Determinacy of Infinitely Long Games

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The main subject of this book is games in which two players are given a set A of infinite sequences of natural numbers and take turns choosing natural numbers, producing an infinite sequence. The player who moves first wins if this sequence belongs to A; otherwise the opponent wins. Such a game is *determined* if one of the players has a winning strategy.

If A belongs to a set Γ of sets of infinite sequences of natural numbers, then we call the game a Γ game. We will present proofs of theorems of the following form: *Under hypothesis H, all Γ games are determined.* In Chapter 1, the sets Γ are the first few levels of the Borel hierarchy and the hypotheses H are the axioms of second-order arithmetic or slightly more. For most of Chapter 2, Γ is the set of all Borel sets and H is ZFC. In the remaining chapters, the sets Γ get larger and larger, and the hypotheses H are large cardinal hypotheses.

Many of these theorems have converses or quasi-converses. These are presented as exercises with hints that are essentially sketches of proofs.

The reader should have basic familiarity with set theory, but the book assumes no familiarity with games, descriptive set theory, or large cardinals.

All of the nine chapters of the book are included in the current posting, but another section may be added later to Chapter 5. Though Chapter 9 is included, the reader should be aware that it has not been seriously proofread, and it—especially the last part of it—might have significant errors. Corrections and suggestions for Chapter 9 (and for the other chapters) would be welcome.

Chapter 1

Elementary Methods

In this chapter we introduce the basic concepts of our subject and prove as much determinacy as, roughly speaking, can be proved without appealing to the existence of infinite sets larger than the sets of legal positions in our games.

Readers interested primarily in the main results may wish to read just the introductory Section 1.1 and the basic Section 1.2, where the determinacy of open games is proved.

The proofs in §1.1 and §1.2 do not really need the Power Set or Replacement Axioms of set theory, though this fact is not mentioned in those sections. In §1.3 and in much of §1.4, we explicitly work in a set theory without the Power Set Axiom and with only a fragmentary Replacement Axiom (adopted mostly to avoid complexities). We try to do this in a sufficiently unobtrusive way that readers unfamiliar with axiomatic set theory should be able to follow the proofs as ordinary proofs. In §1.4 we discuss the optimal determinacy result for this theory, due to Antonio Montalban and Richard Shore. In a slightly stronger theory, we prove the determinacy of all $\mathbf{\Delta}^0_4$ games (games that are both $\mathbf{G}_{\delta\sigma\delta}$ and $\mathbf{F}_{\sigma\delta\sigma}$). In the exercises we discuss Harvey Friedman's methods, which show that the determinacy of all $\mathbf{\Sigma}^0_4$ games is not provable in the usual ZFC set theory if the Power Set Axiom is dropped, and we mention an improvement by Montalban and Shore showing the optimality of their positive results. Later (in §2.3) we will use the results from §1.4 in analyzing level by level how much of the Power Set and Replacement Axioms is needed for our proof of the determinacy of Borel games.

1.1 Basic Definitions

We begin by discussing rules of play of our games and afterward take up such matters as winning, winning strategies, and determinacy.

Plays of our games will be finite or infinite sequences of *moves*. Rules of play are given by specifying a *game tree*. A *game tree* is a nonempty tree of finite sequences, i.e. is a set T of finite sequences such that if $p \in T$ and p extends q then $q \in T$. Members of T are called *legal positions in* T or simply *positions in* T. When there is no danger of confusion, we will call them *legal positions* or *positions*. A position in T is *terminal in* T if it has no proper extension in T. If p is a non-terminal position, then a *legal move at* p *in* T or simply a *move at* p *in* T is an a such that $p^\frown\langle a\rangle \in T$, where $^\frown$ is the concatenation operation on sequences. A *play in* T is a finite or infinite sequence every initial part of which belongs to T and which is a terminal position in T if finite. All our games will have two players, I and II. ("I" and "II" are not very imaginative names, but they have become traditional.) Play of a game in T begins at the *initial* position (the empty sequence $\emptyset$, which must belong to every game tree). I moves first, and moves alternate between the two players. Thus a play of a game is produced as follows:

$$\begin{array}{lccccccc} \mathrm{I} & a_0 & & a_2 & & a_4 & & \dots \\ \mathrm{II} & & a_1 & & a_3 & & \dots & \end{array}$$

Each $\langle a_0, a_1, \dots, a_n\rangle$ must be a position in T. If a terminal position is reached, then we have a play of the game and no further moves are made. If no terminal position is reached, then the play is infinite.

There are various ways in which we could have chosen a more general notion of game tree, even in our context of two-person games of perfect information:

(1) We did not have to require that our players alternate moves. Instead we could have introduced a *move function* M, defined on all non-terminal positions, with $M(p)$ giving the player who moves at p. There are two reasons we did not do this. First, it is not really more general, since we can get the same effect in our more restricted set-up. Suppose, for instance that we want to simulate a game in which player II makes the first two moves. To do so we introduce a new tree in which (a) the first move must be the empty sequence and (b) the second move must be a sequence of length 2 that is a legal position in the original tree. (See Exercise 1.1.5.) The second and more important reason why we do not introduce a move function is that it would

make the notion of a game tree more complicated. A game tree would be a pair $\langle T, M\rangle$ and we would continually have to pay attention to the extra object M in situations where it played no significant role.

(2) There is a more general notion of game tree which we could have chosen. By a *tree* we mean a partial ordering with wellordered initial segments. That is to say, a tree is a pair $\langle T, <\rangle$ such that T is a set and $<$ partially orders T and

$$(\forall p \in T)(< \text{wellorders } \{q \in T \mid q < p\}).$$

A tree of finite sequences becomes a special case of a tree if we define $p < q$ to mean that p is properly extended by q. Why did we not define *game trees* to be arbitrary trees rather than trees of finite sequences? One reason has to do with *finite* sequences. (See (3) below.) The other reason has to do with *sequences*: We want our positions to be sequences of *moves*. With general trees, we have no extra objects to be our moves. This isn't really a serious problem, however. We will always be assuming that our players have complete knowledge about the position whenever they make a move. Thus making a move is essentially the same as choosing the new position that will result when the move has been made. With general trees as game trees, *move* could be defined by changing "essentially the same as" into "identical with." In other words, a *legal move at p* could be defined to be a position q that is an immediate successor of p, i.e. a $q \in T$ with $p < q$ such that there is no r with $p < r < q$. In this chapter such a solution would be quite satisfactory for us. Indeed it would work somewhat more smoothly than our actual definitions (and so we do after all keep it as an "actual" definition, as the reader will see two paragraphs hence). In later chapters, however, we will often be concerned with properties of individual moves in our official sense. For example, it may be important that certain moves are chosen from a countable set or that other moves come from a space that carries a measure. Our choice was thus made in view of these later chapters. We confess that we were also influenced by a desire to conform to real games: in chess a move involves changing the placement of one or two individual pieces; it does not involve the complete history of the game.

(3) As we indicated above, our game trees are special not only in that they are trees of sequences but also in that we demand that the sequences be *finite*. (In the context of general trees the corresponding restriction would be a requirement that each member of T have only finitely many predecessors.) If we removed this restriction we would be dealing not merely with games

of infinite length—games that take forever to play—but also with games of *transfinite* length—games that aren't finished even after the players have played forever. Such games are indeed of interest and a good deal of theory about them has been developed. We will occasionally discuss these longer games, both in the text and in exercises. Nevertheless, such games are in some ways essentially more complex than merely infinite games, and we chose in this case simplicity in our subject matter.

(4) Other possible generalizations of our notion of game tree allow for such things as *simultaneous moves* by the two players. Though there are ways to get the effect of such generalizations, we actually use simultaneous moves when we study games of imperfect information in §2.4.

We said in (2) above that there is a possible definition of "move" according to which a move is a position. Though we did not choose this definition, there will be a number of occasions at which it would have been notationally simpler if we had chosen it. Let us then compromise and define a *Move* at p in T to be a position q such that, for some move a at p in T, $q = p^\frown\langle a\rangle$. This usage would produce ambiguity if we were ever to write "Move" at the beginning of a sentence, but we will have no reason to do so.

It is time to be more precise about some of our terminology and to introduce some basic notation. By a *finite sequence* we mean a function whose domain is the set of all predecessors of some natural number. We adopt the convention from set theory that a natural number *is* the set of all its predecessors, so that a finite sequence is a function whose domain is a natural number. We also adopt the set-theoretic notion of function, identifying a function with its graph. With this convention, a finite sequence p is extended by a finite sequence q if and only if $p \subseteq q$, and so "$\subseteq$" will be our standard notation for "is extended by." The *length* of a finite sequence p is the domain of p. We denote the length of p by $\ell\mathrm{h}(p)$. Infinite sequences will be treated similarly. An *infinite sequence* is a function with domain ω, the set of all natural numbers. The *length* of an infinite sequence is ω. Infinite ordinal numbers are also considered to be the set of all their predecessors. If x is a finite or infinite play in T, then $p \subseteq x$ just means that p is extended by x. We will denote the set of all plays in T by $\lceil T\rceil$.

The most important example of a game tree is ${}^{<\omega}\omega$, the set of all finite sequences of natural numbers. The set of all plays in this tree is ${}^{\omega}\omega$, the set of all infinite sequences of natural numbers. Note that ${}^{x}y$ is the set of all functions $f : x \to y$. (It is sometimes important to distinguish, e.g. ${}^{\omega}\omega$ from the ordinal number ω^ω.) The notation ${}^{<x}y$, for ordinal numbers x, stands

for $\bigcup_{x'<x} {}^{x'}y$. The tree ${}^{<\omega}\omega$ is an example of a tree all of whose plays are infinite. If we wished, we could deal only with such trees, extending what are now terminal positions by adjoining infinitely many irrelevant moves.

A *strategy for* I *in* T is a function σ whose domain is

$$\{p \in T \mid \ell\mathrm{h}(p) \text{ is even and } p \text{ is not terminal}\}$$

such that $\sigma(p)$ is always a legal move in T at p. A *strategy for* II *in* T is similarly a function τ with domain $\{p \in T \mid \ell\mathrm{h}(p) \text{ is odd and } p \text{ is not terminal}\}$ such that $\tau(p)$ is always a legal move at p. By $\mathcal{S}_{\mathrm{I}}(T)$ we mean the set of all strategies for I in T; by $\mathcal{S}_{\mathrm{II}}(T)$ we mean the set of all strategies for II in T. We let $\mathcal{S}(T) = \mathcal{S}_{\mathrm{I}}(T) \cup \mathcal{S}_{\mathrm{II}}(T)$. Just as we defined Moves as well as ordinary moves, we could define *Strategies* which are like strategies except that their values are Moves instead of moves. We refrain from doing so: it turns out not to be as useful as the Move move. A position p in T is *consistent with* a strategy σ for I if $p(n) = \sigma(p \restriction n)$ for every even $n < \ell\mathrm{h}(p)$. (Here $p \restriction n$ is the *restriction* of the function p to the set $n = \{0, 1, \ldots, n-1\}$, i.e. $p \restriction n$ is the initial part of p of length n.) A play x in T is *consistent with* σ if every position $p \subseteq x$ is consistent with σ. Being *consistent with* a strategy for II is similarly defined.

For each game tree T and each $A \subseteq \lceil T \rceil$, i.e., for each set of plays in T, we have a *game* $G(A;T)$. I *wins* a play x of $G(A;T)$ just in case $x \in A$. Otherwise II *wins* x. It will be convenient to have $G(A;T)$ defined even for sets A that are not subsets of $\lceil T \rceil$. In this case $G(A;T)$ will be the same game as $G(A \cap \lceil T \rceil; T)$. A strategy σ for I is a *winning strategy for* $G(A;T)$ if I wins each play consistent with σ. *Winning strategies* for II are similarly defined. $G(A;T)$ is *determined* if either I or II has a winning strategy for $G(A;T)$. Note that it is impossible for both players to have winning strategies, since there would be a play consistent with both strategies.

In Chapter 2 we will find it useful to introduce a variant notion of a game tree in which there are some built-in winning conditions: Certain terminal positions are designated as losing for one or the other of the players *independently of* A. We defer making the definition until we have some use for it.

Not all games in our sense are determined. (See [Gale and Stewart, 1953], [Mycielski, 1964], page 114 of [Mauldin, 1981], and also Exercises 1.1.2 and 1.1.4.) To get determinacy results it is necessary to impose conditions of some kind on the games.

One way to do this is to impose conditions of size on the game tree. As we will see in the next section, all games $G(A;T)$ with T finite are determined. There are, however, undetermined games $G(A;T)$ with T countable (See Exercise 1.1.2.) Most of the concern in this book will be with determinacy results in the case T is countable.

Remark. All known proofs of the existence of undetermined games in countable trees make use of the Axiom of Choice. [Mycielski and Steinhaus, 1962] proposes as an alternative to the Axiom of Choice an assertion, there called the *Axiom of Determinateness* and now called the *Axiom of Determinacy* or simply AD.

AD: *All games in countable trees are determined.*

Large cardinal axioms imply that the Axiom of Determinacy is consistent with the axioms of set theory other than Choice. This will be proved in Chapter 9. (In this book we make free use of the Axiom of Choice, though we will make occasional remarks about whether or not particular theorems require it.)

Are there other conditions on T implying determinacy? In §1.2 we will see that the absence of infinite plays is such a condition. But the absence of infinite plays in T is for practical purposes equivalent with a simple topological condition on I's winning set A. Such topological conditions will be the hypotheses of almost all our determinacy theorems, and so it is to topology that we now turn.

For $p \in T$ let

$$T_p = \{q \in T \mid q \subseteq p \ \vee \ p \subseteq q\}.$$

T_p is a *game subtree* of T, i.e. T_p is a *subtree* of T (a subset of T that is a game tree), and every position terminal in T_p is terminal in T. Games in T_p are played just as are those in T, except that the first $\ell\text{h}(p)$ moves are fixed in advance so as to produce the position p. We give $\lceil T \rceil$ a topology by taking as basic open sets the $\lceil T_p \rceil$ for $p \in T$. For $A \subseteq \lceil T \rceil$, let us say that the game $G(A;T)$ is *open*, *closed*, etc. just in case A is open, closed, etc. respectively.

Remark. If p is a position in T, we will never create ambiguity by using the notation "T_p" with any meaning other than that given it in the preceding paragraph. The reader should be warned, however, that we will take such liberties as denoting elements of an infinite sequence of game trees by "T_i."

In this chapter we will prove determinacy results for games in low levels of the Borel hierarchy. We now define that hierachy and prove some basic facts about it.

We use the logical notation for the Borel hierarchy in a topological space. $\mathbf{\Sigma}^0_1$ is the class of open sets; $\mathbf{\Pi}^0_1$ is the class of closed sets; $\mathbf{\Delta}^0_1$ the class of clopen (closed and open) sets. For ordinals $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha$ is the class of all countable unions of sets belonging to $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$, $\mathbf{\Pi}^0_\alpha$ is the class of complements of sets belonging to $\mathbf{\Sigma}^0_\alpha$, and $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$. A set is *Borel* if it belongs to the smallest class containing the open sets and closed under countable unions and complements. The following lemma gives some basic facts about the Borel hierarchy in $\lceil T \rceil$.

Lemma 1.1.1. *(1) The following hold in spaces $\lceil T \rceil$ for every ordinal number $\alpha \geq 1$:*

(a) $(\forall \beta)(\alpha < \beta \to \mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Delta}^0_\beta)$.

(b) $\mathbf{\Sigma}^0_\alpha$ *is closed under countable unions and finite intersections.*

(c) $\mathbf{\Pi}^0_\alpha$ *is closed under countable intersections and finite unions.*

(2) A set is Borel if and only if it belongs to $\bigcup_{1\leq\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$, where ω_1 is the least uncountable ordinal number.

Proof. (1)(a). If $A \in \mathbf{\Pi}^0_\alpha$, then $A = \bigcup\{A\}$; thus $A \in \mathbf{\Sigma}^0_\beta$ for all $\beta > \alpha$. This shows that $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$ for all $1 \leq \alpha < \beta$. It follows directly that, for all such α and β, $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta$. If $1 < \alpha < \beta$, then it is immediate from the definition that $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$. Let $A \in \mathbf{\Sigma}^0_1$. Since A is open,

$$A = \bigcup\{\lceil T_p \rceil \mid p \in T \wedge \lceil T_p \rceil \subseteq A\}.$$

For $n \in \omega$, let

$$A_n = \bigcup\{\lceil T_p \rceil \mid p \in T \wedge \ell\mathrm{h}(p) = n \wedge \lceil T_p \rceil \subseteq A\}.$$

Each A_n is closed as well as open, since

$$\neg A_n = \bigcup\{\lceil T_p \rceil \mid p \in T \wedge \ell\mathrm{h}(p) = n \wedge \lceil T_p \rceil \not\subseteq A\}.$$

($\neg A$ is $\lceil T \rceil \setminus A$.) Since $A = \bigcup_{n\in\omega} A_n$, we have that A is a countable union of $\mathbf{\Pi}^0_1$ sets and so that $A \in \mathbf{\Sigma}^0_\beta$ for every $\beta > 1$. We have now shown that

$\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$ whenever $1 \leq \alpha < \beta$. Combining this with our first observation, we have that $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Delta}^0_\beta$ for all such α and β. Since complements of $\mathbf{\Delta}^0_\beta$ sets are also $\mathbf{\Delta}^0_\beta$, we get the other half of (1)(a).

(1)(b) and (1)(c). The open sets of any space are closed under arbitrary unions. For $\alpha > 1$, the closure of $\mathbf{\Sigma}^0_\alpha$ under countable unions is immediate from the definition. The closure of $\mathbf{\Pi}^0_\alpha$ under countable intersections follows from the closure of $\mathbf{\Sigma}^0_\alpha$ under countable unions. The open sets of any space are closed under finite intersections. Let $\alpha > 1$ and $j \in \omega$ and let $A_i \in \mathbf{\Sigma}^0_\alpha$ for $i < j$. For each $i < j$, let $A_{i,n}$, $n \in \omega$, be such that each $A_{i,n} \in \bigcup_{1 \leq \gamma < \alpha} \mathbf{\Pi}^0_\gamma$ and each $A_i = \bigcup_{n\in\omega} A_{i,n}$. Then

$$\bigcap_{i<j} A_i = \bigcap_{i<j} \bigcup_{n\in\omega} A_{i,n} = \bigcup_{s\in {}^j\omega} \bigcap_{i<j} A_{i,s(i)}.$$

To show that $\bigcap_{i<j} A_i \in \mathbf{\Sigma}^0_\alpha$, it thus suffices to show that each $\bigcap_{i<j} A_{i,s(i)} \in \bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$. For this fix $s \in {}^j\omega$. By (1)(a) there is a $\gamma < \alpha$ such that $A_{i,s(i)} \in \mathbf{\Pi}^0_\gamma$ for every $j < i$. By the closure of all $\mathbf{\Pi}^0_\gamma$ under countable, and so under finite, intersections, it follows that $\bigcap_{i<j} A_{i,s(i)} \in \mathbf{\Pi}^0_\gamma$. The closure of $\mathbf{\Pi}^0_\alpha$ under finite unions follows from the closure of $\mathbf{\Sigma}^0_\alpha$ under finite intersections.

