

Completeness or Incompleteness of Basic Mathematical Concepts

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Draft 2/12/18

0 Introduction

I am addressing the topic of the EFI workshop through a discussion of basic mathematical structural concepts, in particular those of natural number and set. I will consider what they are, what their role is in mathematics, in what sense they might be complete or incomplete, and what kind of evidence we have or might have for their completeness or incompleteness.

I share with Kurt Gödel and Solomon Feferman the view that mathematical concepts, not mathematical objects, are what mathematics is about.³ Gödel, in the text of his 1951 Gibbs Lecture, says:

Therefore a mathematical proposition, although it does not say anything about space-time reality, still may have a very sound objective content, insofar as it says something about relations of concepts.⁴

Though probably neither Feferman nor I would put it this way, we both basically agree. There are some differences that I have with each of the them. My view is closer to Gödel's than to Feferman's. Nevertheless, there is a great deal that I agree with in Feferman's *conceptual structuralism*, which he describes in, e.g., [8].

One point of disagreement is that Feferman takes mathematical concepts to be human creations. He calls them objective, but the objectivity seems ultimately only intersubjective. Gödel thinks that mathematical concepts are genuine objects, part of the basic furniture of the world. For him they are non-spatio-temporal entities. I would say that I stand in between the two, but in fact I don't have much

¹I would like to thank Peter Koellner for numerous corrections and for valuable comments and suggestions about the earlier version of this paper that was posted at the EFI website.

²I would like to thank Penelope Maddy for probing questions about another earlier version, questions that have led me to try to make the current version of the paper more explicit and more clear about what my views are.

³Actually Gödel counts concepts as objects. What I am calling objects he calls "things."

⁴Gödel [14], p. 321.

to say about the ontology of mathematical concepts. I think that it is correct to call them objective and that it is more correct to say that they were discovered than that they were created by us. I don't think that this is incompatible with our having epistemic access to them.

Gödel admits two kinds of evidence for truths about mathematical concepts. In current terminology, these are *intrinsic* evidence and *extrinsic* evidence. I think he is right in admitting both. Feferman seems to allow only intrinsic evidence, though perhaps he just has very high standards for extrinsic evidence. At least superficially, I am more with Feferman than with Gödel on how we get intrinsic evidence. Gödel says that it is through a kind of non-sensory perception, which he calls "mathematical intuition." It is not hard to understand how Gödel could be led to such a view. Having placed mathematical concepts in another world, he is impelled to come up with a mechanism for our getting in contact with them. It is not clear, though, how literally we should take the word "perception." Except for the use of this word, everything he says makes it seem that our direct knowledge of mathematical concepts comes from garden-variety grasping or understanding them.

The question on which Feferman, Gödel, and I most clearly and directly disagree is that of the status of the Continuum Hypothesis CH. Feferman is sure that CH has no truth-value. Gödel is sure that it has a truth-value. I believe that the question of whether it has a truth-value is open, and one of the goals of this paper is to understand both possible answers.

The basic mathematical concepts I will be discussing are concepts of structures. The specific concepts that I will consider are the concept of the natural numbers, that of the natural numbers and the sets of natural numbers, etc., the general iterative concept of the sets, and extensions of this concept. These are examples of concepts with two properties. (1) We normally construe them as concepts of single structures. (2) Each of them can be understood and studied without a background knowledge or assumptions about other structural concepts. In saying that the concepts mentioned in the second sentence of this paragraph have property (1), I don't want to imply that they are not at bottom concepts of *kinds* of structures. As I will formulate these concepts, each of them is a concept of a kind of structure, and we regard the structures of this kind as being of a single isomorphism type. To what extent regarding them in this way is justified is one of the topics of this paper.

Some of the main points of my view are the following:

(1) Mathematical concepts differ from other concepts mainly in that they are

amenable to mathematical study. Basic mathematical concepts, insofar as they are taken as basic rather than as being defined from other mathematical concepts, do not come with anything like certifiably precise characterizations. Gödel thinks that basic mathematical concepts are not definable in any reductive way.⁵ He also thinks that they have to be objects in something like Frege's third world, and he thinks that our knowledge of them comes from a kind of perception. My views about mathematics have a lot in common with Gödel's, but his reification of mathematical concepts is one of two main points of difference (the other main point concerning instantiation of basic concepts). His talk of perception is a third difference, but—as I suggested above—it might be a somewhat superficial difference.

(2) The concept of the natural numbers and that of the sets are both concepts of kinds of structures. When I speak of a "structure," I mean some objects and some relations and functions on the objects. Hence my structures are like models, except that I don't require that the objects of a structure form a set.

Thinking that, for example, the concept of the sets determines what it is for an object to be a set is very common but, I believe, wrong. I will thus treat the concept as does (one kind of) structuralist, but nothing important will turn on this.⁶

(3) A fundamental question about a basic structural concept is the question of which statements are *implied by the concept*. I get the phrase "implied by the concept" from Gödel. An example of his use of the term is his saying, on page 182 of [12], "there may exist, besides the ordinary axioms, the axioms of infinity and the axioms mentioned in footnote 17, other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts." I intend to use the phrase in the same way as Gödel does. On my reading of the quoted passage and other similar ones, Gödel does not treat the notion as an epistemic one. A statement could, in principle, be implied by a concept without this being known—or even knowable. One of my reasons for reading Gödel in this way

⁵For some evidence that he thinks this, see the following: the remark about "set of x 's" quoted on page 16 below; the statement on page 321 of [14] that the comprehension axiom for sets of integers, which he expresses in terms of well-defined properties, "cannot be reduced to anything substantially simpler"; the discussion on page 139 of [11] of the sense of "analytic" as true in virtue of meaning, "where this meaning may perhaps be undefinable (i.e., irreducible to anything more fundamental)".

⁶The reader will notice that I use somewhat odd terminology for basic concepts. E.g., I say "the concept of the sets" instead of the usual "the concept of set." This done to stress that it is structures, not individual objects, that may instantiate the concept. This terminology has a downside. For example, the phrase "the concept of the natural numbers" might suggest misleadingly suggest that at most one structure could instantiate the concept.

is that he thinks that the concept of the sets determines truth-values for, e.g., all first-order set-theoretic statements,⁷ and thus he seems to think that, for every such sentence σ , either σ or its negation is implied by the concept. Whether I am right or wrong about Gödel's use, I will always use "implied by the concept" in a non-epistemic sense.⁸

I will treat "implied by the concept" as a primitive notion. In Section 3, I will discuss whether and in what way it might be correct to say that a statement is implied by a basic concept just in case it would have to be true in any structure that instantiated the concept.

One important point is that I regard the question of whether a statement is implied by a basic concept to be meaningful independently of whether there are any structures that instantiate the concept.

(4) The concept of the natural numbers is *first-order complete*: it determines truth-values for all sentences of the usual first-order language of arithmetic. That is, it implies each first-order sentence or its negation but not both. (Other kinds of completeness, such as second-order completeness or quantifier-free completeness, are analogously defined.) In fact I think that the concept of the natural numbers has a stronger property than first-order completeness. I will discuss this property, which I call "full determinateness" in the next section. I regard it as an open question whether the concept of the sets—or even, say, the concept of the natural numbers and the sets of natural numbers—is first-order complete.

(5) The concepts of the sets and of the natural numbers are both *categorical*: neither has non-isomorphic instantiations. (A more conservative statement would, for the concept of the sets, replace "categorical" with "categorical except for the length of the rank hierarchy.")

(6) We do not at present know that the concept of the sets—or even just the concept of the sets of sets of natural numbers—is instantiated. I do not have an opinion as to whether it is known that the concept of the natural numbers is instantiated. But I have no real quarrel with those who say it is known that that the concept is instantiated. I will explain this in the next section.

⁷See, e.g., the first full paragraph of page 262 of [13]. In the last part of Section 1, I will discuss this paragraph and its implied assertion that the concept of the sets is instantiated.

⁸I don't know of any passages in which Gödel uses "implied by the concept" in a way that suggests that it has an epistemic component. However, there is one passage in each CH paper where Gödel uses the seemingly related phrase "intrinsic necessity" in what seems to be an epistemic sense. I will quote one of these two nearly identical passages right after the second paragraph of Section 4. I will discuss, on page 26 and again on page 31, what Gödel might mean in these passages by "intrinsic necessity."

(7) It is irrelevant to pure mathematics whether either of these concepts is instantiated. My main reason for making this assertion is that our number-theoretic and set-theoretic knowledge—including our axioms—is based entirely on concepts.

(7) is something I believe, but it will not really play a central role in this paper. My main concern in the paper is (3). I will be investigating what statements are implied by the basic mathematical concepts, and this seems an important question even if mathematics is ultimately about abstract objects.

