LOCAL DIMENSION OF NORMAL SPACES

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Introduction

LET $\dim X$ be the covering dimension of a space X and let $\operatorname{ind} X$ and $\operatorname{Ind} X$ be the dimensions defined inductively in terms of the boundaries of neighbourhoods of points and closed sets respectively. The local dimension $\operatorname{loc}\dim X$ is the least number n such that every point has a closed neighbourhood $\bar{U}$ with $\dim \bar{U} \leqslant n$. The local inductive dimension $\operatorname{loc}\operatorname{Ind} X$ is defined analogously, while $\operatorname{ind} X$ is already a local property.

The subset theorem, that $\dim A \leqslant \dim X$ for $A \subset X$, which was proved by E. Čech (3) for perfectly normal spaces is here extended to totally normal spaces. Čech's problem (4) of whether the subset theorem holds for completely normal Hausdorff spaces is reduced to the problem of whether the local dimension of a completely normal Hausdorff space is always equal to its dimension: that is, whether $\operatorname{loc}\dim X = \dim X$ for X completely normal and Hausdorff. But it follows from [3.7] below that a completely normal space X such that $\operatorname{loc}\dim X < \dim X$, if any such exists, must be neither paracompact nor the union of a sequence of closed paracompact sets nor the union of two paracompact sets one of which is closed. Thus most of the usual methods of constructing counter-examples are excluded.

However, a normal space M is constructed for which

$$\operatorname{loc}\dim M < \dim M.$$

Though this example is not completely normal, it is not clear that the lack of complete normality plays any significant role.

It is well known [(6) appendix] that a normal Hausdorff space may have a non-normal subspace of higher dimension. An example is given below of a normal Hausdorff space N with a normal subspace M such that $\dim N = \operatorname{Ind} N = 0$ but $\dim M = \operatorname{Ind} M = 1$.

The normal space M also has the property that $\operatorname{ind} M < \dim M$. Examples are known (8, 9) of normal Hausdorff spaces X such that $\operatorname{ind} X > \dim X$. Thus for normal Hausdorff spaces there are the known relations $\operatorname{ind} X \leqslant \operatorname{Ind} X$ and $\dim X \leqslant \operatorname{Ind} X$ and no others.

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A lemma which proves useful in several of the proofs below is the following (see [2.1]): *If a closed set A of a normal space X is at most n-dimensional, and if every closed set which does not meet A is at most n-dimensional, then* $\dim X \leqslant n$.

1. Definitions and elementary relations

A *covering* of a topological space X is a collection of open sets of X whose union is X. A covering β is called a *refinement* of a covering α if each member of β is contained in some member of α.

The *order* of a collection of subsets of a space X is -1 if all the subsets are empty; otherwise the *order* is the largest integer n such that some $n+1$ members of the collection have a non-empty intersection, or is ∞ if there is no such largest number.

The *dimension* of a space X, $\dim X$, is the least integer n such that every finite covering of X has a refinement of order not exceeding n, or the dimension is ∞ if there is no such integer.

[1.1] *For any space X, $\dim X \leqslant n$ if and only if, for every finite covering $\{U_1,..., U_k\}$ of X, there is a covering $\{V_1,..., V_k\}$ of order not exceeding n with each $V_i \subset U_i$.*

Proof. The condition is clearly sufficient, for $\{V_i\}$ is a refinement of $\{U_i\}$. To show necessity, let $\dim X \leqslant n$. Then the covering $\{U_i\}$ has some refinement β of order not exceeding n. Let each member of β be associated with one of the sets U_j containing it and let V_i be the union of the sets of β thus associated with U_i. Then V_i is open, $V_i \subset U_i$, and each point of X is in some member of β, and hence in some V_i. Each point p is in at most $n+1$ members of β, each of which is associated with a unique U_j, and hence p is in at most $n+1$ members of V_i. Thus V_i is a covering of order not exceeding n, as was to be shown.

The inductive dimensions $\operatorname{ind} X$ and $\operatorname{Ind} X$ are defined inductively as follows. If X is the empty set, $\operatorname{ind} X = \operatorname{Ind} X = -1$. If $\operatorname{ind} X \leqslant n-1$ has already been defined, $\operatorname{ind} X \leqslant n$ means that, for each point p and open set U with $p \in U$, there is an open set V with $p \in V \subset U$ for which $\operatorname{ind}(\bar{V}-V) \leqslant n-1$. Similarly, $\operatorname{Ind} X \leqslant n$ means that, for each closed set F and open set U with $F \subset U$, there is an open set V with $F \subset V \subset U$ for which $\operatorname{Ind}(\bar{V}-V) \leqslant n-1$. And $\operatorname{ind} X = \infty$ [$\operatorname{Ind} X = \infty$] means that there is no integer n for which $\operatorname{ind} X \leqslant n$ [$\operatorname{Ind} X \leqslant n$].

It is known [(6) appendix] that, for an arbitrary space X, $\dim X = 0$ if and only if $\operatorname{Ind} X = 0$. If A is any subset of a space X, then $\operatorname{ind} A \leqslant \operatorname{ind} X$ [(6) appendix]. If A is a closed subset of X, then

$$\dim A \leqslant \dim X \text{ [(3) §4]} \quad \text{and} \quad \operatorname{Ind} A \leqslant \operatorname{Ind} X \text{ [(2) §16]}.$$

If p is a point of a space X, the *dimension of* X at p, $\dim_p X$, is defined as follows: $\dim_p X$ is the least integer n such that, for some open set U containing p, $\dim \bar{U} = n$ or, if there is no such integer, $\dim_p X = \infty$.

Similarly $\operatorname{ind}_p X$ [$\operatorname{Ind}_p X$] is defined to be the least integer n such that, for some open set U containing p, $\operatorname{ind} \bar{U} = n$ [$\operatorname{Ind} \bar{U} = n$] or, if there is no such integer, $\operatorname{ind}_p X = \infty$ [$\operatorname{Ind}_p X = \infty$]. It should be noted that the dimension at a point as defined here is entirely different from that defined by Menger (**10**) and Hurewicz and Wallman (**6**).

The *local dimension* of a space X, $\operatorname{loc}\dim X$, is defined as follows. If X is empty, $\operatorname{loc}\dim X = -1$. Otherwise, $\operatorname{loc}\dim X$ is the least integer n such that, for every point $p \in X$, $\dim_p X \leqslant n$ or, if there is no such integer, $\operatorname{loc}\dim X = \infty$.

The *local inductive dimensions*, $\operatorname{loc}\operatorname{ind} X$ and $\operatorname{loc}\operatorname{Ind} X$, are defined similarly. If X is empty, $\operatorname{loc}\operatorname{ind} X = \operatorname{loc}\operatorname{Ind} X = -1$. Otherwise, $\operatorname{loc}\operatorname{ind} X$ [$\operatorname{loc}\operatorname{Ind} X$] is the least integer n such that, for every point $p \in X$, $\operatorname{ind}_p X \leqslant n$ [$\operatorname{Ind}_p X \leqslant n$] or, if there is no such integer,

$$\operatorname{loc}\operatorname{ind} X = \infty \quad [\operatorname{loc}\operatorname{Ind} X = \infty].$$

Thus $\operatorname{loc}\dim X$ [$\operatorname{loc}\operatorname{ind} X$, $\operatorname{loc}\operatorname{Ind} X$] is the least integer n such that there exists a covering $\{U_\lambda\}$ of X with each $\dim \bar{U}_\lambda \leqslant n$ [$\operatorname{ind} \bar{U}_\lambda \leqslant n$, $\operatorname{Ind} \bar{U}_\lambda \leqslant n$], or, if there is no such integer, $\operatorname{loc}\dim X = \infty$ [$\operatorname{loc}\operatorname{ind} X = \infty$, $\operatorname{loc}\operatorname{Ind} X = \infty$].

