Using Correlated Subset Structure for Compressive Sensing Recovery

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The mathematical problem (notation)

- Wish to construct high-resolution image $x \in \mathbb{C}^{N \times N}$ from low resolution $y \in \mathbb{C}^{n \times n}$

- Model: $y = SHx + \eta$
  - $S \in \mathbb{C}^{n^2 \times N^2}$: downsampling matrix
  - $H \in \mathbb{C}^{N^2 \times N^2}$: filtering (antialiasing) matrix
  - η: sensor noise

- Formulation: $x = \Psi c$
  - $\Psi \in \mathbb{C}^{N^2 \times N^2}$: sparsifying basis (ONB or frame)
  - $y = SH\Psi c + \eta = \Phi c + \eta$

- *Problem:* Reconstruct signal c from measurements y

Sampling matrix Φ

- Φ tyically assumed to be random/incoherent/RIP
- Here, Φ has structure and correlated columns
  - Assume H imperfect filter $\rightarrow$ Φ preserves enough high frequency info
  - Hope: SH and Ψ have sufficient incoherency
  - For typical sparsifiers Ψ, Φ has spatial/structured incoherence

Sampling matrix Φ

- Ψ: Haar basis, SH a 128×256 downsampler and filter

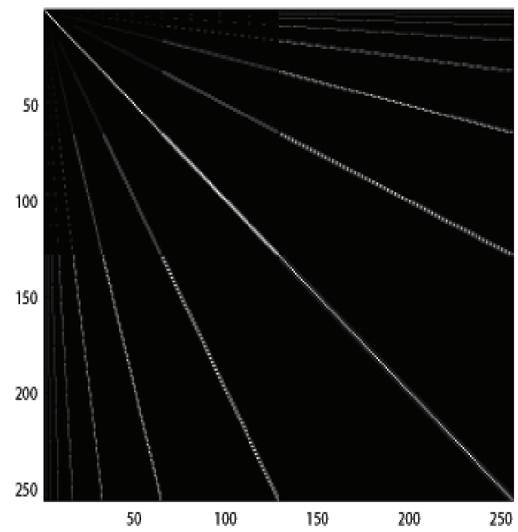

Figure 1: Absolute values of $\Phi^*\Phi$.

Filtered wavelet basis correlated with spatially overlapping bases, but uncorrelated with spatially distant ones.

Sampling matrix structure

- General problem: Sparse reconstruction from sampling operator with groups of correlated atoms

- Not necessarily due to some redundant dictionary

- How to exploit such structure?

- Simple modification of existing greedy algorithms?

CoSaMP

COSAMP (N-Tropp)

input: Sampling operator Φ, measurements y, sparsity level s
initialize: Set $x^0 = 0$, $i = 0$.
repeat
signal proxy: Set $p = \Phi^*(y - \Phi x^i)$, $\Omega = \text{supp}(p_{2s})$, $T = \Omega \cup \text{supp}(x^i)$.
signal estimation: Using least-squares, set $b|_T = \Phi_T^\dagger y$ and $b|_{T^c} = 0$.
prune and update: Increment i and to obtain the next approximation, set $x^i = b_s$.
output: s-sparse reconstructed vector $\widehat{x} = x^i$

Partial Inversion

PARTINV (Divekar-N)

input: Sampling operator Φ, measurements y, sparsity level s
initialize: $c^0 = \Phi^* y$, $\Omega = \mathrm{supp}(c^0_s)$, $i = 0$
repeat
  signal proxy: Set $c^i_\Omega = \Phi^\dagger_\Omega y$, $r = y - \Phi_\Omega c^i_\Omega$, $T = \Omega^c$
  signal estimation: $c^i_T = \Phi^*_T r$.
  prune and update: Set $\Omega = \mathrm{supp}(c^i_s)$, increment i.
  output: s-sparse reconstructed vector $\widehat{c} = c^i$

Motivation

- PartInv:

$$\hat{c}_\Omega = \Phi_\Omega^\dagger y = c_\Omega + (\Phi_\Omega^* \Phi_\Omega)^{-1} \Phi_\Omega^* \Phi_{\Omega^c} c_{\Omega^c}.$$

- CoSaMP:

$$\hat{c}_\Omega = \Phi_\Omega^* y = \Phi_\Omega^* \Phi_\Omega c_\Omega + \Phi_\Omega^* \Phi_{\Omega^c} c_{\Omega^c} = c_\Omega + (\Phi_\Omega^* \Phi_\Omega - I) c_\Omega + \Phi_\Omega^* \Phi_{\Omega^c} c_{\Omega^c}$$

- *High mutual interference* whereas $(\Phi_\Omega^* \Phi_\Omega)^{-1}$ can be controlled by tuning $|\Omega|$ $(= s)$.