I will say a bit about (7) now and in the next section but I won't say much more about it in the rest of the paper. I believe that mathematical objects (e.g., numbers and sets) are not what mathematics is about, that the truth or falsity of mathematical statements does not depend on mathematical objects or even on whether they exist.

A partly superficial difference between my views and Gödel's is in the role of mathematical objects in mathematics. Gödel believed that they play an important role. For one thing, he considered mathematical concepts to be a species of mathematical objects. Since he characterized mathematical truth in terms of relations of concepts, his view has to count as object-based. But the role of he ascribes to non-concept mathematical objects such as numbers and sets is limited. It is, I believe, more limited than I said it was in [18]. I will discuss Gödel's views about the role of such objects in the next section.

1 Mathematical Objects

Most philosophical accounts of mathematics are *object-based*. They take the subject matter of mathematics to be mathematical objects. They characterize mathematical truth in terms of structures composed of objects.

What seems to me the strongest argument in favor of object-based accounts is that they—or, at least, some of them—allow one to take mathematical discourse at face value. Euclid's theorem that there are infinitely many prime numbers is, on face value, about a particular domain of objects, the positive integers. What makes it true is, on face value, that infinitely many of these objects have a certain property.

Of course, many object-based accounts involve taking mathematical discourse at something other than face value. Some structuralist accounts are examples. But such accounts take one aspect of mathematical language at face

value: its existential import. The statement that there are infinitely many prime numbers seems to assert the existence of some things, and pretty much all object-based accounts construe such statements as genuinely assertions of existence.

There are major problems that object-based accounts must face. There the problem, raised in Benacerraf [1], of how we can know truths about objects with which we seemingly do not interact. There is the problem of how we know even that these objects exist. There is the problem of just what objects such things as numbers and sets are.

Dealing with these problems has led philosophers of mathematics to come up with what seem to me strange sounding notions about mathematical objects. Here are some of them:

- The natural numbers are (or are being) created by us.
- Mathematical objects are “thin” objects.
- Specifying the internal identity conditions for a supposed kind of mathematical objects can be sufficient for determining what these objects are.
- Mathematical objects are “logical” objects, and this guarantees their existence.

For me the main problem with assuming as a matter of course that the existence of mathematical objects instantiating our mathematical concepts is that such assumptions are not innocent. They can have consequences that we have good reasons to question. Assume, for example, that we know that the concept of the sets of sets of natural numbers is instantiated. I will argue later that this concept is categorical. (This is an old argument, due to Zermelo.) But instantiation and categoricity together imply that the concept is first-order complete, it would seem—and I believe. This means that if we know instantiation then we know that CH, which is a first-order statement about the concept in question, has a definite truth-value. Do we really know that it has a truth-value? I don’t think so.⁹

There is a property of concepts short of being instantiated that has all the important consequences of instantiation. Say that a concept of a kind of structure is *fully determinate* if is determined, in full detail, what a structure instantiating it would be like. By “what it would be like” I mean what it would be like *qua*

⁹It is important to note that I do not deny that the concept of the sets is instantiated. I deny only that we know at present that it is instantiated.

structure. I don't mean that it would be determined what the *objects* of an instantiation would be. Most mathematicians—and I am included—think of the concept of the natural numbers as having this property. Full determinateness follows from categoricity plus instantiation. Full determinateness is what we want our basic concepts to have. It does all the important work of instantiation. I do not see any reason that full determinateness implies instantiation. However, all my worries about too easily assuming instantiation apply just as well to too easily assuming full determinateness. Hence I don't mind if someone asserts that every fully determinate mathematical concept is instantiated.

Gödel on mathematical objects

Gödel classifies objects into two sorts, *things* and *concepts*. The role he ascribes to mathematical concepts is a central one. Mathematical propositions are about the relations of concepts. A true mathematical proposition is *analytic*, true in virtue of meaning. In one place¹⁰ he says that the meaning in question is the meaning of the concepts occurring in the proposition. In another place¹¹—thinking of propositions as linguistic—he says that it is the meaning of the terms occurring in the proposition, where the meaning of the terms is the concepts they denote.

What role Gödel ascribes to mathematical things, e.g., to sets and numbers, is less clear.

On the one hand, he says of classes (i.e., sets) and concepts:

It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the “data”, i.e., in the latter case the actually occurring sense perceptions.

On the other hand, his account of mathematical truth makes it puzzling what role sets and other mathematical “things” are supposed to play. The revised and expanded version of his paper on the Continuum Hypothesis has passages that look relevant to this puzzle. Here is one of them.

¹⁰Gödel [11], p. 139

¹¹Gödel [14], p. 346

But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception.¹²

It would be natural to suppose that Gödel is talking about both perception of concepts and perception of sets. Nevertheless nothing he says in the paper (or elsewhere, so far as I know) suggests that perception of sets could yield significant mathematical knowledge. The ZFC axioms' forcing themselves on us is surely intended as evidence that we perceive the *concept* of the sets. When elsewhere in the paper he discusses actual or imagined new axioms, the source of our certain knowledge of their truth is always characterized as the concept of the sets and other concepts. He talks of large cardinal axioms that are suggested by the "very concept of set" on which the ZFC axioms are based; of new axioms that "only unfold the content of the concept of set"; of new axioms that "are implied by the general concept of set"; of the possibility that new axioms will be found via "more profound understanding of the concepts underlying logic and mathematics."¹³ There is nothing to suggest that perception of *sets* could help in finding new axioms or played a role in finding the old ones.

A second relevant-looking passage is the following.

For if the meanings of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality.¹⁴

What he explained on page 262 (of Benacerraf and Putnam [2]) and in footnote 14 was the iterative concept of the sets. The quoted passage thus seems to be saying that if that concept of the sets is sound then it is instantiated by some structure

¹²Gödel [13], p. 268.

¹³See pp. 260-261.

¹⁴Gödel [13], p. 260.

and, moreover, the instantiation is unique.¹⁵ In what sense the instantiation is supposed to be unique is not clear. No doubt he at least intends uniqueness up to isomorphism.

The argument given in the quoted passage seems to be the only argument given in the Continuum Hypothesis papers for why the CH must have a truth-value. The corresponding passage¹⁶ in the original version of the paper has what is probably supposed to be the same argument. Instead of the assumption that the explained meanings of the primitive terms of set theory are sound, there is the assumption that “the concepts and axioms” have a “well-defined meaning.”¹⁷

Some naturally arising questions about the argument are:

- (1) Is soundness of meaning the same as well-definedness of meaning? I.e., are the two versions of the argument the same? A related question is: Why did Gödel replace the first version by the second?
- (2) Do these assumptions imply *by definition* the existence of an instantiation (the well-determined reality)? I.e., is existence of a unique instantiation part of what is being assumed?
- (3) Does Gödel have reasons for thinking that the assumptions are true? Evidently he does think they are true.

Whatever the answers to these individual questions are, the important question is: *Do we have good reasons for believing that the concept of the sets has a unique instantiation?* As I have already indicated, I think that the answer is yes for uniqueness and no for existence, and I will say why later in this paper.

¹⁵There is a way of reading this passage on which no uniqueness is asserted and there is no implication that the Continuum Hypothesis has a definite (instantiation-independent) truth-value. But this is not the reading Gödel intends. He intends that the concept of the sets determines a unique instantiation—at least, one that is unique enough to determine a truth-value for CH.

¹⁶Gödel [12], p.181.

¹⁷The real reason for my qualification “seems to be” in the first sentence of this paragraph is not this earlier version of the argument. It is another passage in the revised version that, on one reading, gives a different argument for there being a truth-value for the CH. On page 268 of Gödel [13], he says, “The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor’s continuum hypothesis.” If the existence of “meaning to the question of the truth or falsity of propositions like Cantor’s continuum hypothesis” is understood as implying that such propositions have truth-values, then the argument seems very weak, so charity suggests that Gödel intended something weaker. See Parsons [19] for a discussion of the passage.

2 The Concept of the Natural Numbers

Despite the title of this section, I will mainly discuss not the concept of the natural numbers but the perhaps more general concept of an ω -sequence. The concept of the natural numbers is often taken to be the concept of a single structure, a concept that determines not just the isomorphism type of a structure but also the objects that form the structure's domain. Whether the concept of the natural numbers determines, e.g., what object is the number 3 has long been debated. I don't believe the concept does this. Nor do I—an agnostic about the existence of such mathematical objects—believe that the concept of the natural numbers determines what 3 has to be if it exists. But probably the concept does determine some properties of the numbers. Perhaps, for example, it is part of the concept that numbers have to be abstract objects and that being cardinalities has to be part of their essences. By talking mainly about the concept of an ω -sequence, I will avoid these issues.