[1.2] *For any point p of a space X, $\dim_p X \leqslant n$ if and only if each neighbourhood U of p contains a neighbourhood V of p with $\dim \bar{V} \leqslant n$.*

Proof. By the definition of $\dim_p X$, if there is any open set V with $p \in V$ and $\dim \bar{V} \leqslant n$, then $\dim_p X \leqslant n$. On the other hand, if

$$\dim_p X \leqslant n,$$

there exists a neighbourhood W of p with $\dim \bar{W} \leqslant n$. If $V = U \cap W$, then $p \in V \subset U$ and, since $\bar{V}$ is a closed subset of $\bar{W}$, $\dim \bar{V} \leqslant n$.

[1.3] *For any space X, $\operatorname{loc}\dim X \leqslant n$ if and only if every covering of X has a refinement $\{U_\lambda\}$ with each $\dim \bar{U}_\lambda \leqslant n$.*

This follows immediately from [1.2]. Analogous propositions are clearly true of $\operatorname{loc}\operatorname{ind} X$ and $\operatorname{loc}\operatorname{Ind} X$.

[1.4] *For any space X, $\operatorname{loc}\dim X \leqslant \dim X$, $\operatorname{loc}\operatorname{ind} X \leqslant \operatorname{ind} X$, and $\operatorname{loc}\operatorname{Ind} X \leqslant \operatorname{Ind} X$.*

Proof. If $\dim X \leqslant n$, then every point $p \in X$ has the neighbourhood X for which $\dim \bar{X} = \dim X \leqslant n$, and hence $\dim_p X \leqslant n$. Thus $\operatorname{loc}\dim X \leqslant n$. Similarly $\operatorname{ind} X \leqslant n$ implies $\operatorname{loc}\operatorname{ind} X \leqslant n$ and

$$\operatorname{Ind} X \leqslant n \quad \text{implies} \quad \operatorname{loc}\operatorname{Ind} X \leqslant n.$$

[1.5] *For any space* X, $\operatorname{loc}\operatorname{ind} X = \operatorname{ind} X$.

Proof. Let $\operatorname{loc}\operatorname{ind} X \leqslant n$ and let $p \in U \subset X$ with U open. Then there is some open set W with $p \in W$ and $\operatorname{ind} \bar{W} \leqslant n$. Hence $p \in U \cap W$ with $U \cap W$ open in $\bar{W}$. Hence, by the definition of $\operatorname{ind} \bar{W}$, there is a set V open in $\bar{W}$ with $p \in V \subset U \cap W$ and $\operatorname{ind} B \leqslant n-1$, where B is the boundary of V in $\bar{W}$. But, since V is open in $\bar{W}$, it is open in W and hence in X. Also the closure of V in $\bar{W}$ is its closure $\bar{V}$ in X and hence $B = \bar{V}-V$. Thus $p \in V \subset U$ with V open and $\operatorname{ind}(\bar{V}-V) \leqslant n-1$. Therefore

$$\operatorname{ind} X \leqslant n.$$

Thus $\operatorname{ind} X \leqslant \operatorname{loc}\operatorname{ind} X$, and so $\operatorname{loc}\operatorname{ind} X = \operatorname{ind} X$, as was to be shown.

Thus the inductive dimension $\operatorname{ind} X$ is already a local property of X. But $\dim X$ and $\operatorname{Ind} X$ are not in general local properties, as will be seen in § 6 below.

[1.6] *If* X *is a regular space*, $\operatorname{ind} X \leqslant \operatorname{loc}\operatorname{Ind} X$.

Proof. This is shown inductively. Clearly $\operatorname{loc}\operatorname{Ind} X = -1$ implies $\operatorname{ind} X = -1$. Assume it proved that

$$\operatorname{loc}\operatorname{Ind} X \leqslant n-1 \quad \text{implies} \quad \operatorname{ind} X \leqslant n-1.$$

Let $\operatorname{loc}\operatorname{Ind} X \leqslant n$ and let $p \in U \subset X$ with U open. Then, for some open set W, $p \in W$ and $\operatorname{Ind} \bar{W} \leqslant n$. Since X is regular, there is an open set G with $p \in G \subset \bar{G} \subset U \cap W$. Since $\operatorname{Ind} \bar{W} \leqslant n$, there is an open set V of $\bar{W}$ with $\bar{G} \subset V \subset U \cap W$ and $\operatorname{Ind} B \leqslant n-1$, where B is the boundary of V in $\bar{W}$. Since V is open in $\bar{W}$, it is open in W and hence in X. The closure of V in $\bar{W}$ is its closure $\bar{V}$ in X, and hence $B = \bar{V}-V$. Since $B \subset X$, B is regular and, since $\operatorname{Ind} B \leqslant n-1$, $\operatorname{loc}\operatorname{Ind} B \leqslant n-1$. Hence by the induction hypothesis $\operatorname{ind} B \leqslant n-1$. Thus $p \in V \subset U$ with V open and $\operatorname{ind}(\bar{V}-V) \leqslant n-1$. Hence $\operatorname{ind} X \leqslant n$, as was to be shown.

The above result, $\operatorname{ind} X \leqslant \operatorname{loc}\operatorname{Ind} X$, is in fact clearly true for any space in which the closure of each one-point set has no proper closed subset, and hence true for T_1 spaces as well as regular spaces.

It is known (**11**) that, if X is a normal space, $\dim X \leqslant \operatorname{Ind} X$. It follows that, for a normal space X, $\operatorname{loc}\dim X \leqslant \operatorname{loc}\operatorname{Ind} X$. For, if $\operatorname{loc}\operatorname{Ind} X \leqslant n$, then each point $p \in X$ has a neighbourhood U with

$\operatorname{Ind}\overline{U} \leqslant n$, and the closed set $\overline{U}$ of X is normal; hence $\dim\overline{U} \leqslant n$. Thus $\operatorname{loc}\dim X \leqslant n$.

Thus, using [1.4], [1.5], and [1.6], we have the following:

[1.7] *If X is a normal regular space, then*

$$\operatorname{ind} X \leqslant \operatorname{loc}\operatorname{Ind} X \leqslant \operatorname{Ind} X,$$
$$\operatorname{loc}\dim X \leqslant \operatorname{loc}\operatorname{Ind} X \leqslant \operatorname{Ind} X,$$
$$\operatorname{loc}\dim X \leqslant \dim X \leqslant \operatorname{Ind} X.$$

Since every normal Hausdorff space is regular, the inequalities of [1.7] hold in particular for normal Hausdorff spaces.

2. Properties of dimension

[2.1] *Let A be a closed set of a normal space X. If* $\dim A \leqslant n$ *and if* $\dim F \leqslant n$ *for every closed set F of X which does not meet A, then*

$$\dim X \leqslant n.$$

Proof. Let $\{U_1,\ldots, U_k\}$ be a covering of X. Then, by (3) § 22, since X is normal and A is closed and $\dim A \leqslant n$, there exists a collection $\{V_i\}$ of open sets of X of order not exceeding n with each $V_i \subset U_i$ and $A \subset \bigcup_{i=1}^{k} V_i$. Let $V = \bigcup_{i=1}^{k} V_i$; then V is open and $A \subset V$.

Since X is normal, there exist open sets P and Q such that

$$X-V \subset P \subset \overline{P} \subset Q \subset \overline{Q} \subset X-A.$$

Since $\overline{Q}$ is closed and $\overline{Q} \cap A = 0$, $\dim \overline{Q} \leqslant n$. The sets $U_i \cap P$ together with the sets V_i form a collection of open sets of X covering the closed set $\overline{Q}$. Hence, by (3) § 22, $\overline{Q}$ is covered by a collection $\{G_i, H_j\}$ of open sets of X of order not exceeding n with $G_i \subset U_i \cap P$ and $H_j \subset V_j$.

