Lattices from equiangular tight frames with applications to lattice sparse recovery

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The compressed sensing problem

1. Signal of interest $f \in \mathbb{C}^d (= \mathbb{C}^{N \times N})$
2. Measurement operator $\mathcal{A} : \mathbb{C}^d \to \mathbb{C}^m$ $(m \ll d)$
3. Measurements $y = \mathcal{A}f + \xi$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} \quad & \mathcal{A} & \quad \end{bmatrix} \begin{bmatrix} f \\ \\ \end{bmatrix} + \begin{bmatrix} \xi \end{bmatrix}$$

4. *Problem:* Reconstruct signal f from measurements y

Sparsity

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Assume f is *sparse*:

- In the coordinate basis: $\|f\|_0 \stackrel{\text{def}}{=} |\text{supp}(f)| \leq s \ll d$
- In orthonormal basis: $f = Bx$ where $\|x\|_0 \leq s \ll d$

In practice, we encounter *compressible* signals.

- ⋆ f_s is the best s-sparse approximation to f

Many applications

- Radar, Error Correction
- Computational Biology, Geophysical Data Analysis
- Data Mining, classification
- Neuroscience
- Imaging
- Sparse channel estimation, sparse initial state estimation
- Topology identification of interconnected systems
- ...

Reconstruction approaches

⋆ ℓ_1-minimization [Candès-Romberg-Tao]
Let A satisfy the *Restricted Isometry Property* and set:

$$\hat{f} = \underset{g}{\operatorname{argmin}} \|g\|_1 \quad \text{such that} \quad \|\mathcal{A}f - y\|_2 \leq \varepsilon,$$

where $\|\xi\|_2 \leq \varepsilon$. Then we can stably recover the signal f:

$$\|f - \hat{f}\|_2 \lesssim \varepsilon + \frac{\|x - x_s\|_1}{\sqrt{s}}.$$

This error bound is optimal.
⋆ Other methods (iterative, greedy) too (OMP, ROMP, StOMP, CoSaMP, IHT, ...)

Restricted Isometry Property

- $\mathcal{A}$ satisfies the Restricted Isometry Property (RIP) when there is $\delta < c$ such that

$$(1-\delta)\|f\|_2 \leq \|\mathcal{A}f\|_2 \leq (1+\delta)\|f\|_2 \quad \text{whenever } \|f\|_0 \leq s.$$

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$\star$ Related to dimension reduction and the Johnson-Lindenstrauss Lemma (dimension reduction with preserved geometry).

Sparsity plus other structures?

What if signal is also *lattice*-valued?

- Wireless communications
- Radar (massive MIMO) [Rossi et.al.]
- Wideband spectrum sensing [Axell et.al.]
- Error correcting codes [Candès et.al.]
- ...

Lattices

What is a lattice?

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A **lattice** $\Lambda \subset \mathbb{R}^n$ of rank $1 \leq k \leq n$ is a free $\mathbb{Z}$-module of rank k, which is the same as a discrete co-compact subgroup of $V := \operatorname{span}_{\mathbb{R}} \Lambda$. If $k = n$, i.e. $V = \mathbb{R}^n$, we say that Λ is a lattice of **full rank** in $\mathbb{R}^n$. Hence

$$\Lambda = \operatorname{span}_{\mathbb{Z}}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_k\} = A\mathbb{Z}^k,$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k \in \mathbb{R}^n$ are $\mathbb{R}$-linearly independent **basis** vectors for Λ and $A = (\boldsymbol{a}_1 \ \dots \ \boldsymbol{a}_k)$ is the corresponding $n \times k$ basis matrix.

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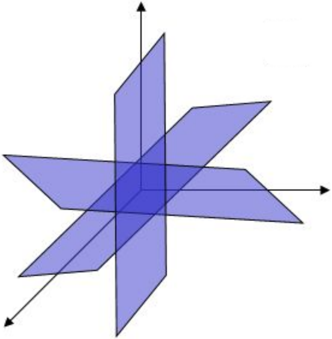
$\star$ A *sparse lattice-valued signal* is $v \in \Lambda$ with $\|v\|_0 \leq s$. Alternatively can consider $v = Aw$ where $w \in \mathbb{Z}^k$ and $\|w\|_0 \leq s$.