The concept of an ω -sequence—or that of the natural number sequence—may be taken not as *basic* but as *defined*, usually as defined from the concept of the sets. Throughout this section, I will take the concept as basic. An instantiation of the concept will consist of some objects and a function, the successor function. One can, if one wishes, think of the successor function as merely a relation, not as an additional object. As I will explain shortly, ordering and basic arithmetical functions are determined by the successor function, and so we might as well think of them as belonging to the instantiation proper.

There are various ways in which we can explain to one another the concept of the sequence of all the natural numbers or, more generally, the concept of an ω -sequence. We can push upward the problem of explaining it, defining it in terms of the stronger concept of the sets. Direct attempts at explaining it often involve metaphors: counting forever; an endless row of telephone poles or cellphone towers; etc. If we want to avoid metaphor, we can talk of an unending sequence or of an infinite sequence. If we wish not to pack so much into the word “sequence,” then we can say that that an ω -sequence consists of some objects ordered so that there is no last one and so that each of them has only finitely many predecessors. This explanation makes the word “finite” do the main work. We can shift the main work from one word to another, but somewhere we will use a word that we do not explicitly define or we define only in terms of other words in the circle. One might worry—and in the past many did worry—that all these concepts are incoherent or at least vague and perhaps non-objective.

The fact that we can latch onto and communicate to one another concepts that we cannot precisely define is not easily explained. It is a remarkable fact

about us. Note, though, that this fact does not imply anything about what kinds of *computation* we are capable of.

Is the concept of an ω -sequence a clear and precise one? In particular, is it clear and precise enough to determine a truth-value for every every sentence expressible in the language of first-order arithmetic? (To be specific, let's declare this to be the first-order language with 0 , S , $+$ and \cdot .) As I indicated earlier, I will call the concept *first-order complete* if the answer is yes—if it does determine truth-values for all these statements. First-order completeness does not mean that the answers to all arithmetical questions are knowable by us. In terminology (of Gödel) that I introduced earlier, it means that an answer is implied by the concept and the opposite answer is not also implied by the concept.

I will use the phrase “first-order complete” in in a similar way in discussing other concepts. E.g., by the question of whether the concept of the sets is *first-order complete* I mean the question of whether that concept determines truth-values for all sentences of the usual first-order language of set theory.

Of course, what one means by “first-order completeness” of a concept depends on what functions and relations one includes. Since I am (mostly) taking the concept of ω -sequence to be a concept of structures with only one unary operation, it would perhaps seem more correct to define first-order completeness for that concept in terms of the language with only with only S as the only non-logical symbol. But order, addition, and multiplication are recursively definable from successor, so it makes sense to include them. Indeed, it makes sense to take them to be part of the concept. I won't worry about whether doing so would yield a different, or just an equivalent, concept.

The question of the first-order completeness of a concept may not be a clear and precise one. If one is unsure about the answer in the ω -sequence case, one may worry even about whether the notion of a first-order formula is clear and precise.

I suspect that most mathematicians believe that the concept of an ω -sequence is first-order complete. I believe that it is. I also suspect that most mathematicians believe—as I do—that the concept is clear and precise in a stronger way, that it has the property of full determinateness that was introduced on page 6.

It may be impossible to give a clear description of this property, but I will try again here. Say that a structural concept is *fully determinate* if it fully determines what any instantiation would be like. Another way to state this is to say that a structural concept is fully determinate if and only if the concept fully determines a single *isomorphism type*. I don't think of isomorphism types as equivalence classes of structures. As I conceive of them, isomorphism types are fully deter-

minate ways that a structure could be. In the language of one sort of structuralist, one might say that a fully determinate concept determines a single “structure.” All it lacks for being a structure in my sense is having its “places” filled by objects. I will make the following assumption.

Modal Assumption: *Every very isomorphism type is or could have been the isomorphism type of a structure in my sense.*

The idea behind the Modal Assumption is that (i) an isomorphism type will be instantiated if there are enough objects to form an instantiation, and (ii) it is possible that there are enough objects. The Modal Assumption is intended to be less an assumption than a partial specification of the notion of possibility that I am using.

I don’t take full determinateness to imply that there are such *objects* as isomorphism types or structuralist structures, but I don’t mind too much if it is taken in that way. I don’t even mind if one says that the full determinateness of the concept of an ω -sequence implies that such a sequence exists or even that the natural numbers exist. My objection to assuming that there are instantiations of, e.g., the concept of the sets is entirely based on uncertainty about whether the concept is fully determinate.

I am now going to discuss some questions about the ω -sequence concept that are related to first-order completeness and full determinateness but are—I believe, importantly different questions.

One question that is not the same as the full determinateness question or the first-order completeness question for a concept is the question of whether the concept is a genuine mathematical concept. The concept of an ω -sequence is the paradigm of a fundamental mathematical concept. It supports rich and intricate mathematics. It is also fully determinate, but that is an additional fact about it. There could be genuine basic mathematical concept that was not fully determinate or even first-order complete. Some think that the concept of the sets is such a concept.

Another question that is different from those of full determinateness and first-order full completeness is the question of categoricity: Are any two structures instantiating the concept isomorphic? Obviously a concept can be clear, precise, and first-order complete without being categorical. The the concept of a dense linear ordering without endpoints is an example. But I also think it possible that a concept be categorical without being first-order complete. The concept of an ω -sequence is not an example, but I do contend that (a) we know the concept of

an ω -sequence to be categorical, but (b) this knowledge does not *per se* tell us that the concept is first-order complete, and (c) we know the concept of the subsets of $V_{\omega+1}$ to be categorical, but we do not know whether it is first-order complete. Justifying each of these perhaps surprising assertions will take me some time.

The Peano Axioms

In arguing for categoricity of the concept of an ω -sequence, the first thing I want to note is that the concept implies a version of the Peano Axioms, what I will call the *Informal Peano Axioms*. These axioms apply to structures with a unary operation S and a distinguished object 0 . Nothing significant for my purposes would be affected if we included binary operations $+$ and \cdot and axioms for them, as in the usual first-order Peano Axioms.) The axioms of Informal Peano Axioms are:

- (1) 0 is not a value of S .
- (2) S is one-one.
- (3) For any property P , if 0 has P and if $S(x)$ has P whenever x has P , then everything has P .

Axiom (3), the Induction Axiom, is framed in terms of the notion of a property. (Peano framed his Induction Axiom in terms of classes.) I have followed Bertrand Russell in using the word “any” and not the word “all” in stating Induction. Russell’s distinction between *any* and *all* is—if I understand it—at heart a distinction between *schematic* universal quantification and *genuine* universal quantification. In the way I intend (3) to be taken, it is equivalent with the following *schema*.

- (3') If 0 has property P and if $S(x)$ has P whenever x has P , then everything has P .

Here there is no restriction on what may be substituted for “ P ” to get an instance of the schema—i.e., no restriction to any particular language. In the future, I will speak of the Induction Axiom as the “Induction Schema” or—to distinguish it from first-order induction schemas—as the “Informal Induction Schema.”

One might worry that the general notion of property is vague, unclear, or even incoherent, and so that we do not have a precise notion of what counts as an instance of the Induction Schema. Perhaps this is so. But as far as *using* the schema is concerned, all that the worry necessitates is making sure that the instances one uses all involve clear cases of properties.

Understanding the open-ended Induction Schema does not involve treating properties as objects. In particular, it does not involve an assumption that the notion of property is definite enough to support genuine quantification over properties. Contrast this with the Second Order Induction Axiom, the induction axiom of the Second Order Peano Axioms, i.e., the Peano Axioms as usually formulated in the formal language of full second-order logic (with non-logical symbols “0” and “ S ”). The language of full second order logic allows one to define properties by quantification—including nested quantification—over properties (or sets or whatever else one might take the second-order quantifiers to range over).

Of course, if one is working in a background set theory and if one is considering only structures with domains that are sets, then quantifiers over properties can be replaced by quantifiers over subsets of the domain. In this situation, the Informal Peano Axioms and the Second Order Peano Axioms are essentially the same. But that is not our situation. In arguing for categoricity, the only objects whose existence I want to assume are those belonging to the domains of the two given structures satisfying the axioms. I do not even want to treat the two structures as objects. Rather I will assume that are determined by their objects, properties and relations.

