha_2 = 1$. This is again expected since $w_2 = 0.25$ is applied on \tilde{T}_2 which contains the *correctly* identified elements in T_0 and $w_1 = 0.5$ is applied on \tilde{T}_1 which contains the *incorrectly* estimated elements in T_0^c . As α_1 increases to 1 and α_2 decreases to 0 the recovery error increases since fewer correctly identified elements in T_0 receive the smaller weight w_2 , but rather the larger weight $w = 0.5$. The recovery results when using a single constant weight of either $w = 0.5$ or $w = 0.25$ or two weights with $\alpha_1 = \alpha_2 = 0.5$ all seem similar, however, there is a subtlety. The recovery tends to be *slightly* better when the single weight $w = 0.25$ is used than when $w = 0.5$ is used, and the two weight result with $\alpha_1 = \alpha_2 = 0.5$ seems to fall in between the $w = 0.25$ and $w = 0.5$ curves. Turning to the theory, when $\alpha = 0.5$ the quantity $b = 1$ and when $\alpha_1 = \alpha_2 = 0.5$ the quantity $K_2 = 1$, in which case $C_0 = C'_0$ and $C_1 = C'_1$ and the RIP condition (11) of Theorem 2 is identical to (5) in Theorem 1. Thus, the relationship between recovery error for these settings can be explained by the terms $w\|x - x_s\|_1 + (1 - w)\|x_{\tilde{T}^c \cap T_0^c}\|_1$ and

$$(w_1 + w_2)\|x - x_s\|_1 + (1 - w_1 - w_2)\|x_{\tilde{T}^c \cap T_0^c}\|_1 - w_2\|x_{\tilde{T}_1 \cap T_0^c}\|_1 - w_1\|x_{\tilde{T}_2 \cap T_0^c}\|_1$$

from the error bounds in Theorems 1 and 2, respectively. Specifically, since $\tilde{T} = \tilde{T}_1 \cup \tilde{T}_2$, when $w = 0.25$, $w_1 = 0.5$, $w_2 = 0.25$, and noting that $\|x - x_s\|_1 = \|x_{T_0^c}\|_1$ and $\|x_{\tilde{T}^c \cap T_0^c}\|_1 \leq \|x_{T_0^c}\|_1$, we have

$$0.25\|x_{T_0^c}\|_1 + 0.75\|x_{\tilde{T}^c \cap T_0^c}\|_1 \leq 0.75\|x_{T_0^c}\|_1 + 0.25\|x_{\tilde{T}^c \cap T_0^c}\|_1 - 0.25\|x_{\tilde{T}_1 \cap T_0^c}\|_1 - 0.5\|x_{\tilde{T}_2 \cap T_0^c}\|_1.$$

Similarly, when $w = 0.5$, we have

$$0.75\|x_{T_0^c}\|_1 + 0.25\|x_{\tilde{T}^c \cap T_0^c}\|_1 - 0.25\|x_{\tilde{T}_1 \cap T_0^c}\|_1 - 0.5\|x_{\tilde{T}_2 \cap T_0^c}\|_1 \leq 0.5\|x_{T_0^c}\|_1 + 0.5\|x_{\tilde{T}^c \cap T_0^c}\|_1.$$

Although satisfying the above relationships, these terms are quite close, which is reflected in the closeness of the curves in Figure 2.