Let $W_i = G_i \cup H_i \cup (V_i-\overline{P})$; then W_i is open in X and $W_i \subset U_i$. Each point of $\overline{P}$ is in at most $n+1$ of the sets $\{G_i, H_j\}$ and in none of the sets $V_i-\overline{P}$; hence it is in at most $n+1$ of the sets W_i. Each point of $X-\overline{P}$ is in none of the sets G_i and in at most $n+1$ of the sets V_i and hence, since $H_i \subset V_i$, it is in at most $n+1$ of the sets W_i. Thus $\{W_i\}$ is of order not exceeding n. Each point of $\overline{P}$ is contained in $\overline{Q}$ and hence in some G_i or H_j; hence it is in some W_i. And each point of $X-\overline{P}$ is in some $V_i-\overline{P}$ and hence in some W_i. Thus $\{W_i\}$ is a covering of X. Thus $\{W_i\}$ is a refinement of $\{U_i\}$ of order not exceeding n. Hence $\dim X \leqslant n$, as was to be shown.

[2.2] *If a normal space X is the union of two sets A and B with A closed and* $\dim A \leqslant n$ *and* $\dim B \leqslant n$, *then* $\dim X \leqslant n$.

Proof. If F is a closed set of X which does not meet A, then F is a closed subset of B and $\dim F \leqslant \dim B \leqslant n$. Hence, by [2.1], $\dim X \leqslant n$, as was to be shown.

[2.3] *If A is a closed set of a normal space X, then*

$$\dim X \leqslant \max(\dim A, \dim(X-A)).$$

Proof. This follows immediately from [2.2] on setting $B = X-A$.

A normal space X is called *totally normal* (5) if each open subspace Y of X has a locally finite covering by open subsets each of which is an F_σ set of X. As will be shown in [2.6] below, the inequality in [2.3] becomes equality if X is totally normal.

[2.4] *Each totally normal space is regular.*

Proof. If X is totally normal, let $p \in Y \subset X$ with Y open. Then p is in some open F_σ set U contained in Y and, since U is an F_σ set, there is a closed set F with $p \in F \subset U$. Since X is normal, there is an open set V with $F \subset V \subset \bar{V} \subset U$; then $p \in V \subset \bar{V} \subset Y$. Thus X is regular.

[2.5] *Let a space X be the union of disjoint sets A each of which is open and closed in X. If each $\dim A_\lambda \leqslant n$, then $\dim X \leqslant n$.*

Proof. Let $\{U_1,\ldots, U_k\}$ be any finite covering of X. Then

$$\{U_1 \cap A_\lambda,\ldots, U_k \cap A_\lambda\}$$

is a covering of A_λ and $\dim A_\lambda \leqslant n$; hence there is a covering $\{V_{\lambda i}\}$ of A_λ of order not exceeding n with $V_{\lambda i} \subset U_i \cap A_\lambda$. Let $V_i = \bigcup_\lambda V_{\lambda i}$; then $\{V_i\}$ is a covering of X of order not exceeding n and $V_i \subset U_i$. Therefore

$$\dim X \leqslant n,$$

as was to be shown.

[2.6] *If Y is an open set of a totally normal space X, then* $\dim Y \leqslant \dim X$.

Proof. Let $\dim X \leqslant n$; it is sufficient to show that $\dim Y \leqslant n$. Since Y is an open set of a totally normal space X, we know [(5) proposition 4.3] that for each $i = 1, 2,\ldots$ there is a collection $\{W_{i\lambda}\}$, locally finite in Y, of disjoint open sets and a corresponding collection $\{F_{i\lambda}\}$ of closed sets of X such that $F_{i\lambda} \subset W_{i\lambda} \subset Y$ and $\bigcup_{i=1}^{\infty} \bigcup_\lambda F_{i\lambda} = Y$. Since $F_{i\lambda}$ is closed in X, $\dim F_{i\lambda} \leqslant \dim X \leqslant n$. Let $F_i = \bigcup_\lambda F_{i\lambda}$; then since, for fixed i, $\{F_{i\lambda}\}$ is locally finite in Y, F_i is closed in Y. Likewise $\bigcup_{\mu\neq\lambda} F_{i\mu}$ is closed in Y, and hence $F_{i\lambda} = F_i - \bigcup_{\mu\neq\lambda} F_{i\mu}$ is open in F_i. Therefore, by [2.5], $\dim F_i \leqslant n$. Since X is totally normal, it is completely normal [(5) proposition 4.6], and hence Y is normal. By the sum theorem [(3) § 23], since $Y = \bigcup_{i=1}^{\infty} F_i$ with F_i closed in Y and $\dim F_i \leqslant n$, therefore $\dim Y \leqslant n$, as was to be shown.

[2.7] *If, for each open set Y of a space X, $\dim Y \leqslant n$, then, for each set A of X, $\dim A \leqslant n$.*

Proof. Let $\{G_1,..., G_k\}$ be a covering of A. Each G_i is open in A and hence there is an open set U_i of X such that $G_i = U_i \cap A$. Let $Y = \bigcup_{i=1}^{k} U_i$; then $A \subset Y$ and Y is open in X, and hence $\dim Y \leqslant n$. Hence there is a covering $\{V_i\}$ of Y of order not exceeding n with each $V_i \subset U_i$. Then $\{V_i \cap A\}$ is a covering of A of order not exceeding n and

$$V_i \cap A \subset U_i \cap A = G_i.$$

Hence $\dim A \leqslant n$ as was to be shown.

The subset theorem, which was proved by Čech [(3) § 28] for perfectly normal spaces can be extended to totally normal spaces as follows.

[2.8] *If A is a set in a totally normal space X, then* $\dim A \leqslant \dim X$.

Proof. This follows immediately from [2.6] and [2.7].

It is also true [(5) Theorem 2] that, if A is any subset of a totally normal space X, then $\operatorname{Ind} A \leqslant \operatorname{Ind} X$.

3. Relation of local dimension to dimension

For any space X one has the trivial relation $\operatorname{loc}\dim X \leqslant \dim X$, and an example is given below of a normal Hausdorff space M with

$$\operatorname{loc}\dim M < \dim M.$$

But, as is now to be shown, in [3.3], [3.5], and [3.6], there is a wide class of normal spaces for which $\operatorname{loc}\dim X = \dim X$.

[3.1] *If A is a closed set of a space X,* $\operatorname{loc}\dim A \leqslant \operatorname{loc}\dim X$.

Proof. Let $\operatorname{loc}\dim X \leqslant n$ and let x be a point of A. Then there is a neighbourhood U of x in X with $\dim \bar{U} \leqslant n$. Then $U \cap A$ is a neighbourhood of x in A and the closure of $U \cap A$ in A is a closed subset of $\bar{U}$ and hence has dimension not exceeding n. Hence $\operatorname{loc}\dim A \leqslant n$. Thus

$$\operatorname{loc}\dim A \leqslant \operatorname{loc}\dim X,$$

as was to be shown.

[3.2] *If $\{U_\lambda\}$ is any covering of a paracompact normal space X, then X is the union of a sequence of closed sets $\{H_i\}$ each of which is the union of a collection $\{H_{i\mu}\}$ of disjoint sets with each $H_{i\mu}$ open and closed in H_i and with each $H_{i\mu}$ contained in some U_λ.*

Proof. Since X is paracompact, $\{U_\lambda\}$ has a locally finite refinement $\{V_\mu\}$. Since X is normal, $\{V_\mu\}$ can be shrunk [(7) (I, 33, 4)] to a covering $\{W_\mu\}$ with $\overline{W}_\mu \subset V_\mu$. There exist real continuous functions f_μ $(0 \leqslant f_\mu(x) \leqslant 1)$ such that $f_\mu(x) = 0$ if $x \in X - V_\mu$ and $f_\mu(x) = 1$ if $x \in \overline{W}_\mu$. Let $F_{i\mu}$ be the set of points x for which $f_\mu(x) \geqslant 1/i$ and let G_μ be the set of points for which $f_\mu(x) > 0$.