Do the Informal Peano Axioms fully axiomatize the concept of an ω -sequence? Would any structure satisfying the axioms have to instantiate the concept? In so far as these are definite questions, the answer is yes. Consider a possible structure M satisfying the axioms. Let P be the property of being an object of M that comes from the 0 of M by finitely many applications of the S function of M . By the instance of the Induction Schema given by P , every object of M has P . Hence M is an ω -sequence. Since I think that P is a clear example of a property, I think this argument is valid.

Of course, the axioms are not an axiomatization of the concept the way one normally talks about axiomatization. They are not first-order axioms. It is not precisely specified exactly what the axioms are: what would count as an instance of the Informal Induction Schema. As a tool for proving theorems about the concept, they don't seem to go much beyond the first-order axioms.

In any case, what will be used in proving categoricity of the concept is only that the Informal Peano Axioms are implied by the concept, not the converse.

Categoricity.

The categoricity of the ω -sequence concept has been proved in more than one way, and I will not be presenting a new way to prove it. But I do want to be care-

ful about what I assume. In particular, I want to avoid non-necessary existential assumptions. Dedekind’s proof (see [5]), is done in terms of sets (which he calls “systems”), and uses various existence principles for sets.

Let \mathbf{M} and \mathbf{N} be structures satisfying the Informal Peano Axioms. We specify a function f sending objects of \mathbf{M} to objects of \mathbf{N} as follows:

$$\begin{aligned} f(0)_{\mathbf{M}} &= 0_{\mathbf{N}}; \\ f(S_{\mathbf{M}}(a)) &= S_{\mathbf{N}}(f(a)). \end{aligned}$$

Using the Informal Induction Schema in \mathbf{M} , we can show that these clauses determine a unique value of $f(a)$ for every object a of \mathbf{M} .¹⁸ By more uses of the Informal Induction Schema in \mathbf{M} , we can prove that this defined function is one-one and is a homomorphism. Using the Informal Induction Schema in \mathbf{N} , we can prove that the defined function is a surjection. The properties involved in the instances of Informal Induction are definable from the two models, so there is nothing problematic about them.

Note that categoricity of the Informal Peano Axioms does not by itself imply the first-order completeness of the axioms or the ω -sequence concept, for the trivial reason that categoricity implies nothing if there is no structure satisfying the axioms. Dedekind¹⁹ was well aware that categoricity by itself is worthless, and that led him to his often maligned existence proof. What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers. Indeed, I believe that full determinateness of the concept is the only possibly legitimate justification we have for saying that the concept is instantiated or that the natural numbers exist.

3 The Concept of the Sets

The modern iterative concept has four important components:

- (1) the concept of the natural numbers;
- (2) the concept of the sets of x ’s;

¹⁸Coding finite sequences of elements of each model by objects of the model, we apply Informal Induction in \mathbf{M} for the property of being an element b of \mathbf{M} such there is a unique (code of) a function from the \mathbf{M} -predecessors of b into \mathbf{N} that satisfies the inductive clauses. These functions piece together to define f .

¹⁹Dedekind [5]

- (3) the concept of transfinite iteration;
- (4) the concept of absolute infinity.

Perhaps we should include the concept of extensionality as Component (0). Component (1) might be thought of as subsumed under the other three, but I have treated it separately. In the way I am thinking of the concept of the sets, it is a concept of a *kind* of structure, and so one does not have to add anything about what kind of objects a set is.

The concept of absolute infinity comes from Cantor. He held that sequence of the ordinal numbers is absolutely infinite, whereas sets are merely *transfinite*. Cantor's assertion justifies the Axiom of Infinity, the Axiom of Replacement, and some large cardinal axioms. The concept—or some substitute for it—is an important ingredient in the concept of the sets. Nevertheless, it will not play a major role in this paper.

Sets of x 's.

The phrase “set of x 's” comes from Gödel. The x 's are some objects that form a set and the sets of x 's are the sets whose members are x 's. It might have been better, given what Gödel says in the quotation below, to speak of “class of x 's.”²⁰ Gödel says

The operation “set of x 's” cannot be defined satisfactorily (at least in the present state of knowledge), but only be paraphrased by other expressions involving again the concept of set, such as: “multitude of x 's”, “combination of any number of x 's”, “part of the totality of x 's”; but as opposed to the concept of set in general (if considered a primitive) we have a clear notion of the operation.²¹

For Gödel (and for me), this concept is—like the concept of the natural numbers—typical of the basic concepts of mathematics. It is not definable in any straightforward sense. We can understand it and communicate it to one another, though what we literally say in communicating it by no means singles out the concept in any clear and precise way. As I said earlier, our ability to understand and communicate

²⁰Gödel doesn't say whether the concept applies if the x 's do not form a set. I am assuming that it does not apply. It would be okay to let it apply, but then only “class of x 's” would give the intended meaning.

²¹Gödel [12], p. 180.

such concepts is a striking and important fact about us. Gödel says that the concept is of the sets of x 's is "clear." But Feferman says that the concept is unclear for the case when the x 's are the natural numbers, so—one would presume—for any case when there are infinitely many x 's. Feferman says that the concept of an arbitrary set of natural numbers is "vague." My own view is that the clarity—or, in my language, the full determinateness—of the concept of the sets of x 's is an open question, and that we cannot rule out that the answer varies with what the x 's are, even when there are infinitely many.²²

Let us first look at the general concept of the sets of x 's. This is a concept of structures with two sorts of objects and a relation that we call membership that can hold between objects of the first sort, the x 's, and objects of the second sort, the sets of x 's. Clearly the following axioms are implied by the concept.

- (1) If sets α and β have the same members, then $\alpha = \beta$.
- (2) For any property P , there is a set whose members are those x 's that have P .

Axiom (1) is, of course, the Axiom of Extensionality. Axiom (2) is a Comprehension Axiom, which I will interpret as an open-ended schema and call the Informal Comprehension Schema, analogous to the Informal Induction Schema.

Do these axioms fully axiomatize the concept of the sets of x 's? It is very plausible to say they do. Gödel seems to have thought so. In the posthumously published version of his Gibbs Lecture, he says, of the case when the x 's are the integers:

For example, the basic axiom, or rather axiom schema, for the concept of set of integers says that, given a well-defined property of integers (that is, a propositional expression $\varphi(n)$ with an integer variable n) there exists the set M of those integers which have the property φ .

It is true that these axioms are valid owing to the meaning of the term "set"—one might even say that they express the very meaning of

²²One argument in favor of the clarity of the concept is that it is not easy to see what could be the source of unclarity. Vagueness does not seem to be the answer. What are generally regarded as the two main characteristics of vagueness, borderline cases and the absence of sharp boundaries, are nowhere to be seen. The cause of our worrying about the concept of the sets of x 's is not examples of properties about whose definiteness we are unsure, and it is not because we see that there is an absence of a sharp boundary. One proposed source of a lack of clarity is that the concept depends upon the notion of *all* definite properties of x 's. The only way I could understand an unclarity about what is meant by "all definite properties of x 's" would be if there were an unclarity in what is meant by "definite property of x 's."

the term “set”—and therefore they might fittingly be called analytic; however, the term “tautological”, that is, devoid of content, for them is entirely out of place.²³

I have omitted a few sentences between the two parts of the quotation, sentences about why the axioms of the schema are not tautologies. The quotation occurs in the midst of a section in which Gödel argues that mathematical truths are analytic but are not mere tautologies.

There is a similar section in Gödel’s earlier “Russell’s mathematical logic.” In it there is a passage like the one I have just quoted, except that Gödel there adds the Axiom of Choice, saying that “nothing can express better the meaning of the term ‘class’ than the axiom of classes... and the axiom of choice.” (The “...” replaces a reference to the number of an earlier page on which Russell’s axiom of classes is discussed.)

Does one need to add Choice to fully axiomatize the concept of the sets of x ’s? I suppose that depends on how one construes the term “property” occurring in the Informal Comprehension Schema. I will return to this issue below.

Gödel does not mention Extensionality, but clearly it is necessary for a full axiomatization of “the sets of x ’s.”

To “fully express” the concept, do we need to specify something more, for example, *what object* the set whose only member is the planet Mars is? People who think that the natural numbers can be any ω -sequence often think that sets have to be particular objects. I do not think this is so, and I also don’t think there is any way to make the specification, but I won’t argue these points here. I will simply ignore any constraints the concept might put on what counts as a set and what counts as membership other than structural constraints such as those imposed by (1) and (2).

Axioms (1) and (2) are *categorical for fixed x ’s*. I.e., any two structures satisfying the axioms and having the the same x ’s are isomorphic by unique isomorphism that is the identity on the x ’s. Here is the proof (essentially due to Zermelo, whose Separation Axiom should, I believe, be viewed as an open ended schema).