(2). By (1)(a),

$$\bigcup_{1\leq\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha \subseteq \bigcup_{1\leq\alpha<\omega_1} \mathbf{\Pi}^0_\alpha.$$

Hence $\bigcup_{1\leq\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ is closed under complements. If $\mathcal{A}$ is a countable subset of $\bigcup_{1\leq\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$, then there is a countable ordinal δ such that $\mathcal{A} \subseteq \bigcup_{1\leq\alpha<\delta} \mathbf{\Sigma}^0_\alpha \subseteq \bigcup_{1\leq\alpha<\delta+1} \mathbf{\Pi}^0_\alpha$. Hence $\bigcup \mathcal{A} \in \mathbf{\Sigma}^0_{\delta+1}$. We have then that $\bigcup_{1\leq\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ is a class containing the open sets and closed under countable unions and complements. By definition, this means that every Borel set belongs to $\bigcup_{1\leq\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$.

The fact that every $\mathbf{\Sigma}^0_\alpha$, $1 \leq \alpha < \omega_1$, is Borel is proved by an easy induction on α. □

It follows from part (2) of the lemma that, for all $\alpha \geq \omega_1$, $\mathbf{\Sigma}^0_\alpha = \mathbf{\Pi}^0_\alpha = \mathbf{\Delta}^0_\alpha =$ the class of all Borel sets. If, e.g., $T = {}^{<\omega}X$ and the cardinal number $|X|$ of X is at least 2, then the Borel hierarchy does not collapse before ω_1, i.e. $\mathbf{\Delta}^0_\alpha \subsetneq \mathbf{\Sigma}^0_\alpha \subsetneq \mathbf{\Delta}^0_\beta$ whenever $1 \leq \alpha < \beta < \omega_1$. (See Exercise 1.F.6 of [Moschovakis, 2009].)

The "0" in "$\mathbf{\Sigma}^0_\alpha$" means that the sets in the class are definable by quantification over objects of type 0, i.e. natural numbers: Countable unions correspond to existential quantification over natural numbers; countable intersections correspond to universal quantification over natural numbers. The "α" in "$\mathbf{\Sigma}^0_\alpha$" means that there are α alternations of universal and existential quantifiers, and the "$\mathbf{\Sigma}$" means that the first quantifier is existential. For example the $\mathbf{\Sigma}^0_2$ sets are just those sets A such that there is a clopen $B \in \lceil T \rceil \times {}^2\omega$ such that

$$(\forall x \in \lceil T \rceil)(x \in A \leftrightarrow (\exists m_1)(\forall m_2)\, \langle x, m_1, m_2 \rangle \in B).$$

In later chapters we will introduce classes $\mathbf{\Sigma}^1_n$.

In the exercises we will sometimes deal with the *effective Borel hierarchy* of subsets of ${}^\omega\omega$. (The reader not familiar with recursion theory can skip the definition that follows and skip also the relevant exercises.) For simplicity we stick to finite levels of that hierarchy. $A \subseteq {}^\omega\omega$ belongs to Σ^0_n, for $n \geq 1$, if there is a recursive $B \subseteq {}^\omega\omega \times {}^n\omega$ such that

$$(\forall x)(x \in A \leftrightarrow (\exists m_1)(\forall m_2)(\exists m_3) \cdots (\mathrm{Q} m_n)\langle x, m_1, m_2, m_3, \ldots m_n \rangle \in B).$$

$A \in \Pi^0_n$ if $\neg A \in \Sigma^0_n$. $\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$. It is fairly easy to see that if we replace "recursive" by "clopen" in this definition, we get the ordinary finite Borel hierarchy. If $x \in {}^\omega\omega$, then we define $\Sigma^0_n(x)$, $\Pi^0_n(x)$, and $\Delta^0_n(x)$ by replacing "recursive" by "recursive in x". It is fairly easy to see that, e.g., $\mathbf{\Sigma}^0_n = \bigcup_{x \in {}^\omega\omega} \Sigma^0_n(x)$. (See page 160 of [Moschovakis, 1980].)

We end this section by listing the formal ZFC (Zermelo–Fraenkel, with Choice) axioms for set theory. These axioms will play no explicit role until §1.3, and even there and in §1.4, all the proofs in the text should be readable by someone unfamiliar with formal set theory and ZFC.

First order logic has the symbols

$$(\ ,\ )\ ,\ \neg\ ,\ \wedge\ ,\ \exists\ ,\ =\ ,$$

together with *variables*

$$v_0, v_1, v_2, \ldots .$$

We assume the reader has enough familiarity with symbolic logic to know that, e.g., "$\wedge$" is interpreted to mean "and." We will often be careless about what are our official variables, connectives, and quantifiers. One make think of use of symbols other than the official ones as abbreviation. We will also be careless about parentheses.

The *language of set theory* has, in addition to the symbols of first order logic, the two-place predicate symbol $\in$. The formulas of the language of set theory are defined inductively as the smallest class satifying the following.

(a) If x and y are variables, then $x = y$ and $x \in y$ are (atomic) formulas.

(b) If φ is a formula, then so is $\neg\varphi$.

(c) If φ and ψ are formulas, then $(\varphi \wedge \psi)$ is a formula;

(d) If φ is a formula and x is a variable, then $(\exists x)\,\varphi$ is a formula.

An occurrence of a variable in a formula is *free* if it is not in the scope of a quantifier, i.e., if it is not in a subformula of the form $(\exists x)\,\varphi$. When we write, e.g., $\varphi(v_1, \ldots, v_n)$, we imply that only variables among $v_1, \ldots, v_n$ occur free in φ.

Following are the formal ZFC axioms. In stating them we make use of some standard abbreviations, whose definitions the reader should be able to give. For example, we write $\emptyset$ for the empty set (whose existence and uniqueness follows from the Axioms of Empty Set, Comprehension, and Extensionality), so that $x = \emptyset$ abbreviates $\neg(\exists y)\, y \in x$. A perhaps less familiar abbreviation is $(\exists! x)\,\varphi(x, z_1, \ldots, z_n)$, which abbreviates

$$(\exists x)(\,\varphi(x, z_1, \ldots z_n) \;\wedge\; (\forall y)(\varphi(y, z_1, \ldots, z_n) \rightarrow y = x)).$$

We precede the statement of each formal axiom by a parenthetical informal version of the axiom.

Empty Set: (There is a set with no members.)

$$(\exists x)(\forall y)\, y \notin x.$$

Extensionality: (Two sets with the same members are identical.)

$$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \;\rightarrow\; x = y).$$

Comprehension (Axiom Schema)**:** (Every definable subcollection of a set is a set.) For formulas $\varphi(x, u, w_1, \ldots, w_n)$,

$$(\forall w_1) \cdots (\forall w_n)(\forall u)(\exists v)(\forall x)(x \in v \;\leftrightarrow\; x \in u \;\wedge\; \varphi).$$

Foundation: (Every nonempty set has a $\in$-minimal member.)

$$(\forall x)(x \neq \emptyset \;\rightarrow\; (\exists y \in x)\, y \cap x = \emptyset)).$$

Pairing: (For any sets x and y, there is a set whose members are precisely x and y.)

$$(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \;\leftrightarrow\; (w = x \;\vee\; w = y)),$$

Union: (For any set x, there is a set of all the members of members of x.)

$$(\forall x)(\exists u)(\forall z)(z \in u \leftrightarrow (\exists y)(z \in y \wedge y \in x)).$$

Infinity: There is a set x such that $\emptyset \in x$ and such that x is closed under the operation $y \mapsto y \cup \{y\}$.)

$$(\exists x)(\emptyset \in x \wedge (\forall y \in x)\, y \cup \{y\} \in x).$$

Replacement (Axiom Schema)**:** (If F is a definable operation and the domain of F is a set, then the range of F is a set.) For formulas $\varphi(x, y, u, w_1, \ldots, w_n)$,

$$\begin{aligned}(\forall w_1) \cdots (\forall w_n)(\forall u)((\forall x \in u)(\exists! y)\, \varphi \\ \rightarrow (\exists v)(\forall y)(y \in v \leftrightarrow (\exists x \in u)\, \varphi)).\end{aligned}$$

Power Set: For any set x, there is a set of all the subsets of x.)

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \subseteq x).$$

Choice: If x is any set of disjoint nonempty sets, then there is a set u that has exactly one member in common with each member of x.

$$\begin{aligned}(\forall x)((\forall y \in x)(y \neq \emptyset \wedge (\forall z \in x)(y \neq z \rightarrow y \cap z = \emptyset))) \\ \rightarrow (\exists u)(\forall y \in x)(\exists! w)\, w \in y \cap u))).\end{aligned}$$

In formal logic the Empty Set Axiom is superfluous; for the existence of some object is provable, and so the existence of $\emptyset$ follows by Comprehension.

Exercise 1.1.1. Let $A \subseteq {}^{\omega}\omega$ with $|A| < 2^{\aleph_0}$. Prove that II has a winning strategy for $G(A; {}^{<\omega}\omega)$.

Exercise 1.1.2. Prove that not every game $G(A; {}^{<\omega}\omega)$ is determined.

Hint. Use the Axiom of Choice to wellorder the set of all strategies in ${}^{<\omega}\omega$ in a sequence of order type $2^{\aleph_0}$. Now diagonalize to get an A for which no strategy is winning. This is the proof in [Gale and Stewart, 1953], and it is the most direct one. There are many other proofs. Unpublished work of Banach and Mazur gives a proof which proceeds by showing that AD implies that all sets of reals have the property of Baire. See pages 298-300 of [Moschovakis, 1980], page 114 of [Mauldin, 1981], and [Oxtoby, 1957]. For another proof, see Exercise 1.1.3.

Exercise 1.1.3. Let T^* be the game tree plays in which are as follows:

$$\begin{array}{lccccccc} \mathrm{I} & s_0 & & s_1 & & s_2 & & \dots \\ \mathrm{II} & & a_0 & & a_1 & & \dots, & \end{array}$$

where each $s_i \in {}^{<\omega}\omega$ and each $a_i \in \{0,1\}$. For any $A \subseteq {}^{\omega}2$, let Let

$$A^* = \{\langle s_0, a_0, s_1, a_1, \dots\rangle \mid s_0{}^\frown\langle a_0\rangle{}^\frown s_1{}^\frown\langle a_1\rangle{}^\frown \dots \in A\}$$

and let $G^*(A) = G(A^*; T^*)$.

(a) Prove that I has a winning strategy for $G^*(A)$ if and only if A has a perfect subset (a non-empty closed subset without isolated points).

(b) Prove that II has a winning strategy for $G^*(A)$ if and only if A is countable.

(c) Use the Axiom of Choice to construct an uncountable subset of ${}^{\omega}2$ with no perfect subset.

Remark. This is a result of [Davis, 1964].

Exercise 1.1.4. Prove, in ZF (i.e., in ZFC without the Axiom of Choice) that not every game $G(A; {}^{<\omega}\omega_1)$ is determined. (Recall that ω_1 is the least uncountable ordinal number, i.e. the set of all countable ordinal numbers.) This result appears in [Mycielski, 1964].

Hint. Use Exercise 1.1.3 to show that it follows from AD that there is no one-one $f : \omega_1 \to {}^{\omega}2$. (Assume such an f exists and get a one-one $g : \omega_1 \to \mathbb{R}$. Then use the existence of a perfect subset of the range of g to get a one-one $h : \mathbb{R} \to \omega_1$, and show that this contradicts AD.) Now consider the game $G(A; {}^{<\omega}\omega_1)$, where A is the set of all $x : \omega \to \omega_1$ such that $x(0) \geq \omega$ and $\{\langle m, n\rangle \mid x(2m+1) < x(2n+1)\}$ is not a wellordering of ω of order type $x(0)$.

Exercise 1.1.5. Assume that all $\mathbf{\Sigma}^0_7$ games in countable trees are determined, and prove that this still holds when we broaden our notion of games to allow a move function as on page 4 above. (Obviously $\mathbf{\Sigma}^0_7$ is just an example.)

1.2 Open Games

The main result of this section is Theorem 1.2.4, the important basic theorem of [Gale and Stewart, 1953] that all open games are determined. We will also introduce and study the technical concept of a *quasistrategy*, a concept that will be the main tool in the rest of this chapter.

The fact that all games in finite trees are determined is usually attributed to [Zermelo, 1913]. (See page 371 of [Kanamori, 1994] for a discussion.) The proof of this fact works with very little change to give a proof of determinacy for the case of trees without infinite plays.

Theorem 1.2.1. *If there are no infinite plays in T, then $G(A;T)$ is determined for every $A \subseteq \lceil T \rceil$.*

Proof. The Theorem follows easily from the following lemma.

Lemma 1.2.2. *If $G(A;T_p)$ is not determined, then there is a Move q at p such that $G(A;T_q)$ is not determined. (Recall the definition of "Move" on page 4 and recall that $G(A;T_q)$ is $G(A \cap \lceil T_q \rceil; T_q)$.)*

Proof of Lemma. Assume that $G(A;T_p)$ is not determined. Assume for definiteness that p has even length. (The other case is similar.)

If q is a legal Move at p, then I does not have a winning strategy for $G(A;T_q)$. If he had such a strategy σ', then that strategy together with the move q would give him a winning strategy σ for $G(A;T_p)$:

$$\sigma(r) = \begin{cases} q(\ell\mathrm{h}(r)) & \text{if } r \subseteq p; \\ \sigma'(r) & \text{if } q \subseteq r. \end{cases}$$

(Technically we should also define $\sigma(r)$ in the third case: $p \subseteq r \wedge q \not\subseteq r$. The reason we omitted this case is that such positions r are not consistent with σ. We could have defined *strategy* so that strategies take as arguments only positions consistent with them. See Exercise 1.2.3 for a minor reason for doing so.)

It suffices then to show that there is a Move q at p such that II does not have a winning strategy for $G(A;T_q)$. If there is no such q, then for each Move q at p there is a winning strategy τ_q for II for $G(A;T_q)$. We then get a winning strategy τ for II for $G(A;T_p)$ by setting

$$\tau(r) = \begin{cases} p(\ell\mathrm{h}(r)) & \text{if } r \subseteq p; \\ \tau_q(r) & \text{if } q \subseteq r \text{ and } q \text{ is a Move at } p. \end{cases}$$

(We can describe τ more briefly as $\bigcup_q \tau_q$.) This contradiction shows that q must exist and completes the proof of the lemma. $\square$

Now let us prove the theorem by proving its contrapositive. Suppose that $G(A;T)$ is not determined. Repeated applications of the lemma give us a sequence

$$p_0 \subsetneq p_1 \subsetneq p_2 \subsetneq \dots$$

of elements of T. There is a an infinite play x such that $x \supseteq p_i$ for all i. $\square$

Remark. Both the proof of the lemma and the proof of the theorem from the lemma use the Axiom of Choice. If we strengthen the hypothesis of the theorem to make T wellfounded (equivalent in the presence of Choice to our hypothesis that T has no infinite plays), then the latter use of the Axiom of Choice is avoided. (See Exercise 1.2.1.) The former use is necessary even for trees which contain only positions of length ≤ 2. (See Exercise 1.2.2.) Of course, Choice is not needed to prove the theorem for a T that has a canonical wellordering, as does our main example ${}^{<\omega}\omega$.

Corollary 1.2.3. *All clopen games are determined.*

Proof Let $A \subseteq \lceil T \rceil$ with A clopen. For each $x \in \lceil T \rceil$ there is a $p \subseteq x$ such that $\lceil T_p \rceil \subseteq A$ or $\lceil T_p \rceil \subseteq \neg A$. This is because both A and $\neg A$ are open and so are the unions of their basic open subsets. Let

$$T^* = \{q \in T \mid (\forall p \subsetneq q)(\lceil T_p \rceil \cap A \neq \emptyset \ \wedge \ \lceil T_p \rceil \cap \neg A \neq \emptyset)\}.$$

The game tree T^* has no infinite plays: If x is an infinite play in T^*, then x is a play in T and so there is a $p \subseteq x$ such that $\lceil T_p \rceil \subseteq A$ or $\lceil T_p \rceil \subseteq \neg A$. But the definition of T^* gives the contradiction that p is terminal in T^*. Let

$$A^* = \{x \in \lceil T^* \rceil \mid (\exists y \in \lceil T \rceil)(x \subseteq y \wedge y \in A)\}.$$

By Theorem 1.2.1, $G(A^*;T^*)$ is determined. Assume for definiteness that σ^* is a winning strategy for I for $G(A^*;T^*)$. Let σ be any strategy for I in T such that σ agrees with σ^* on non-terminal positions in T^*. We show that σ is a winning strategy for $G(A;T)$. Let $x \in \lceil T \rceil$ be consistent with σ. There is a $p \subseteq x$ that is terminal in T^*. Either $\lceil T_p \rceil \subseteq A$ or else $\lceil T_p \rceil \subseteq \neg A$. But p is consistent with σ^*, so $\lceil T_p \rceil \cap A \neq \emptyset$ and this means that $x \in A$. $\square$

The following terminology will be convenient in many of the proofs that follow. By *G is a win for* I we mean that there is a winning strategy for I for G. Similarly define *G is a win for* II.

Theorem 1.2.4. ([Gale and Stewart, 1953]) *All open games are determined. All closed games are determined.*

Proof The first assertion implies the second: If $A \subseteq \lceil T \rceil$ is closed, let

$$T' = \{\emptyset\} \cup \{\langle 0 \rangle ^\frown p \mid p \in T\}; \qquad A' = \{\langle 0 \rangle ^\frown x \mid x \notin A\}.$$

The open game $G(A'; T')$ is just $G(A; T)$ with the roles of the players reversed via the dummy initial move 0. If the former is determined then so is the latter.

Lemma 1.2.5. *Let T, A, and $p \in T$ be arbitrary and assume that $G(A; T_p)$ is not a win for* I.

(i) If $\ell\mathrm{h}(p)$ is even then there is no Move q at p such that $G(A; T_q)$ is a win for I.

(ii) If $\ell\mathrm{h}(p)$ is odd then there is a Move q at p such that $G(A; T_q)$ is not a win for I.

