RESEARCH STATEMENT

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1. BACKGROUND AND HISTORY

My area of interest is Algebraic Geometry. I am mainly interested in the birational geometry and the moduli of higher dimensional varieties, namely, the Minimal Model Program (MMP) and its singularities in characteristic \( p > 0 \), the abundance conjecture, vanishing theorems, varieties of log Fano and log general types and their boundedness in positive characteristic, etc. I am also interested in the birational geometry of varieties over imperfect fields and their possible geometric and arithmetic applications.

One of the main goals of algebraic geometry is to classify algebraic objects, namely algebraic varieties. In birational geometry we want to classify algebraic varieties up to birational isomorphisms, i.e., we say that two algebraic varieties are “Birationally Equivalent” if their function fields are isomorphic or equivalently, they have isomorphic open sets. This defines an equivalence relation on the class of all algebraic varieties of fixed dimension. Next we want to find a “Good” representative or a Model in each of these equivalence classes.

In dimension 1, for any curve \( C \) we know that it’s normalization \( \tilde{C} \) is smooth and it is birational to \( C \), so we can pick \( \tilde{C} \) as the “Good” representative for the equivalence class of \( C \). Here \( \tilde{C} \) is smooth and this is as good as it gets for a “Good” representative.

In dimension 2, for a surface \( S \), its normalization is not smooth in general, so choosing a good representative in this case is not as easy as in the curve case. In the 20th century the Italian School of Algebraic Geometry showed that it is possible to choose a “Good” model for surfaces. At that time Castelnuovo showed that given a smooth projective algebraic surface \( X \), if \( C \) is a \((-1)\)-curve on \( X \), i.e., \( C \cong \mathbb{P}^1 \) and \( C^2 = -1 \), then \( C \) can be contracted to get another smooth projective surface \( Y \) which is birational to \( X \). We can continue this process of contracting \((-1)\)-curves. It can be shown that this process stops after a finitely many steps producing a smooth projective surface \( X^* \) which does not contain any \((-1)\)-curves. Now if the canonical divisor \( K_X \) of the surface \( X \) is (pseudo) effective, then \( K_X \) becomes nef. This \( X^* \) enjoys a special property: Every birational morphism \( f : X^* \to Z \) to another smooth projective surface \( Z \) is necessarily an isomorphism. We call \( X^* \) a Minimal Model and it is our candidate for the “Good” representative in the surface case.

In higher dimension (\( \dim \geq 3 \)) one would like to use similar methods to find minimal models. It is not hard to see that in higher dimension we can not work in the category of smooth varieties alone, we must allow some mild singularities. Constructing minimal models in higher dimensions eluded mathematicians for more than half of the 20th century until 1988 when Mori gave an explicit construction of flips, the last ingredient necessary to construct the minimal model for threefolds. The corresponding algorithm is called the “Minimal Model Program (MMP)” or “Mori Program”, which aims to obtain a minimal model for varieties in
arbitrary dimension via a sequence of simple birational maps, called divisorial contractions and flips. By blowing-down $K_X$-negative curves, we may get a divisorial contraction (a birational morphism whose exceptional locus is a prime Weil divisor) or a flipping contraction (a birational map whose exceptional locus has codimension at least 2 in $X$). To deal with the case of a flipping contraction $f : X \to Z$, we need to show that flip exists, i.e., that the canonical algebra
\[ \bigoplus_{m \geq 0} f_* O_X(m(K_X + \Delta)) \]
is a finitely generated $O_Z$-algebra. Two obstacles arise in this program:

1. Existence of flips,
2. Termination of a sequence of flips.

In [Mor88] Mori showed the existence of flips for 3-dimensional varieties over $\mathbb{C}$, answering the first question and hence finishing the MMP in dimension 3 (the second problem is not too difficult in this case). Existence of flips for $\dim X \geq 4$ remained open until 2006 when Birkar, Cascini, Hacon, McKernan proved the existence of flips in arbitrary dimension over $\mathbb{C}$ in their celebrated paper [BCHM10], hence answering the First Question once and for all over $\mathbb{C}$. In the same article they also proved the termination flips with scaling for varieties of log general type and hence proving the existence of the minimal model for those varieties in arbitrary dimensions over $\mathbb{C}$. This answers the second question for a large class of varieties over $\mathbb{C}$, but the general question of termination of flips remain open till this date.