Let \mathfrak{M}_1 and \mathfrak{M}_2 be structures satisfying (1) and (2) and having the same x ’s. Let \in_1 and \in_2 be the relations of the two structures. With each α that is a set in the sense of \mathfrak{M}_1 , we associate $\pi(\alpha)$, a set in the sense of \mathfrak{M}_2 . To do this, let P be the property of being an x such that $x \in_1 \alpha$. By the Informal Comprehension Axiom

²³Gödel [14], p. 321.

for \mathfrak{M}_2 , there is a set β in the sense of \mathfrak{M}_2 such that, for every x of \mathfrak{M}_2 ,

$$x \in_2 \beta \leftrightarrow P(\beta).$$

By Extensionality for \mathfrak{M}_2 , there is at most one such β . Let $\pi(\alpha) = \beta$. Using Informal Comprehension and Extensionality for \mathfrak{M}_1 , we can show that π is one-one and onto, and so is an isomorphism.

Here are some comments on the proof.

(i) Since the axioms are implied by the concept of the sets of x 's, that concept is *categorical for fixed x 's*, i.e., any two instantiations of the concept with the same x 's are isomorphic.

(ii) The properties P used in the proof were defined from the given structures. Hence there is no problem about the legitimacy of the instances of Informal Comprehension that were used, and there was no use of the Axiom of Choice.

(iii) The proof can obviously be modified to get an isomorphism when \mathfrak{M}_1 has x 's, \mathfrak{M}_2 has y 's, and we are given a one-one correspondence between the x 's and the y 's. The modified proof defines the unique isomorphism extending the given correspondence. In particular, the x 's and y 's could be the objects of isomorphic structures instantiating some categorical concept (e.g., the concept of the natural numbers), and the given correspondence could be an isomorphism between the two structures.

(iv) We could have only one sort of variable, with the language being the language of set theory. The x 's and the y 's could be the objects of isomorphic instantiations of the concept of some V_α , if there are such instantiations. Then \mathfrak{M}_1 and \mathfrak{M}_2 would be isomorphic instantiations of the concept of $V_{\alpha+1}$.

As with the ω -sequence concept, categoricity does not *by itself* guarantee first-order completeness. I.e., categoricity for fixed x 's does not by itself guarantee that the concept of the sets of x 's determines, for any fixed x 's, a truth-value for every first-order sentence in the associated two-sorted language. In order for it to have such an effect, the concept has to have an instantiation with these as the x 's, and the concept must determine a truth-value for every first-order sentence true in all such instantiations. (Satisfying the second of these two requirements for a sentence σ would rule out its being an accident that all instantiations give the same truth-value to σ .)

We could think of a structure instantiating the general concept of the sets as what is gotten by starting with the natural numbers and iterating the sets of x 's

operation absolutely infinitely many times.²⁴ The usual construction, which gets only pure sets, starts with the empty set: $V_0 = \emptyset$; $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$ for limit ordinals λ ; $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ = the set of all sets of x 's, where the x 's are the members of V_α .

Using comment (iv) above, one can show that, for any ordinal α , the categoricity of the concept of V_α implies the categoricity of the concept of $V_{\alpha+1}$.²⁵ One can also show how to get categorical axioms implied by the latter concept from any given categorical axioms implied by the former.

Using the categoricity of the concept of an ω -sequence, one can show that the concept of V_ω is categorical. Categorical axioms implied by it are easily found with Zermelo's Separation Axiom (what I would call the "Informal Separation Axiom") as the one non-first-order axiom.

Note that everything said in the last two paragraphs remains true if "categoricity" is replaced by "necessary categoricity" and "categorical" is replaced by "necessarily categorical."

There are, of course, more categoricity results involving the concept of the sets. Zermelo's categoricity-except-for-hierarchy-length theorem is one. In [17], I argue that the general concept of the sets is categorical.

Here are two key concepts.

- (1) the concept C_1 of $V_{\omega+1}$;
- (2) the concept C_2 of $V_{\omega+2}$.

Both these concepts are necessarily categorical. Are they fully determinate? Are they first-order complete?

I will first consider the concept C_2 . (Consistency of terminology would require me to call it something like "the concept of the sets of rank $\leq \omega + 1$," but I won't be this consistent.) Much of what I say about this concept would also apply to the concepts of higher V_α 's and, arguably, even to the concept of V , the general concept of the sets.

CH is a first-order statement about $V_{\omega+2}$. Hence C_2 determines a truth-value for CH if C_2 is first-order complete.

Assume for definiteness that C_2 has an instantiation \mathfrak{M}_1 in which CH is true.²⁶ By categoricity, CH is true in every instantiation of C_2 . Under our assumptions, it seems very likely that C_2 implies CH.

²⁴On page 180 of [12], Gödel talks of a set as being "anything obtainable from the integers (or some other well-defined objects) by iterated application of the operation of 'set of'."

²⁵Talking about the "concept" of V_α for undefinable ordinals α is stretching the concept of concept.

²⁶The case that CH is false in some instantiation of C_2 is treated similarly.

How might C_2 fail to imply CH? Clearly C_2 fails to imply CH if there is a fully determinate way that an instantiation of C_2 falsifying CH could have been. The way I am construing possibility, another way to state this is as follows. C_2 does not imply CH if there is an isomorphism type that, had there been enough objects, would have been the isomorphism type of an instantiation of C_2 in which CH is false. I will argue that this cannot happen.

I will use the “isomorphism type” terminology, but bear in mind that for me an isomorphism type is just a fully determinate way for structures to be, whether or not there are any structures of that isomorphism type.

Our Modal Assumption²⁷ implies that, for any two isomorphism types Θ and Θ' , it is possible that there are structures of both isomorphism types. To see this, apply the Modal Assumption to the isomorphism type of structures that are the disjoint unions of a type- Θ structure and a type- Θ' structure.

Let us say that an isomorphism type Θ *conforms to* a structural concept C if structures of isomorphism type Θ are or would be instantiations of C .

In the situation that I am trying to rule out, there are two isomorphism types, the isomorphism type Θ_1 of \mathfrak{M}_1 and another isomorphism type Θ_2 . Both Θ_1 and Θ_2 conform to C_2 . While Θ_1 satisfies CH, Θ_2 does not. By our Modal Assumption, there could have been instantiations of C_2 of both isomorphism types. Since C_2 is necessarily categorical, Θ_1 and Θ_2 are the same. Hence we get the contradiction that they cannot differ about satisfying CH.

All our argument used about C_2 was the assumption that C_2 is a necessarily categorical structural concept. Hence the argument shows that the Modal Assumption has the following consequence.

Isomorphism Type Uniqueness: *For any necessarily categorical structural concept C , there is at most one isomorphism type that conforms to C .*

I regard Isomorphism Type Uniqueness as justifying the the following assumption.

Concept Implication Assumption: *Let C be a necessarily categorical structural concept. If Θ is an isomorphism type conforming to C , then C is fully determinate and every sentence satisfied by Θ whose truth is preserved under isomorphism is implied by C .*

What the Concept Implication Assumption says about a necessarily categorical structural concept C is that if there is one and only one way that an instan-

²⁷See page12.

tiation of C could be, then everything that would be true in such an instantiation is implied by C . It seems clear that this would be true for any reasonable way sharpening the meaning of “implied by a concept.”

Does the Concept Implication Assumption provide a necessary and sufficient condition? I.e., is *does a necessarily categorical structural concept C imply a sentence σ if and only if there is an isomorphism type (which has to be unique) that conforms to C and satisfies σ .*

To see why I say no, assume that the answer is yes and consider C_2 in the case when C_2 does not determine a truth-value for CH. This is a case that I don’t think we can at present rule out. Since there is at most one isomorphism type that conforms to C_2 , there is no isomorphism type that conforms to C_2 ; for if some isomorphism type conformed to C_2 , then C_2 would determine a truth-value for CH. Since we are assuming that the answer to the italicized question is yes, there is no sentence that is implied by C_2 . Hence C_2 does not imply, e.g., the Axiom of Extensionality. But Extensionality seems a paradigm of a sentence implied by C_2 . Gödel would surely count it as implied by C_2 , and I am trying to agree with his notion of being implied by a concept.

On page 4, I hinted that, for example, that there might be a sense in which “CH is implied by C_2 ” is equivalent with “Necessarily CH is true in any instantiation of C_2 .” Here are two candidate versions of such an equivalence.

- (1) CH is implied by C_2 if and only if (a) there is no isomorphism type that satisfies \neg CH and conforms to C_2 .
- (2) CH is implied by C_2 if and only if (b) there could not have been an isomorphism type that satisfied \neg CH and conformed to C_2 .

Let us first consider version (1).

