Signal Recovery with Regularized OMP

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Setup

1. Suppose x is an unknown s-sparse signal in $\mathbb{R}^d$.
   - $\|x\|_0 \stackrel{\text{def}}{=} |\text{supp}(x)| \leq s \ll d$.
2. Design measurement matrix $\Phi : \mathbb{R}^d \to \mathbb{R}^m$.
3. Collect noisy measurements $u = \Phi x + e$.

$$\begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} \quad \Phi \quad \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} e \end{bmatrix}$$

4. Problem: Reconstruct signal x from measurements u

Designing an algorithm

$$\begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} & \Phi & \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} e \end{bmatrix}$$

Important Questions

- What kind(s) of measurement matrices?
- How many measurements needed?
- Are the guarantees uniform?
- Is algorithm stable?
- Fast runtime?

Orthogonal Matching Pursuit

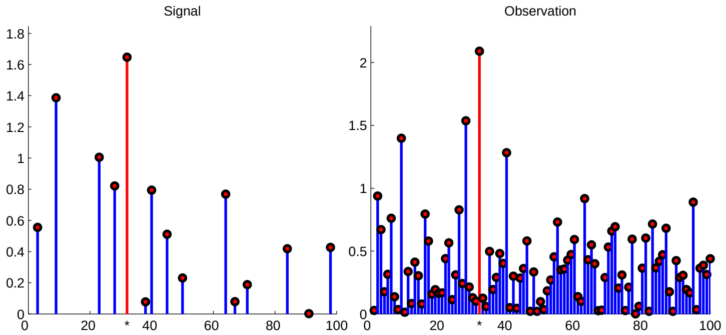
Idea

- Noiseless case: When Φ is (sub)Gaussian, $y \overset{\text{def}}{=} \Phi^* u = \Phi^* \Phi x$ is a good approximation to x.
- At each iteration, select largest component of y to be in support.
- Support of $x \Rightarrow x$.

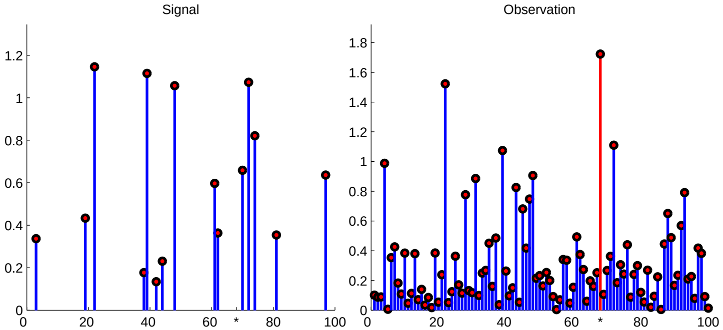
Theorem (Gilbert-Tropp)

When Φ is (sub)Gaussian with $m \gtrsim s \log d$, OMP correctly recovers each fixed signal with high probability.

OMP: Good case



OMP: Bad case



Restricted Isometry Property

- The s^{th} restricted isometry constant of Φ is the smallest δ_s such that

$$(1-\delta_s)\|x\|_2 \le \|\Phi x\|_2 \le (1+\delta_s)\|x\|_2 \quad \text{whenever } \|x\|_0 \le s.$$

- For Gaussian or Bernoulli measurement matrices, with high probability
$$\delta_s \le c < 1 \quad \text{when } m \gtrsim s \log d.$$
- Random Fourier and others with fast multiply have similar property.
- Convex optimization methods use the RIP and provide uniform and stable guarantees, but lack the speed of the greedy approach.

Gap in the approaches

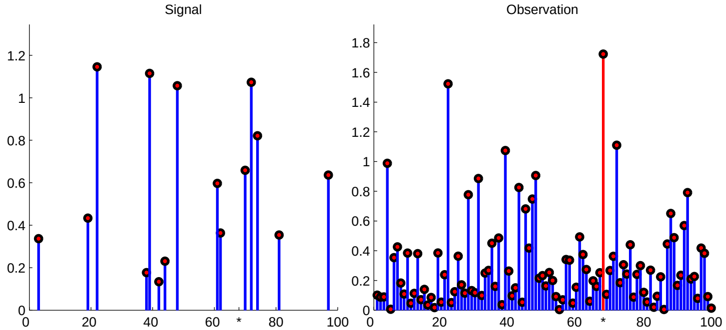
	Convex Opt.		OMP
Uniform?	yes		no
Stable?	yes		no
Runtime?	(LP)		$O(smd)$

Insight of Regularized OMP - Needell, Vershynin

- The RIP guarantees that every s columns of Φ is close to an orthonormal system.
- Thus $y = \Phi^*\Phi x$ is locally like x.
- Why not choose the s largest components of y, instead of the largest?
- Allow ourselves to make mistakes, as long as we don't make too many.
- A regularization step is needed to ensure the indentified energy translates to identified support.

$\Rightarrow$ use RIP in a greedy algorithm!