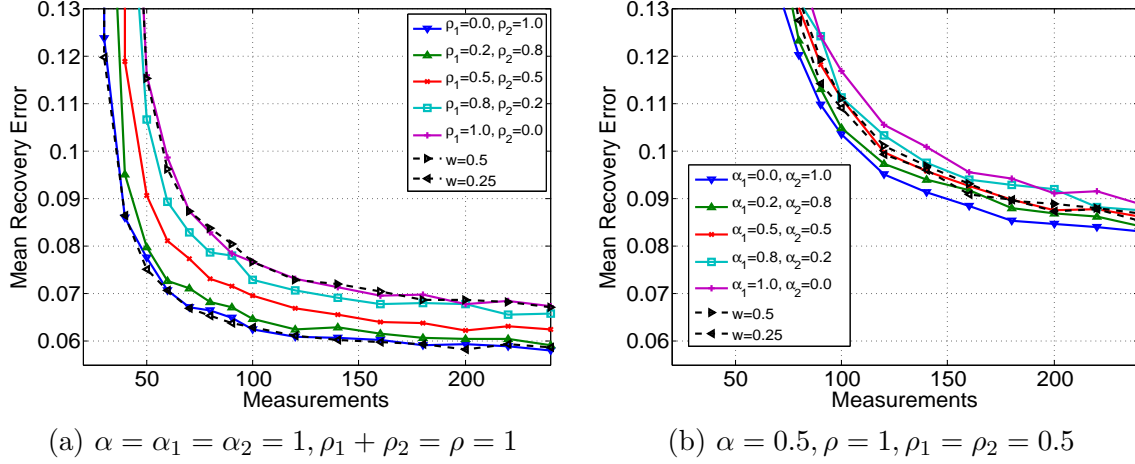


Figure 2: Comparison of the mean recovery error over 500 trials versus the number of measurements taken while using weighted ℓ_1 -minimization with a single support estimate and a single weight (black dashed) and two disjoint support estimates with two distinct weights (solid, marker as indicated). We set $w_1 = 0.5$, $w_2 = 0.25$ and (a) vary the sizes of ρ_1 and ρ_2 for fixed ρ , α , α_1 , and α_2 and (b) vary the sizes of α_1 and α_2 for fixed α , ρ , ρ_1 , and ρ_2 .

In our next set of experiments, the signal x is synthetically generated with a given signal distribution (i.e., with a specified probability of each entry being nonzero). Again, the signal is of dimension $n = 256$ with standard Gaussian nonzero values, the measurement matrix is Gaussian, the measurement noise z is i.i.d. Gaussian with mean zero and standard deviation 0.01, and 50 trials are performed. The optimal choice of weight given the signal distribution is not obvious, and the determination of the optimal relationship is beyond the scope of this paper. We find, however, that the following method empirically performs well for the signal models tested. Let P_i denote the probability that entry i in x is nonzero for $i = 1, \dots, n$. Then, we take the weights to be¹ $w_i = \frac{e^{-5P_i} - e^{-5}}{1 - e^{-5}}$ (see Figure 3). For comparison with weighted ℓ_1 -minimization with a single weight, all indices i with $P_i > 0$ are assigned the same weight w .

Figure 4 displays the recovery results for signals with a power law distribution. That is, $P_i = \frac{1}{i}$ for $i = 1, \dots, n$. However, for any i such that $\frac{1}{i} < 0.025$, we set $P_i = 0$ so that the same weight is not applied on *all* indices in the single weight case (if the same weight were applied on all indices, the result would be the same as the standard un-weighted ℓ_1 -minimization in (2)). The probabilities P_i and the non-uniform weights w_i are shown in Figure 4. Over all trials, the average signal sparsity was 4.33. We compare the non-uniform weight approach to using a uniform weight of $w \in \{0.1, 0.3, 0.5, 0.7, 0.9, 1\}$ (note that $w = 1$ is equivalent to solving (2)). We see that although the recovery error tends to decrease as the weight w decreases in the single weight setting, the non-uniform weight approach outperforms all the others.

¹The relationship $w_i = 1 - P_i$ may seem more natural, however, we found that the more aggressive relationship between w_i and P_i as implemented in the experiments provided superior performance.

Figure 5 displays the recovery results for signals with a binary tree distribution. That is, the support is organized on a binary tree (plus an extra root node at the top). The first index has just one child; the second and further indices have two children each. An s -sparse support is filled by choosing the first index location, and then in each of the $s - 1$ remaining rounds, choosing one index randomly among the unselected indices which currently have a selected parent. This type of model is characteristic of natural images (see [20, 17] for similar constructions). The probabilities P_i for such a model were calculated experimentally over 10,000 trials of the described tree-sparse support generation, where the sparsity $s = 24$ and the dimension $n = 256$. Again, for any support index i such that the experimentally calculated probability $P_i < 0.025$, we set $P_i = 0$. The probabilities P_i and the non-uniform weights w_i are shown in Figure 5². We again compare the non-uniform weight approach to using a uniform weight of $w \in \{0.1, 0.3, 0.5, 0.7, 0.9, 1\}$, and see that the non-uniform weight approach is outperforming all the others. Interestingly, in the single weight setting, the recovery error decreases as the weight decreases from $w = 1$ to $w = 0.5$, but then increases for $w = 0.3$ and increases even further for $w = 0.1$. This illustrates that being overly aggressive in the weight assignment can worsen the reconstruction performance.

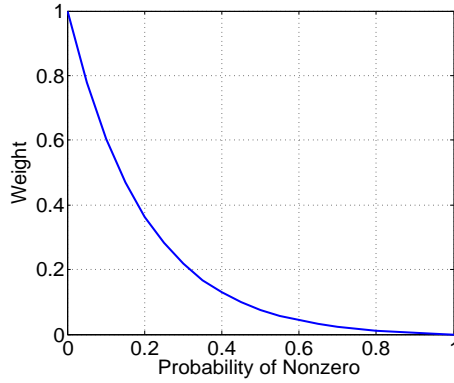


Figure 3: Relationship between probability P_i of an index being nonzero and non-uniform weight w_i used in our numerical experiments: $w_i = \frac{e^{-5P_i} - e^{-5}}{1 - e^{-5}}$.