The next obvious question one can ask is: Does MMP work in positive characteristic? Very little answers are known for this question. In dimension 1 and 2 and for smooth varieties, the same methods as characteristic 0 work in positive characteristic. For the MMP on singular surfaces in char $p > 0$ see [KK] and [FT12].

There are two main technical difficulties that appear in the positive characteristic MMP:

1. Resolution of singularities in higher dimension ($\geq 4$) is not known in characteristic $p > 0$.
2. The Kawamata-Viehweg vanishing theorem fails in higher dimension ($\geq 2$) in characteristic $p > 0$.

Fortunately the resolution of singularities is known in dimension 3 and in char $p > 0$ due to [Abh98], [Cut04], [CP08, CP09]. Kawamata-Viehweg vanishing theorem is a very essential tool for char 0 MMP, almost any proof related to MMP uses this vanishing theorem to lift sections of a line bundle from a divisor. For example, lifting sections is in the heart of the proof of the main theorem in [BCHM10]. Their technique is based on ideas of Siu and a repeated and systematic use of the Kawamata-Viehweg vanishing theorem. Recently in a series of papers by Hara, Hochster, Huneke, Mustaţă, Schwede, Smith, Tucker and others, it has become clear that one can sometimes replace the vanishing theorems by use of test ideals, Frobenius maps and the Serre vanishing theorem. The $F$-singularity techniques coming from the tight closure theory in commutative algebra have proved to be a powerful tool in studying birational geometry in char $p > 0$. 
Good progress has been made recently towards the minimal model program on 3-folds in char $p > 5$, due to Hacon, Xu, Birkar, Waldron, Schwede, Gongyo, Tanaka, Cascini and others (see [HX15], [Bir16b], [BW17], [CGS14] and [Wal17]). They successfully managed to use the existence of the resolution of singularities for 3-folds and the $F$-singularity techniques to run the MMP on 3-folds in char $p > 5$. In dimension $\geq 4$ almost nothing is known towards the minimal model program in char $p > 0$. Most of research is influenced by the questions arising from the minimal model program in positive characteristic.

2. Past Research

My past research involved the studies of the singularities of MMP (adjunction, inversion of adjunction, canonical bundle formula, etc.), and other MMP related problems (Abundance problem, cohomological vanishing theorems, etc.).

(1) Inversion of Adjunction. In [Das15] we showed that a char $p > 0$ analog of the log terminal inversion of adjunction holds in arbitrary dimension. More specifically we proved the following theorem:

**Theorem 2.1.** [Das15, Theorem 4.1, Corollary 5.4] Let $(X, S + B)$ be a pair where $X$ is a normal variety in characteristic $p > 0$, $S + B \geq 0$ is a $\mathbb{Q}$-divisor, $K_X + S + B$ is $\mathbb{Q}$-Cartier and $S = [S + B]$. Let $\nu : S^n \to S$ be the normalization and write $(K_X + S + B)|_{S^n} = K_{S^n} + B_{S^n}$. If $(S^n, B_{S^n})$ is strongly $F$-regular then $S$ is normal, furthermore $S$ is a unique center of sharp $F$-purity of $(X, S + B)$ in a neighborhood of $S$ and $(X, S + B)$ is purely $F$-regular near $S$.

Inversion of adjunction is a powerful and essential tool to run the MMP. It allows us to get specific information about the singularities of the ambient varieties from the singularities of its subvarieties. This inversion of adjunction statement above is used in [HX15] and [Bir16b] to construct flips for 3-folds in char $p > 5$.

In the same article, we also proved the equality of the ‘Different’ and the ‘$F$-Different’ conjectured by Schwede in [Sch09]. The equality of these two technical terms create a bridge between the MMP singularities and the $F$-singularities in char $p > 0$. It allows one to apply the results obtained in $F$-singularities to the MMP singularities and vice-versa. We proved the following theorem:

**Theorem 2.2.** [Das15, Theorem 5.3] Let $(X, S + \Delta \geq 0)$ be a pair, where $X$ is a normal excellent scheme of pure dimension over a $F$-finite field $k$ of characteristic $p > 0$ and $S + \Delta \geq 0$ is a $\mathbb{Q}$-divisor on $X$ such that $(p^e - 1)(K_X + S + \Delta)$ is Cartier for some $e > 0$. Also assume that $S$ is a reduced Weil divisor and $S \wedge \Delta = 0$. Then the $F$-Different, $F$-$\text{Diff}_{S^n}(\Delta)$ is equal to the Different, $\text{Diff}_{S^n}(\Delta)$, i.e., $F$-$\text{Diff}_{S^n}(\Delta) = \text{Diff}_{S^n}(\Delta)$, where $S^n \to S$ is the normalization morphism.