Lattices



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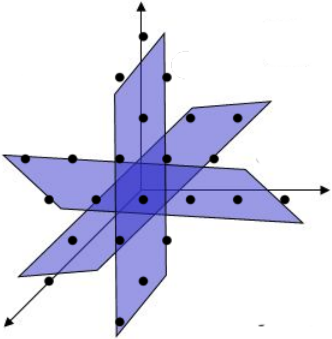

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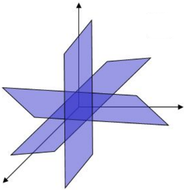

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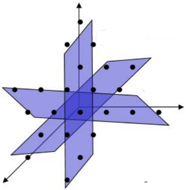

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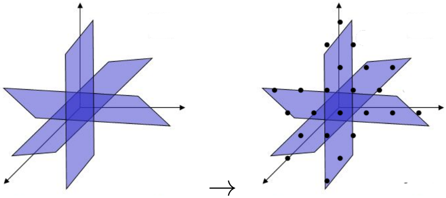

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- Question: when is lattice knowledge helpful??

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- Or suppose instead that $x \in \Lambda = \mathbb{Z}^d$. ??
- The point: sometimes lattice info can give a huge savings. Sometimes maybe not?

Some results

- Dense ± 1 signals [Mangasarian-Recht '11] :
  $\min \|x\|_\infty \quad \text{s.t.} \quad \mathcal{A}x = y$

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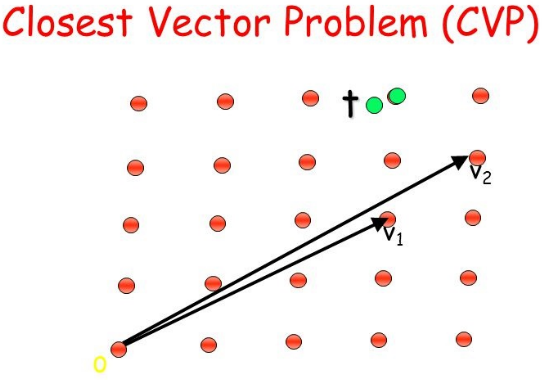
- Sparse lattice signals [Flinth-Kutyniok '16] : OMP with initialization step (PROMP)

Some results [Sphere decoders]

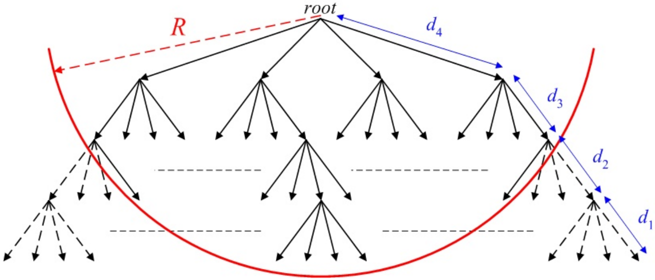
⋆ The closest point problem:



⋆ Find point in lattice closest to a given vector in some metric (e.g. $\|x - y\|_2$).

Some results [Sphere decoders]

⋆ Sphere decoder:



⋆ Using some ordering of the lattice (recursively), prune the search tree using spheres of specified radius.

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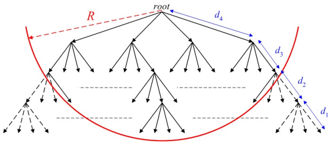

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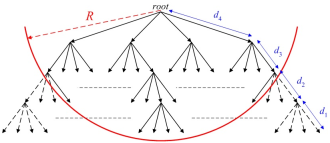

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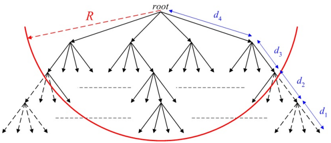

- Use sphere decoder method with metric $\|y - Ax\|_2 + \lambda\|x\|_0$
- Lattice pruning/ordering no longer clear in this metric
- Lack of rigorous theory

Some results [Flinth-Kutyniok '16]

⋆ PROMP:

- Run least squares $\hat{x} = \text{argmin}_x \|\mathcal{A}x - y\|_2$
- "Carefully" threshold and keep support estimate S
- Run OMP initialized with this support estimate and $\hat{x}_S$

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⋆ Some theory about accuracy of initialization

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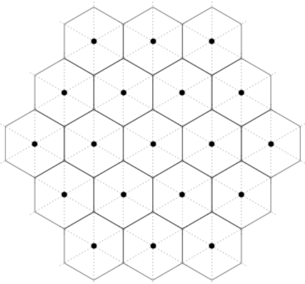
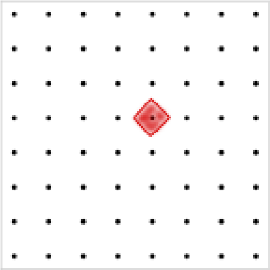
⋆ Same is true for lattices whose Voronoi region Ω satisfies $A^{-1}\Omega \subset (-1,1)^k$.

Voronoi: $\Omega \stackrel{\text{def}}{=} \{v : \forall z \in A\mathbb{Z}^k, \|v\|_2 \leq \|v - z\|_2\}$. (e.g. diamond)





Lattices: minimal vectors

Minimal norm of a lattice Λ is

$$|\Lambda| = \min\left\{\|\boldsymbol{x}\| : \boldsymbol{x} \in \Lambda \setminus \{\mathbf{0}\}\right\},$$

where $\| \ \|$ is Euclidean norm. The set of **minimal vectors** of Λ is

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- If $\operatorname{rk} \Lambda > 4$, a strictly stronger condition is that Λ is **generated by minimal vectors**, i.e. $\Lambda = \operatorname{span}_{\mathbb{Z}} S(\Lambda)$.
- It has been shown by Conway & Sloane (1995) and Martinet & Schürmann (2011) that there are lattices of rank ≥ 10 generated by minimal vectors which do not contain a **basis of minimal vectors**.

Lattices: eutaxy and perfection

Let $k = \operatorname{rk} \Lambda$ and

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This lattice is called **eutactic** if there exist positive real numbers $c_1, \dots, c_m$ such that

$$\|\boldsymbol{v}\|^2 = \sum_{i=1}^{m} c_i \langle \boldsymbol{v}, \boldsymbol{x}_i \rangle^2$$

for every vector $\boldsymbol{v} \in \operatorname{span}_{\mathbb{R}} \Lambda$, where $\langle \cdot, \cdot \rangle$ is the usual inner product.

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for every vector $\boldsymbol{v} \in \operatorname{span}_{\mathbb{R}} \Lambda$, where $\langle \cdot, \cdot \rangle$ is the usual inner product. If $c_1 = \dots = c_m$, we say that Λ is **strongly eutactic** (e.g. $\mathbb{Z}^d$).

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This lattice is called **perfect** if the set of symmetric matrices

$$\{\boldsymbol{x}_i \boldsymbol{x}_i^t : \boldsymbol{x}_i \in S(\Lambda)\}$$

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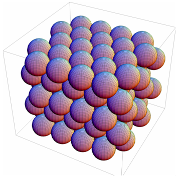
⋆ These properties arise in classifying lattices and sphere packing problems.



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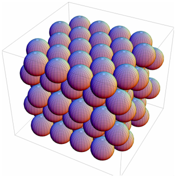

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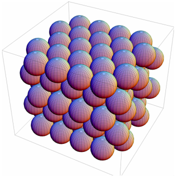

⋆ If a lattice is strongly eutactic, but not perfect, then it is a local minimum of the packing density function. Extremal lattices are local maxima.

Equiangular frames

Another interesting construction of lattices comes from *frames*. A collection of $n \geq k$ unit vectors $\boldsymbol{f}_1, \dots, \boldsymbol{f}_n \in \mathbb{R}^k$ is called an (real) (k, n)-**equiangular tight frame** (ETF) if it spans $\mathbb{R}^k$ and

1. $|\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle| = c$ for all $1 \leq i \neq j \leq n$, for some constant $c \in [0, 1]$,
2. $\sum_{i=1}^{n} \langle \boldsymbol{f}_i, \boldsymbol{x} \rangle^2 = \gamma \|\boldsymbol{x}\|^2$ for each $\boldsymbol{x} \in \mathbb{R}^k$, for some absolute constant $\gamma \in \mathbb{R}$.