²Note that a tree-sparse support is generated using the described method. This means, although unlikely, it is possible for a support to contain indices such that P_i has been set to zero.

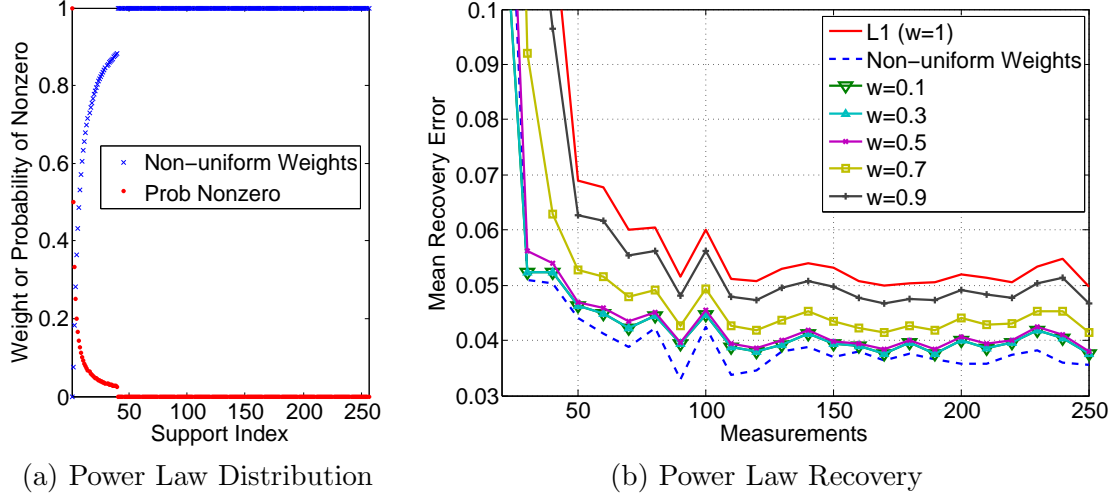


Figure 4: (a) The probability of an index being nonzero (red dot) and the weight assigned (blue x) in the weighted ℓ_1 -minimization with non-uniform weights, where $w_i = \frac{e^{-5P_i} - e^{-5}}{1 - e^{-5}}$, versus the support index. (b) Mean recovery error over 50 trials versus the number of measurements taken for signals with a power law distribution. We compare the mean recovery error for weighted ℓ_1 -minimization with non-uniform weights (blue dashed), standard ℓ_1 -minimization (red solid), and weighted ℓ_1 -minimization with a single weight (marker as indicated).