Let the indices μ be well ordered. Let $H_{i\mu}$ be the set $F_{i\mu} - \bigcup_{\nu<\mu} G_\nu$ and let $H_i = \bigcup_\mu H_{i\mu}$. The sets $F_{i\mu}$ are closed, G_ν is open, and each $H_{i\mu}$ is closed. If $\nu < \mu$, $H_{i\nu} \subset G_\nu$ and $H_{i\nu} \cap H_{i\mu} = 0$. Thus, for each i, the sets $H_{i\mu}$ are disjoint. Since $H_{i\mu} \subset V_\mu$, the collection $\{H_{i\mu}\}$ for fixed i is locally finite; hence H_i is closed and $\bigcup_{\nu \neq \mu} H_{i\nu}$ is closed, whence $H_{i\mu}$ is open in H_i. Thus H_i is the union of disjoint sets $H_{i\mu}$, each open and closed in H_i, and $H_{i\mu} \subset V_\mu$, which is contained in some U_λ.

Each point $x \in X$ is in some W_μ and hence in some G_μ. Then, if we take the first μ for which $x \in G_\mu$, $x \in F_{i\mu}$ for some i while x is in none of the sets G_ν with $\nu < \mu$. Hence $x \in H_{i\mu}$ for some i, and therefore $x \in H_i$. Thus $X = \bigcup_{i=1}^{\infty} H_i$, as was to be shown.

[3.3] *If X is a paracompact normal space, then* $\operatorname{loc}\dim X = \dim X$.

Proof. Let $\operatorname{loc}\dim X \leqslant n$. Then each point $x \in X$ is in some open set U_x such that $\dim \overline{U}_x \leqslant n$. Then, by [3.2], X is the union of a sequence of closed sets $\{H_i\}$ each of which is the union of a collection $\{H_{i\mu}\}$ of disjoint sets with each $H_{i\mu}$ contained in some U_x. Then $H_{i\mu}$ is a closed subset of some $\overline{U}_x$, and hence $\dim H_{i\mu} \leqslant n$. Hence, by [2.5], $\dim H_i \leqslant n$. Hence, by the sum theorem [(3) § 23], since $X = \bigcup_{i=1}^{\infty} H_i$ with H_i closed and $\dim H_i \leqslant n$, we have $\dim X \leqslant n$. Thus $\dim X \leqslant \operatorname{loc}\dim X$, and hence $\operatorname{loc}\dim X = \dim X$, as was to be shown.

[3.4] *If X is a paracompact totally normal space, then*

$$\operatorname{loc}\operatorname{Ind} X = \operatorname{Ind} X.$$

Proof. Let $\operatorname{loc}\operatorname{Ind} X \leqslant n$. Then each point $x \in X$ is in some open set U_x such that $\operatorname{Ind} \overline{U}_x \leqslant n$. Then, by [3.2], X is the union of a sequence of closed sets $\{H_i\}$ each of which is the union of a collection $\{H_{i\mu}\}$ of disjoint sets with each $H_{i\mu}$ open and closed in H_i and with each $H_{i\mu}$ contained in some U_x. Then $H_{i\mu}$ is a closed subset of some $\overline{U}_x$, and hence $\operatorname{Ind} H_{i\mu} \leqslant n$. Hence, by (5) proposition 5.1, $\operatorname{Ind} H_i \leqslant n$. By the sum theorem [(5) Theorem 4] for the inductive dimension of totally normal spaces, since $X = \bigcup_{i=1}^{\infty} H_i$ with H_i closed and $\operatorname{Ind} H_i \leqslant n$, therefore $\operatorname{Ind} X \leqslant n$. Thus $\operatorname{Ind} X \leqslant \operatorname{loc} \operatorname{Ind} X$, and hence $\operatorname{loc} \operatorname{Ind} X = \operatorname{Ind} X$, as was to be shown.

[3.5] *If a normal space X is the union of a sequence $\{A_i\}$ of closed paracompact subsets, then* $\operatorname{loc} \dim X = \dim X$.

Proof. Let $\operatorname{loc} \dim X \leqslant n$. Then, by [3.1], since A_i is closed,

$$\operatorname{loc} \dim A_i \leqslant n.$$

Since X is normal, the closed set A_i is normal. Hence, by [3.3], since A_i is paracompact, $\dim A_i \leqslant n$. By the sum theorem [(3) § 23], since $X = \bigcup_{i=1}^{\infty} A_i$ with A_i closed and $\dim A_i \leqslant n$, therefore $\dim X \leqslant n$. Thus $\dim X \leqslant \operatorname{loc} \dim X$, and hence $\operatorname{loc} \dim X = \dim X$.

[3.6] *If a normal space X is the union of two paracompact sets A and B with A closed in X, then* $\operatorname{loc} \dim X = \dim X$.

Proof. Let $\operatorname{loc} \dim X \leqslant n$. Then, by [3.1], since A is closed,

$$\operatorname{loc} \dim A \leqslant n.$$

Since X is normal, the closed set A is normal. Hence, by [3.3], since A is paracompact, $\dim A \leqslant n$. Let F be any closed set of X which does not meet A; then F is normal and $\operatorname{loc} \dim F \leqslant n$. Since F is a closed subset of B, F is paracompact and hence, by [3.3], $\dim F \leqslant n$. Hence, by [2.1], $\dim X \leqslant n$. Thus $\dim X \leqslant \operatorname{loc} \dim X$, and hence

$$\operatorname{loc} \dim X = \dim X.$$

[3.7] *Let X be an n-dimensional normal space. If X is paracompact, or the union of two paracompact sets one of which is closed, or the union of a sequence of closed paracompact sets, then the set of points of X at which X is n-dimensional is an n-dimensional closed set of X.*

Proof. Let D be the set of points of X at which X is n-dimensional and let $p \in X - D$. Since $\operatorname{loc} \dim X \leqslant \dim X = n$, $\dim_p X < n$ and, for some neighbourhood U of p, $\dim \overline{U} \leqslant n-1$. Then, for each point $x \in U$, $\dim_x X \leqslant n-1$, and hence $x \in X - D$. Thus $X - D$ is open and D is closed.

Let F be any closed set of X which does not meet D. Then each point x of F has a neighbourhood U in X such that $\dim \overline{U} \leqslant n-1$. Then $U \cap F$ is a neighbourhood of x in F and its closure in F is a closed subset of $\overline{U}$ and hence has dimension not exceeding $n-1$. Thus

loc dim $F \leqslant n-1$. If X is paracompact, then the closed set F is paracompact. If $X = A \cup B$ with A closed and both A and B paracompact, then
$$F = (A \cap F) \cup (B \cap F)$$
with $A \cap F$ closed and, since $A \cap F$ is closed in A and $B \cap F$ is closed in B, $A \cap F$ and $B \cap F$ are paracompact. If $X = \bigcup_{i=1}^{\infty} A_i$ with each A_i closed and paracompact, then $F = \bigcup_{i=1}^{\infty} A_i \cap F$ and each $A_i \cap F$ is closed and paracompact. The closed set F of X is normal; hence, by [3.3] or [3.5] or [3.6], $\dim F \leqslant n-1$.

By [2.1], if the dimension of D were $\leqslant n-1$, then we should have $\dim X \leqslant n-1$, which is absurd. Thus $\dim D \geqslant n$ and, since D is closed in X, $\dim D \leqslant \dim X = n$. Hence $\dim D = n$, as was to be shown.