- Improved error when Ω and Ω^c sufficiently uncorrelated.

- Better estimate → more accurate Ω → better estimate.

Experiments

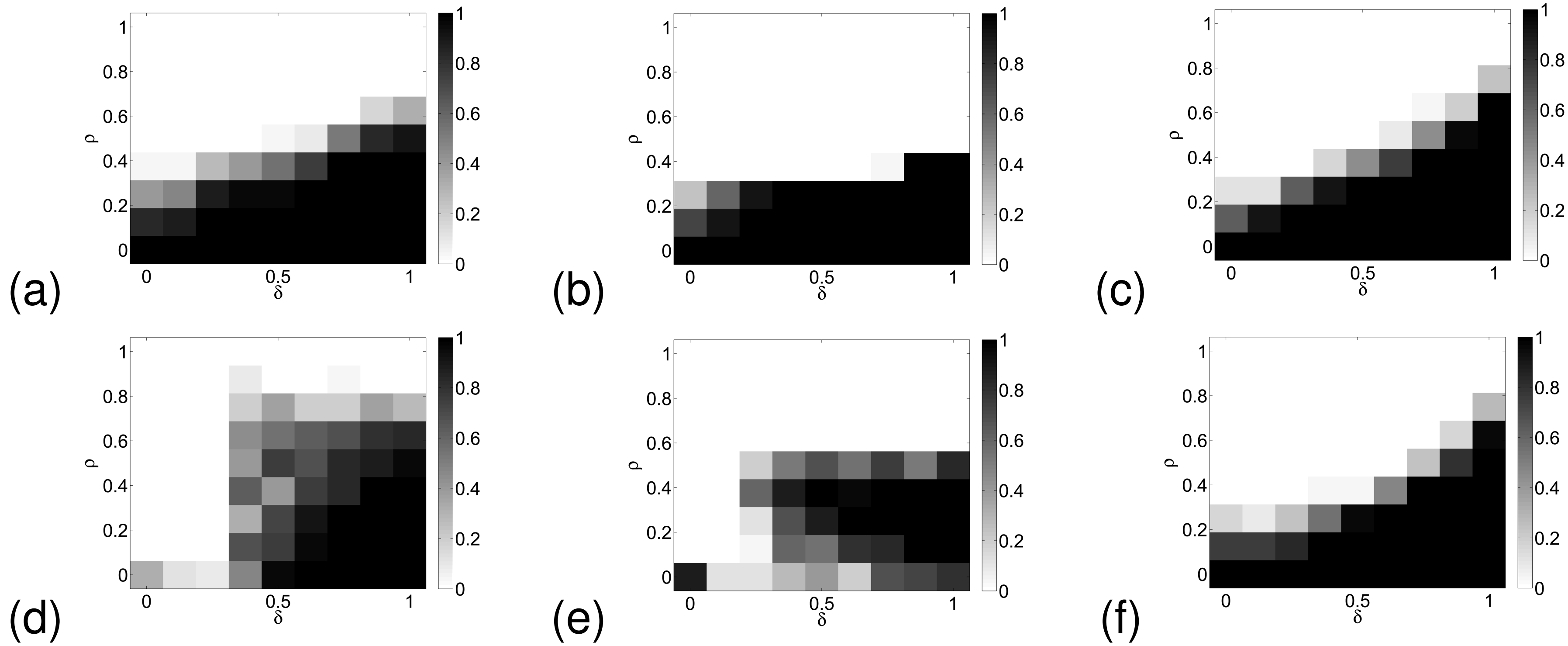

Figure 2: *Proportion of successes on Gaussian matrices using (a) PartInv, (b) CoSaMP and (c) ℓ_1-minimization, and proportion of successes on correlated column subset matrices using (d) PartInv, (e) CoSaMP and (f) ℓ_1-minimization for various values of* $\delta = \frac{M}{N} \in (0,1)$ *(horizontal axis) and* $\rho = \frac{s}{M} \in (0,1)$ *(vertical axis).*

Wavelet tree structured sparsity

- Suppose Ω is index set of wavelet basis belonging to a tree rooted at a coarse scale.
- Set $z_\Omega = \Phi_\Omega^* y = \Phi_\Omega^* \Phi_\Omega c_\Omega + \Phi_\Omega^* \Phi_{\Omega^c} c_{\Omega^c}$.
  - Ω and Ω^c uncorrelated → second term small
  - c_Ω has non-zero entries & Ω correlated → first term large
- Therefore, $s_\Omega = \sum_{j \in \Omega} |z_j|$ a good proxy for strength of non-zeros in tree Ω.