If there is an isomorphism type that conforms to C_2 and satisfies CH, then it follows from (1) that CH is implied by C_2 and Isomorphism Type Uniqueness. The additional Concept Implication Assumption is not needed. The analogous fact holds the obvious generalization of (1) to necessarily categorical structural concepts.

Nevertheless I am not fully satisfied with (1), To see why, consider what happens if there is no isomorphism type that conforms to C_2 . By the Concept Implication Assumption, this case occurs if CH has no truth-value, and I believe that this case is, at present, a genuine epistemic possibility. It, together with (1), has the consequence that both CH and \neg CH are implied by C_2 , and so that C_2 is inconsistent. Thus (1) implies that either CH has a truth-value or else C_2 is inconsistent.

The natural generalization of (1) implies that *a necessarily categorical structural concept is first-order incomplete if and only if it is inconsistent*. While I don't rule this out, it seems important to make the best possible attempt at finding a way for incompleteness to occur without inconsistency.

Another worry about (1) is the negative character it assigns to the notion of being implied by C_2 . (1) makes CH to follow from C_2 not because of evidence that it is somehow part of that concept but rather because of a lack of strong evidence that its negation is part of the concept. For first-order concepts (concepts given wholly by first-order theories), the proofs of Completeness show—even for those us who are agnostic about instantiation—that there is an isomorphism type (in my sense) conforming to the concept. Nevertheless, it is worth investigating whether concepts like C_2 might be consistent without there being isomorphism types conforming to them.

Now let's consider version (2).

If our modal logic satisfies S4 and if our Modal Assumption²⁸ holds necessarily, then (b) is equivalent with

(b') There could not have been an instantiation of C_2 that satisfied \neg CH.

(2)²⁹ To see this, assume that S4 holds. Note first that (b') follows from (b). For the other direction, assume that there could have been an isomorphism type that satisfied \neg CH and conformed to C_2 . By the necessity of our Modal Assumption, it could have been the case that there could have been an instantiation of C_2 satisfying \neg CH. The falsity of (b') follows by S4.

With version (2), the Concept Implication Assumption seems needed to deduce that C_2 implies CH if some isomorphism type conforming to C_2 satisfies CH.

In the case in which there is no isomorphism type that conforms to C_2 , the inconsistency argument that applies to (1) may not go through for version (2). The non-existence of an actual isomorphism type conforming to C_2 may not imply that there couldn't have been such an isomorphism type—even an instantiated one. Thus (2) and its natural generalization seem to allow for genuine incompleteness of necessarily categorical concepts, not just for incompleteness via inconsistency.

My reasons for being uncomfortable with (2) are: (i) like (1), it negatively characterizes the concept of being implied by C_2 and (ii) it involves the questionable notion of possible, non-actual ways that structures could be. It seems

²⁸See page 12.

²⁹Something like (b') version of (2) was what I had in mind in earlier versions of this paper.

plausible that possible ways for structures to be are actual ways for structures to be, in which case (2) is equivalent with (1).³⁰

Nevertheless, I regard (2) is a better candidate than (1) for a notion of implication by concepts that allows for incompleteness of set-theoretical concepts.

If CH does not have a truth-value, then there is something defective about the concept of the sets. Either it—even C_2 —is inconsistent, or C_2 is consistent but has no instantiations and there are no isomorphism types that could have been isomorphism types of instantiations of C_2 .

Is the question of whether CH has a truth-value a concept-dependent question? E.g., could the full concept of the sets determine a truth-value for CH without that truth-value's being implied by C_2 ? I will consider questions of this kind in Section 5, concluding that the answers are likely negative.

What about the case of C_1 , the concept of the subsets of V_ω (or, equivalently, the concept of the natural numbers and the sets of natural numbers)? Is this concept fully determinate? If we had the sort of direct, intuitive evidence that we have for the full determinateness of the concept of the natural numbers, then wouldn't this evidence apply to the concept of the sets of x 's in general? Wouldn't intuitive evidence allow us to see that whenever the concept of the x 's is fully determinate then so is the concept of the sets of x 's? There are those who are convinced that the general concept is fully determinate in this way. If they are right, then CH has a definite truth-value.

There is room to try to separate, with respect to direct intuitive evidence of full determinateness, the concept of $V_{\omega+1}$ from from the concepts of the V_α for $\alpha > \omega + 1$. Since the concept of an ω -sequence is fully determinate, isn't the concept of an ω -sequence of, say, 0's and 1's a clear one? If so, what could be wrong with the concept of all ω -sequences of 0's and 1's'? I don't think that this argument is without force, but I'm not quite convinced by it.

I do think that, given the evidence that we have at present, the case of C_1 looks very different from that of C_2 . A major part of the difference is that for the former concept we have available a first-order theory for C_1 that is supported by much evidence and that seems as complete for the C_1 as first-order PA is for the concept of the natural numbers. The main contenders for problem cases analogous to CH are provable or refutable by this theory. The evidence that supports the theory is mainly *extrinsic*. I will discuss the theory and extrinsic evidence in the next

³⁰I have somewhat the same feeling about instantiations of, e.g., the concept of the sets, because it seems plausible that the possible purely abstract objects should be the same as the actual purely abstract objects, and if the concept of the sets can be instantiated (necessarily by a proper class of objects) then purely abstract objects should be enough to instantiate it.

section.

As I am construing the third important component of the concept of set, the concept of transfinite iteration, that concept is essentially the concept of ordinal numbers or simply that of wellordering. It is also intimately related to the concept of L —more specifically to the concept of a proper or non-proper initial segment of L . There seems to be nothing that creates worries about it as CH does about the concept of the sets of x 's. The intuition that supports confidence in the full determinateness of the ω -sequence concept extends at least to small transfinite ordinals and to the associated initial segments of L . Since I am leaving length of iteration out of the concept of transfinite iteration, that concept is not fully determinate, but it—and so the concept of L —might well be fully determinate except for length.

The concept of an initial segment of L has as much claim to be (informally) axiomatized as the concepts of the natural numbers and the sets of x 's. An open-ended Informal Wellfoundedness Axiom plays the role analogous to that of Informal Induction and Informal Comprehension. These axioms are categorical except for length.

Cantor described the sequence of all the ordinal numbers as “absolutely infinite,” so I am using the term “absolute infinity” for the concept that is the fourth component of the concept of the sets. One can argue that the concept is categorical, and that any two instantiations of the concept of the sets (of the concept of an absolutely infinite iteration of the sets of x 's operation) have to be isomorphic.³¹ But it is hard to see how there could be a full informal axiomatization of the concept of the sets. There are also worries about the coherence of the concept. People worry, e.g., that if the universe of sets can be regarded as a “completed” totality, then the cumulative set hierarchy should go even further. Such worries are one of the reasons for the currently popular doubts that it is possible to quantify over absolutely everything. I am also dubious about the notion of absolute infinity, but this would not by itself make me question quantification over everything.

4 Extrinsic Evidence

By *intrinsic evidence* for the truth of, say, a statement about the the concept of the sets, I mean direct evidence for the statement's being implied by the concept of the sets. Such evidence provides pretty conclusive support for the ZFC axioms,

³¹See [17].

the informal ones as well as the first-order ones. It seems reasonable to say that existence of some large cardinals—for example, inaccessible cardinals—follows from the absolute infinity of the class of ordinals.

A hypothetical example of *extrinsic evidence* occurs in the following oft-quoted statement of Gödel from the 1947 version of “What is Cantor’s Continuum Problem?”

Furthermore, even disregarding the intrinsic necessity of a new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible in another way, namely, by studying its “success”, that is, its fruitfulness in consequences and in particular its “verifiable” consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs.³²

Gödel goes on to say that say that the axioms for the real numbers are “to some extent” an example, because statements of number theory are sometimes first proved using analysis and are later shown to have elementary proofs. He goes on to imagine axioms having a “much higher degree of verification.”

One thing to note about this passage is that it is hard to see how “even in case it had no intrinsic necessity at all” fits with the rest of what Gödel says in this paper. On the preceding page, Gödel says, “For if the meanings of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows that the concepts and axioms of set theory describe some well-determined reality, in which CH must be either true or false.” This seems to imply that Gödel believes that all set-theoretic truths are implied by the concept of the sets. If Gödel is using “intrinsically necessity” in the passage above as synonymous with “being implied by the concept of the sets,” then one wonders why he didn’t say “even if we did not know that it had intrinsic necessity” instead of “even it had no intrinsic necessity”? Such thoughts suggest that Gödel takes intrinsic necessity to have an epistemic component. On page 31, I will suggest what that epistemic sense might be.