In reference to the hypotheses of [3.5], [3.6], and [3.7], note that Bing's example H [see (**1**)] is a normal space which is the union of a countable number of discrete and hence paracompact (even metrizable) closed subsets but it is not paracompact. Either of his examples G or H is a non-paracompact normal space which is the union of two discrete and hence paracompact subsets, one of which is closed.

Nor does the normality of X follow from the other properties. For example, let R be a non-countable set of points, one of which is called r_0. The open sets of R are the sets not containing r_0 and the sets containing all but a finite number of points of R. Then R is a normal space: in fact it is a compact Hausdorff space. Let S have a countably infinite set of points and let its open sets be sets not containing a special point s_0 and sets containing all but a finite number of points. Let X be the subset of $R \times S$ formed by removing the point (r_0, s_0). Then X is a Hausdorff space.

If
$$A = ((R \times s_0) \cup (r_0 \times S)) \cap X,$$
then A is closed in X and both A and $X - A$ are discrete and hence paracompact. Also the sets $R \times s$ for $s \neq s_0$ are compact and $(R \times s_0) \cap X$ is discrete; thus X is the union of a countable collection of closed paracompact subsets. But X is neither normal nor paracompact.

4. Properties of local dimension

As has been shown in [3.1] above, the closed-subset theorem holds for the local dimension of arbitrary spaces. I now show that the open-subset theorem holds for the local dimension, though not necessarily for the dimension (see § 7 below), of regular spaces. And the subset theorem of local dimension holds for totally normal spaces. The finite-sum theorem, but not the countable-sum theorem (see § 7 below), holds for the local dimension of normal spaces.

[4.1] *If Y is an open set of a regular space X, then* $\operatorname{loc}\dim Y \leqslant \operatorname{loc}\dim X$ *and* $\operatorname{loc}\operatorname{Ind} Y \leqslant \operatorname{loc}\operatorname{Ind} X$.

Proof. Let $\operatorname{loc}\dim X \leqslant n$ and let x be a point of Y. There is a neighbourhood U of x in X such that $\dim \bar{U} \leqslant n$. Since X is regular, there is an open set V containing x whose closure $\bar{V}$ is contained in the open set $U \cap Y$. Then V is a neighbourhood of x in Y, $\bar{V}$ is the closure of V in Y, and, since $\bar{V}$ is a closed subset of $\bar{U}$, $\dim \bar{V} \leqslant n$. Thus $\operatorname{loc}\dim Y \leqslant n$. Hence $\operatorname{loc}\dim Y \leqslant \operatorname{loc}\dim X$. The proof that $\operatorname{loc}\operatorname{Ind} Y \leqslant \operatorname{loc}\operatorname{Ind} X$ is similar and is omitted.

[4.2] *If A is a subset of a totally normal space X, then*

$$\operatorname{loc}\dim A \leqslant \operatorname{loc}\dim X \quad \textit{and} \quad \operatorname{loc}\operatorname{Ind} A \leqslant \operatorname{loc}\operatorname{Ind} X.$$

Proof. Let $\operatorname{loc}\dim X \leqslant n$ and let x be a point of A. There is a neighbourhood U of x in X such that $\dim \bar{U} \leqslant n$. Then $U \cap A$ is a neighbourhood of x in A whose closure in A is a subset of the totally normal space $\bar{U}$ and hence, by [2.8], has dimension not exceeding n. Thus $\operatorname{loc}\dim A \leqslant n$. Hence $\operatorname{loc}\dim A \leqslant \operatorname{loc}\dim X$. The proof that $\operatorname{loc}\operatorname{Ind} A \leqslant \operatorname{loc}\operatorname{Ind} X$ is similar but uses (5), Theorem 2, instead of [2.8].

[4.3] *If a normal space X is the union of two closed sets A and B and if* $\operatorname{loc}\dim A \leqslant n$ *and* $\operatorname{loc}\dim B \leqslant n$, *then* $\operatorname{loc}\dim X \leqslant n$.

Proof. If $x \in X - A$, then $x \in B$ and there is an open set $U \cap B$ of B, where U is open in X, such that $\dim \overline{U \cap B} \leqslant n$. If $W = U \cap (X-A)$, then W is open in X, $x \in W$, and, since $\bar{W} \subset \overline{U \cap B}$, $\dim \bar{W} \leqslant n$. Similarly, if $x \in X - B$, there is an open set of X containing x whose closure has dimension not exceeding n. If $x \in A \cap B$, then there exist open sets U and V containing x such that $\dim \overline{U \cap A} \leqslant n$ and $\dim \overline{V \cap B} \leqslant n$. Let

$$W = X - (A-U) - (B-V);$$

then W is open, $x \in W$, and $W \subset (U \cap A) \cup (V \cap B)$. Then

$$\dim \bar{W} \leqslant \dim(\overline{U \cap A} \cup \overline{V \cap B}) \leqslant \max(\dim \overline{U \cap A}, \dim \overline{V \cap B}) \leqslant n$$

since the sum theorem [(3) § 23] holds in the normal space $\bar{W}$. Thus $\operatorname{loc}\dim X \leqslant n$, as was to be shown.

5. The subset theorem and local dimension

In this section the problem of whether the subset theorem of dimension holds for all completely normal regular spaces is reduced to the apparently simpler problem of whether the local dimension of every completely normal regular space is equal to its dimension.

If one drops the condition of regularity, there are trivial counterexamples to the subset theorem. For example, let I be a line segment and let X be the space consisting of I together with one additional point x_0, the open sets of X being the open sets of I and the whole space X. Then X is completely normal but not regular, and $\dim X = 0$ while $\dim I = 1$.

[5.1] *If X is any normal regular space, there is a normal regular space X^* containing X as an open subset such that* $\dim X^* \leqslant \operatorname{loc}\dim X$. *If X is a Hausdorff space or a completely normal space, so is X^*.*

Proof. If X is empty, let $X^* = X$. Otherwise the points of the space X^* are the points of X together with one new point x_0. A set U of X^* is to be open if either (i) $U \subset X$ and U is open in X or (ii) $x_0 \in U$ and $X^* - U$ is a closed set of X which is contained in an open set V of X such that $\dim \bar{V} \leqslant \operatorname{loc}\dim X$.

It is clear that the empty set is an open set of type (i) and the whole space X^* is an open set of type (ii). The intersection of two open sets, one of which is of type (i), is an open set of type (i). If U_1 and U_2 are open sets of type (ii), then $x_0 \in U_1 \cap U_2$ and

$$X^* - (U_1 \cap U_2) = (X^* - U_1) \cup (X^* - U_2),$$

which is the union of two closed sets and hence is a closed set of X.

If $X^* - U_1 \subset V_1$ and $X^* - U_2 \subset V_2$ with V_1 and V_2 open in X and

$$\dim \bar{V}_1 \leqslant \operatorname{loc}\dim X \quad \text{and} \quad \dim \bar{V}_2 \leqslant \operatorname{loc}\dim X,$$

then $X^* - (U_1 \cap U_2) \subset V_1 \cup V_2$ and $\overline{V_1 \cup V_2} = \bar{V}_1 \cup \bar{V}_2$ is a closed set of X and hence is normal. Therefore, by the sum theorem,

$$\dim(\bar{V}_1 \cup \bar{V}_2) = \max(\dim \bar{V}_1, \dim \bar{V}_2) \leqslant \operatorname{loc}\dim X.$$

Thus the intersection $U_1 \cap U_2$ is an open set of type (ii).

The union of any collection of open sets of type (i) is again an open set of type (i). If the collection contains an open set U_1 of type (ii), then the union U contains x_0, and $U - (x_0)$ is a union of open sets of X, and hence is open in X. Therefore $X^* - U$ is closed in X and, if

$$X^* - U_1 \subset V_1$$

with V_1 open in X and $\dim \bar{V}_1 \leqslant \operatorname{loc}\dim X$, then $X^* - U$ is also contained in V_1. Therefore U is an open set of type (ii). Thus X^* is a topological space, and clearly X is a subspace. The set X is an open set of type (i) in X^*.