OMP's Bad Case:



ROMP Algorithm

ROMP(Φ, u, s)
Input: Measurement matrix Φ, noisy measurements u, sparsity level s
Output: Index set I containing support of x with $|I| \leq 2s$.

$I = \emptyset$, $r = u$ { Initialization }
while $|I| < 2s$ or s times

Identify: Choose a set J of the s biggest coordinates in magnitude of the observation vector $y = \Phi^* r$, or all of its nonzero coordinates, whichever set is smaller.

Regularize: Among all subsets $J_0 \subset J$ with comparable coordinates:

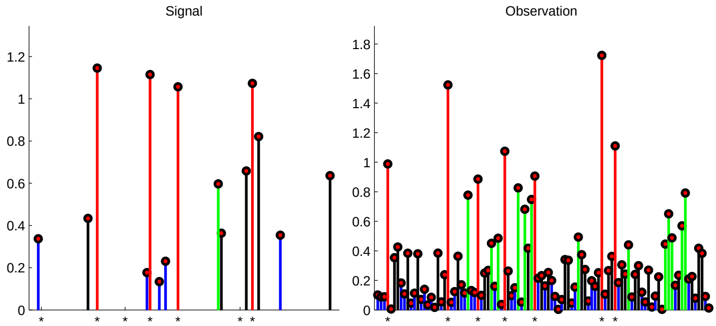
$$|y(i)| \leq 2|y(j)| \quad \text{for all } i, j \in J_0,$$

choose J_0 with the maximal energy $\|y|_{J_0}\|_2$.

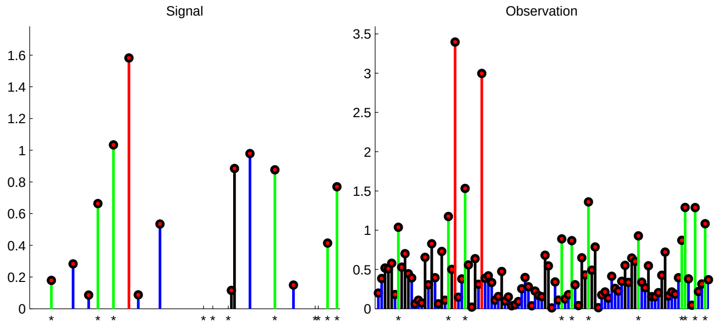
Update: Add the set J_0 to the index set: $I \leftarrow I \cup J_0$, and update the residual:

$$w = \operatorname*{argmin}_{z \in \mathbb{R}^I} \|u - \Phi z\|_2; \qquad r = u - \Phi w.$$

How ROMP works:



How ROMP works:



Iteration Invariant

The Key Idea

We show that the following holds at each iteration:

- Each iteration selects at least one coordinate.
- All the selected coordinates have not been selected previously.
- For each incorrect coordinate chosen, a correct one is also chosen.

Guarantees

Theorem: Needell-Vershynin

For any measurement matrix with Restricted Isometry constant $\delta_{8s} \leq c/\sqrt{\log s}$, ROMP approximately reconstructs any arbitrary signal x from its noisy measurements $u = \Phi x + e$ in at most s iterations:

$$\|\hat{x} - x\|_2 \leq C\sqrt{\log s}\Big(\|e\|_2 + \frac{\|x - x_s\|_1}{\sqrt{s}}\Big).$$

Breakthrough

ROMP is the first greedy algorithm with strong guarantees similar to those of convex optimization methods! Note also that ROMP requires no prior knowledge about the error vector e.

Answering the questions

Important Questions

1. What kind(s) of measurement matrices?
   - Any that satisfy RIP (Generic)
2. How many measurements needed?
   - Approximately $s \log s \log d$
3. Are the guarantees uniform?
   - Uniform guarantees (via RIP)
4. Is algorithm stable?
   - Is stable.
5. Fast runtime?
   - Runtime is $O(smd)$.

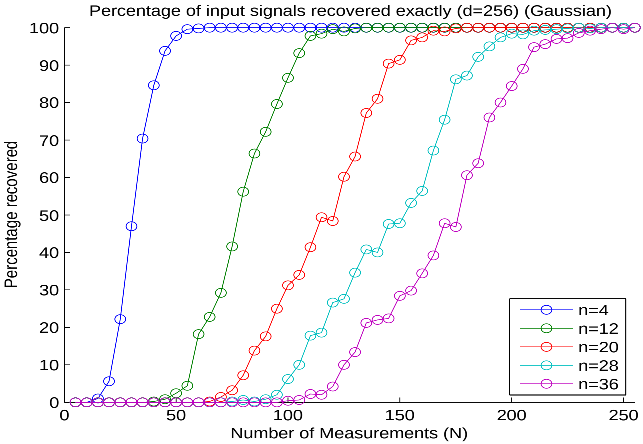


Figure: Sparse signals with noiseless measurements.