Proof of Lemma. The proof of Lemma 1.2.2 essentially contains the proof of the present lemma, so we will be brief. (i) If there is a Move q at p such that $G(A; T_q)$ is a win for I, then I can win $G(A; T_p)$ by first playing q and then playing (the moves given by) a winning strategy for $G(A; T_q)$. (ii) If σ_q is a winning strategy for I for $G(A; T_q)$ for each Move q at p, then $\bigcup_q \sigma_q$ is a winning strategy for I for $G(A; T_p)$. □

Returning to the proof of the theorem, let us assume that $A \subseteq \lceil T \rceil$ is open and that $G(A; T)$ is not a win for I. We will prove that there is a winning strategy τ for II for $G(A; T)$. For each position p of odd length such that $G(A; T_p)$ is not a win for I, choose a move $\tau(p)$ at p such that $G(A; T_{p ^\frown \langle \tau(p) \rangle})$ is not a win for I. Part (ii) of the lemma gives the existence of such a move. For other positions of odd length, let $\tau(p)$ be arbitrary. Let x be a play consistent with τ. By induction, using part (i) of the lemma, we get that each $p \subseteq x$ is such that $G(A; T_p)$ is not a win for I. But A is open. If $x \in A$ then $x \in \lceil T_p \rceil$ for some p such that $\lceil T_p \rceil \subseteq A$. For any such p, $p \subseteq x$ and $G(A; T_p)$ is obviously a win for I. This contradiction gives that $x \notin A$. This in turn shows that τ is a winning strategy for II. □

Lemma 1.2.5 has other applications besides Theorem 1.2.4. For making such applications, it will be useful to reformulate the lemma, which we now do.

A *quasistrategy for* II *in* T is a game subtree T' of T such that

(a) if $p \in T'$ and $\ell\mathrm{h}(p)$ is even, then every Move in T at p belongs to T';

(b) if $p \in T'$, $\ell\mathrm{h}(p)$ is odd, and p is not terminal in T, then some Move at p in T belongs to T'.

(Note that a subtree T' of T satisfying (a) and (b) is automatically a game subtree of T and so a quasistrategy for II in T.) *Quasistrategies for* I are similarly defined.

Every strategy τ for II in T gives rise to a quasistrategy for II in T: Let $T' = \{p \in T \mid p \text{ is consistent with } \tau\}$. Except for irrelevancies, T' determines τ: T' determines $\tau(p)$ for all positions p consistent with τ. The special property distinguishing the quasistrategy determined by a strategy from a general quasistrategy is that in (b) "some" can be replaced by "one and only one." Thus we may think of a quasistrategy as a many-valued strategy. Quasistrategies are often useful in situations where one is not assuming the Axiom of Choice. But they are also useful in proofs of determinacy, as the rest of this chapter will show. Quasistrategies for II in T are sometimes called II-*imposed subtrees* (or *subgames*) *of* T.

The following Lemma is really just a reformulation of Lemma 1.2.5 (and its dual).

Lemma 1.2.6. *(1) If $G(A;T)$ is not a win for* I*, then*

$$\{q \in T \mid (\forall p \subseteq q)\ G(A;T_p) \text{ is not a win for I}\}$$

is a quasistrategy for II*.*

(2) If $G(A;T)$ is not a win for II*, then*

$$\{q \in T \mid (\forall p \subseteq q)\ G(A;T_p) \text{ is not a win for II}\}$$

is a quasistrategy for I*.*

Proof. For (1), assume that $G(A;T)$ is not a win for I and let $T' = \{q \in T \mid (\forall p \subseteq q)\, G(A;T_p) \text{ is not a win for I}\}$. Clearly T' is a subtree of T. Property (a) for T' follows from (i) of Lemma 1.2.5. Property (b) follows from (ii). (2) similarly follows from the obvious variant of Lemma 1.2.5. □

Whenever $G(A;T)$ is not a win for I, let us call

$$\{q \in T \mid (\forall p \subseteq q)\ G(A;T_p) \text{ is not a win for I}\}$$

II*'s non-losing quasistrategy for* $G(A;T)$, . Similarly define I*'s non-losing quasistrategy for* $G(A;T)$ when $G(A;T)$ is not a win for II. The proof of Theorem 1.2.4 from Lemma 1.2.5 amounted to showing that, for A open, II's non-losing quasistrategy for $G(A;T)$ is a *winning quasistrategy*, in the obvious sense.

Lemma 1.2.7. *(1) If $G(A;T)$ is not a win for* I *and T' is* II*'s non-losing quasistrategy, then $G(A;T')$ is not a win for* I.

(2) If $G(A;T)$ is not a win for II *and T' is* I*'s non-losing quasistrategy, then $G(A;T')$ is not a win for* II.

Proof. We prove (1). Suppose that σ is a winning strategy for I for $G(A;T')$. Then I can win $G(A;T)$ by playing σ until (if ever) II first departs from T' at some position p and then playing a winning strategy for $G(A;T_p)$. $\square$

Exercise 1.2.1. A game tree T is *wellfounded* if for every nonempty $Y \subseteq T$ there is a terminal element p of $T \cap Y$, i.e. a $p \in T \cap Y$ such that no q properly extending p belongs to $T \cap Y$.

(a) Prove that T is wellfounded if and only if there are no infinite plays in T. (The "if" direction will require the Axiom of Choice.)

(b) Assume that Lemma 1.2.2 holds for A and T and prove in ZF (i.e., don't use the Axiom of Choice) that if T is wellfounded then $G(A;T)$ is determined.

Exercise 1.2.2. Show that the Axiom of Choice is equivalent in ZF with the determinacy of all games in trees T such that every $p \in T$ has length ≤ 2.

Exercise 1.2.3. Working in ZF, assume that the Axiom of Choice is false. Prove that there are A and T such that (i) there is a play of length 1 belonging to A but (ii) $G(A;T)$ is not determined. (This shows that, in the absence of Choice, it would be more natural to define *strategy* as suggested during the proof of Lemma 1.2.2.)

Exercise 1.2.4. Let $A \subseteq \lceil T \rceil$ be open. For each ordinal number α, we define P_α, a set of positions of even length in T. The definition proceeds by transfinite induction on α. Let $p \in P_0$ if and only if $\lceil T_p \rceil \subseteq A$. For $\alpha > 0$, $p \in P_\alpha$ if and only if $p \in P_0$ or there is a Move q at p such that either $q \in A$ or q is not terminal and, for every Move r at q, $r \in \bigcup_{\beta<\alpha} P_\beta$. First show that there is an α such that $(\forall \gamma \geq \alpha) P_\gamma = P_\alpha$. Now let P_∞ be this limiting value of P_α. Show that $G(A;T)$ is a win for I if the initial position $\emptyset \in P_\infty$ and that $G(A;T)$ is a win for II if $\emptyset \notin P_\infty$. (This is a more constructive proof of Theorem 1.2.4. It was independently noticed by several people. See Blass [1972] for a related result.)

Exercise 1.2.5. Use the construction of Exercise 1.2.4 to prove that, if $A \subseteq {}^\omega\omega$, $A \in \Sigma^0_1$, and $G(A; {}^{<\omega}\omega)$ is a win for I, then there is a winning strategy for I belonging to $L(\beta)$ for β the least admissible ordinal greater than ω. Prove also for such A and for the same β, that if $G(A; {}^{<\omega}\omega)$ is a win for II then there is a winning strategy for II belonging to $L(\beta + 1)$. (The literal construction of Exercise 1.2.4 doesn't quite work; modify the definition of P_0 to get $P_0 \in L(\beta)$.)

1.3 The Theorems of Wolfe and Davis

In §2.1 we will prove that all Borel games are determined. Nevertheless, the remaining two sections of this chapter will be devoted to proofs of partial results that will not be used in the proof in §2.1. What is of interest about these proofs is that in essence they do not use the Power Set and Replacement Axioms of ZFC (though one of them does use something that goes beyond the other standard ZFC axioms). A striking result of [Friedman, 1971], proved before Borel determinacy, implies that both Power Set and Replacement are needed to prove that all Borel games (even in countable trees) are determined. This is surprising because almost all theorems of mathematics can be proved in *Zermelo Set Theory* (ZC): ZFC without the Axiom of Replacement but with Comprehension. Moreover the assertion that all Borel games in countable trees are determined concerns only countable objects, whereas Friedman's result might be described as implying that Borel determinacy cannot be proved without invoking principles about uncountable objects.

In the next two sections and in §2.3, we want to avoid using Power Set and Replacement whenever we can. However, in order not to get embroiled in technicalities, it is convenient to have available always a small part of the Axiom of Replacement. To describe the appropriate theory, we need to introduce the *Lévy hierarchy* of formulas of the language of set theory.

First we define the *bounded* formulas as constituting the smallest class satisfying the following:

(a) Every atomic formula is bounded.

(b) If φ is bounded, then so are $(\exists x)(x \in y \wedge \varphi)$ and $(\forall x)(x \in y \to \varphi)$.

A formula is called Σ_0 and also Π_0 if it is bounded. For $n \in \omega$, a formula is Σ_{n+1} if it is $(\exists x)\,\varphi$ for some variable x and some Π_n formula φ; it is Π_{n+1} if it is $(\forall x)\,\varphi$ for some variable x and some Σ_n formula φ.

The theory in which we will work in most of the next two sections is $\mathrm{ZC}^- + \Sigma_1$ Replacement: ZFC without the Axiom of Power Set and with the Axiom of Replacement only for Σ_1 formulas. Another way to describe this theory is that it is Kripke-Platek set theory with Choice (KPC) plus Comprehension. The point of Σ_1 Replacement is that it gives us cartesian products, enough ordinal numbers, and some simple definitions by transfinite recursion. We could get by without Σ_1 Replacement, but then we would have to be careful how we *formulate* some of our theorems as well as how we prove them. With respect to the absence of the Power Set axiom, the reader not familiar with formal axiomatic set theory should notice that the sets we deal with in proofs about games in a tree T are subsets of T or are other sets of no greater size than T. Sometimes we mention larger sets, e.g. $\lceil T \rceil$ and subsets of $\lceil T \rceil$. Talk of $\lceil T \rceil$ is eliminable in simple ways: for example, instead of "$(\forall x)(x \in \lceil T \rceil \to \ldots)$," we can say "$(\forall x)((\forall p \subseteq x)\, p \in T) \to \ldots)$." Our talk of subsets of $\lceil T \rceil$ will be almost always be eliminable because the subsets in question will be Borel sets, and therefore they can be specified by countable systems of subsets of T: To specify a Borel set, it is enough to describe how it is built up a countable family of open sets; the open sets A themselves are given by the set of $p \in T$ such that $\lceil T_p \rceil \subseteq A$. Lemma 1.4.1 gives another way to specify a Borel set: via a clopen subset of $\lceil T \rceil \times \lceil S \rceil$, with S a countable tree.

The proofs of all results in §1.1 and §1.2 go through in $\mathrm{ZC}^- + \Sigma_1$ *Replacement.*

In this section we will prove, in $\mathrm{ZC}^- + \Sigma_1$ Replacement, determinacy for Borel levels through $\boldsymbol{\Sigma}^0_3$.

Theorem 1.3.1. ([Wolfe, 1955]; $\mathrm{ZC}^- + \Sigma_1$ Replacement) *All* $\boldsymbol{\Sigma}^0_2$ *(*$\mathbf{F}_\sigma$*) games are determined.*

Proof. We first prove the following lemma.

Lemma 1.3.2. *Let $B \subseteq A \subseteq \lceil T \rceil$ with B closed. If $G(A;T)$ is not a win for* I, *then there is a strategy τ for* II *such that every play consistent with τ contains a position p with these properties:*

(i) $\lceil T_p \rceil \cap B$ *is empty.*

(ii) $G(A; T_p)$ *is not a win for* I.

Proof of Lemma. Assume that $G(A;T)$ is not a win for I. Let C be the set of all $x \in \lceil T \rceil$ such that no $p \subseteq x$ satisfies both (i) and (ii). The lemma asserts precisely that $G(C;T)$ is a win for II. Assume for a contradiction that this is false. C is closed, so Theorem 1.2.4 implies that $G(C;T)$ is a win for I. Let T' be II's non-losing quasistrategy for $G(A;T)$. By Lemma 1.2.7, $G(A;T')$ is not a win for I. But T' does not restrict I's moves in T, so $G(C;T')$ is a win for I. Let σ be a winning strategy for I for $G(C;T')$. Let $x \in \lceil T' \rceil$ be consistent with σ. For every $p \in T'$, and so for every $p \subseteq x$, $G(A;T_p)$ is not a win for I; i.e., (ii) holds for every $p \subseteq x$. Thus (i) fails for every $p \subseteq x$. In other words $\lceil T_p \rceil \cap B$ is nonempty for every $p \subseteq x$. But B is closed, so this implies that $x \in B$. $B \subseteq A$ and hence $x \in A$ also. Since x was an arbitrary play consistent with σ, we have derived the contradiction that σ is a winning strategy for I for $G(A;T')$. $\square$

For the proof of the theorem, let $A \subseteq \lceil T \rceil$ with $A \in \mathbf{\Sigma}^0_2$. Then A can be written as $A = \bigcup_{i \in \omega} A_i$ with each A_i closed. Assume that $G(A;T)$ is not a win for I. We get a winning strategy τ for II as follows. Here and on other occasions, we describe (the essential part of) a strategy by describing an arbitrary play consistent with the strategy. Let τ_0 be as given by the lemma with $B = A_0$. Let τ agree with τ_0 until a position p_0 is first reached satisfying (i) and (ii). Now apply the lemma with $B = A_1$ and T_{p_0} for T, getting τ_1. Let τ agree with τ_1 from p_0 until a p_1 is first reached satisfying (i) and (ii). Continue in this way. If $\bigcup_{i \in \omega} p_i$ is finite and non-terminal, let τ be arbitrary on positions extending $\bigcup_{i \in \omega} p_i$. Let x be consistent with τ. For each i, there is a $p_i \subseteq x$ with $\lceil T_{p_i} \rceil \cap A_i = \emptyset$. Hence $x \notin \bigcup_{i \in \omega} A_i$; i.e., $x \notin A$. $\square$

Theorem 1.3.3. ([Davis, 1964]; ZC$^-$+ Σ_1 Replacement) *All* $\mathbf{\Sigma}^0_3$ *(*$\mathbf{G}_{\delta\sigma}$*) games are determined.*

Proof We first prove a lemma analogous to Lemma 1.3.2.

Lemma 1.3.4. *Let* $B \subseteq A \subseteq \lceil T \rceil$ *with* $B \in \mathbf{\Pi}^0_2$*. If* $G(A;T)$ *is not a win for* I*, then there is a quasistrategy* T^* *for* II *with the following properties:*

(i) $\lceil T^* \rceil \cap B$ *is empty.*

(ii) $G(A; T^*)$ *is not a win for* I.

Proof of Lemma. Assume that $G(A; T)$ is not a win for I. Let T' be II's non-losing quasistrategy for $G(A; T)$. Note that, for each $p \in T'$, T'_p is II's non-losing quasistrategy for $G(A; T_p)$; thus by Lemma 1.2.7 every $p \in T'$ is such that $G(A; T'_p)$ is not a win for I.

Let us call a position p in T' *good* if there is a quasistrategy T^* for II in T'_p such that (i) $\lceil T^* \rceil \cap B$ is empty and (ii) $G(A; T^*)$ is not a win for I. The lemma will be proved if we can show that the initial position $\emptyset$ is good, since a T^* witnessing that $\emptyset$ is good is also a quasistrategy for II in T. Now $B \in \mathbf{\Pi}^0_2$, so let $B = \bigcap_{n \in \omega} D_n$ with each D_n open. For each n let

$$E_n \;=\; A \cup \{x \in \lceil T' \rceil \mid (\exists p \subseteq x)(\lceil T'_p \rceil \subseteq D_n \;\wedge\; p \text{ is not good})\}.$$

Fix n and assume that $G(E_n; T')$ is not a win for I. We show that $\emptyset$ is good. Define a quasistrategy T^* for II in T' as follows: T^* agrees with II's non-losing quasistrategy T'' for $G(E_n; T')$ until first (if ever) a position p is reached with $\lceil T'_p \rceil \subseteq D_n$. Consider a first such p reached. Since p belongs to T'', p must be good. Choose a quasistrategy $\hat{T}(p)$ for II witnessing that p is good. Let T^* agree with $\hat{T}(p)$ for $q \supseteq p$. We will show that T^* witnesses that $\emptyset$ is good. If $x \in \lceil T^* \rceil$ then either $x \notin D_n$ or else x belongs to some $\lceil \hat{T}(p) \rceil$ and so $x \notin B$ by (i). Thus $\lceil T^* \rceil \subseteq \neg D_n \cup \neg B = \neg B$, and we need only show that $G(A; T^*)$ is not a win for I. Suppose to the contrary that σ is a winning strategy for I for $G(A; T^*)$. If there is a position p consistent with σ such that $\lceil T'_p \rceil \subseteq D_n$, then there is such a p such that $T^*_p = \hat{T}(p)$. $\hat{T}(p)$ has property (ii) and so $G(A; T^*_p)$ is not a win for I. But then σ cannot be a winning strategy for $G(A; T^*)$. Hence no such p can exist, and so every play consistent with σ belongs to $\lceil T'' \rceil$. By Lemma 1.2.7, $G(E_n; T'')$ is not a win for I. Thus there is an $x \in \lceil T'' \rceil$ such that x is consistent with σ and $x \notin E_n$. $A \subseteq E_n$, and so $x \notin A$. Therefore σ is not a winning strategy. This contradiction completes the proof that T^* witnesses that $\emptyset$ is good.

The argument just given has shown that $\emptyset$ is good unless, for each $n \in \omega$, $G(E_n; T')$ is a win for I. For $p \in T'$ and $n \in \omega$, let

$$E^p_n \;=\; A \cup \{x \in \lceil T'_p \rceil \mid (\exists q \subseteq x)(p \subseteq q \;\wedge\; \lceil T'_q \rceil \subseteq D_n \;\wedge\; q \text{ is not good})\}.$$

The same argument shows that, for all $p \in T'$ and all $n \in \omega$, p is good unless $G(E^p_n; T'_p)$ is a win for I.

Assume that the lemma is false. We get a strategy σ for I as follows. Let σ_0 be a winning strategy for I for $G(E_0; T')$. σ agrees with σ_0 until first (if ever) a p_0 is reached with $\lceil T'_{p_0} \rceil \subseteq D_0$ and p_0 not good. If such a p_0 is reached, choose a winning strategy σ_1 for I for $G(E_1^{p_0}; T'_{p_0})$. Let σ agree with σ_1 from p_0 until a p_1 is first reached with $\lceil T'_{p_1} \rceil \subseteq D_1$ and p_1 not good. Continue in this way, letting σ be arbitrary on positions extending $\bigcup_{n\in\omega} p_n$ if the latter is a non-terminal position. If some p_n does not exist, then the play x belongs to $E_n^{p_{n-1}}$ (E_n if $n = 0$) but there is no $p \subseteq x$ with $p_{n-1} \subseteq p$ if $n > 0$ and $\lceil T'_p \rceil \subseteq D_n$ and p not good. By the definition of E_n, $x \in A$. If all p_n exist, then the play belongs to $\bigcap_{n\in\omega} D_n = B \subseteq A$. Thus every play consistent with σ belongs to A, contrary to the hypothesis that $G(A; T')$ is not a win for I.