I will use the term “extrinsic evidence” for any kind of evidence that is not intrinsic. It is often used in a narrower sense, so that extrinsic evidence for an axiom or theory is evidence based on the consequences of that axiom or theory.

³²[12], pp. 182-83. The same passage, with small changes in wording, occurs in the 1964 version of the paper.

Is consequence-based evidence of the sort Gödel describes evidence for truth? The intuition that it is evidence for truth is, at least for me, a strong one. For mathematics as for empirical sciences, it is hard to find an argument that justifies this intuition. Moreover it might seem that how well it is justified could depend on one's account of mathematical truth: that the degree of justification might be different if truth is about a hypothetical domain of objects than if truth is something like being implied by a concept. In any case, I think that the intuition should be trusted more than any account of mathematical truth, so I regard Gödel-style extrinsic evidence as a clear example of genuine evidence. Furthermore, I don't see why one of the two mentioned sorts of accounts of truth makes it better evidence than the other would.

Let us finally turn briefly to a discussion of the concept C_1 of $V_{\omega+1}$ —or, equivalently the concept of the natural numbers and the subsets of the natural numbers.

At present this case looks very different from the case of C_2 . The standard first-order axioms for C_1 are what I have been calling Second-Order Arithmetic. By adding to these axioms the schema of Projective Determinacy, one gets a first-order theory that (1) seems as complete for C_1 as the first-order Peano Axioms are for the concept of the natural numbers and (2) for whose truth there is a large, diverse and—to many of us—convincing body of evidence.³³

The question of whether the evidence for determinacy axioms and large cardinal axioms justifies belief that these axioms are implied by the concept of the sets is a difficult one. I would like to point out, though, that a lot of the evidence for large cardinals and determinacy (and also much of the evidence Woodin has cited in his endeavor to solve the continuum problem) does indeed feel like evidence for *truth* and not just for satisfying methodological desiderata. Examples of evidence of this kind are diverse. One example that I am particularly fond of involves prediction and confirmation. This is the example of the Wadge degrees. Wadge proved that determinacy for a class of subsets of Baire space implies that the sets in that class are essentially linearly ordered by the relation “is a continuous preimage of.” This ordering was later shown to be a well-founded. Wadge's proof from determinacy is about one line long. Wadge's theorem for the special case of Borel sets is a statement about $V_{\omega+1}$. Several years after Wadge's proof, that special case was proved from the ZFC axioms, by a fairly complex proof. Several years after that, the Borel case was proved in Second-Order Arithmetic, by a very long and complex proof. These facts seem to me a significant piece of evidence for the truth of general determinacy hypotheses (and Projective Determinacy in

³³See Koellner [15] for a statement of projective determinacy and material on (1) and (2).

particular), and the body of extrinsic evidence for these hypotheses seems more solid than any view about what such truth consists in.

Indeed projective determinacy seems a good example of the kind of axiom Gödel envisioned in the passage quoted above. I believe that (1) and (2) provide strong extrinsic evidence that that C_1 is fully determinate and so that it is first-order complete.

5 A Puzzling Phenomenon

I could have called this section “Intrinsic Evidence.” In it I will be talking about what statements about a basic concept can, from intrinsic evidence alone, be known to be implied by that concept. These are the statements that can be directly seen to be implied by the concept plus the statements that can be deduced from such statements. I will call deductions from statements of the former kind *proofs directly from the concept*. I will discuss what seem to be great limitations on this method of getting knowledge. I will also discuss how extrinsic evidence about basic concepts can be often be gotten by proofs directly from stronger concepts.

In this paper I have suggested that mathematical truth has to do with concepts and not with objects, but I have not yet put forth a concept-based account of mathematical truth. A good try at giving such an account is to say that an *arithmetical truth* is an arithmetical statement that is implied by the concept of the natural numbers, that a *set-theoretical truth* is a set-theoretical statement that is implied by the concept of the sets, etc. As we will see, the puzzling phenomenon of the section title might make one worry that this account makes mathematical truth a *relative* notion. E.g., the puzzling phenomenon might make one worry that there are arithmetical statements that are not arithmetical truths but are set-theoretical truths. (This is not a worry if one believes that the concept of the natural numbers is fully determinate, but analogous worries about higher order concepts are serious.)

Here is one kind of example of the puzzling phenomenon in the special case of arithmetical statements. There many sentences σ in the language of First-Order (Peano) Arithmetic such that:

- (i) σ is provable in Second-Order Arithmetic,³⁴ whose axioms can be directly

³⁴Recall that by, e.g., “Second-Order Arithmetic,” I mean the standard two-sorted, first-order theory of the natural numbers and sets of natural numbers. I will sometimes call n th-Order arith-

seen to be implied by the concept of the natural numbers and the sets of natural numbers;

- (ii) σ is not provable in First-Order Arithmetic and is not known to follow from the Informal Peano Axioms without using (in applications of the induction schema) properties defined using higher-order concepts.

Such a sentence σ is thus known by intrinsic evidence to be implied by the concept of the natural numbers and the sets of natural numbers (equivalently, from the concept C_1 of $V_{\omega+1}$), but there is no known way to prove σ directly from the concept of the natural numbers.

One species of examples is that of consistency statements, which are equivalent to Π_1 sentences of the language of First-Order Arithmetic. The consistency of First-Order Arithmetic itself is probably not an example. While it is not provable in First-Order Arithmetic, one can plausibly argue that it can be proved directly from the concept of the natural numbers. But the consistency statements for various fragments of Second-Order Arithmetic provide examples, as do the consistency statements for ZFC and fragments of it.

The phenomenon occurs at every level. For any positive integers n and $m > n$, there are sentences σ in the language of n th-Order Arithmetic with the following properties. (a) σ is provable in m th-Order Arithmetic. (b) σ is not provable in Arithmetic of Order $m-1$. (c) there is no known way to prove σ directly from the concept C_{m-2} .

The phenomenon also occurs with respect to the full concept of the sets. For example, $V \neq L$ is provable in ZFC + “There is a measurable cardinal” but not in ZFC alone. While there is some intrinsic evidence that “There is a measurable cardinal” is implied by the concept of the sets, this intrinsic evidence seems not enough by itself to yield *knowledge*.

One might cling to the dream that the phenomenon is caused by our incomplete understanding of our concepts. For example, one might hope that some radically new method will be found that yields proofs that (a) are directly from the concept of the natural numbers and (b) whose conclusions are arithmetical sentences of the kind we have been talking about, including statements whose truth are equivalent to the consistency of ZFC or even of strong large cardinal axioms. But this seems a good example of a pipedream, and so do analogous dreams about higher-order concepts. Thus it seems highly probable that, for each of our basic concepts, there are questions about the concept that are not answerable on the basis of intrinsic

metic “Arithmetic of Order n .”

evidence alone. We will see that this implies, in particular, that those of us who believe that the concept of the natural numbers is fully determinate have to live with the high probability that many first-order arithmetical sentences are implied by the concept of the natural numbers and can be directly shown to follow from higher-order concepts but will not—and perhaps cannot—be directly shown to follow from the concept of the natural numbers.

The phenomenon we are discussing is also discussed in Gödel [14], in the passage about axiomatizing the concept of the sets that begins near the bottom of page 305 and continues through page 307.

Gödel says that we should regard the axiomatization of set theory as being done stepwise, with the axioms arranged in levels. Consider first the finite levels. In the sample account he gives of these levels, the 0th level contains the axioms about the integers (construed as sets in some way). Level 1 adds the axioms about the sets of integers, and so on. For general finite n , the n th level adds the axioms for the sets belonging to $\mathcal{P}^n(\omega)$. (It would be more natural to let the levels $0, \dots, n$ provide together the axioms for, say, $V_{\omega+n}$.)

Gödel says nothing explicit for any n about what “the axioms” of level n are. One might initially think that he is leaving this open and perhaps that he allows it change with time. But this does not fit well with the idea of a stepwise, level-by-level process, and it also does not fit with the facts Gödel cites on page 307. It is very likely that Gödel is thinking of the level-0 axioms as being those of first-order Peano Arithmetic and the axioms of levels 0 through n as combining to give the first-order theory I am calling “Arithmetic of Order $n + 1$.” The open-ended sequence of limit-level axioms give closure conditions on the ordinals, conditions whose purpose is to keep the set-theoretic hierarchy going. These axioms amount to Infinity, Replacement, and what we would now regard as fairly weak large cardinal axioms.

Gödel remarks on page 307 that each new level of axioms yields proofs of arithmetical sentences not provable from the axioms for the lower levels.³⁵ He has great hopes for “set-theoretical number theory,” which, as he remarks, had so far made use only of the level 1 axioms, in analytic number theory.