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If this is the case, it is known that

$$k \leq n \leq \frac{k(k+1)}{2}, \ c = \sqrt{\frac{n-k}{k(n-1)}}, \ \gamma = \sqrt{\frac{n}{k}}.$$

Mercedes

Here is a (2, 3)-ETF $\mathcal{F} := \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix} \right\}$:

Mercedes

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Notice that $\pm\mathcal{F} = S(\Lambda_h)$, the set of minimal vectors of the hexagonal lattice $\Lambda_h = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix} \mathbb{Z}^2$.

Lattices: questions

- When does the (integer) span of an ETF form a lattice?
- If so, does it have a basis of minimal vectors?
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Consequences:

- If the span is a lattice, the frame viewed as a sensing matrix yields an image that is a discrete set.
- If the frame atoms are minimal vectors, we can guarantee separation between sample vectors in its image.
- Johnson-Lindenstrauss may then be used for reconstruction guarantees?
- When is reconstruction impossible?

Lattice construction

Let $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$ be a (k, n)-ETF, and define

$$\Lambda(\mathcal{F}) = \operatorname{span}_{\mathbb{Z}} \mathcal{F}.$$

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Proposition 2 (Böttcher, Fukshansky, Garcia, Maharaj, N- '16)

If $\Lambda(\mathcal{F})$ is a lattice and

$$S(\Lambda(\mathcal{F})) = \{\pm\boldsymbol{f}_1, \dots, \pm\boldsymbol{f}_n\},$$

then $\Lambda(\mathcal{F})$ is strongly eutactic.

Main results on ETF lattices

Theorem 3 (Böttcher, Fukshansky, Garcia, Maharaj, N- '16)

1. *For every $k \geq 2$, there are $(k, k+1)$-ETFs $\mathcal{F}$ such that $\Lambda(\mathcal{F})$ is a full-rank lattice. This lattice has a basis of minimal vectors, is non-perfect and strongly eutactic.*

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1. *For every* $k \geq 2$*, there are* $(k, k+1)$*-ETFs* $\mathcal{F}$ *such that* $\Lambda(\mathcal{F})$ *is a full-rank lattice. This lattice has a basis of minimal vectors, is non-perfect and strongly eutactic.*
2. *There are infinitely many* k *for which there exist* $(k, 2k)$*-ETFs* $\mathcal{F}$ *such that* $\Lambda(\mathcal{F})$ *is a full-rank lattice, e.g.* $(5, 10)$, $(13, 26)$.

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5. *There is a $(6, 16)$-ETF $\mathcal{F}$ for which $\Lambda(\mathcal{F})$ is a full-rank lattice that has a basis of minimal vectors.*

Remarks

- There are often multiple ETFs with the same parameters (k, n). For instance, we exhibit two lattices from $(5, 10)$-ETFs, three lattices from $(13, 26)$-ETFs, and ten lattices from $(25, 50)$-ETFs. We also compute determinants of all our examples.

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- Perfection of the lattice from $(7, 28)$-ETF was previously (2015) established by Roland Bacher, however he constructed this lattice differently and then remarked that its minimal vectors comprise a set of equiangular lines.
- Minimal vectors of ETF lattices often are precisely $\pm$ frame vectors (this is the case with all our examples). In this case, the set of corresponding symmetric matrices has *at most* $k(k+1)/2$ matrices, which is the *least* possible number required to span all symmetric matrices. Hence ETF lattices are unlikely to be perfect (and hence extremal) – the $(7, 28)$ case is likely an exception.

Future directions

- Further study *geometric* properties of ETFs to decipher when they create a lattice.
- Given a lattice ETF whose atoms are minimal vectors, *how* can we reconstruct lattice signals?
- How can we incorporate sparsity? $\rightarrow$ need Johnson-Lindenstrauss
- Computationally efficient reconstructions that beat classical CS methods?

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Thank you!

Now "lattice" take any questions...