(2) Sub-adjunction in Positive Characteristic. In a joint paper [DH15] with Christopher Hacon we proved a (relative) vanishing theorem of Kawamata-Viehweg type for 3-folds in char $p > 5$.

**Theorem 2.3.** [DH15, Theorem 3.5] Let $f : (X, S + B \geq 0) \to Z$ be either a pl-divisorial contraction or a pl-flipping contraction. If the maximum dimension of the fibers of $f$ is 1, then $R^i f_* \mathcal{O}_X(-S) = 0$ for all $i > 0$. 


As an easy consequence we prove that the codimension 2 minimal LC centers of 3-folds in char \( p > 5 \) are smooth.

**Corollary 2.4.** [DH15, Theorem 3.6] Let \((X, \Delta)\) be a \(\mathbb{Q}\)-factorial 3-fold log canonical pair such that \(X\) has KLT singularities. If \(W\) is a minimal log canonical center of \((X, \Delta)\), then \(W\) is normal.

We also prove that the Kawamata’s sub-adjunction formula holds for codimension 2 minimal LC centers in char \( p > 5 \) under some mild hypothesis [DH15, Theorem 4.9].

(3) F-Different and Canonical Bundle Formula. In a joint paper [DS17] with Karl Schwede we studied the F-Different and its relation to the Canonical Bundle Formula in char \( p > 0 \).

Let \((X, \Delta \geq 0)\) be a LC pair and \(W\) a LC center of codimension at least 2. In an ‘Ideal World’ we would like to have the following adjunction formula:

\[
(K_X + \Delta)|_{W^N} \sim_{\mathbb{Q}} K_{W^N} + \text{Diff}_{W^N} + M_{W^N}, \quad W^N \to W \text{ is the normalization,}
\]

satisfying

(a) \(\text{Diff}_{W^N}\) is effective and \((W^N, \text{Diff}_{W^N})\) is LC.

(b) \(M_{W^N}\) is semi-ample.

While in reality we can not always achieve the relation (2.1) satisfying the two conditions above (even in char 0), there is a natural candidate for the same formula coming from the F-adjunction theory in char \( p > 0 \) (see [Sch09]) which could be useful in many situations: If \((X, \Delta \geq 0)\) is a sharply F-pure pair and \(W\) a LC center, then there exists a canonically determined unique divisor \(\Delta_{W,F-diff}\) on \(W^N\) such that

\[
(K_X + \Delta)|_{W^N} \sim_{\mathbb{Q}} K_{W^N} + \Delta_{W,F-diff} \geq 0 \text{ on } W^N \text{ such that}
\]

Observe that unlike \(\Delta_{W,F-diff} \geq 0\), the divisor \(\text{Diff}_{W^N} + M_{W^N}\) is not unique (since \(M_{W^N}\) is not unique).

Theorem 2.5. [DS17, Corollary 4.4] Suppose that \((X, \Delta)\) is a sharply F-pure (and hence log canonical) pair and \(W \subseteq X\) is a log canonical center (and hence an F-pure center). Let \(\nu : W^N \to W\) be the normalization of \(W\) and let \(\text{Diff}_{W^N}\) and \(\Delta_{W,F-diff}\) be the different and F-different respectively. Then \(\text{Diff}_{W^N} \leq \Delta_{W,F-diff} \).

One should think of the difference \(\Delta_{W,F-diff} - \text{Diff}_{W^N}\) as a char \( p > 0 \) analog of the moduli part of the adjunction formula (like \(M_{W^N}\) in (2.1)). In char 0 the moduli part \(M_{W^N}\) comes from analyzing the fibers of certain fiber space. More specifically it is attached to some canonical bundle formula. In [DS17] we found an analogous formula called F-Canonical Bundle Formula derived from the Frobenius splitting theory.