□

Now let us prove the theorem. Let $A \subseteq \lceil T \rceil$ with $A \in \mathbf{\Sigma}^0_3$. Let $A = \bigcup_{i\in\omega} A_i$ with each $A_i \in \mathbf{\Pi}^0_2$. We get a strategy τ for II as follows. Apply the Lemma with $B = A_0$ to get $T^*(\emptyset)$. For positions $p_1 \in T$ of length 1, let $\tau(p_1)$ be an arbitrary move legal in II's non-losing quasistrategy for $G(A; T^*(\emptyset))$. For any position p_2 consistent with τ and with $\ell\mathrm{h}(p_2) = 2$, apply the lemma with $B = A_1$ and with $(T^*(\emptyset))_{p_2}$ for T, getting $T^*(p_2)$. For any position $p_3 \in T^*(\emptyset)$ with $\ell\mathrm{h}(p_3) = 3$, let $\tau(p_3)$ be an arbitrary move legal in II's non-losing quasistrategy for $G(A; T^*(p_3))$. Continue in this way. Let x be a play consistent with τ. If x is finite, then x belongs to II's non-losing quasistrategy for $G(A; T^*(\emptyset))$, hence $(T^*(\emptyset))_x \not\subseteq A$, and so $x \notin A$. If x is infinite, then $x \in \bigcap_{n\in\omega} \lceil T^*(x \restriction n) \rceil \subseteq \bigcap_{n\in\omega} \neg A_n$, so $x \notin A$. Thus τ is a winning strategy for II for $G(A; T)$. □

Exercise 1.3.1. Let $A \subseteq \lceil T \rceil$ with $A \in \mathbf{\Sigma}^0_2$. Let $A = \bigcup_{n\in\omega} A_n$ with each A_n closed. For each ordinal number α, we define P_α, a set of positions of even length in T. For $p \in T$ and $\lceil T_p \rceil \cap A \neq \emptyset$, let $n(p)$ be the least n such that $\lceil T_p \rceil \cap A_n \neq \emptyset$. For each ordinal α, let

$$B^p_\alpha = \{x \in \lceil T_p \rceil \mid x \in A_{n(p)} \vee (\exists q)(p \subseteq q \subseteq x \wedge q \in \bigcup_{\beta<\alpha} P_\beta)\}.$$

Let $p \in P_\alpha$ if and only if $n(p)$ is defined and $G(B^p_\alpha; T_p)$ is a win for I. As with Exercise 1.2.4, first show that there is an α such that $(\forall \gamma \geq \alpha)\, P_\gamma = P_\alpha$, and let P_∞ be this limiting value of P_α. Now show that $G(A; T)$ is a win for I if $\emptyset \in P_\infty$ and that $G(A; T)$ is a win for II if $\emptyset \notin P_\infty$.

Exercise 1.3.2. Use the construction of Exercise 1.3.1 to prove Solovay's result (see pages 414–415 of [Moschovakis, 1980]) that, if $A \subseteq {}^\omega\omega$, $A \in \Sigma^0_2$, and $G(A;T)$ is a win for I, then there is a winning strategy for I belonging to L_β for β the closure ordinal for Σ^1_1 monotone inductive definitions. Prove also that, for such A and for the same β, that if $G(A;T)$ is a win for II then there is a winning strategy for II belonging to $L(\beta')$, where β' the least admissible ordinal $> \beta$.

1.4 $\mathbf{\Delta}^0_4$ Games

In this section we prove the determinacy of all $\mathbf{\Delta}^0_4$ games. For countable trees, $\mathbf{\Delta}^0_4$ coincides with the *difference hierarchy on* $\mathbf{\Pi}^0_3$. (Theorem 1.4.2, a result in [Kuratowski, 1958]). For uncountable trees, $\mathbf{\Delta}^0_4$ coincides with what we call the *generalized difference hierarchy* on $\mathbf{\Pi}^0_3$. Because the proofs in this section are somewhat complicated, we first deal fully with the case of countable trees. We prove, in the countable case, the equality of the difference hierarchy with $\mathbf{\Delta}^0_4$, and we prove (in the general case) determinacy for the difference hierarchy. Then we take up general trees, showing how to modify the definitions and proofs from the countable case to make them work in the general case.

In earlier versions of this chapter, we mistakenly claimed that our proof of determinacy for the difference hierarchy on $\mathbf{\Pi}^0_3$ went through in ZC$^-$+ Σ_1 Replacement. In [Montalban and Shore, 2012], the authors point out that only for fixed finite levels of that difference hierarchy does our proof go through in ZC$^-$+ Σ_1 Replacement. They go on to demonstrate that the assertion that determinacy holds in countable trees for all finite levels cannot be proved in ZFC$^-$ (ZFC minus Power Set). This improves the known theorem, proved using the methods of [Friedman, 1971], that the determinacy of all $\mathbf{\Sigma}^0_4$ games in countable trees cannot be proved in ZFC$^-$. (See Exercise 1.4.1.) Before giving our proof of determinacy for the full difference hierarchy, we present the simplification of that proof that results from adapting it to the case of a fixed finite level of the hierarchy. For this proof, we will not need to treat the case of countable trees separately.

For the both the fixed-finite-level case and the full difference hierarchy case, we first give determinacy proofs without paying attention to what hypotheses are being used. Afterward we discuss hypotheses. The determinacy of the full difference hierarchy needs a theory stronger than ZC$^-$+ Σ_1 Replacement, but this theory does not imply the existence of uncountable sets. Moreover it, like ZC$^-$+ Σ_1 Replacement, does not imply that all $\mathbf{\Sigma}^0_4$ games in countable trees are determined. (See Exercise 1.4.6.)

If $\mathbf{\Gamma}$ is a class of sets (e.g., $\mathbf{\Pi}^0_3$) and $\alpha > 0$ is a countable ordinal, then a set $A \subseteq \lceil T \rceil$ belongs to α-$\mathbf{\Gamma}$, the *αth level of the difference hierarchy on* $\mathbf{\Gamma}$, just in case there is a sequence $\langle A_\beta \mid \beta < \alpha \rangle$ with each $A_\beta \in \mathbf{\Gamma}$ and such that

$$(\forall x \in \lceil T \rceil)(x \in A \leftrightarrow \mu\beta(x \notin A_\beta \vee \beta = \alpha) \text{ is odd}),$$

where "μ" means "the least" and "odd" is defined in the natural way. (Limit ordinals are even.) The difference hierarchy is ordinarily defined only for classes $\mathbf{\Gamma}$ closed under countable intersections (so that $\bigcap_{\gamma<\beta} A_\gamma \in \mathbf{\Gamma}$) and we will consider it only for such classes. 1-$\mathbf{\Gamma} = \mathbf{\Gamma}$, 2-$\mathbf{\Gamma}$ is the class of differences of sets belonging to $\mathbf{\Gamma}$, etc. Let $\mathbf{Diff}(\mathbf{\Gamma}) = \bigcup_{\alpha<\omega_1} \alpha$-$\mathbf{\Gamma}$. We are interested in the difference hierarchy because of Theorem 1.4.2, which states that $\mathbf{Diff}(\mathbf{\Pi}^0_\alpha) = \mathbf{\Delta}^0_{\alpha+1}$. Before proving this result of Kuratowski we prove a characterization of $\mathbf{\Sigma}^0_\alpha$ that will be useful for the proof.

A game tree T is *wellfounded* if, for every nonempty $Y \subseteq T$, there is a p in $T \cap Y$ such that no $q \supsetneq p$ belongs to $T \cap Y$. Using the Axiom of Choice, one can show that T is wellfounded if and only if there are no infinite plays in T. (Exercise 1.2.1.) For wellfounded T, the plays in T are thus exactly the same as the terminal positions in T. If T is wellfounded, then we can define functions with domain T by *transfinite recursion.* (See Theorem 5.6 of [Kunen, 1980] or pages 82–83 of [Moschovakis, 1980].) To define an f with domain$(f) = T$, it is enough to define $f(p)$ in terms of the restriction of f to the proper extensions of p, i.e. to define an operation G (which may be a proper class) and let

$$f(p) = G(f \restriction \{q \mid p \subsetneq q\})$$

for each $p \in T$. (Since the existence of f is proved by using the Axiom of Replacement, we will in this section use definition by transfinite recursion only for G's definable by a simple enough formula that Σ_1 Replacement suffices to get f.) When we make such a definition, we will say that we are defining *f by induction on T*. One can talk in more general terms of *wellfounded relations*: A relation $\prec$ is *wellfounded* if, for every nonempty set Y, there is an element of Y minimal with respect to $\prec$. Thus a game tree T is wellfounded if and only if $\supsetneq \restriction T$ is a wellfounded relation. In general, the wellfoundedness of a relation is equivalent with the non-existence of infinite descending chains with respect to $\prec$. Definition by transfinite recursion is applicable to general wellfounded relations. When we use it for $\prec$ we will say we are making a definition *by induction on* $\prec$.

Let Ord be the (proper) class of all ordinal numbers, and let T be a wellfounded game tree. By induction on T, we define $\|\ \|^T : T \to \text{Ord}$ by

$$\|p\|^T \;=\; \sup\{\|q\|^T + 1 \mid p \subsetneq q\}.$$

Note that the supremum in this definition might as well be restricted to q with $\ell\text{h}(q) = \ell\text{h}(p) + 1$. Define also

$$\|T\| = \|\emptyset\|^T.$$

Note that $\|T\|$ and each $\|p\|^T$ are ordinals smaller than $|T|^+$, the least cardinal number greater than $|T|$, the cardinal number of T. (Of course, since we now are working in the weak theory $\text{ZC}^- + \Sigma_1$ Replacement, we don't know that $|T|^+$ exists. Nevertheless, we will use the notation $|T|^+$, construing it as proper class if it is not a set.)

Lemma 1.4.1. ($\text{ZC}^-+\ \Sigma_1$ Replacement) *$A \subseteq \lceil T \rceil$ is Borel if and only if there is a tree $S \subseteq {}^{<\omega}\omega$ and there is a clopen $B \subseteq \lceil T \rceil \times \lceil S \rceil$ such that S is wellfounded and*

$$(\forall x \in \lceil T \rceil)(x \in A \leftrightarrow G(B(x); S) \textit{ is a win for } I),$$

where $B(x) = \{q \in \lceil S \rceil \mid \langle x, q \rangle \in B\}$. Moreover, for $\alpha > 0$ and $A \subseteq \lceil T \rceil$, $A \in \mathbf{\Sigma}^0_\alpha$ if and only if such an S and B exist with $\|S\| \le \alpha$.

Proof. If such S and B exist with $\|S\| \le \alpha$, define $A_q \subseteq \lceil T \rceil$ for $q \in S$ by

$$x \in A_q \;\leftrightarrow\; G(B(x); S_q) \text{ is a win for } I.$$

We prove by induction on $\|q\| = \|q\|^S > 0$ that

$$A_q \;\in\; \begin{cases} \mathbf{\Sigma}^0_{\|q\|} & \text{if } \ell\text{h}(q) \text{ is even;} \\ \mathbf{\Pi}^0_{\|q\|} & \text{if } \ell\text{h}(q) \text{ is odd.} \end{cases}$$

If $\|q\| = 0$ then q is terminal, and so $x \in A_q \leftrightarrow \langle x, q \rangle \in B$, so A_q is clopen. Let $\|q\| > 0$. If $\ell\text{h}(q)$ is even, then

$$A_q \;=\; \bigcup\{A_{q'} \mid q \subseteq q' \,\wedge\, \ell\text{h}(q') = \ell\text{h}(q) + 1\};$$

If $\ell\mathrm{h}(q)$ is odd, then

$$A_q = \bigcap\{A_{q'} \mid q \subseteq q' \wedge \ell\mathrm{h}(q') = \ell\mathrm{h}(q)+1\}.$$

In both cases, the desired conclusion follows directly by induction.

We prove the converse by induction on α. If $\alpha = 1$ then A is open. Let $S = {}^1\omega$ and let $\langle x, \langle k\rangle\rangle \in B \leftrightarrow \lceil T_{x\restriction k}\rceil \subseteq A$. Let $\alpha > 1$ and assume $A \in \mathbf{\Sigma}^0_\alpha$. Then there are A_i, $i \in \omega$, such that $A = \bigcup_{i\in\omega} A_i$ and such that each $A_i \in \mathbf{\Pi}^0_{\beta_i}$ for some $\beta_i < \alpha$. Let $S_i \subseteq {}^{<\omega}\omega$ and B_i be given by induction for $\neg A_i$ and β_i, for each i. Let

$$\begin{aligned} S &= \{\langle i\rangle^\frown q \mid q \in S_i\}; \\ B &= \{\langle x, \langle i\rangle^\frown q\rangle \mid q \in \lceil S_i\rceil \wedge \langle x, q\rangle \notin B_i\}. \end{aligned}$$

It is easy to see that S and B are as required. $\square$

Theorem 1.4.2. ([Kuratowski, 1958], §33 III; ZC$^-$+ Σ_1 Replacement) *For countable T, $\mathbf{Diff}(\mathbf{\Pi}^0_\xi) = \mathbf{\Delta}^0_{\xi+1}$, for each countable ordinal $\xi \geq 1$.*

Proof We first show $\mathbf{Diff}(\mathbf{\Pi}^0_\xi) \subseteq \mathbf{\Delta}^0_{\xi+1}$. (For this part, we do not need the assumption that T is countable.) Let $\langle A_\beta \mid \beta < \alpha\rangle$ witness that $A \in \mathbf{Diff}(\mathbf{\Pi}^0_\xi)$. Assume for definiteness that α is even.

$$\begin{aligned} A &= \bigcup_{\substack{\beta<\alpha \\ \beta \text{ odd}}} (\neg A_\beta \cap \bigcap_{\gamma<\beta} A_\gamma) \\ &= \bigcup_{\beta<\alpha} \neg A_\beta \;\cap \bigcap_{\substack{\beta<\alpha \\ \beta \text{ even}}} (A_\beta \cup \bigcup_{\gamma<\beta} \neg A_\gamma) \end{aligned}$$

Since each $A_\gamma \in \mathbf{\Pi}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ is closed under countable intersections, $\bigcap_{\gamma<\beta} A_\gamma \in \mathbf{\Pi}^0_\xi \subseteq \mathbf{\Sigma}^0_{\xi+1}$. Furthermore, $\neg A_\beta \in \mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Sigma}^0_{\xi+1}$. Since $\mathbf{\Sigma}^0_{\xi+1}$ is closed under finite intersections, $\neg A_\beta \cap \bigcap_{\gamma<\beta} A_\gamma \in \mathbf{\Sigma}^0_{\xi+1}$. The first equation then gives that $A \in \mathbf{\Sigma}^0_{\xi+1}$. An analogous calculation using the second equation gives that A is the intersection of a member of $\mathbf{\Sigma}^0_\xi$ and a member of $\mathbf{\Pi}^0_{\xi+1}$; hence $A \in \mathbf{\Pi}^0_{\xi+1}$ also. By the definition of $\mathbf{\Delta}^0_{\xi+1}$, $A \in \mathbf{\Delta}^0_{\xi+1}$.

We now turn to the more difficult other half of the theorem. Let $A \in \mathbf{\Delta}^0_{\xi+1}$. There are sets C_i, $i \in \omega$, such that each $C_i \in \mathbf{\Pi}^0_\xi$ and such that

$$\begin{aligned} \neg A &= \bigcup_{i\in\omega} C_{2i}; \\ A &= \bigcup_{i\in\omega} C_{2i+1}. \end{aligned}$$

For each $i \in \omega$ let S_i and B_i be given by Lemma 1.4.1 with $\neg C_i$ for the A of that lemma and ξ for the α. Let

$$\begin{aligned} S &= \{\langle i\rangle^\frown q \mid q \in S_i\}; \\ B &= \{\langle x, \langle i\rangle^\frown q\rangle \mid q \in \lceil S_i\rceil \wedge \langle x, q\rangle \notin B_i\}. \end{aligned}$$

Note that $G(B(x); S)$ is a win for I if and only if $x \in \bigcup_{n\in\omega} C_n = A \cup \neg A = \lceil T\rceil$, and so each $G(B(x); S)$ is a win for I. For $n \in \omega$ let

$$S^n = \{q \in S \mid \ell\text{h}(q) \leq 2n \wedge (\forall n' < n)(n' \text{ even } \rightarrow q(n') < n)\}.$$

and let

$$\begin{aligned} U_n = \{\langle p, t\rangle \mid\, & p \in T \wedge \ell\text{h}(p) = n \wedge t \in \mathcal{S}_{\text{II}}(S^n) \wedge \\ & (\forall q \in \lceil S\rceil \cap S^n)(q \text{ consistent with } t \rightarrow \lceil T_p\rceil \times \{q\} \not\subseteq B)\}. \end{aligned}$$

Recall (page 5) that $\mathcal{S}_{\text{II}}(S^n)$ is the set of all strategies for II in S^n.

Let $U = \bigcup_{n\in\omega} U_n$. Partially order U by

$$\langle p, t\rangle \prec \langle p', t'\rangle \;\leftrightarrow\; (p' \subsetneq p \wedge t' \subsetneq t).$$

The relation $\prec$ is wellfounded, since if

$$\cdots \prec \langle p_2, t_2\rangle \prec \langle p_1, t_1\rangle \prec \langle p_0, t_0\rangle$$

is an infinite descending chain with respect to $\prec$, then $\bigcup_{n\in\omega} t_n$ is a winning strategy for II for $G(B(\bigcup_{n\in\omega} p_n); S)$, and this game is a win for I. By induction on $\prec$ define $\text{ord}(p,t) = \sup\{\text{ord}(p', t') + 1 \mid \langle p', t'\rangle \prec \langle p, t\rangle\}$. Note that the supremum in this definition might as well be restricted to $\langle p', t'\rangle \in U_{n+1}$ if $\langle p, t\rangle \in U_n$. The unique member of U_0 is $\langle\emptyset, \emptyset\rangle$. Set $\gamma = \text{ord}(\langle\emptyset, \emptyset\rangle)$. By the hypothesis that T is countable, it follows that U is countable also; hence γ is a countable ordinal.