The space X^* is normal. For, if E and F are disjoint closed sets of X^*, at least one of them, say F, does not contain x_0. Then X^*-F is an open set of X^* containing x_0; hence it is an open set of type (ii). Therefore there is an open set V of X with $F \subset V$ and $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Then $V \cap (X^*-E)$ is an open set of X containing the closed set F. Since X is normal, there exists an open set W of X such that

$$F \subset W \subset \bar{W} \subset V \cap (X^*-E).$$

Let $U = X^*-\bar{W}$; then $x_0 \in U$, the set $X^*-U = \bar{W}$ is closed in X, $\bar{W} \subset V$ with V open in X, and $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Therefore U is an open set of type (ii) while the set W is open of type (i) in X^*. We have $F \subset W$ and, since $\bar{W} \subset X^*-E$,

$$E \subset X^*-\bar{W} = U, \qquad U \cap W = 0.$$

Therefore X^* is normal.

The space X^* is regular. For, if $x \in U \subset X^*$ with U open in X^*, then either $x = x_0$ and U is open of type (ii) or $x \neq x_0$ and $x \in U \cap X$, which is open of type (i) in X^*. If $x = x_0 \in U$, then, since X is open in X^*, (x_0) is closed and, by the normality of X^*, there is an open set W with $x_0 \in W \subset \bar{W} \subset U$, where $\bar{W}$ is the closure of W in X^*. If $x \neq x_0$, then, by the definition of local dimension, there is some neighbourhood V of x in X such that $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Since X is regular, there is an open set W of X with $x \in W \subset \bar{W} \subset V \cap U$. Since $\bar{W}$ is closed in X and $\bar{W} \subset V$ with V open and $\dim \bar{V} \leqslant \operatorname{loc}\dim X$, therefore $X^*-\bar{W}$ is open of type (ii) and $\bar{W}$ is closed in X^*. Thus $x \in W \subset \bar{W} \subset U$ with W open and $\bar{W}$ closed in X^*. Therefore X^* is regular.

The set (x_0) is closed in X^* and $\dim(x_0) = 0 \leqslant \operatorname{loc}\dim X$ since X is non-empty. Let F be any closed set of X^* which does not meet (x_0). Then X^*-F is an open set of type (ii) and hence F is closed in X and $F \subset V$ for some open set V of X with $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Therefore $\dim F \leqslant \operatorname{loc}\dim X$. Hence, by [2.1], since X^* is normal,

$$\dim X^* \leqslant \operatorname{loc}\dim X.$$

Let X be a Hausdorff space. Then, if $x \in X$, (x) is a closed set of X. And, since $\dim_x X \leqslant \operatorname{loc}\dim X$, there is an open set V of X with $x \in V$ and $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Hence $X^*-(x)$ is an open set of type (ii) and (x) is a closed set of X^*. Since X is open in X^*, (x_0) is a closed set of X^*. Thus all one-point sets of X^* are closed. Hence, since X^* is normal, X^* is a Hausdorff space.

Let X be completely normal. If U is an open set of type (ii) in X^*, then X^*-U is closed in X and $X^*-U \subset V$ with V open in X and

$\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Since X is normal, there is an open set W of X with $X^* - U \subset W \subset \bar{W} \subset V$. Then $X^* - \bar{W}$ is open of type (ii) and $\bar{W}$ is closed in X^*. Since $X^* - W$ is closed in the normal space X^*, $X^* - W$ is normal. Since $\bar{W} \cap U$ is a subset of the completely normal space X, $\bar{W} \cap U$ is normal. Then U is the union of two relatively closed normal subsets $X^* - W$ and $\bar{W} \cap U$; hence [(**12**) 186, lemma] U is normal.

And, if U is an open set of type (i), then $U \subset X$ and hence U is normal. Thus every open set of X^* is normal and hence [(**5**) proposition 1.1] X^* is completely normal. This completes the proof of [5.1].

[5.2] *If X is a normal regular space such that* $\dim X > \operatorname{loc}\dim X$, *then there is a normal regular space X^* containing X as an open subset such that* $\dim X > \dim X^*$. *If X is a Hausdorff space or a completely normal space, so is X^*.*

Proof. By [5.1] we have $\dim X^* \leqslant \operatorname{loc}\dim X$. Then, since

$$\dim X > \operatorname{loc}\dim X,$$

therefore $\dim X > \dim X^*$. The remaining conclusions follow from [5.1].

[5.3] *If X is a completely normal regular space with a subset A such that* $\dim A > \dim X$, *then X has an open subset Y such that*

$$\dim Y > \operatorname{loc}\dim Y.$$

Proof. Since $\dim X < \dim A$, $\dim X$ is finite. Since $\dim A > \dim X$, there is a covering $\{G_1,\ldots, G_k\}$ of A which has no refinement of order not exceeding $\dim X$. Let $G_i = A \cap U_i$ with U_i open in X, and let $Y = \bigcup_{i=1}^{k} U_i$. Then Y is open in X and the covering $\{U_i\}$ of Y has no refinement of order not exceeding $\dim X$. Therefore $\dim Y > \dim X$. But, by [4.1],

$$\operatorname{loc}\dim Y \leqslant \operatorname{loc}\dim X \leqslant \dim X.$$

Therefore $\dim Y > \operatorname{loc}\dim Y$, as was to be shown.

6. An example

We now construct an example of a normal Hausdorff space M such that $\operatorname{loc}\dim M < \dim M$. Let T be the space consisting of the ordinal numbers less than ω_1 with the usual order topology [(**6**) appendix]. For each $\alpha \in T$, let

$$T_\alpha = \{\beta : \beta \leqslant \alpha\}, \qquad T'_\alpha = \{\beta : \beta \in T, \beta > \alpha\}.$$

Then, for each α, T_α and T'_α are disjoint closed sets of T whose union is T.

[6.1] *If $\{U_i\}$ is a countable (or finite) covering of T, then, for some integer j and some $\alpha \in T$, $T'_\alpha \subset U_j$.*

Proof. Assume on the contrary that for each α and j there is some $\beta \in T$ with $\beta > \alpha$ such that the interval $(\alpha, \beta) = \{\xi: \alpha < \xi < \beta\}$ is not contained in U_j. Let the least such β be $\beta_j(\alpha)$. Let $\gamma(\alpha)$ be the least upper bound of the sequence of ordinal numbers $\beta_j(\alpha)$. Then

$$\alpha < \gamma(\alpha) < \omega_1 \quad \text{and} \quad \gamma(\alpha) \in T.$$

Let $\alpha_1 = \gamma(0)$, $\alpha_2 = \gamma(\alpha_1), \ldots, \alpha_{r+1} = \gamma(\alpha_r), \ldots$. Then the sequence $\{\alpha_r\}$ has a least upper bound δ in T and $\delta > a_r$ since $\alpha_{r+1} > \alpha_r$. But δ is in some set U_j of the covering and is in some interval (α, β) contained in U_j. Then $\alpha < \delta$, and hence, for some r, $\alpha < \alpha_r$ and

$$\alpha_{r+1} = \gamma(\alpha_r) \geqslant \beta > \delta,$$

which is absurd. This completes the proof.

Let I be the space of real numbers, $0 \leqslant p \leqslant 1$, and let the numbers $p \in I$ be divided into congruence classes *modulo* the rational numbers. There are $\mathbf{c}$ such classes and $\mathbf{c} \geqslant \aleph_1$. Let $\aleph_1$ of these classes Q_α be chosen and indexed by the ordinal numbers $\alpha \in T$.

Example M. Let M be the subspace of the product space $T \times I$ consisting of those pairs (α, p) for which $p \notin \bigcup_{\beta \geqslant \alpha} Q_\beta$.