PartInv for wavelet tree structured sparsity

PARTINV II (Divekar-N)

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input: Sampling operator Φ, measurements y, sparsity level s, # trees t
initialize: $c^0 = \Phi^* y,\ i = 0$
* **For each** $j = 1 \ldots t$ **:** $s_j \leftarrow \sum_{l \in T_j} |c_l^i|$

* **Selection:** $\Omega \leftarrow$ indices of columns in the sets with the largest s_j, to include at least s

repeat
signal proxy: Set $c_\Omega^i = \Phi_\Omega^\dagger y,\ r = y - \Phi_\Omega c_\Omega^i,\ T = \Omega^c$
signal estimation: $c_T^i = \Phi_T^* r$.
prune and update: Repeat steps *, increment i.

output: s-sparse reconstructed vector $\widehat{c} = c^i$

Experiments

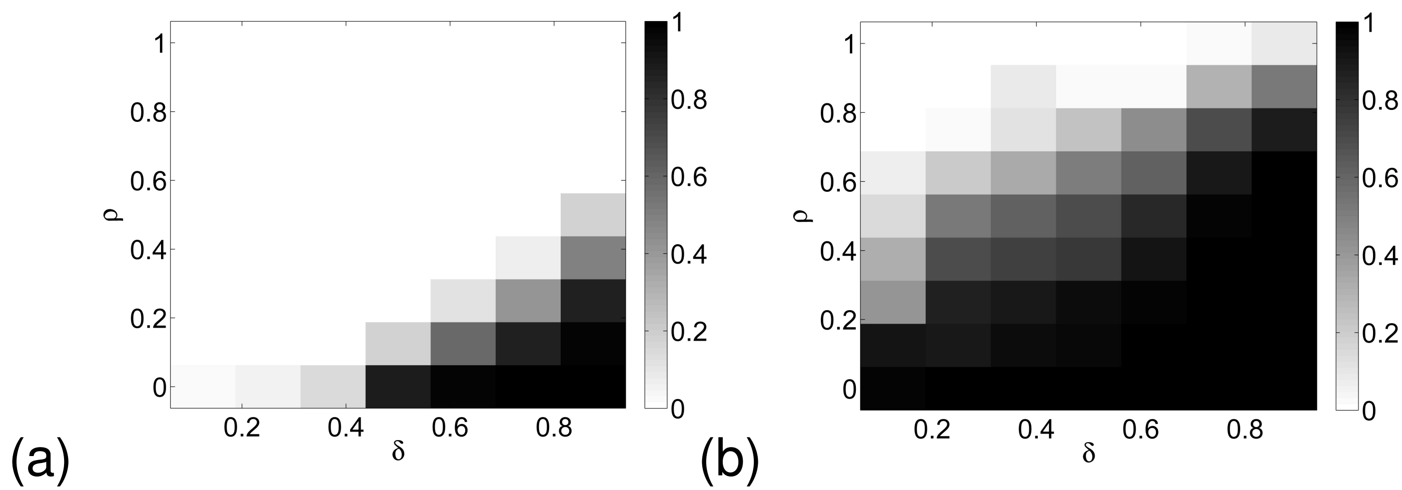

Figure 3: *Proportion of successes with nonzero coefficients concentrated on wavelet trees from (a) ℓ_1-minimization and (b) PartInv. Daubechies-5 wavelet basis using 32×32 patches with 5 levels of decomposition, using $t = 49$ tree sets.*

More

- Theoretical Results?
- Need to control $(\Phi_\Omega^* \Phi_\Omega^*)^{-1}$
- Can bound for certain signal/sampling schemes
- Can adapt to other sparsity structures
- Block
- Level sets

Thank you!

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References:

- A. Divekar and D. Needell. Using Correlated Subset Structure for Compressive Sensing Recovery. SAMPTA 2013.
- D. Needell and J. A. Tropp. CoSaMP: Iterative signal recovery from incomplete and inaccurate samples. Applied and Computational Harmonic Analysis, 26(3):301-321, 2008.