If $x \in \lceil T \rceil$, $t \in \mathcal{S}_{\mathrm{II}}(S^n)$, and $i < n$, then t gives a fragment s_i^t of a strategy for I for $G(B_i(x); S_i)$: Let $s_i^t(q) = t(\langle i \rangle ^\frown q)$ for each q such that $\langle i \rangle ^\frown q \in S^n$. Let us say that $t \in \mathcal{S}_{\mathrm{II}}(S^n)$ is *n-wrong for* $x \in \lceil T \rceil$ if there is an $i < n$ such that, for all $\sigma \in \mathcal{S}_{\mathrm{I}}(S_i)$, if $s_i^t \subseteq \sigma$ then σ is not a winning strategy for I for $G(B_i(x); S_i)$. Note that if $t \in \mathcal{S}_{\mathrm{II}}(S^n)$ is not n-wrong for x then $\langle x \restriction n, t \rangle \in U_n$.

Remark. The reason we had to define "n-wrong" and not simply "wrong" is that it is possible to have $S^n = S^{n'}$ with $n \neq n'$.

Now define, for $\alpha \leq \gamma$,

$$\begin{aligned} A_{2\alpha} \;=\; \{x \in \lceil T \rceil \mid (\forall n)(\forall t)((&n \text{ even} \wedge \langle x \restriction n, t \rangle \in U_n \wedge \mathrm{ord}(x \restriction n, t) = \alpha) \\ &\rightarrow\; t \text{ is } n\text{-wrong for } x)\}. \end{aligned}$$

Similarly define $A_{2\alpha+1}$ for $\alpha < \gamma$, with "odd" replacing "even."

We show that $\langle A_\beta \mid \beta \leq 2\gamma \rangle$ witnesses that $A \in \mathbf{Diff}(\mathbf{\Pi}^0_\xi)$. Let $x \in \lceil T \rceil$ and let α be the least ordinal such that there exist n and $t \in \mathcal{S}_{\mathrm{II}}(S^n)$ with t not n-wrong for x and with $\mathrm{ord}(x \restriction n, t) = \alpha$. Such α, n, and t exist with $\alpha \leq \gamma$, for $\emptyset \in \mathcal{S}_{\mathrm{II}}(S^n)$, and the fact that there is no $i < 0$ guarantees that $\emptyset$ is not 0-wrong for x. Let n and $t \in \mathcal{S}_{\mathrm{II}}(S^n)$ be such that t is not n-wrong for x and $\mathrm{ord}(x \restriction n, t) = \alpha$. Choose n to be even, if possible. Note that $\alpha < \gamma$ if n is odd, since $n > 0$. We show that

$$\begin{aligned} n \text{ even} \;&\rightarrow\; x \in \bigcap_{\beta < 2\alpha} (A_\beta \setminus A_{2\alpha}) \;\wedge\; x \notin A; \\ n \text{ odd} \;&\rightarrow\; x \in \bigcap_{\beta \leq 2\alpha} (A_\beta \setminus A_{2\alpha+1}) \;\wedge\; x \in A. \end{aligned}$$

We do the case that n is even; the other case is similar. Since n is even, t witnesses that $x \notin A_{2\alpha}$. But the minimality of α implies that, for all $\alpha' < \alpha$, x belongs both to $A_{2\alpha'}$ and to $A_{2\alpha'+1}$. Thus $x \in \bigcap_{\beta<2\alpha} A_\beta \setminus A_{2\alpha}$. Since t is not wrong for x, there is for each $i < n$ a winning strategy σ_i for I for $G(B_i(x); S_i)$ such that $s_i^t \subseteq \sigma_i$. This means, first of all, that $G(B_i(x); S_i)$ is a win for I for each $i < n$ and so that $(\forall i < n)\, x \notin C_i$. Suppose for a contradiction that also $x \notin C_n$. Let σ_n be a winning strategy for I for $G(B_n(x); S_n)$. Let $t' \in \mathcal{S}_{II}(S^{n+1})$ be such that $s_i^{t'} \subseteq \sigma_i$ for each $i \leq n$. Clearly t' is not n-wrong for x. This implies, in particular, that $\langle x \restriction n+1, t' \rangle \in U_{n+1}$. But $\langle x \restriction n+1, t' \rangle \prec \langle x \restriction n, t \rangle$, and so $\mathrm{ord}(x \restriction n+1, t') < \mathrm{ord}(x \restriction n, t) = \alpha$. This contradiction gives us that $x \in C_n$. Since n is even, it follows that $x \notin A$.

To complete the proof of the theorem, we show that each $A_\beta \in \mathbf{\Pi}^0_\xi$. Assume for definiteness that $\beta = 2\alpha$. Since $\mathbf{\Pi}^0_\xi$ is closed under countable intersections, it is enough to show that, for any fixed t and even n, the set $A_{2\alpha,n,t}$ defined by

$$A_{2\alpha,n,t} \;=\; \{x \mid (\langle x \restriction n, t\rangle \in U_n \;\wedge\; \mathrm{ord}(x \restriction n, t) = \alpha) \;\rightarrow\; t \text{ is } n\text{-wrong for } x\}$$

belongs to $\mathbf{\Pi}^0_\xi$. We apply Lemma 1.4.1. Fix such n and t. For $i < n$ let

$$S'_i \;=\; \{q \in S_i \mid q \text{ is consistent with } s^t_i\}.$$

Let

$$B'_i \;=\; \{\langle x, q\rangle \in B_i \mid \langle x \restriction n, t\rangle \in U_n \;\wedge\; \mathrm{ord}(x \restriction n, t) = \alpha\}.$$

Now $x \in A_{2\alpha,n,t}$ if and only if there is an $i < n$ such that $G(B'_i(x); S'_i)$ is not a win for I. Since $\|S'_i\| \leq \|S_i\| \leq \xi$ for each i and since each B'_i is clopen, the fact that $A_{2\alpha,n,t} \in \mathbf{\Pi}^0_\xi$ follows by Lemma 1.4.1. □

We will prove the determinacy of $\mathbf{\Delta}^0_4$ games by proving that all $\mathbf{Diff}(\mathbf{\Pi}^0_3)$ games are determined and then applying Theorem 1.4.2. This proof is rather complicated, so we first deal with the simpler case of α-$\mathbf{\Pi}^0_3$ games for α finite. The following well known fact gives an equivalent characterization of the games of this form.

Lemma 1.4.3. *Let $\mathbf{\Gamma}$ be a collection of subsets of $\lceil T\rceil$ closed under finite unions and intersections. A subset A of $\lceil T\rceil$ is k-$\mathbf{\Gamma}$ with $k \in \omega$ if and only if A is a Boolean combination of finitely many $\mathbf{\Gamma}$ sets.*

Proof. The "only if" direction is clear.

For the "if" direction, let A be a finite Boolean combination of finitely many sets in $\mathbf{\Gamma}$. For some positive integer k, $A = \bigcup_{i<k} E_i$ with each E_i the intersection of finitely many sets each a member of $\mathbf{\Gamma}$ or the complement of member of $\mathbf{\Gamma}$. Since $\mathbf{\Gamma}$ is closed under finite unions and intersections, we may assume that

$$A = \bigcup_{i<k} (B_i \setminus C_i),$$

where $1 \leq k \in \omega$ and where the B_i and C_i belong to $\mathbf{\Gamma}$. We may also assume that $B_i \supseteq C_i$ for each $i < k$.

For $x \in \lceil \mathbf{\Gamma} \rceil$ and $j < k$ let

$$\begin{array}{lcl} x \in A_{2j} & \leftrightarrow & x \text{ belongs to at least } j+1 \text{ of the sets } B_i; \\ x \in A_{2j+1} & \leftrightarrow & x \text{ belongs to at least } j+1 \text{ of the sets } C_i. \end{array}$$

We will show that $\langle A_j \mid i < 2k \rangle$ witnesses that $A \in 2k$-$\mathbf{\Gamma}$. Let $x \in \lceil T \rceil$. Let m be least such that $m = 2k$ or $x \notin A_m$.

First assume that m is an odd number $2j + 1$. Then x belongs to exactly j of the C_i and to at least $j + 1$ of the B_i. Thus there is an i such that $x \in B_i \setminus C_i$, and so $x \in A$.

Next assume that m is an even number $2j$. Then x belongs to exactly j of the B_i and to at least j of the C_i. Since $B_i \supseteq C_i$ for each i, it follows that $\{i \mid x \in B_i\} = \{i \mid x \in C_i\}$. Thus there is no i such that $x \in B_i \setminus C_i$, and so $x \notin A$. □

Theorem 1.4.4. *For each finite k, ZC$^-$+ Σ_1 Replacement $\vdash$ "All k-$\mathbf{\Pi}^0_3$ games are determined, and therefore $G(A;T)$ is determined for every A in the Boolean algebra generated by $\mathbf{\Pi}^0_3$ subsets of $\lceil T \rceil$."*

Remark. The proof that follows was gotten by specializing—and so somewhat simplifying—the proof of Theorem 1.4.10 to the case $\gamma = k$. The proof of Theorem 1.1 of [Montalban and Shore, 2012] was gotten in the same way, and at heart the two proofs are the same. Montalban's and Shore's context differs from ours in that they are dealing with second order arithmetic instead of set theory, and so only countable trees are involved. Also they pay attention to what fragment of the axiomatic theory of second order arithmetic is needed for the proof. We won't pay attention to this in giving our proof, but we will discuss it afterward.

Proof. We give the proof for the case that k is odd. From this, the proof for the case of even k can be gotten by exchanging "I" and "II."

Let $A \subseteq \lceil T \rceil$ with $A \in k$-$\mathbf{\Pi}^0_3$. Let $\langle A_n \mid n < k \rangle$ witness that $A \in k$-$\mathbf{\Pi}^0_3$. Also without loss of generality, we assume that

$$m \leq n < k \rightarrow A_m \supseteq A_n.$$

For each $n < k$, let $A_{n,i}$, $i \in \omega$, be $\mathbf{\Sigma}^0_2$ sets such that $A_n = \bigcap_{i \in \omega} A_{n,i}$. For each $n < k$ and each $i \in \omega$, let $A_{n,i,j}$, $j \in \omega$, be closed sets such that $A_{n,i} = \bigcup_{j \in \omega} A_{n,i,j}$.

For $s \in {}^{\leq k}\omega$ and for game subtrees S of T, we define, by induction on $\ell\mathrm{h}(s)$, the assertion $P^s(S)$.

(1) $P^{\emptyset}(S)$ holds if and only if $G(A;S)$ is a win for II;

(2) If $\ell\text{h}(s) = n + 1$ and n is even, then $P^s(S)$ holds if and only if there is a quasistrategy U for I in S such that

(a) $\lceil U \rceil \subseteq A \cup A_{k-n-1,s(n)}$;
(b) $P^{s \restriction n}(U)$ fails;

(3) If $\ell\text{h}(s) = n + 1$ and n is odd, then $P^s(S)$ holds if and only if there is a quasistrategy U for II in S such that

(a) $\lceil U \rceil \subseteq \neg A \cup A_{k-n-1,s(n)}$;
(b) $P^{s \restriction n}(U)$ fails;

Note that $k - n - 1$ is even if and only if n is even.

For $\ell\text{h}(s) > 0$, we say that U *witnesses* $P^s(S)$ if the obvious conditions hold. For $\ell\text{h}(s) = 0$, U *witnesses* $P^s(S)$ if U is (the quasistrategy corresponding to) a winning strategy for II for $G(A; S)$.

For $\ell\text{h}(s) = n+1$, we say that a quasistrategy U (for I if n is even and for II if n is odd) *locally witnesses* $P^s(S)$ if there is a subset $\mathcal{D}$ of S and there are for each $d \in \mathcal{D}$ quasistrategies R^d in S_d, for II if n is even and for I if n is odd, such that

(i) for each $d \in \mathcal{D} \cap U$, $U_d \cap R^d$ witnesses $P^s(R^d)$;

(ii) $\lceil U \rceil \setminus \bigcup_{d \in \mathcal{D}} \lceil R^d \rceil \subseteq \begin{cases} A & \text{if } n \text{ is even;} \\ \neg A & \text{if } n \text{ is odd;} \end{cases}$

(iii) for each $p \in S$, there is at most one $d \in \mathcal{D}$ such that $d \subseteq p$ and $p \in R^d$.

If U witnesses $P^s(S)$, then U locally witnesses P^s: Let $\mathcal{D} = \{\emptyset\}$ and let $R^d = S$. The next lemma, which will be an important technical tool, is the converse.

Lemma 1.4.5. *Let $s \in {}^{\leq k}\omega$ with $0 < \ell\text{h}(s) = n + 1$. Assume that U locally witnesses $P^s(S)$. Then U witnesses $P^s(S)$.*

Proof of Lemma. We prove the lemma by induction on n (actually on odd and even n separately).

Note first that U cannot fail to have property (a) (i.e. (2)(a) or (3)(a), whichever is appropriate). To see this, assume for definiteness that n is even and let $x \in \lceil U \rceil$. By (ii), we may assume that $x \in \lceil R^d \rceil$ for some $d \in \mathcal{D}$. But then (i) implies that $x \in A \cup A_{k-n-1,s(n)}$.

We now turn to property (b), for which we really need induction.

Suppose first that $n = 0$. Assume for a contradiction that (b) fails. Since $n = 0$, let τ be a winning strategy for II for $G(A; U)$.

We show that there is a $d \in \mathcal{D}$ consistent with τ such that if $x \supseteq d$ is a play consistent with τ then x belongs to $\lceil R^d \rceil$. Assume this is false. Then for each $d \in \mathcal{D}$ such that d is consistent with τ, let $f(d) \supsetneq d$, $f(d)$ consistent with τ, $f(d) \notin R^d$, and $(\forall q)(d \subseteq q \subsetneq f(d) \rightarrow q \in R^d)$. By (iii) there are no members d and d' of $\mathcal{D}$ that are consistent with τ and such that $d \subsetneq d' \subseteq f(d)$. It follows that there is a play x consistent with τ such that $f(d) \subseteq x$ whenever $d \subseteq x$. Clearly x cannot belong to $\bigcup_{d \in \mathcal{D}} \lceil R^d \rceil$. But (ii) gives the contradiction that $x \in A$.

Let then d be consistent with τ such that x belongs to $\lceil R^d \rceil$ for every play $x \supseteq d$ such that x is consistent with τ. Then the obvious restriction of τ is a winning strategy for II for $G(A; U_d \cap R^d)$. Hence $P^{\emptyset}(U_d \cap R^d)$, contradicting (i).

Next suppose that $n > 0$ is even. Assume for a contradiction that (b) fails. Let S' witness $P^{s \restriction n}(U)$. We define $\mathcal{D}' \subseteq S'$ as follows:

$$d \in \mathcal{D}' \leftrightarrow \begin{cases} d \in S' \wedge \\ d \in \mathcal{D} \wedge \\ G(\neg \lceil R^d \rceil; S'_d) \text{ is a win for II.} \end{cases}$$

For $d \in \mathcal{D}'$, let R'^d be II's non-losing quasistrategy for $G(\neg \lceil R^d \rceil; S'_d)$. Note that $R'^d \subseteq R^d$.

Let $d \in \mathcal{D}'$. Since R'^d is a quasistrategy for II in S'_d and S'_d is a quasistrategy for II in U_d, it follows that R'^d is a quasistrategy for II in U_d. Since $\lceil R'^d \rceil \subseteq \lceil S' \rceil \subseteq \neg A \cup A_{k-n+,s(n-1)}$, condition (3)(a) holds for R'^d. By (i), R'^d cannot witness $P^{s \restriction n}(U_d)$, so (3)(b) must fail for R'^d. Let then U'^d witness $P^{s \restriction n-1}(R'^d)$.

We define a quasistrategy U' for I in S' as follows:

(1) If $p \in U'$ and there is no $d \in \mathcal{D}$ such that $d \subseteq p$ and $p \in R^d$, then let any move legal in S' at p be legal in U' at p.

(2) For each $q \in S'$ and $d \in \mathcal{D}$ such $d \subseteq q$, such that $q \in R^d \setminus R'^d$ (taking $R'^d = \emptyset$ for $d \notin \mathcal{D}'$), and such that every $q' \subsetneq q$ belongs to R'^d, let σ_q be a winning strategy for I for $G(\neg \lceil R^d \rceil; S'_d)$. Whenever such a q belongs to U', we let U'_q agree with σ_q until a position $p \notin R^d$ is reached.

(3) For $d \in \mathcal{D}' \cap U'$, let $U'_d \cap R'^d = U'^d$.

Using $\mathcal{D}'$ and $\langle R'^d \mid d \in \mathcal{D}'\rangle$, we now show that U' locally witnesses $P^{s\restriction n-1}(S')$. Induction will then give that U' witnesses $P^{s\restriction n-1}(S')$, contradicting property (3)(b) of S'. Property (i) follows from clause (3) in the definition of U' and the fact that U'^d witnesses $P^{s\restriction n-1}(R'^d)$. For (ii), note first that clause (2) in the definition of U' guarantees that, for each $d \in \mathcal{D}$,

$$\lceil U' \rceil \cap \lceil R^d \rceil \subseteq \lceil R'^d \rceil.$$

Thus $\lceil U' \rceil \setminus \bigcup_{d\in\mathcal{D}'} \lceil R'^d \rceil = \lceil U' \rceil \setminus \bigcup_{d\in\mathcal{D}} \lceil R^d \rceil \subseteq A$. (iii) follows from the facts that $\mathcal{D}' \subseteq \mathcal{D}$ and that $(\forall d \in \mathcal{D}')\, R'^d \subseteq R^d$.

For the remaining case, that of an odd $n > 0$, we make the same definitions as for the case of even $n > 0$, except that we exchange I and II, A and $\neg A$, and $G(\neg\lceil R^d \rceil; S'_d)$ and $G(\lceil R^d \rceil; S'_d)$. The argument is exactly the same, except for a minor change in the case $n = 1$: In that case, the U'^d are the quasistrategies corresponding to winning strategies for II for $G(A; R'^d)$, and we must prove that $\lceil U' \rceil \subseteq \neg A$. As before, $\lceil U' \rceil \setminus \bigcup_{d\in\mathcal{D}'} \lceil R'^d \rceil \subseteq \neg A$. Moreover for $d \in \mathcal{D}'$ we have that $\lceil U' \rceil \cap \lceil R'^d \rceil = \lceil U'^d \rceil \subseteq \neg A$.

We say that $P^s(S)$ *fails everywhere* if $P^s(S_p)$ fails for every $p \in S$.