We define a *special* covering σ of M as follows. For some irreducible covering $\{W_1, \ldots, W_k\}$ of I by intervals open in I and for some $\alpha \in T$, σ consists of the covering of $M'_\alpha = (T'_\alpha \times I) \cap M$ by the sets $(T'_\alpha \times W_i) \cap M$, together with a covering of $M_\alpha = (T_\alpha \times I) \cap M$ by a finite number of disjoint open (and closed) sets. We may assume that $0 \in W_1$, $1 \in W_k$, and, for $i = 1, \ldots, k-1$, $W_i \cap W_{i+1}$ is not empty.

[6.2] *For each finite covering $\{U_i\}$ of M and $p \in I$ there is a neighbourhood (open interval) W of p in I and an $\alpha \in T$ such that, for some U_j of the covering,*

$$(T'_\alpha \times W) \cap M \subset U_j.$$

Proof. For each $p \in I$ there exists some $\beta \in T$ such that $T'_\beta \times p \subset M$; if $p \in Q_\alpha$, it is sufficient to take $\beta > \alpha$ while, if p is in no Q_α, one may take $\beta = 0$. Let $W_n(p)$ be the n^{-1}-neighbourhood of p in I and let $V(j, n)$ be the set of points α of T'_β such that, for some $\gamma < \alpha$,

$$((\gamma, \alpha+1) \times W_n(p)) \cap M \subset U_j.$$

Clearly $V(j, n)$ is an open set in T'_β.

For each $\alpha \in T'_\beta$, $(\alpha, p) \in M$, and hence $(\alpha, p) \in U_j$ for some U_j of the covering. There is an open set G_j of $T \times I$ such that $U_j = G_j \cap M$.

Then $(\alpha, p) \in U_j$, and hence there is some product neighbourhood $(\gamma, \alpha+1) \times W_n(p)$ of (α, p) contained in G_j. Then

$$((\gamma, \alpha+1) \times W_n(p)) \cap M \subset U_j,$$

and hence $\alpha \in V(j, n)$. Thus $\{V(j, n)\}$ is a covering of T'_β, and, since j and n take a finite and countable number of values respectively, the covering is countable. Adding the open set T_β, we get a countable covering of T.

By [6.1] there exist $j(p)$, $n(p)$, and $\alpha(p)$ such that $T'_{\alpha(p)} \subset V(j(p), n(p))$, and hence $(T'_{\alpha(p)} \times W_{n(p)}(p)) \cap M \subset U_j$. Thus it is sufficient to take $\alpha = \alpha(p)$ and $W = W_{n(p)}(p)$.

[6.3] *Every finite covering $\{U_i\}$ of M has a special refinement.*

Proof. By [6.2], for each $p \in I$ there is a neighbourhood $W(p)$ and an element $\alpha(p)$ of T such that $(T'_{\alpha(p)} \times W(p)) \cap M \subset U_j$ for some j. Since I is compact, the covering $\{W(p)\}$ of I contains an irreducible finite covering $\{W_k\}$ with $W_k = W(p_k)$. Let α be the greatest of the corresponding ordinal numbers $\alpha(p_k)$. Then for each W_k there is some U_j such that

$$(T'_\alpha \times W_k) \cap M \subset U_j.$$

The space $M_\alpha = (T_\alpha \times I) \cap M$ is a subspace of $T_\alpha \times (I - Q_\alpha)$, which is a zero-dimensional separable metrizable space. Hence the covering of M_α by the sets $U_i \cap M_\alpha$ has a finite refinement which is a covering by disjoint open sets. This, together with the collection of sets $(T'_\alpha \times W_k) \cap M$ which cover $M'_\alpha = (T'_\alpha \times I) \cap M$, forms the required special covering of M. This completes the proof.

A covering $\{U_i\}$ of a space X is called *shrinkable* if there is a covering $\{V_i\}$ of X such that $\bar{V}_i \subset U_i$. A space X is normal if and only if each finite covering of X is shrinkable [(7) 26], or, equivalently, if and only if each finite covering of X has a shrinkable finite refinement. In particular the covering $\{W_l\}$ of I is shrinkable, and hence each special covering of M is shrinkable. Therefore, since every finite covering has a special refinement, M is a normal space.

Since T and I are Hausdorff spaces, the subspace M of $T \times I$ is a Hausdorff space. Hence, since M is normal, it is a regular space.

It can easily be shown that M is countably paracompact but is not paracompact, not countably compact, and not completely normal.

[6.4] *For the normal Hausdorff space M we have*

$$\operatorname{ind} M = \operatorname{loc} \dim M = \operatorname{loc} \operatorname{Ind} M = 0.$$

Proof. Each point $(\alpha, p) \in M$ is contained in the open and closed set $M_\alpha \subset M$, and M is a subset of the zero-dimensional separable metrizable space $T_\alpha \times (I - Q_\alpha)$. Thus

$$\dim M_\alpha = \operatorname{ind} M_\alpha = \operatorname{Ind} M_\alpha = 0,$$

and hence $$\operatorname{ind} M = \operatorname{loc}\dim M = \operatorname{loc}\operatorname{Ind} M = 0,$$

as was to be shown.

[6.5] *For the space* M, $\dim M = 1$.

Proof. Since each finite covering of M has a special refinement and since a special covering has order not exceeding 1, therefore $\dim M \leqslant 1$.

Let G_0 be the set of points (α, p) of M with $p < 1$, and let G_1 be the set of points with $p > 0$. Then $\{G_0, G_1\}$ is a covering of M. Let $\{U_1, \ldots, U_r\}$ be any refinement of $\{G_0, G_1\}$.

Choose a special refinement of $\{U_i\}$. The set $(T'_\alpha \times W_1) \cap M$ is contained in some set U_i, and $U_i \subset G_0$. The set $(T'_\alpha \times W_k) \cap M$ is not contained in G_0 and hence is not contained in U_i. Hence there is a first j such that $(T'_\alpha \times W_j) \cap M \not\subset U_i$; let $(T'_\alpha \times W_j) \cap M \subset U_h$. Then, for any $p \in W_{j-1} \cap W_j$ and any β so large that $\beta > \alpha$ and $(\beta, p) \in M$, we have

$$(\beta, p) \in (T'_\alpha \times W_j) \cap M \subset U_h, \qquad (\beta, p) \in (T'_\alpha \times W_{j-1}) \cap M \subset U_i.$$

Thus $(\beta, p) \in U_i \cap U_h$, and the order of $\{U_i\}$ is at least one. Therefore $\dim M \geqslant 1$, and hence $\dim M = 1$.

[6.6] $\operatorname{Ind} M = 1$.

Proof. Let $F \subset U$ with F closed in M and U open in M. Choose a special refinement of the covering $\{M - F, U\}$ of M. Let V be the union of the sets of the special refinement which meet F; then $F \subset V \subset U$.

For each $j = 1, \ldots, k$, $\overline{W}_j - W_j$ consists of at most two points. Let $E = \bigcup_i (\overline{W}_j - W_j)$; then E is a finite subset of I. It is known that $\operatorname{Ind} T = 0$, and it follows that $\operatorname{Ind}(T \times E) = 0$. But $(\overline{V} - V) \cap M$ is a closed subset of $T \times E$; hence $\operatorname{Ind}((\overline{V} - V) \cap M) \leqslant 0$. Hence $\operatorname{Ind} M \leqslant 1$. Therefore $1 = \dim M \leqslant \operatorname{Ind} M \leqslant 1$, and hence $\operatorname{Ind} M = 1$, as was to be shown.

7. More examples

Example N. Let N be the space M^* formed from M by adding a single point x_0 as in § 5 above. A basic set of neighbourhoods of x_0 in N consists of the sets $(x_0) \cup M'_\alpha$ for $\alpha \in T$, where $M'_\alpha = (T'_\alpha \times I) \cap M$.