If I am right about what axioms Gödel is calling *the* axioms of set theory, then it seems plausible that at the time of writing Gödel hoped that every mathematical question is in principle answerable using this transfinite, open-ended sequence of

³⁵One might take “each” to mean “each finite.” Gödel possibly intends something general, thinking that each new level of axioms will yield a proof of the consistency of the set of all axioms of the lower levels.

axioms. In other words, it is plausible that he then thought that ZFC plus what we now call weak large weak large cardinal axioms could answer every set-theoretical question that we might ask. We now know that this is false.

Something that is still not obviously false is that such axioms can answer every set-theoretical question whose answer can be known from intrinsic evidence alone. Of course, Gödel does not use such terminology. But when he, in the passage I quoted on page 26, talks of *intrinsic necessity* as a property that set-theoretical statements might have or lack, I suspect that he has in mind what I am calling “being knowable from intrinsic evidence about the concept of the sets” or “being provable directly from the concept of the sets.” If this is so, then Gödel seems to be allowing for the possibility that there are set-theoretical statements that are implied by the concept of the sets but cannot be known by intrinsic evidence to be implied by that concept.

Did Gödel believe, e.g., that there are arithmetical statements provable in set-theoretical number theory but not provable directly from the concept of the natural numbers? He certainly says nothing to suggest that this is not a possibility.

Consider the case of a proof in Second-Order Arithmetic of a sentence σ of First-order Arithmetic. This proof counts as a proof of σ for at least the purely sociological reason that any proof formalizable in ZFC counts as a mathematical proof. Almost all mathematicians would count it as a proof of σ , though many of them would be happier with an “elementary” proof, which would probably amount to a proof in first-order Peano Arithmetic. Gödel clearly thinks that it counts as a proof of σ .

What does the conceptualist notion of truth suggested on page 28 say about such a proof. Since it is a proof directly from the concept C_1 , it demonstrates that σ is implied by the the concept C_1 and by stronger concepts such as that of the sets. Hence it proves that σ counts as what we might call a *second-order* arithmetic truth and as a set-theoretic truth. But does the proof show that it is an arithmetic truth? Does it show that σ is implied by the concept of the natural numbers, or does it prove only that σ is implied by the stronger concept of the natural numbers and the sets of natural numbers? It provides what I would classify as *extrinsic* evidence that σ is implied by the natural number concept. (I will discuss below the strength of this evidence.) Should we demand that a *proof* that it is implied by that concept be a proof directly from that concept? Should we demand of a *proof* that a set-theoretic sentence is a set-theoretic truth that it be a proof directly from that concept of the sets? I.e., it should use only axioms directly seen to follow from that concept, e.g., the informal ZFC axioms plus further principles justified on the basis of the absolute infinity of the sequence of ordinal numbers?

Whatever the answer to the questions just asked, there is a more important question. Can proofs from higher-order concepts whose conclusions are sentences of n th-order arithmetic show—or provide strong evidence—that these sentences are implied by the n th order arithmetic concept? Can one justify, from the conceptualist point of view, the claim that a sentence σ of n th-order arithmetic is implied by the concept C_{n+1} of $V_{\omega+n+1}$ if one knows that σ is a theorem of, say, arithmetic of order $n + 3$. Or is this just a fact about the stronger concept C_{n+2} . At least in the special case $n = 1$, I think we can. This because we know, or at least have strong evidence for, the following two assertions.

- (1) The concept C_2 is consistent.
- (2) The concept of the natural numbers is fully determinate, and so it is first-order complete.

(It would not seem unfair to say that we know (2) from intrinsic evidence.)

Assume that σ is a sentence of the first-order language of arithmetic, and assume that

$$\text{Third-Order Arithmetic} \vdash \sigma.$$

By the first-order completeness of the concept of the natural numbers, that concept implies σ or implies $\neg\sigma$. The concept of the natural numbers is contained in C_2 , and so C_2 implies whichever of σ and $\neg\sigma$ is implied by the concept of the natural numbers. We know that the axioms of Third Order arithmetic are implied by C_2 . Hence C_2 implies σ . By the consistency of C_2 , σ is implied by the concept of the natural numbers.

To apply this kind of argument at a higher level, e.g., with the subsets of V_ω in the place of the natural numbers, we need to have evidence of full determinateness—or at least first-order completeness—of the of the higher level concept. Do we, then, have evidence for the full determinateness or the first-order completeness of higher level concepts? For the concept of the subsets of $V_{\omega+1}$, the status of CH keeps me from thinking we have evidence for first-order completeness would justify a claim of knowledge.

What about the concept C_1 ?

In the case of the natural numbers, what I regard as the strongest—though far from the only—evidence for full determinateness comes from directly considering the concept. We feel we know exactly what a structure instantiating it would have to be like. Do we have such a feeling in the case of the subsets of V_ω ? I have certainly heard people say that we do—or, at least, that *they* do. A strong version of this claim would be that the concept of the sets of x 's can be directly seen to

be fully determinate whenever there is a fully determinate concept of the x 's. The truth of this claim would imply that we know that the concept of $V_{\omega+2}$ is fully determinate, and hence that CH has a truth-value. Since I do not think we know at present whether or not the CH has a truth-value, I do not think we now know that the claim is true.

Is the claim be justified in the special case when the x 's are the subsets of V_ω or, equivalently, the sets of natural numbers? I.e., can we see directly that the concept C_1 is fully determinate? I said in the preceding section that we have strong evidence for the truth of a theory, Second-Order Arithmetic + Projective Determinacy, that is as complete for the natural numbers and the sets of natural numbers as first-order Peano Arithmetic is for first-order the natural numbers. I believe that this evidence, which is mainly extrinsic, counts as evidence for the full determinateness—of the concept C_1 . Therefore I believe that therefore, examples of our phenomenon whose conclusions are second-order arithmetical sentences as evidence that those sentences are implied by the concept C_1 .

What should we say about the examples of the phenomenon whose conclusions are higher-order sentences or general set-theoretic sentences?

Is the following a logical possibility? Our concepts of many of the levels of the of the set-theoretic hierarchy and that of the full concept of the sets (and of stronger concepts, such as those involving large cardinals) are not first-order complete. Furthermore, for any of these concepts there are first-order questions about the concept to which the concept implies no answer, but which are answered by the standard first-order theory of a stronger concept from our stock of basic concepts. One consequence of this would be that the suggested conceptualist notion of truth is a relative notion. A sentence about, e.g., C_2 might be neither a third-order arithmetical truth or a third-order arithmetical falsehood but be, e.g., a 7th-order arithmetical truth.

The situation just described is obviously incompatible under (the generalization of) version (1) of implication by concepts, but it does not seem incompatible with version (2).

While I don't see how to rule out the described situation, I think that, e.g., the provability in 7th-Order Arithmetic of a sentence of third-order arithmetic provides extrinsic evidence that the sentence is implied by the concept C_2 . Even though our direct epistemic access to C_2 may come from stronger concepts like C_7 , it seems a reasonable practice to treat that what we can prove about C_2 using C_7 is implied by C_2 .

Here is a (perhaps distorted account of) a suggestion of Peter Koellner of a way that we might try to justify such a practice. We might make a methodological

defeasible assumption that our basic concepts are fully determinate or even that they are instantiated.

This methodology fits nicely with regarding such evidence as evidence for truth. If we follow it, we are assuming, for example, that all statements about C_2 are third-order arithmetical truths. Our goal is to determine which ones are true. Whether or not our assumption is correct, it is clear what, on the assumption, we are looking for evidence of.

At the beginning of [7], Feferman quotes a statement I made in 1976:

Throughout the latter part of my discussion, I have been assuming a naive and uncritical attitude toward CH. While this *is* in fact my attitude, I by no means wish to dismiss the opposite viewpoint. Those who argue that the concept of set is not sufficiently clear to fix the truth-value of CH have a position which is at present difficult to assail. As long as no new axiom is found which decides CH, their case will continue to grow stronger, and our assertion that the meaning of CH is clear will sound more and more empty.³⁶

Perhaps the statement, “This is in fact my viewpoint,” could be understood as a declaration that I was following the methodology just discussed.

I still think what was said in the last quoted sentence is true, but I am actually more optimistic now than I was then about there being truth values for statements like CH. One of the main reasons for my optimism is the difficulty of getting a plausible account of the absence of such truth-values. Categoricity rules out multiple, non-isomorphic instantiations of concepts like C_2 . The simplest account of concept implication, version (1) on page 22, counts C_2 as inconsistent if CH has no truth-value. Version (2) seems to avoid such inconsistency, but the assumptions needed for avoidance are perhaps implausible. The obvious way to explain this situation is to propose that there is no absence of truth-values.

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