Lemma 1.4.6. *Let $s \in {}^{\leq k}\omega$ and let $m = lh(s)$. If $P^s(S)$ fails, then there is a quasistrategy W in S for I if m is even and for II if m is odd such that $P^s(W)$ fails everywhere.*

Proof of Lemma. The case $m = 0$ is Lemma 1.2.7, so assume $m = n + 1$. Suppose for definiteness that n is even; the other case is similar. Let $\mathcal{D}$ be the set of all $d \in S$ such that $P^s(S_d)$ but such that, for every $p \subsetneq d$, $P^s(S_p)$ fails. For each $d \in \mathcal{D}$, let U^d witness $P^s(S_d)$. Let

$$B = \{x \in \lceil S \rceil \mid (\exists d \in \mathcal{D})\, d \subseteq x\}.$$

First assume for a contradiction that the open game $G(B; S)$ is a win for I. Let σ be a winning strategy for I for $G(B; S)$. We define a quasistrategy U for I in S as follows: U agrees with σ until a position $d \in \mathcal{D}$ is reached. Then $U_d = U^d$. For $d \in \mathcal{D}$, let $R^d = S_d$. It is easy to see, using $\mathcal{D}$ and $\langle R^d \mid d \in \mathcal{D}\rangle$, that U locally witnesses $P^s(S)$, so Lemma 1.4.5 gives the contradiction that U witnesses $P^s(S)$.

We know then that $G(B; S)$ is a win for II. Let W be II's non-losing quasistrategy. Assume for a contradiction that $q \in W$ and that U^* witnesses

$P^s(W_q)$. Let U be a quasistrategy for I in S_q defined as follows: Let $U \cap W_q = U^*$. When first (if ever) a position $p \notin W$ is reached, let U agree with a winning strategy σ_p for I for $G(B;S)$ until a position $d \in \mathcal{D}$ is reached. Then let $U_d = U^d$. Let $\mathcal{D}' = \mathcal{D} \cup \{q\}$. Let $R^q = W_q$ and let $R^d = S_d$ for $d \in \mathcal{D}$. It is easy to see, using $\mathcal{D}'$ and $\langle R^d \mid d \in \mathcal{D}' \rangle$, that U locally witnesses $P^s(S_q)$. Lemma 1.4.5 gives us the contradiction that some $d \subseteq q$ belongs to $\mathcal{D}$. $\square$

For $n+1 = \ell\mathrm{h}(s)$, we say that W *strongly witnesses* $P^s(S)$ if, for all $p \in W$, W_p witnesses $P^s(S_p)$, i.e. if W witnesses $P^s(S)$ and $P^{s \restriction n}(W)$ fails everywhere.

Lemma 1.4.7. *Let $s \in {}^{\leq k}\omega$ with $0 < \ell\mathrm{h}(s) = n+1$. If $P^s(S)$, then there is a W that strongly witnesses $P^s(S)$.*

Proof of Lemma. Assume for definiteness that n is even. Let U witness that $P^s(S)$. By property (2)(b) of U, $P^{s \restriction n}(U)$ fails. By Lemma 1.4.6, let W be a quasistrategy for I in U such that $P^{s \restriction n}(W)$ fails everywhere. Since W is a quasistrategy for I in S and W inherits property (2)(a) from U, it follows that W strongly witnesses $P^s(S)$. $\square$

Lemma 1.4.8. *Let $s \in {}^{\leq k}\omega$ with $0 < \ell\mathrm{h}(s) = n+1$. At least one of $P^s(S)$ and $P^{s \restriction n}(S)$ holds.*

Proof of Lemma. We prove the lemma by induction on $k - n$, simultaneously for all s and S.

Suppose for definiteness that n is even. (The case that n is odd is slightly simpler, since $n+1 = k$ is impossible.) Assume that $P^s(S)$ fails. We will define a quasistrategy U for II, and also $\mathcal{D} \subseteq S$ and $\langle R^d \mid d \in \mathcal{D} \rangle$. Simultaneously we will define the notion of a position $q \in U$ *marking stage* j, for $j \in \omega$. For any play $x \in \lceil U \rceil$, the set of j such that some $q \subseteq x$ marks stage j will be a (not necessarily proper) initial segment of ω, and, whenever $q \subsetneq x$ marks stage j and $q' \subseteq x$ marks stage j', we will have $q \subsetneq q' \leftrightarrow j < j'$.

The initial position $\emptyset$ marks stage 0. By induction, $P^{s ^\frown \langle 0 \rangle}(S)$ holds if $n+1 < k$; let $W^\emptyset$ be a quasistrategy for II strongly witnessing this. If $n+1 = k$, let $W^\emptyset$ be a quasistrategy for II in S such that $P^s(W^\emptyset)$ fails everywhere.

Assume inductively that $q \in U$ marks stage j and that q belongs to a quasistrategy W^q for II in S_q such that $P^s(W^q)$ fails everywhere and such that W^q strongly witnesses $P^{s ^\frown \langle j \rangle}(S_q)$ if $n+1 < k$.

Assume first that $G(A_{k-n-1,s(n),j}; W^q)$ is a win for I. Then $q \in \mathcal{D}$. Let $\hat{R}^q$ be I's non-losing quasistrategy for for $G(A_{k-n-1,s(n),j}; W^q)$. Let $R^q \cap W^q = \hat{R}^q$ and, for $p \in S_q \setminus W^q$, let $R^q_p = S_p$. Let U^q witness $P^{s\restriction n}(\hat{R}^q)$. ($U^q$ exists since $\lceil \hat{R}^q \rceil \subseteq A_{k-n-1,s(n),j} \subseteq A_{k-n-1,s(n)}$, and so the non-existence of U^q would imply $P^s(W^q)$, whereas $P^s(W^q)$ fails everywhere.) We let U agree with U^q on $\hat{R}^q$. No $p \in \hat{R}^q$ with $q \subsetneq p$ belongs to $\mathcal{D}$ or marks any stage.

Suppose that either $\hat{R}^q$ exists and $p \supseteq q$ is a first position in U_q not belonging to $\hat{R}^q$ or else $p = q$ and $\hat{R}^q$ does not exist (i.e., $G(A_{k-n-1,s(n),j}; W^q)$ is a win for II). Let U agree with a winning strategy τ_p for II for $G(A_{k-n-1,s(n),j}; W^q)$ until a position $q' \supseteq p$ is first reached with $\lceil W^q_{q'} \rceil \cap A_{k-n-1,s(n),j} = \emptyset$ and $q \supsetneq p$ if p is not terminal. No q^* with $p \subseteq q^* \subsetneq q'$ belongs to $\mathcal{D}$ or marks any stage. The position q' marks stage $j+1$. $P^s(W^q_{q'})$ fails everywhere, because $P^s(W^q)$ fails everywhere. If $n = k$, let $W^{q'} = W^q_{q'}$. If $n < k$ then, by induction, $P^{s^\frown\langle j+1\rangle}(W^q_{q'})$ holds; let $W^{q'}$ strongly witness this. The position q' marks stage $j+1$. Note that $W^{q'}$ strongly witnesses $P^{s^\frown\langle j+1\rangle}(S_{q'})$ if $n < k$, as required.

This completes the definition of U.

Suppose first that $n > 0$. We will show, using $\mathcal{D}$ and $\langle R^d \mid d \in \mathcal{D}\rangle$, that U locally witnesses $P^{s\restriction n}(S)$. By Lemma 1.4.5, this will show that U witnesses $P^{s\restriction n}(S)$.

Since $U_d \cap R^d = U_d \cap \hat{R}^d = U^d$ for $d \in \mathcal{D}$, condition (i) holds. For (ii), suppose that x is a play in U such that $x \notin \bigcup_{d\in\mathcal{D}} \lceil R^d \rceil$. From the definition it follows that we have either

$$\emptyset = q_0 \subsetneq q_1 \subsetneq q_2 \subsetneq \cdots \subseteq x$$

or

$$\emptyset = q_0 \subsetneq \cdots \subsetneq q_k = q_{k+1} = \cdots = x$$

such that each q_j marks stage j. From the definition we also get that

$$j < j' \rightarrow W^{q_j} \supseteq W^{q_{j'}}.$$

Hence $x \in \bigcap_{j\in\omega} \lceil W^{q_j} \rceil$. Since W^{q_j} witnesses $P^{s^\frown\langle j\rangle}(S_{q_j})$ if $n+1 < k$, it follows in that case that

$$\begin{aligned} x \in \bigcap_{j\in\omega} (\neg A \cup A_{k-n-2,j}) &= \neg A \cup \bigcap_{j\in\omega} A_{k-n-2,j} \\ &= \neg A \cup A_{k-n-2}. \end{aligned}$$

From the definition we also get that, for each j,

$$\lceil W^{q_{j+1}} \rceil \cap A_{k-n-1,s(n),j} \;=\; \emptyset.$$

Hence $x \notin \bigcup_{j\in\omega} A_{k-n-1,s(n),j} = A_{k-n-1,s(n)}$. Since $A_{k-n-1,s(n)} \supseteq A_{k-n-1}$, we have that $x \notin A_{k-n-1}$. If $n+1=k$, this means that $x \notin A_0$, and so $x \notin A$. If $n+1<k$, then

$$x \in (\neg A \cup A_{k-n-2}) \setminus A_{k-n-1}.$$

By our assumption that the A_m are monotonely decreasing with m,

$$x \in (\neg A \cup \bigcap_{m<k-n-1} A_m) \setminus A_{k-n-1}.$$

Since $k-n-1$ is even, $x \in \neg A$, as required by (ii). It is easy to see that (iii) holds.

Now suppose that $n=0$. The argument for (ii) in the case $n>0$ still works, so $\lceil U \rceil \setminus \bigcup_{d\in\mathcal{D}} \lceil R^d \rceil \supseteq \neg A$. Since $U_d \cap R^d = U^d$ and $\lceil U^d \rceil \subseteq \neg A$, we have that $\bigcup_{d\in\mathcal{D}} \lceil R^d \rceil \subseteq \neg A$. Thus U is a winning quasistrategy for $G(S;A)$, and so $P^{\emptyset}(S)$ holds. □

We can now prove the theorem. Assume that $G(A;T)$ is not a win for II. This means that $P^{\emptyset}(T)$ fails. By Lemma 1.4.6, let $W^{\emptyset}$ be a quasistrategy for I in T such that $P^{\emptyset}(W^{\emptyset})$ fails everywhere.

We define a quasistrategy U for I in $W^{\emptyset}$. Assume inductively that we have defined $\{p \mid p \in U \wedge \ell\mathrm{h}(p) \leq j\}$. Let $p \in U$ with $\ell\mathrm{h}(p) = j$. Assume inductively also that $p \in W^p$, where W^p is a quasistrategy for I in $W^{\emptyset}_p$ such that $P^{\emptyset}(W^p)$ fails everywhere. For each $q \supsetneq p$ with $\ell\mathrm{h}(q) = j+1$, let $q \in U \leftrightarrow q \in W^p$. By Lemma 1.4.8, $P^{\langle j\rangle}(W^p_q)$ holds for all such q. Let W^q be a quasistrategy for I in W^p_q strongly witnessing $P^{\langle j\rangle}(W^p_q)$.

We show that every play $x \in \lceil U \rceil$ belongs to A, and so that U is a winning quasistrategy for I for $G(A;T)$. Let $x \in \lceil U \rceil$. If x is finite, the fact that $x \in \lceil W^{\emptyset} \rceil$ implies that $x \in A$. Assume then that x is infinite. Since $x \in \lceil W^{x\restriction j+1} \rceil$ for each $j \in \omega$, it follows that

$$x \in A \cup \bigcap_{j\in\omega} A_{k-1,j} \;=\; A \cup A_{k-1} \;=\; A.$$

The last equality holds because k is odd and the A_n are monotonely decreasing with n. □

To get sharper form of Theorem 1.4.4, we need to pay attention to what fragment of the Comprehension schema is used for a given k. As we remarked on page 19, ZC^-+ Σ_1 Replacement is the same as KPC + Comprehension. Similarly Z^- + Σ_1 Replacement is the same as KP + Comprehension. If, e.g., we restrict Comprehension to Π_n formulas for $n \leq k$, then we get the theory KP + Π_k Comprehension.

Theorem 1.4.9. ([Montalban and Shore, 2012]) *For each finite k, KP + Π_{k+2} Comprehension $\vdash$ "All k-$\mathbf{\Pi}^0_3$ games are in countable trees are determined."*

Proof. See [Montalban and Shore, 2012]. The proof is basically like that of Theorem 1.4.4, with careful attention to how much Comprehension the steps of that proof use. In addition to Comprehension, the proof of Theorem 1.4.4, also uses Σ_{k+2} Dependent Choice. To justify this, Montalban and Shore use L and absoluteness to prove that Σ_{k+2} DC is conservative over KP + Π_{k+2} Comprehension for formulas that are Π_4 over the reals. $\square$

In the case of uncountable trees, the most useful version of their theorem is probably one that adds "+ $V = L$" to "KP + Π_{k+2} Comprehension." The author has not checked whether adding this hypothesis to the hypotheses of Theorem 1.4.9 and dropping the word "countable" yields a theorem (without increasing "$k + 2$)."

Theorem 1.4.10. *All* $\mathbf{Diff}(\mathbf{\Pi}^0_3)$ *games are determined.*

Proof. Let $A \subseteq \lceil T \rceil$ with $A \in \mathbf{Diff}(\mathbf{\Pi}^0_3)$. Let $\langle A_\alpha \mid \alpha < \gamma \rangle$ witness that $A \in \mathbf{Diff}(\mathbf{\Pi}^0_3)$. Without loss of generality, we assume that γ is odd. Also without loss of generality, we assume that

$$\alpha \leq \beta < \gamma \rightarrow A_\alpha \supseteq A_\beta.$$

For each $\alpha < \gamma$, let $A_{\alpha,i}$, $i \in \omega$, be $\mathbf{\Sigma}^0_2$ sets such that $A_\alpha = \bigcap_{i \in \omega} A_{\alpha,i}$. For each $\alpha < \gamma$ and each $i \in \omega$, let $A_{\alpha,i,j}$, $j \in \omega$, be closed sets such that $A_{\alpha,i} = \bigcup_{j \in \omega} A_{\alpha,i,j}$.

Let $\mathcal{Q}$ be the set of all pairs $\langle r, s \rangle$ such that

(i) $r \in {}^{<\omega}\gamma$;

(ii) $s \in {}^{<\omega}\omega$;

(iii) $\ell\mathrm{h}(r) = \ell\mathrm{h}(s)$;

(iv) $(\forall i < \ell\mathrm{h}(r))(i \text{ even} \leftrightarrow r(i) \text{ even})$.

(v) $i < j < \ell\mathrm{h}(r) \to r(i) > r(j)$.

For $\langle r, s\rangle \in \mathcal{Q}$ and for game subtrees S of T, we define, by induction on $\ell\mathrm{h}(r)$ the assertion $P^{r,s}(S)$:

(1) $P^{\emptyset,\emptyset}(S)$ holds if and only if $G(A; S)$ is a win for II;

(2) If $\ell\mathrm{h}(r) = n + 1$ and n is even, then $P^{r,s}(S)$ holds if and only if there is a quasistrategy U for I in S such that

  (a) $\lceil U\rceil \subseteq A \cup A_{r(n),s(n)}$;
  (b) $P^{r\restriction n,s\restriction n}(U)$ fails;

(3) If $\ell\mathrm{h}(r) = n + 1$ and n is odd, then $P^{r,s}(S)$ holds if and only if there is a quasistrategy U for II in S such that

  (a) $\lceil U\rceil \subseteq \neg A \cup A_{r(n),s(n)}$;
  (b) $P^{r\restriction n,s\restriction n}(U)$ fails.

For $\langle r, s\rangle \in \mathcal{Q}$ and $\ell\mathrm{h}(r) > 0$, U *witnesses* $P^{r,s}(S)$ if the obvious conditions hold. For $\ell\mathrm{h}(r) = 0$, U *witnesses* $P^{r,s}(S)$ if U is (the quasistrategy corresponding to) a winning strategy for II for $G(A; S)$.

For $\langle r, s\rangle \in \mathcal{Q}$ and $\ell\mathrm{h}(r) = n + 1$, we say that a quasistrategy U (for I if n is even and for II if n is odd) *locally witnesses* $P^{r,s}(S)$ if there is a subset $\mathcal{D}$ of S and there are for each $d \in \mathcal{D}$ quasistrategies R^d in S_d, for II if n is even and for I if n is odd, such that

(i) for each $d \in \mathcal{D} \cap U$, $U_d \cap R^d$ witnesses $P^{r,s}(R^d)$;

(ii) $\lceil U\rceil \setminus \bigcup_{d\in\mathcal{D}}\lceil R^d\rceil \subseteq \begin{cases} A & \text{if } n \text{ is even;} \\ \neg A & \text{if } n \text{ is odd;} \end{cases}$

(iii) for each $p \in S$, there is at most one $d \in \mathcal{D}$ such that $d \subseteq p$ and $p \in R^d$.

As in the special case occurring in the proof of Theorem 1.4.4, if U witnesses $P^{r,s}(S)$, then U locally witnesses $P^{r,s}$: Let $\mathcal{D} = \{\emptyset\}$ and let $R^d = S$. The next lemma, the analogue of Lemma 1.4.5, is the converse.

Lemma 1.4.11. *Let $\langle r, s\rangle \in \mathcal{Q}$ with $0 < \ell\mathrm{h}(r) = n + 1$. Assume that U locally witnesses $P^{r,s}(S)$. Then U witnesses $P^{r,s}(S)$.*

Proof of Lemma. We prove the lemma by induction on n (actually on odd and even n separately).

Note first that U cannot fail to have property (a) (i.e. (2)(a) or (3)(a), whichever is appropriate). To see this, assume for definiteness that n is even and let $x \in \lceil U \rceil$. By (ii), we may assume that $x \in \lceil R^d \rceil$ for some $d \in \mathcal{D}$. But then (i) implies that $x \in A \cup A_{r(n),s(n)}$.

We now turn to property (b), for which we really need induction.

Suppose first that $n = 0$. Assume for a contradiction that (b) fails. Since $n = 0$, let τ be a winning strategy for II for $G(A; U)$.

We show that there is a $d \in \mathcal{D}$ consistent with τ such that if $x \supseteq d$ is a play consistent with τ then x belongs to $\lceil R^d \rceil$. Assume this is false. Then for each $d \in \mathcal{D}$ such that d is consistent with τ, let $f(d) \supsetneq d$, $f(d)$ consistent with τ, $f(d) \notin R^d$, and $(\forall q)(d \subseteq q \subsetneq f(d) \rightarrow q \in R^d)$. By (iii) there are no members d and d' of $\mathcal{D}$ that are consistent with τ and such that $d \subsetneq d' \subseteq f(d)$. It follows that there is a play x consistent with τ such that $f(d) \subseteq x$ whenever $d \subseteq x$. Clearly x cannot belong to $\bigcup_{d \in \mathcal{D}} \lceil R^d \rceil$. But (ii) gives the contradiction that $x \in A$.