[7.1] *The space* N *is a normal Hausdorff space such that*

$$\dim N = \operatorname{Ind} N = 0.$$

Proof. It follows from [5.1] that N is a normal Hausdorff space and that $\dim N \leqslant \operatorname{loc}\dim M$. Hence, by [6.4], $\dim N \leqslant 0$ and hence, since N is not empty, $\dim N = 0$. This implies [(6) appendix] that $\operatorname{Ind} N = 0$.

Example N shows that the subset theorem does not hold for all normal Hausdorff spaces, even if the subset is required to be normal. For N is a normal Hausdorff space with $\dim N = \operatorname{Ind} N = 0$, having as an open subspace a normal space M with $\dim M = \operatorname{Ind} M = 1$.

Example Q. Let Q be a space consisting of a sequence $\{N_i\}$ of different copies of the space N together with a special point y_0. A basis for the open sets of Q is formed by the open sets of each N_i together with the sets $(y_0) \cup \bigcup_{i>j} N_i$ for $j = 1, 2, \dots$.

[7.2] *The space Q is a normal Hausdorff space and* $\dim Q = \operatorname{Ind} Q = 0$.

Proof. If p and q are two points of N_j, then, since N_j is a Hausdorff space, p and q have disjoint neighbourhoods in N_j. If $p \in N_i$ and $q \in N_j$, then N_i and N_j are open, and $N_i \cap N_j = 0$. If $p = y_0$ and $q \in N_j$, then p and q have the disjoint neighbourhoods $(y_0) \cup \bigcup_{i>j} N_i$ and N_j. Thus Q is a Hausdorff space.

If E and F are disjoint closed sets of Q, then one of them, say F, does not contain y_0. Then y_0 has a neighbourhood which does not meet F, and hence $F \subset N_i \cup \dots \cup N_j$ for some finite j. Since N_i is normal, there exist disjoint open sets U_i and V_i of N_i with $E \cap N_j \subset U_i$ and $F \cap N_i \subset V_i$. Let

$$U = U_i \cup \dots \cup U_j \cup (y_0) \cup \bigcup_{i>j} N_j, \qquad V = V_1 \cup \dots \cup V_j.$$

Then U and V are open, $E \subset U$, $F \subset V$, and $U \cap V = 0$. Thus Q is a normal space.

If F is any closed set of Q which does not meet (y_0), then F is a closed set of $N_1 \cup \dots \cup N_j$ for some j and hence

$$\dim F \leqslant \dim(N_1 \cup \dots \cup N_j).$$

Hence, by [7.1] and [2.5], $\dim F \leqslant 0$. Therefore, by [2.1], since

$$\dim(y_0) = 0,$$

we have $\dim Q = 0$. It follows [(6) appendix] that $\operatorname{Ind} Q = 0$, which completes the proof.

Example P. Let P be a space consisting of a sequence $\{M_i\}$ of different copies of the space M together with a special point y_0. A basis for the open sets of P is formed by the open sets of each M_i together with the sets $(y_0) \cup \bigcup_{i>j} M_i$ for $j = 1, 2, \dots$.

[7.3] *The space P is a normal Hausdorff space and* $\operatorname{ind} P = 0$ *while* $\operatorname{loc\,dim} P = \operatorname{loc\,Ind} P = \dim P = \operatorname{Ind} P = 1$.

Proof. That P is a normal Hausdorff space is shown as in the proof of [7.2]. Since $\operatorname{ind} M = 0$, each point of M_i has an arbitrarily small open and closed neighbourhood in M_i. The point y_0 has arbitrarily small open and closed neighbourhoods of the form $(y_0) \cup \bigcup_{i>j} M_i$. Thus $\operatorname{ind} P = 0$.

The point y_0 has a neighbourhood U such that $\dim \overline{U} \leqslant \operatorname{loc\,dim} P$. And the neighbourhood U contains a neighbourhood of the form $(y_0) \cup \bigcup_{i>j} M_i$, and hence contains the closed set M_{j+1}. Therefore

$$\dim \overline{U} \geqslant \dim M_{j+1} = 1.$$

Thus $\operatorname{loc\,dim} P \geqslant 1$.

If $F \subset U$ with F closed and U open in P, then y_0 has a neighbourhood $(y_0) \cup \bigcup_{i>j} M_i$ which either does not meet F or is contained in U. Since $\operatorname{Ind} M_i = 1$, there is an open set V_i with boundary $B_i = \overline{V}_i - V_i \subset M_i$ such that $F \cap M_i \subset V_i \subset U \cap M_i$ and $\operatorname{Ind} B_i \leqslant 0$. Let V be the union of the sets V_i for $i \leqslant j$ together with the open and closed set $(y_0) \cup \bigcup_{i>j} M_i$ in case the latter meets F. Then $F \subset V \subset U$ and the boundary of V is $B = B_1 \cup \ldots \cup B_j$. Thus B is the union of disjoint relatively open and closed sets B_i with each $\operatorname{Ind} B_i \leqslant 0$. Hence [(**5**) proposition 5.1] $\operatorname{Ind} B \leqslant 0$. Therefore $\operatorname{Ind} P \leqslant 1$. Hence

$$1 \leqslant \operatorname{loc\,dim} P \leqslant \dim P \leqslant \operatorname{Ind} P \leqslant 1,$$

$$1 \leqslant \operatorname{loc\,dim} P \leqslant \operatorname{loc\,Ind} P \leqslant \operatorname{Ind} P \leqslant 1.$$

This completes the proof.

Clearly P is a subspace of Q. Thus the subset theorem does not hold for the local dimension of normal Hausdorff spaces. For Q is a normal Hausdorff space with $\operatorname{loc\,dim} Q = \operatorname{loc\,Ind} Q = 0$, and P is a normal subspace of Q with $\operatorname{loc\,dim} P = \operatorname{loc\,Ind} P = 1$.

Also, though, by [4.3], the finite-sum theorem holds for the local dimension of normal spaces, the countable-sum theorem does not hold. For the normal space P is the union of a sequence of closed sets

$$(y_0), M_1, M_2, \ldots$$

with $\operatorname{loc\,dim}(y_0) = 0$, $\operatorname{loc\,dim} M_i = 0$, but $\operatorname{loc\,dim} P = 1$.

Example S. O. V. Lokucievskiĭ (**8**) has given an example of a normal Hausdorff compact space S which is the union of two closed subsets S_1 and S_2 such that $\operatorname{ind} S_1 = \operatorname{Ind} S_1 = 1$, $\operatorname{ind} S_2 = \operatorname{Ind} S_2 = 1$, and $\operatorname{ind} S = \operatorname{Ind} S = 2$. Hence, by [1.7], $\operatorname{loc\,Ind} S_1 = \operatorname{loc\,Ind} S_2 = 1$, and

$\operatorname{loc\,ind} S = 2$. Thus not even the finite-sum theorem holds for ind, loc Ind, and Ind.

Since $\dim S_1 \leqslant \operatorname{Ind} S_1 = 1$ and $\dim S_2 \leqslant \operatorname{Ind} S_2 = 1$, therefore, by the sum theorem, $\dim S \leqslant 1$. Also, S contains a closed set homeomorphic to a line segment; hence $\operatorname{loc\,dim} S \geqslant 1$. Therefore

$$\operatorname{loc\,dim} S = \dim S = 1.$$

[7.4] *No relations between* ind, loc dim, loc Ind, dim, *and* Ind, *other than those listed in* [1.7] *above, hold for all normal regular spaces.*

Proof. This is shown by the properties of Examples M, P, and S above, as is more clearly seen in the following table:

Space	ind	loc dim	loc Ind	dim	Ind
M	0	0	0	1	1
P	0	1	1	1	1
S	2	1	2	1	2

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