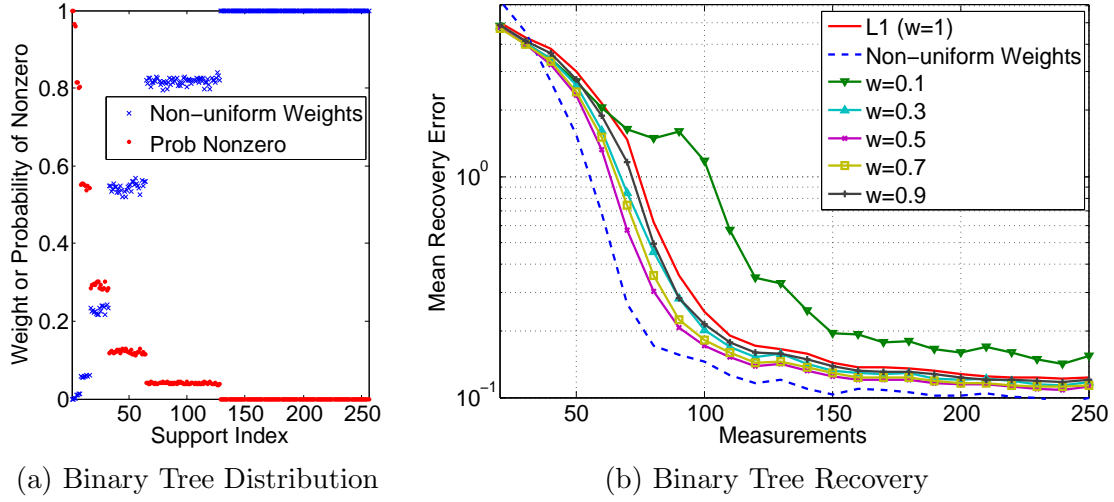


Figure 5: (a) The probability of an index being nonzero (red dot) and the weight assigned (blue x) in the weighted ℓ_1 -minimization with non-uniform weights, where $w_i = \frac{e^{-5P_i} - e^{-5}}{1 - e^{-5}}$, versus the support index. (b) Mean recovery error over 50 trials versus the number of measurements taken for signals following a binary tree sparsity pattern. We compare the mean recovery error for weighted ℓ_1 -minimization with non-uniform weights (blue dashed), standard ℓ_1 -minimization (red solid), and weighted ℓ_1 -minimization with a single weight (marker as indicated).

4.2 Application to the Recovery of Video Signals

One important application where a prior support, or even a prior distribution, can be reasonably estimated is the recovery of video signals since there is often little variation from one frame to the next. In this section, we perform a similar video recovery experiment as in [23], but we also include weighted ℓ_1 -minimization recovery options where the weight does not need to be constant across the entire support estimate. As in [23], we utilize the Foreman sequence at QCIF resolution (i.e., each frame contains 144×176 pixels), and consider only the luma (grayscale) component of the sequence. We split the frames into four blocks of size 72×88 , each of which are processed independently. The measurement matrix for each frame is $A = RD$, where R is an $m \times n$ restriction matrix (i.e., a matrix with m rows from the $n \times n$ identity matrix, selected uniformly at random) with $n = 72 \times 88$, and D is the two-dimensional Discrete Cosine Transform (DCT) (i.e., D is the sparsifying basis). In our experiment, we set $m = \frac{n}{2} = 3168$.

We perform the reconstruction of each frame block using standard ℓ_1 -minimization, weighted ℓ_1 -minimization with a constant weight, and weighted- ℓ_1 minimization with non-uniform weights. For weighted ℓ_1 -minimization with a constant weight w , standard ℓ_1 -minimization is used for the first frame. Then, at frame j for $j \geq 2$ we determine the top 10% of the DCT coefficients (in magnitude) of the previously recovered frame. Denoting the set of such DCT coefficients by \tilde{T}_j , we set $\tilde{T} = \bigcup_{j=2}^J \tilde{T}_j$, and use a constant weight of w on \tilde{T} in the recovery of frame j . Thus, the size of the support estimate \tilde{T} either grows or remains constant from frame to frame. For weighted- ℓ_1 minimization with non-uniform weights we follow a similar procedure, but construct an estimated probability of a given DCT coefficient to be in the top 10% of the DCT coefficients (in magnitude). That is, at frame j for $j \geq 2$ we determine the top 10% of the DCT coefficients (in magnitude) of the previously recovered frame and set $\hat{P}_i = \frac{\# \text{ of times in top 10\%}}{j-1}$ for each DCT coefficient i . Then the weights are taken to be $w_i = \frac{e^{-5\hat{P}_i} - e^{-5}}{1 - e^{-5}}$ (note that for this application, setting $w_i = 1 - \hat{P}_i$ also performed well, but the included relationship between \hat{P}_i and w_i is even more advantageous). As a final oracle-type comparison, we also include reconstruction results for weighted ℓ_1 -minimization with non-uniform weights when the *true empirical* coefficient probabilities were calculated. For each true frame block, we determine the top 10% of the DCT coefficients (in magnitude). The empirical probability that coefficient i is nonzero is calculated as $P_i = \frac{\# \text{ of times in top 10\%}}{300}$ (note that the Foreman sequence has 300 frames). Then the weights are taken to be $w_i = \frac{e^{-5P_i} - e^{-5}}{1 - e^{-5}}$ for every frame.

The reconstruction quality is reported in terms of the peak signal to noise ratio (PSNR) given by the expression

$$\text{PSNR}(x, \hat{x}) = 10 \log_{10} \left(\frac{N \times 255^2}{\|x - \hat{x}\|_2^2} \right), \quad (36)$$

where x and \hat{x} are the true and estimated full frames, respectively, expressed as a column vector. Figure 6 displays the recovery of all 300 frames of the Foreman sequence using weighted ℓ_1 -minimization with both uniform and non-uniform weighting strategies, as well as standard ℓ_1 -minimization. The results demonstrate that a dramatic improvement in PSNR can be achieved when using non-uniform weights, with the recovery method using weights determined from \hat{P}_i close to the recovery using weights determined from P_i . Note that being too aggressive and using a fixed weight of $w = 0.2$ eventually results in performance that falls below that of standard ℓ_1 -minimization. For reference, in Figure 6 we also display selected true frames from the Foreman sequence, particularly where we see locally extreme PSNR values. This experiment illustrates a

practical unsupervised situation where weighted ℓ_1 -minimization can be used to improve signal recovery while eliminating the need for the practitioner to explicitly choose which weight to use. Here, we have presented an option that determines non-uniform weights on the fly and outperforms simple implementations of weighted ℓ_1 -minimization with a single, constant weight.