Let then d be consistent with τ such that x belongs to $\lceil R^d \rceil$ for every play $x \supseteq d$ such that x is consistent with τ. Then the obvious restriction of τ is a winning strategy for II for $G(A; U_d \cap R^d)$. Hence $P^{\emptyset,\emptyset}(U_d \cap R^d)$, contradicting (i).

Next suppose that $n > 0$ is even. Assume for a contradiction that (b) fails. Let S' witness $P^{r \restriction n, s \restriction n}(U)$. We define $\mathcal{D}' \subseteq S'$ as follows:

$$d \in \mathcal{D}' \leftrightarrow \begin{cases} d \in S' \wedge \\ d \in \mathcal{D} \wedge \\ G(\neg \lceil R^d \rceil; S'_d) \text{ is a win for II.} \end{cases}$$

For $d \in \mathcal{D}'$, let R'^d be II's non-losing quasistrategy for $G(\neg \lceil R^d \rceil; S'_d)$. Note that $R'^d \subseteq R^d$.

Let $d \in \mathcal{D}'$. Since R'^d is a quasistrategy for II in S'_d and S'_d is a quasistrategy for II in U_d, it follows that R'^d is a quasistrategy for II in U_d. Since $\lceil R'^d \rceil \subseteq \lceil S' \rceil \subseteq \neg A \cup A_{r(n-1),s(n-1)}$, condition (3)(a) holds for R'^d. By (i), R'^d cannot witness $P^{r \restriction n, s \restriction n}(U_d)$, so (3)(b) must fail for R'^d. Let then U'^d witness $P^{r \restriction n-1, s \restriction n-1}(R'^d)$.

We define a quasistrategy U' for I in S' as follows:

(1) If $p \in U'$ and there is no $d \in \mathcal{D}$ such that $d \subseteq p$ and $p \in R^d$, then let any move legal in S' at p be legal in U' at p.

(2) For each $q \in S'$ and $d \in \mathcal{D}$ such $d \subseteq q$, such that $q \in R^d \setminus R'^d$ (taking $R'^d = \emptyset$ for $d \notin \mathcal{D}'$), and such that every $q' \subsetneq q$ belongs to R'^d, let σ_q be a winning strategy for I for $G(\neg\lceil R^d \rceil; S'_d)$. Whenever such a q belongs to U', we let U'_q agree with σ_q until a position $p \notin R^d$ is reached.

(3) For $d \in \mathcal{D}' \cap U'$, let $U'_d \cap R'^d = U'^d$.

Using $\mathcal{D}'$ and $\langle R'^d \mid d \in \mathcal{D}' \rangle$, we now show that U' locally witnesses $P^{r\restriction n-1, s\restriction n-1}(S')$. Induction will then give that U' witnesses $P^{r\restriction n-1, s\restriction n-1}(S')$, contradicting property (3)(b) of S'. Property (i) follows from clause (3) in the definition of U' and the fact that U'^d witnesses $P^{r\restriction n-1, s\restriction n-1}(R'^d)$. For (ii), note first that clause (2) in the definition of U' guarantees that, for each $d \in \mathcal{D}$,

$$\lceil U' \rceil \cap \lceil R^d \rceil \subseteq \lceil R'^d \rceil.$$

Thus $\lceil U' \rceil \setminus \bigcup_{d \in \mathcal{D}'} \lceil R'^d \rceil = \lceil U' \rceil \setminus \bigcup_{d \in \mathcal{D}} \lceil R^d \rceil \subseteq A$. (iii) follows from the facts that $\mathcal{D}' \subseteq \mathcal{D}$ and that $(\forall d \in \mathcal{D}')\, R'^d \subseteq R^d$.

For the remaining case, that of an odd $n > 0$, we make the same definitions as for the case of even $n > 0$, except that we exchange I and II, A and $\neg A$, and $G(\neg\lceil R^d \rceil; S'_d)$ and $G(\lceil R^d \rceil; S'_d)$. The argument is exactly the same, except for a minor change in the case $n = 1$: In that case, the U'^d are the quasistrategies corresponding to winning strategies for II for $G(A; R'^d)$, and we must prove that $\lceil U' \rceil \subseteq \neg A$. As before, $\lceil U' \rceil \setminus \bigcup_{d \in \mathcal{D}'} \lceil R'^d \rceil \subseteq \neg A$. Moreover for $d \in \mathcal{D}'$ we have that $\lceil U' \rceil \cap \lceil R'^d \rceil = \lceil U'^d \rceil \subseteq \neg A$. □

We say that $P^{r,s}(S)$ *fails everywhere* if $P^{r,s}(S_p)$ fails for every $p \in S$.

Lemma 1.4.12. *Let $\langle r, s \rangle \in \mathcal{Q}$ and let $m = \ell\mathrm{h}(r)$. If $P^{r,s}(S)$ fails, then there is a quasistrategy W in S for I if m is even and for II if m is odd such that $P^{r,s}(W)$ fails everywhere.*

Proof of Lemma. The case $m = 0$ is Lemma 1.2.7, so assume $m = n + 1$. Suppose for definiteness that n is even; the other case is similar. Let $\mathcal{D}$ be the set of all $d \in S$ such that $P^{r,s}(S_d)$ but such that, for every $p \subsetneq d$, $P^{r,s}(S_p)$ fails. For each $d \in \mathcal{D}$, let U^d witness $P^{r,s}(S_d)$. Let

$$B = \{x \in \lceil S \rceil \mid (\exists d \in \mathcal{D})\, d \subseteq x\}.$$

First assume for a contradiction that the open game $G(B; S)$ is a win for I. Let σ be a winning strategy for I for $G(B; S)$. We define a quasistrategy

U for I in S as follows: U agrees with σ until a position $d \in \mathcal{D}$ is reached. Then $U_d = U^d$. For $d \in \mathcal{D}$, let $R^d = S_d$. It is easy to see, using $\mathcal{D}$ and $\langle R^d \mid d \in \mathcal{D}\rangle$, that U locally witnesses $P^{r,s}(S)$, so Lemma 1.4.11 gives the contradiction that U witnesses $P^{r,s}(S)$.

We know then that $G(B;S)$ is a win for II. Let W be II's non-losing quasistrategy. Assume for a contradiction that $q \in W$ and that U^* witnesses $P^{r,s}(W_q)$. Let U be a quasistrategy for I in S_q defined as follows: Let $U \cap W_q = U^*$. When first (if ever) a position $p \notin W$ is reached, let U agree with a winning strategy σ_p for I for $G(B;S)$ until a position $d \in \mathcal{D}$ is reached. Then let $U_d = U^d$. Let $\mathcal{D}' = \mathcal{D} \cup \{q\}$. Let $R^q = W_q$ and let $R^d = S_d$ for $d \in \mathcal{D}$. It is easy to see, using $\mathcal{D}'$ and $\langle R^d \mid d \in \mathcal{D}'\rangle$, that U locally witnesses $P^{r,s}(S_q)$. Lemma 1.4.11 gives us the contradiction that some $d \subseteq q$ belongs to $\mathcal{D}$. $\square$

For $n+1 = \ell\mathrm{h}(r)$, we say that W *strongly witnesses* $P^{r,s}(S)$ if, for all $p \in W$, W_p witnesses $P^{r,s}(S_p)$, i.e. if W witnesses $P^{r,s}(S)$ and $P^{r\restriction n, s\restriction n}(W)$ fails everywhere.

Lemma 1.4.13. *Let $\langle r,s\rangle \in \mathcal{Q}$ with $0 < \ell\mathrm{h}(r) = n+1$. If $P^{r,s}(S)$, then there is a W that strongly witnesses $P^{r,s}(S)$.*

Proof of Lemma. Assume for definiteness that n is even. Let U witness that $P^{r,s}(S)$. By property (2)(b) of U, $P^{r\restriction n, s\restriction n}(U)$ fails. By Lemma 1.4.12, let W be a quasistrategy for I in U such that $P^{r\restriction n, s\restriction n}(W)$ fails everywhere. Since W is a quasistrategy for I in S and W inherits property (2)(a) from U, it follows that W strongly witnesses $P^{r,s}(S)$. $\square$

Lemma 1.4.14. *Let $\langle r,s\rangle \in \mathcal{Q}$ with $0 < \ell\mathrm{h}(r) = n+1$. At least one of $P^{r,s}(S)$ and $P^{r\restriction n, s\restriction n}(S)$ holds.*

Proof of Lemma. We prove the lemma by induction on $r(n)$, simultaneously for all n, r, s, and S. (Recall that r is a strictly decreasing sequence of ordinals.)

Suppose for definiteness that n is even. Assume that $P^{r,s}(S)$ fails. We will define a quasistrategy U for II, and also $\mathcal{D} \subseteq S$ and $\langle R^d \mid d \in \mathcal{D}\rangle$. Simultaneously we will define the notion of a position $q \in U$ *marking stage* j, for $j \in \omega$. For any play $x \in \lceil U \rceil$, the set of j such that some $q \subseteq x$ marks stage j will be a (not necessarily proper) initial segment of ω, and, whenever $q \subsetneq x$ marks stage j and $q' \subseteq x$ marks stage j', we will have $q \subsetneq q' \leftrightarrow j < j'$.

If $r(n) > 0$, let $\langle \beta_j, m_j \rangle$, $j \in \omega$, be an enumeration of all pairs $\langle \beta, m \rangle$ with β odd, $\beta < r(n)$, and $m \in \omega$.

The initial position $\emptyset$ marks stage 0. By induction, $P^{r^\frown \langle \beta_0 \rangle, s^\frown \langle m_0 \rangle}(S)$ holds if $r(n) > 0$; let $W^\emptyset$ be a quasistrategy for II strongly witnessing this. If $r(n) = 0$, let $W^\emptyset$ be a quasistrategy for II in S such that $P^{r,s}(W^\emptyset)$ fails everywhere.

Assume inductively that $q \in U$ marks stage j and that q belongs to a quasistrategy W^q for II in S_q such that $P^{r,s}(W^q)$ fails everywhere and such that W^q strongly witnesses $P^{r^\frown \langle \beta_j \rangle, s^\frown \langle m_j \rangle}(S_q)$ if $r(n) > 0$.

Assume first that $G(A_{r(n),s(n),j}; W^q)$ is a win for I. Then $q \in \mathcal{D}$. Let $\hat{R}^q$ be I's non-losing quasistrategy for $G(A_{r(n),s(n),j}; W^q)$. Let $R^q \cap W^q = \hat{R}^q$ and, for $p \in S_q \setminus W^q$, let $R^q_p = S_p$. Let U^q witness $P^{r\restriction n, s\restriction n}(\hat{R}^q)$. ($U^q$ exists since $\lceil \hat{R}^q \rceil \subseteq A_{r(n),s(n),j} \subseteq A_{r(n),s(n)}$, and so the non-existence of U^q would imply $P^{r,s}(W^q)$, whereas $P^{r,s}(W^q)$ fails everywhere.) We let U agree with U^q on $\hat{R}^q$. No $p \in \hat{R}^q$ with $q \subsetneq p$ belongs to $\mathcal{D}$ or marks any stage.

Suppose that either $\hat{R}^q$ exists and $p \supseteq q$ is some first position in U_q not belonging to $\hat{R}^q$ or else $p = q$ and $\hat{R}^q$ does not exist (i.e., $G(A_{r(n),s(n),j}; W^q)$ is a win for II). Let U agree with a winning strategy τ_p for II for $G(A_{r(n),s(n),j}; W^q)$ until a position $q' \supseteq p$ is first reached with $\lceil W^q_{q'} \rceil \cap A_{r(n),s(n),j} = \emptyset$ and $q \supsetneq p$ if p is not terminal. No q^* with $p \subseteq q^* \subsetneq q'$ belongs to $\mathcal{D}$ or marks any stage. The position q' marks stage $j+1$. $P^{r,s}(W^q_{q'})$ fails everywhere, because $P^{r,s}(W^q)$ fails everywhere. If $r(n) = 0$, let $W^{q'} = W^q_{q'}$. If $r(n) > 0$ then, by induction, $P^{r^\frown \langle \beta_{j+1} \rangle, s^\frown \langle m_{j+1} \rangle}(W^q_{q'})$ holds; let $W^{q'}$ strongly witness this. The position q' marks stage $j+1$. Note that $W^{q'}$ strongly witnesses $P^{r^\frown \langle \beta_{j+1} \rangle, s^\frown \langle m_{j+1} \rangle}(S_{q'})$ if $r(n) > 0$, as required.

This completes the definition of U.

Suppose first that $n > 0$. We will show, using $\mathcal{D}$ and $\langle R^d \mid d \in \mathcal{D} \rangle$, that U locally witnesses $P^{r\restriction n, s\restriction n}(S)$. By Lemma 1.4.11, this will show that U witnesses $P^{r\restriction n, s\restriction n}(S)$.

Since $U_d \cap R^d = U_d \cap \hat{R}^d = U^d$ for $d \in \mathcal{D}$, condition (i) holds. For (ii), suppose that x is a play in U such that $x \notin \bigcup_{d \in \mathcal{D}} \lceil R^d \rceil$. From the definition it follows that we have either

$$\emptyset = q_0 \subsetneq q_1 \subsetneq q_2 \subsetneq \cdots \subseteq x$$

or

$$\emptyset = q_0 \subsetneq \cdots \subsetneq q_k = q_{k+1} = \cdots = x$$

such that each q_j marks stage j. From the definition we also get that

$$j < j' \to W^{q_j} \supseteq W^{q_{j'}}.$$

Hence $x \in \bigcap_{j\in\omega}\lceil W^{q_j}\rceil$. Since W^{q_j} witnesses $P^{r^\frown\langle\beta_j\rangle, s^\frown\langle m_j\rangle}(S_{q_j})$ if $r(n) > 0$, it follows that

$$\begin{aligned} x \in \bigcap_{j\in\omega}(\neg A \cup A_{\beta_j,m_j}) &= \neg A \cup \bigcap_{\substack{\beta<r(n)\\ \beta \text{ odd}}} \bigcap_{m\in\omega} A_{\beta,m} \\ &= \neg A \cup \bigcap_{\substack{\beta<r(n)\\ \beta \text{ odd}}} A_\beta. \end{aligned}$$

From the definition we also get that, for each j,

$$\lceil W^{q_{j+1}}\rceil \cap A_{r(n),s(n),j} = \emptyset.$$

Hence $x \notin \bigcup_{j\in\omega} A_{r(n),s(n),j} = A_{r(n),s(n)}$. Since $A_{r(n),s(n)} \supseteq A_{r(n)}$, we have that $x \notin A_{r(n)}$. Thus we have that

$$x \in (\neg A \cup \bigcap_{\substack{\beta<r(n)\\ \beta \text{ odd}}} A_\beta) \setminus A_{r(n)}.$$

Since $r(n)$ is even, $x \in \neg A$, as required by (ii). It is easy to see that (iii) holds.

Now suppose that $n = 0$. A simplification of the argument for (ii) in the case $n > 0$ still works, so $\lceil U\rceil \setminus \bigcup_{d\in\mathcal{D}}\lceil R^d\rceil \supseteq \neg A$. Since $U_d \cap R^d = U^d$ and $\lceil U^d\rceil \subseteq \neg A$, we have that $\bigcup_{d\in\mathcal{D}}\lceil R^d\rceil \subseteq \neg A$. Thus U is a winning quasistrategy for $G(S; A)$, and so $P^{\emptyset,\emptyset}(S)$ holds. □

We can now prove the theorem. Assume that $G(A; T)$ is not a win for II. This means that $P^{\emptyset,\emptyset}(T)$ fails. By Lemma 1.4.12, let $W^\emptyset$ be a quasistrategy for I in T such that $P^{\emptyset,\emptyset}(W^\emptyset)$ fails everywhere.

Let $\langle\beta_j, m_j\rangle$, $m \in \omega$, be an enumeration of all pairs $\langle\beta, m\rangle$ with β even, $\beta < \gamma$, and $m \in \omega$.

We define a quasistrategy U for I in $W^\emptyset$. Assume inductively that we have defined $\{p \mid p \in U \wedge \ell\mathrm{h}(p) \leq j\}$. Let $p \in U$ with $\ell\mathrm{h}(p) = j$. Assume inductively also that $p \in W^p$, where W^p is a quasistrategy for I in $W^\emptyset_p$ such that $P^{\emptyset,\emptyset}(W^p)$ fails everywhere. For each $q \supsetneq p$ with $\ell\text{h}(q) = j+1$, let $q \in U \leftrightarrow q \in W^p$. By Lemma 1.4.14, $P^{\langle\beta_j\rangle,\langle m_j\rangle}(W^p_q)$ holds for all such q. Let W^q be a quasistrategy for I in W^p_q strongly witnessing $P^{\langle\beta_j\rangle,\langle m_j\rangle}(W^p_q)$.

We show that every play $x \in \lceil U \rceil$ belongs to A, and so that U is a winning quasistrategy for I for $G(A;T)$. Let $x \in \lceil U \rceil$. If x is finite, the fact that $x \in \lceil W^\emptyset \rceil$ implies that $x \in A$. Assume then that x is infinite. Since $x \in \lceil W^{x \restriction j+1} \rceil$ for each $j \in \omega$, it follows that

$$x \in A \cup \bigcap_{\substack{\beta<\gamma \\ \beta \text{ even}}} \bigcap_{i\in\omega} A_{\beta,i} \;=\; A \cup \bigcap_{\substack{\beta<\gamma \\ \beta \text{ even}}} A_\beta \;=\; A.$$

The last equality holds because γ is odd. □

Corollary 1.4.15. *All $\mathbf{\Delta}^0_4$ games in countable trees are determined.*

Proof. The corollary is a direct consequence of Theorems 1.4.2 and 1.4.10. □

Montalban and Shore demonstrate in [Montalban and Shore, 2012] the unprovability in ZC$^-$+ Σ_1 Replacement$^-$—indeed, in ZFC$^-$—of the assertion that all games in ${}^{<\omega}\omega$ that are k-$\mathbf{\Pi}^0_3$ for some $k \in \omega$ are determined. (Theorem 1.4.4 says only that k-$\mathbf{\Pi}^0_3$ determinacy is provable in ZC$^-$+ Σ_1 Replacement for each fixed $k \in \omega$.) Hence neither Theorem 1.4.10 nor Corollary 1.4.15 is provable in ZC$^-$+ Σ_1 Replacement.

As we will demonstrate, our proof of Theorem 1.4.10 does show that every every wellfounded model of ZC$^-$+ Σ_1 Replacement satisfies "All $\mathbf{Diff}(\mathbf{\Pi}^0_3)$ games are determined." A model $(M;E)$ for the language of set theory is *wellfounded* if E is a wellfounded relation. (See page 24 for the definition of wellfoundedness for relations.) By a theorem of Mostowski, a model for the language of set theory that satisfies Extensionality is wellfounded just in case it is isomorphic to a *transitive model*, a model $(M;\in)$ with M a transitive set.