Theorem 2.6. [DS17, Theorem 5.2, Corollary 5.12] Suppose that \(\pi : E \to W\) is a proper morphism between normal F-finite integral schemes with \(\pi_*\mathcal{O}_E = \mathcal{O}_W\). Suppose also that \(\Delta \geq 0\) is a \(\mathbb{Q}\)-divisor on \(E\) such that \((\varphi - 1)(K_E + \Delta)\) is linearly equivalent to the pullback of some Cartier divisor on \(W\). Suppose that the generic fiber of \((E, \Delta)\) over \(W\) is Frobenius split. Then there exists a canonically determined
\(\mathbb{Q}\)-divisor \(\Delta_w \geq 0\) on \(W\) such that \(\pi^*(K_W + \Delta_W) \sim_{\mathbb{Q}} K_X + \Delta\).

Furthermore, we can describe the support of \(\Delta_{\text{vert}}\) as follows. Assume that \(W\) is 1-dimensional and that \(\Delta_{\text{vert}}\) is the vertical part of \(\Delta\). If additionally the fibers of \(\pi\) are geometrically normal, then \(\pi^*\Delta_W - \Delta_{\text{vert}}\) is nonzero precisely over those points \(t \in W\) where the fiber \((E_t, \Delta_t)\) is not Frobenius split.

(4) Abundance Problem for 3-folds in Positive Characteristic. In a joint paper with Joe Waldron, we proved some cases of the abundance conjecture for 3-folds in characteristic \(p > 5\). Namely we proved the following theorem.

Theorem 2.7. [DW16, Theorem A and B] Let \((X, \Delta \geq 0)\) be a KLT pair of dimension 3 in characteristic \(p > 5\). Assume that one of the following conditions is satisfied:

(a) \(\kappa(X, K_X + \Delta) = 1\), or
(b) \(\Delta = 0, K_X \equiv 0\) and \(X\) is not uniruled.

Then \(K_X + \Delta\) is semi-ample.

(5) Kawamata-Viehweg Vanishing Theorem for del Pezzo Surfaces in Positive Characteristic. In [Das17] we showed that the Kawamata-Viehweg vanishing theorem holds for regular del Pezzo surfaces over arbitrary fields (possibly imperfect) of characteristic \(p > 3\). In particular, we proved the following theorem.

Theorem 2.8. [Das17, Theorem 1.1] Let \((X, \Delta \geq 0)\) be a KLT pair of dimension 2 over an arbitrary field (possibly imperfect) of characteristic \(p > 3\). Assume that \(X\) is regular and \(D\) is a \(\mathbb{Z}\)-divisor on \(X\) such that \(D - (K_X + \Delta)\) is nef and big. Then \(H^i(X, \mathcal{O}_X(D)) = 0\) for all \(i > 0\).

These type of vanishing theorem is useful in the classification of regular del Pezzo surfaces over imperfect fields.

3. Current Research

Currently I am working in the following projects.

• Minimal Model Program (MMP) for 3-folds over imperfect fields. This is a joint project with Joe Waldron. MMP over imperfect field is very useful for studying family of varieties (fibrations) in positive characteristic, since the generic fiber of such a family is a variety over an imperfect field (the function field of the base). So far we found that if \((X, \Delta \geq 0)\) is a canonical singularity of dimension 3 in char \(p > 5\) and \(K_X + \Delta\) is pseudo-effective, then the existence and termination of flips works much like [HX15], although some of our arguments are quite different due to the imperfectness of the ground field \(k\). One of the main challenges to generalize these results for a KLT pair over imperfect fields lie with the problem of the Boundedness of the length of \((K_X + \Delta)\)-negative extremal rays. Unlike the 3-folds over algebraically closed field of char \(p > 0\), we can not expect the inequality \(0 < -(K_X + \Delta) \cdot R \leq 6\) to hold over an imperfect field, as evident by [Tan15, Example 7.3]. This inequality is essential for establishing lot of the results for KLT MMP, for example, the finiteness of the number of log minimal models. We propose an alternate inequality over imperfect fields \(k\) which looks something like this: \(-(K_X + \Delta) \cdot C < 6[k' : k]\), where \(C\) is an irreducible curve generating a \((K_X + \Delta)\)-negative extremal ray, \(H^0(C, \mathcal{O}_C) = k'\)
and $[k' : k]$ is extension degree of the field $k'$ over $k$. These types of inequalities may just be enough for our purpose. We are currently working on formulating and proving this inequality. Other known challenges are the base-point free theorem and the projectivity of the contractions of extremal rays.