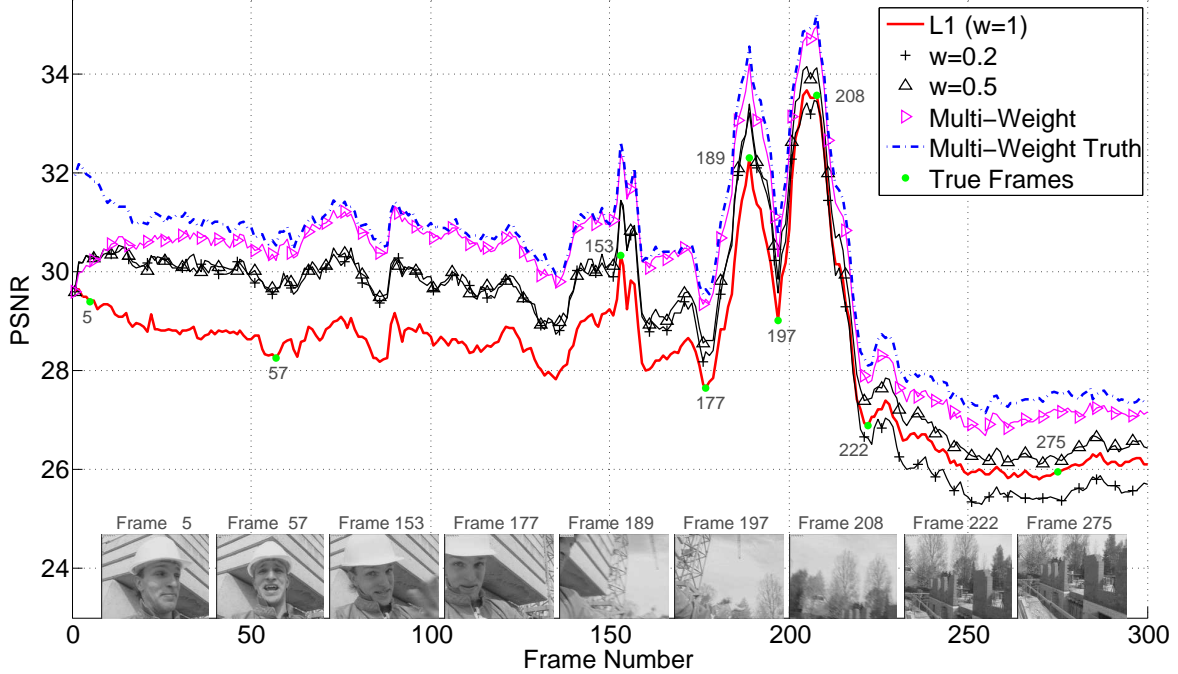


Figure 6: Recovery of frames 1 to 300 of the Foreman sequence at QCIF resolution and processed as four separate blocks of size 72×88 . The sparsity basis is the DCT, the sensing matrix is the restriction operator, and $m = 3168$ measurements are taken. Recovery with standard ℓ_1 -minimization is shown in the solid red curve; recovery with weighted ℓ_1 -minimization with a constant weight in shown in the black curves for $w = 0.2$ and $w = 0.5$ (marker as indicated); recovery with weighted- ℓ_1 minimization with non-uniform weights and the estimated \hat{P}_i is shown in the magenta curve (Multi-Weight, marker as indicated); recovery with weighted- ℓ_1 minimization with non-uniform weights with the oracle-type P_i is shown in the blue dashed curve (Multi-Weight Truth). Selected true frames are also displayed, with markers indicating where these frames correspond to the standard ℓ_1 -minimization PSNR.

5 Discussion

We have generalized the recovery conditions, in terms of the restricted isometry constant of the sensing matrix, of weighted ℓ_1 -minimization for sparse recovery when multiple distinct weights are permitted and arbitrary prior information can be utilized. Our analysis provides both an extension to existing literature that studies weighted ℓ_1 -minimization with a single weight, and an improvement to prior results on weighted ℓ_1 -minimization when multiple distinct weights are allowed. Additionally, we have included simulations that illustrate the theoretical results, and provided examples with synthetic signals and real video data where utilizing many distinct weights is superior to using a single fixed weight.

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