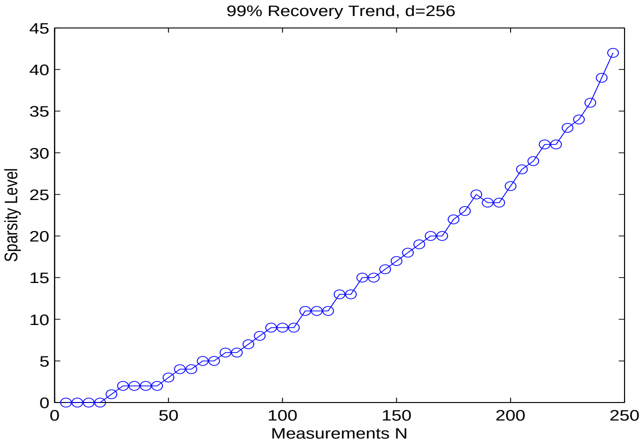


Figure: Sparse flat signals, Gaussian.

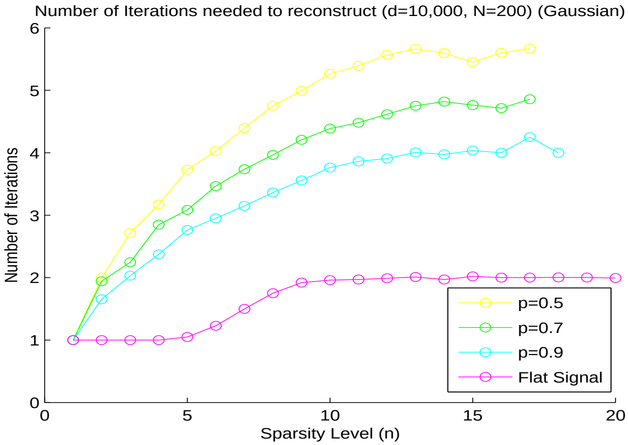


Figure: Iteration count for different kinds of sparse signals.

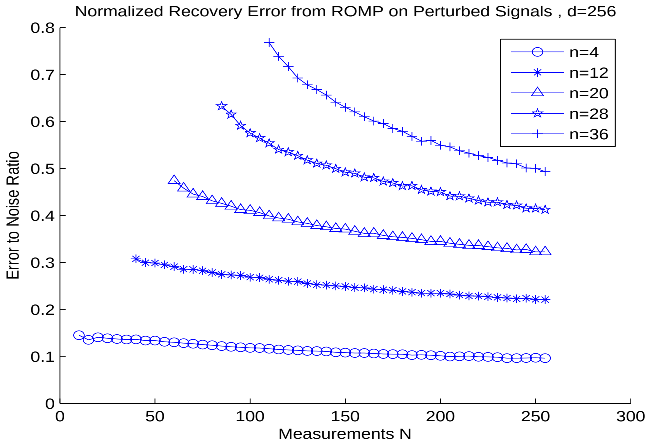


Figure: Sparse flat signals with Gaussian matrix.

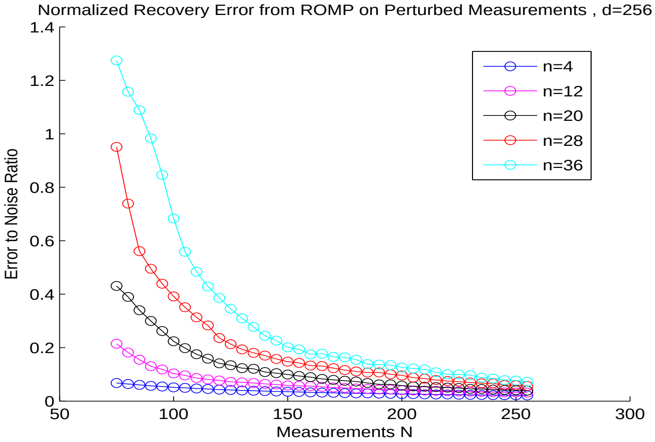


Figure: Error to noise ratio $\frac{\|\hat{v}-v\|_2}{\|e\|_2}$.

Gap in the approaches

	Convex Opt.		OMP
Uniform?	yes		no
Stable?	yes		no
Runtime?	(LP)		$O(smd)$

Bridging the Gap

	Convex Opt.	ROMP	OMP
Uniform?	yes	yes	no
Stable?	yes	yes	no
Runtime?	(LP)	$O(smd)$	$O(smd)$

Finishing remarks

- The logarithmic term $\log s$ appears both in the RIP and in the error bounds.
- Although ROMP posesses the main ideal properties, it is not entirely optimal because of the log factor.
- Compressive Sampling Matching Pursuit (CoSaMP) by Needell-Tropp removes the logarithmic term and provides truly optimal results.
- Both algorithms are efficient in practice.

For more information

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References:

- Needell and Tropp, "CoSaMP: Iterative signal recovery from incomplete and inaccurate samples," *ACHA*, July 2008.
- Needell and Vershynin, "Signal Recovery from Inaccurate and Incomplete Measurements via ROMP,", submitted.
- Tropp and Gilbert, "Signal recovery from random measurements via Orthogonal Matching Pursuit," *Trans. IT*, Dec. 2007.