- **Boundedness of log Fano surfaces over imperfect fields.** This is a joint project with Joe Waldron and it is related to our other joint project. Boundedness of log Fano surfaces have diverse applications in birational geometry, and in algebraic geometry in general. Over algebraically closed field of arbitrary characteristic, this results was first proved by Alexeev in [Ale94]. Following his ideas and some recent results on del Pezzo surfaces over imperfect fields [PW17, Das17] we plan to prove the boundedness statement over imperfect fields. One of the main difficulties here is to bound the Picard rank of the minimal resolution of log del Pezzo surfaces over imperfect fields as in [Ale94, Theorem 5.7, 6.3]. The difficult arise from the complexity of the dual graphs of the minimal resolutions over imperfect fields, as clearly there are more possibilities here than for those over algebraically closed fields. Among other things, the boundedness of log Fano surfaces will help us to construct Mori fiber spaces for 3-folds over imperfect fields as in [BW17] when $K_X + \Delta$ is not pseudo-effective.

- **Boundedness of the negativity of extremal rays for surfaces.** This is a joint project with Cristian Martinez. We are working on the Bounded Negativity Conjecture (BNC), which says that, for any smooth complex projective surface $X$, there exists a positive integer $N > 0$ such that $C^2 \geq -N$ for all irreducible curves $C \subseteq X$. This conjecture is related to lots of other conjectures. For example, BNC implies the positivity of the Seshadri constant of ample line bundles. It is not hard to see that BNC holds for minimal ruled surfaces (which are $K_X$-Mori fiber spaces). We are currently trying to understand whether the constant $N$ in the BNC conjecture is a birational invariant or not, this appears to be an open question.

4. Future Projects

I plan to work in the following problems in the coming days.

**Problem 4.1.** Existence of flips for 3-folds in characteristic $p = 2, 3, \text{ and } 5$.

In a series of recent papers [HX15, Bir16b], [BW17] and [Wal17] by Hacon, Xu, Birkar and Waldron, the existence of flips and the existence of minimal models have been established for KLT 3-folds in characteristic $p > 5$. The restriction on the characteristic of $X$ comes from a theorem of Hara, which says that a normal surface $S$ in char $p > 5$ is strongly $F$-regular if and only if it is KLT. This result is known to fail in char $\leq 5$. In a more recent paper by Cascini and Tanaka [CT16], the authors showed that some expected results related to MMP (which are true in characteristic 0 and $p > 5$) fail to hold in characteristic 2. While their counterexample does not prove the failure of the existence of flips in characteristic 2, it does indicate that there are room for such counterexample.

**Problem 4.2.** Termination of log canonical flips for 3-fold in characteristic $p > 5$. 
This is another major unsolved problem in the minimal model program (MMP) is positive characteristic. We note that in characteristic 0 this problem is solved in dimension 3 and still unknown in higher dimensions.

**Problem 4.3.** Sarkisov links for 3-folds in positive characteristic.

If $K_X + \Delta$ is not pseudo-effective, then there exists a $(K_X + \Delta)$-MMP which terminates with a Mori fiber space. If $f : X \to Y$ and $f' : X' \to Y'$ are two different Mori fiber spaces which are outputs of two runs of $(K_X + \Delta)$-MMP, then in general $X$ and $X'$ are not isomorphic via the maps of the MMPs (although they might be isomorphic as abstract varieties). In characteristic 0, Hacon and McKernan [HM13] showed that two Mori fiber spaces are connected by Sarkisov links. Sarkisov links are also useful for computing birational automorphism groups of Fano varieties. I would like to study these links in positive characteristic.

**Problem 4.4.** Boundedness of three dimensional log Fano varieties in positive characteristic. It is also known as the BAB conjecture.

Fano varieties are one of the central objects of study in the birational classification of varieties, and the boundedness problem is in the heart of it. One of the breakthrough result of last year is Birkar’s proof of the BAB Conjecture [Bir16a]. With this we now know that the $\epsilon$-log canonical log Fano varieties of fixed dimension in characteristic 0 belong to a bounded family. In contrast, we know very little about the same boundedness problem in positive characteristic. The BAB conjecture is known to be true in dimension 2 in positive characteristic due to Alexeev [Ale94]. But almost nothing is known in dimension $\geq 3$ in positive characteristic.

**References**


Bir16a C. Birkar, Singularities of linear systems and boundedness of Fano varieties, ArXiv e-prints (September 2016), 1609.05543.