Our proof of Theorem 1.4.10 also shows that γ-$\mathbf{\Pi}^0_3$ determinacy holds for a certain infinite γ in all ω-models of ZC$^-$+ Σ_1 Replacement. Let us say that a model $(M;E)$ for the language of set theory is an *ω-model* if $\omega \cup \{\omega\} \subseteq M$ and

$$(\forall x \in M)(\forall y \in \omega \cup \{\omega\})(xEy \leftrightarrow x \in y).$$

If $(M; E)$ satisfies a sufficient small fragment of ZF, then $(M; E)$ is an ω-model just in case ω is the ω of $(M; E)$ and the natural numbers are the natural numbers of $(M; E)$.

To prove the facts just mentioned, we need two more definitions. Let $(M; E)$ be a model for the language of set theory. If $x \in M$, then the *transitive closure of x under E* is the smallest set w such that $x \in w$ and such that $yEz \in w \to y \in w$. The *wellfounded part of* $(M; E)$ is the set of all $x \in M$ such that E restricted to the transitive closure of x under E is a wellfounded relation. It is not hard to see that the restriction of E to the wellfounded part of $(M; E)$ is a well-founded relation.

If $(M; E)$ satisfies Extensionality and a small fragment of the Comprehension Schema, then the ordinals of $(M; E)$—i.e., those $x \in M$ such that $(M; E) \models$ "x is an ordinal"—are linearly ordered by E. Thus an ordinal x of $(M; E)$ belongs to the wellfounded part of $(M; E)$ if and only if E wellorders $\{y \mid yEx\}$.

Theorem 1.4.16. *Let $(M; E)$ be an ω-model of ZC$^-$+ Σ_1 Replacement. Assume that T is, in $(M; E)$, a game tree. Assume that T belongs to the wellfounded part of $(M; E)$. Assume that γ is an ordinal of $(M; E)$ and that γ belongs to the wellfounded part of $(M; E)$. Then*

$$(M; E) \models \text{“All } \gamma\text{-}\mathbf{\Pi}^0_3 \text{ games are determined.”}$$

Proof. By Mostowski's theorem, we may assume that the wellfounded part of $(M; E)$ is a transitive set on which E agrees with membership. This implies, in particular, that γ is an ordinal number and that T is a game tree.

Let A be, in $(M; E)$, a γ-$\mathbf{\Pi}^0_3$ subset of $\lceil T \rceil$. Working in $(M; E)$, introduce $\langle A_\alpha \mid \alpha < \gamma\rangle$, $\langle A_{\alpha,n} \mid \alpha < \gamma \wedge n \in \omega\rangle$, and $\mathcal{Q}$ as in the proof of Theorem 1.4.10.

For each $n \in \omega$, define $P^{r,s}_n(S)$ for $S \subseteq T$, $\langle r, s\rangle \in \mathcal{Q}$ and $\ell\text{h}(r) = n$ as $P^{r,s}(S)$ was defined for such objects in the proof of Theorem 1.4.10. The theory ZC$^-$+ Σ_1 Replacement does not allow us to define, by a formula of the language of set theory, the class relation $P^{r,s}(S)$. What we do instead is use the inductive definition of the proof of Theorem 1.4.10 to define, for each $n \in \omega$, a three-place relation $P^{r,s}_n(S)$ for each n. P_n is thus the definable restriction of the undefinable P to triples $\langle S, r, s\rangle$ with $\ell\text{h}(r) = n$.

Working in $(M; E)$ with ZC$^-$+ Σ_1 Replacement we cannot state Lemma 1.4.11. But we can replace it with individual statements for each n, and our inductive proof of the Lemma shows that each of these individual statements is true in $(M; E)$.

The statement of Lemma 1.4.12 also needs to be replaced by individual statements for each n. The proof of Lemma 1.4.12 directly gives, for each n, a proof of the statement for n from the Lemma 1.4.11 replacement statement for n. Similarly the Lemma 1.4.13 replacement statement for n is proved from the Lemma 1.4.12 replacement for n.

Finally, we get replacement statements for Lemma 1.4.14. The nth of these asserts that at least one of $P^{r,s}_{n+1}(S)$ and $P^{r\restriction n, \restriction n}_{n}(S)$ holds for any S and any r and s of length n. Assume that one of these assertions is false in $(M; E)$. Let S, r, and s have the least value of $r(\ell\text{h}(r))$ attained by a triple yielding a statement false in $(M; E)$. There is such a least value, since all values $r(i)$ are ordinals of $\mathfrak{M}$ that are $< \gamma$. The proof of Lemma 1.4.14 shows that if $\ell\text{h}(r) = n + 1$ and $P^{r,s}(S)$ nor $P^{r\restriction n, s\restriction n}(S)$ holds, then there some $P^{r^\frown\langle\beta\rangle, s^\frown\langle m\rangle}(S')$ that does not hold and has $\beta < r(n)\rangle$. Thus that proof, applied to $(M; E)$, contradicts the assumed minimality of $r(\ell\text{h}(r))$.

In the proof of Theorem 1.4.10, the determinacy of $G(A; T)$ is proved using only the $n = 0$ case of Lemma 1.4.14. □

Corollary 1.4.17. *Every wellfounded model of ZC⁻+ Σ_1 Replacement satisfies "All Δ^0_4 games in countable trees are determined."*

Corollary 1.4.18. *If let $(M; E)$ be an ω-model of ZC⁻+ Σ_1 Replacement. Assume that the wellfounded part of $(M; E)$ is a transitive set and that E agrees with $\in$ on the wellfounded part of $(M; E)$. Let γ be an ordinal such that $\in\restriction \gamma$ is isomorphic to a recursive wellordering of ω. Then γ belongs to the wellfounded part of $(M; E)$ and*

$$(M; E) \models \text{"All } \gamma\text{-}\mathbf{\Pi}^0_3 \text{ games are determined."}$$

Proof. All recursive relations on ω belong to M, and recursive wellorderings of ω belonging to M are wellorderings in $(M; E)$. □

Our next goal is to get an upper bound on the hypothesis needed to prove Theorem 1.4.10.

As we have indicated, the point at which our proof of Theorem 1.4.10 goes beyond the resources of ZC⁻+ Σ_1 Replacement is in the definition of $P^{r,s}(S)$. Our definition is by recursion on $\ell\text{h}(r)$, and such recursive definitions are not licensed by ZC$^-$+ Σ_1 Replacement. Our plan is to formulate a theory that does license these definitions.

Let $\mathcal{L}_R$ be the language gotten from the language of set theory by adding R as a two-place predicate symbol.

For $k \geq 1$, let $\psi(v_1, \ldots v_k)$ be a formula of the language of set theory. Let $\chi(v_0, v_1, \ldots, v_k)$ be a formula of $\mathcal{L}_R$ in which every subformula of of the form $R(x, y)$ has the form $R(v_0, y)$. Let $\mathcal{L}_{R_{\psi,\chi}}$ be the language gotten from $\mathcal{L}_R$ by replacing R by a symbol $R_{\psi,\chi}$. Let $\chi_{\psi,\chi}$ be the be the result of replacing R in χ by $R_{\psi,\chi}$. Let $\mathcal{L}_{\mathrm{rec}}$ be the the union, in the obvious sense, of the $\mathcal{L}_{R_{\psi,\chi}}$.

We define a theory Rec(ZC$^-$+ Σ_1 Replacement) in the language $\mathcal{L}_{\mathrm{rec}}$. The axioms of Rec(ZC$^-$+ Σ_1 Replacement) are the same as the axioms of ZC$^-$+ Σ_1 Replacement, with the following additions.

(1) The Comprehension and Σ_1 Replacement Schemas apply to all formulas $\varphi(x, u, w_1, \ldots, w_n)$ of the language $\mathcal{L}_{\mathrm{rec}}$.

(2) There are axioms, described below, for each $R_{\psi,\chi}$.

   (a) $R_{\psi,\chi}(x, y) \to (x \in \omega \wedge (\exists n \in \omega) y$ is an n-tuple).

   (b) $R_{\psi,\chi}(0, \langle v_1, \ldots, v_k\rangle) \leftrightarrow \psi(v_1, \ldots, v_k)$.

   (c) $(\forall n \in \omega)(R_{\psi,\chi}(n+1, \langle v_1, \ldots, v_k\rangle) \leftrightarrow \chi_{\psi,\chi}(n, v_1, \ldots, v_k)$.

The difference between ZC$^-$+ Σ_1 Replacement and Rec(ZC$^-$+ Σ_1 Replacement) is, roughly speaking, that the latter allows formulas to be defined by recursion from formulas of the language of set theory.

Theorem 1.4.19. (Rec(ZC$^-$+ Σ_1 Replacement)) *All* $\mathbf{Diff}(\mathbf{Pi}^0_3)$ *games are determined, and so all* $\mathbf{\Delta}^0_4$ *games in countable trees are determined.*

Proof. We specify formulas $\psi(v_1, v_2, v_3, v_4, v_5)$ and $\chi(v_0, v_1, v_2, v_3, v_4, v_5)$. Each of these formulas will say the following:

(a) v_2 is a countable game tree (which we will call) T.

(b) v_3 is is a $\mathbf{Diff}(\mathbf{\Pi}^0_3)$ subset A of $\lceil T\rceil$.

(c) v_4 is a function $\langle A_\alpha \mid \alpha < \gamma\rangle$ witnessing that $A \in \mathbf{Diff}(\mathbf{\Pi}^0_3)$.

(d) v_5 is a function $\langle A_{\alpha,i} \mid \alpha < \gamma \wedge i \in \omega\rangle$ witnessing that each A_α is a countable intersection of $\mathbf{\Sigma}^0_2$ sets.

The formula $\psi(v_1, v_2, v_3, v_4, v_5)$ will also say the following.

(a) v_1 is a triple $\langle\emptyset, \emptyset, S\rangle$.

(b) S is a game subtree of T.

(c) $G(A; S)$ is a win for II.

Our axioms thus imply that $R_{\psi,\chi}(0, \langle\emptyset, \emptyset, S\rangle) \leftrightarrow G(A; S)$ is a win for II.

The formula $\chi_{\psi,\chi}(n, v_1, v_2, v_3, v_4, v_5)$ will say the following.

(a) v_1 is a triple $\langle r, s, S\rangle$;

(b) $\langle r, s\rangle$ belongs to the set $\mathcal{Q}$ defined as on page 37;

(c) $\ell\mathrm{h}(r) = n + 1$.

(d) If n is even, then there is a quasistrategy U for I in S such that

  (i) $\lceil U\rceil \subseteq A \cup A_{r(n),s(n)}$;
  (ii) $R_{\psi,\chi}(\langle r \restriction n, s \restriction n, U\rangle)$ fails.

(e) If n is odd, then there is a quasistrategy U for II in S such that

  (i) $\lceil U\rceil \subseteq \neg A \cup A_{r(n),s(n)}$;
  (ii) $R_{\psi,\chi}(\langle r \restriction n, s \restriction n, U\rangle)$ fails.

Set

$$P^{r,s}(S) \leftrightarrow R_{\psi,\chi}(\ell\mathrm{h}(r), \langle r, s, S\rangle).$$

It follows by induction from our axioms that $P^{r,s}(S)$ satisfies the definition on page 38. Thus we may repeat the proof of Theorem 1.4.10. □

Richard Shore suggested—or, more accurately, pointed out—to the author that adding a satisfaction predicate to ZC$^-$+ Σ_1 Replacement does the the same thing as Rec(ZC$^-$+ Σ_1 Replacement) does. This is indeed the case, as we will explain briefly.

Fix some reasonable way of construing formulas as sets. We will define a satisfaction predicate by recursion. The defining axioms will be in the language $\mathcal{L}_{\mathrm{Sat}}$ gotten from the language of set theory by adding a three-place predicate symbol Sat. The base clause $\rho_0(x, y)$ says that $\mathrm{Sat}(0, x, y)$ if and only if

(i) x is a a formula of the form $v_i = v_j$ or $v_i \in v_j$ (i.e., is a formula of the language of set theory of length 3);

(ii) y is $\langle z_0, \ldots, z_k\rangle$ for some sets $z_0, \ldots, z_k$ with $k \geq \max(i, j)$.

(iii) either x is $v_i = v_j$ and and $z_i = z_j$ or else x is $v_i \in v_j$ and $z_i \in z_j$.

The recursion clause $\rho(x, y)$ says that, for all $n \in \omega$, $\mathrm{Sat}(n+1, x, y)$ if and only if

(a) x is a formula of the language of set theory of length $\leq n+4$;

(b) y is $\langle z_0, \ldots, z_k \rangle$ for some $z_0, \ldots, z_k$ with $k \geq \max(\{i \mid v_i \text{ is free in } x\})$;

(c) σ_n.

Here σ_n, which we leave to the reader, gives the definition of $\mathrm{Sat}(n+1, x, y)$ in terms of the relation $\mathrm{Sat}(n, -, -)$).

Let Sat(ZC$^-$+ Σ_1 Replacement) be the extension of ZC$^-$+ Σ_1 Replacement (1) with added axioms ρ_0, ρ, and $(\forall z)(\forall x)(\forall y)\mathrm{Sat}(z, x, y) \rightarrow z \in \omega$ and (2) with Comprehension and Σ_1 Replacement for all formulas of $\mathcal{L}_{\mathrm{Sat}}$.

We can define a satisfaction predicate Sat in Rec(ZC$^-$+ Σ_1 Replacement) and prove the axioms of Sat(ZC$^-$+ Σ_1 Replacement). To do this, first let $\psi(v_1)$ be

$$(\exists x)(\exists y)(v_1 = \langle x, y \rangle \ \wedge \ \text{(i)} \ \wedge \ \text{(ii)} \ \wedge \ \text{(iii)}),$$

where (i), (ii), and (iii) are the three clauses above. Next let $\chi(v_0, v_1)$ be

$$(\exists x)(\exists y)(v_1 = \langle x, y \rangle \ \wedge \ (\text{a}') \ \wedge \ (\text{b}) \ \wedge \ (\text{c}')),$$

where (a$'$) is (a) with n replaced by v_0 and (c$'$) is (c) with $\mathrm{Sat}(n, x, y)$ replaced by $R(v_0, \langle x, y \rangle)$. Finally define $\mathrm{Sat}(n, x, y)$ as $R_{\psi,\chi}(n, \langle x, y \rangle)$.

We can also define predicates $R_{\psi,\chi}$ in Sat(ZC$^-$+ Σ_1 Replacement) and prove the axioms of Rec(ZC$^-$+ Σ_1 Replacement). Given ψ and χ, we define by recursion a sequence $\langle \tau_n \mid n \in \omega \rangle$ of formulas of the language of set theory as follows.

(1) $\tau_0(v_1)$ is the formula $(\exists k \in \omega)(v_1$ is a k-tuple and $\wedge\, \psi((v_1)_1, \ldots, (v_1)_k)$. (Here $v_1 = \langle (v_1)_1, \ldots, (v_1)_k \rangle$. We assume that we have a representation of finite sequences that lets us define the $(v_1)_i$ from v_1.

(2) $\tau_{n+1}(v_1)$ is $(\exists k \in \omega)(v_1$ is a k-tuple and $\wedge\ \chi'((v_1)_1, \ldots, (v_1)_k))$, where χ' is the result of replacing each $(R(v_0, x)$ in $\chi(v_0, (v_1)_1, \ldots (v_1)_k))$ by $\tau_n(x)$.

Now we can define $R_{\psi,\chi}(v_0, v_1)$ as $v_0 \in \omega$ and $\mathrm{Sat}(v_0, \tau_n, v_1)$.

We finish this section by extending Corollary 1.4.15 to uncountable trees. This is work of the author, done in 1990.

For classes $\mathbf{\Gamma}$ and ordinals α, we say that a set $A \subseteq \lceil T \rceil$ belongs to the class $(\alpha\text{-}\mathbf{\Gamma})^*$, the αth *level of the generalized difference hierarchy on* $\mathbf{\Gamma}$, just in case there is a sequence $\langle A_\beta \mid \beta < \alpha \rangle$ with each $A_\beta \in \mathbf{\Gamma}$ and there is a function $f : T \to \alpha$ such that both

(a) $(\forall x \in \lceil T \rceil)(x \in A \leftrightarrow \mu\beta(x \notin A_\beta \vee \beta = \alpha)$ is odd);

(b) $(\forall x \in \lceil T \rceil)(\forall \beta < \alpha)(x \notin A_\beta \to (\exists n \in \omega)\, \beta = f(x \restriction n))$.

Let $\mathbf{Diff}^*(\mathbf{\Gamma}) = \bigcup_{\alpha < |T|^+} (\alpha\text{-}\mathbf{\Gamma})^*$. Let T be a game tree. If A and B_j, $j \in J$, are subsets of $\lceil T \rceil$, then A is the *fully open-separated union* of $\{B_j \mid j \in J\}$ if

(a) $A = \bigcup_{j \in J} B_j$;

(b) there are disjoint open sets D_j, $j \in J$, such that $\bigcup_{j \in J} D_j = \lceil T \rceil$ and such that $B_j \subseteq D_j$ for each $j \in J$.

If $\{D_j \mid j \in J\}$ witnesses that A is the fully open-separated union of $\{B_j \mid j \in J\}$, then each $\neg D_j = \bigcup_{j' \in J \setminus \{j\}} D_{j'}$; hence each D_j is in fact a clopen set.

We will use the following lemma in proving that $\mathbf{Diff}^*(\mathbf{\Pi}^0_\xi) \subseteq \mathbf{\Delta}^0_{\xi+1}$.

Lemma 1.4.20. (ZC$^-$+ Σ_1 Replacement) *Let* $1 \le \xi < \omega_1$ *and let* $A \subseteq \lceil T \rceil$ *be the fully open-separated union of* $\{B_j \mid j \in J\}$. *(1) If each* $B_j \in \mathbf{\Sigma}^0_\xi$, *then* $A \in \mathbf{\Sigma}^0_\xi$. *(2) If each* $B_j \in \mathbf{\Pi}^0_\xi$, *then* $A \in \mathbf{\Pi}^0_\xi$.

Proof. We prove the lemma by induction on ξ. Assume that the lemma holds for every $\xi' < \xi$. Let $\{D_j \mid j \in J\}$ witness that A is the fully open-separated union of $\{B_j \mid j \in J\}$.

To prove (1) for ξ, assume that each $B_j \in \mathbf{\Sigma}^0_\xi$. If $\xi = 1$ then the fact that every union of open sets is open gives that $A \in \mathbf{\Sigma}^0_1$. Assume then that $\xi > 1$. If ξ is a limit ordinal, let $\eta_0 < \eta_1 < \ldots$ be such that $\sup_{k \in \omega} \eta_k = \xi$. If $\xi = \eta + 1$, then let $\eta_k = \eta$ for each $k \in \omega$. For $j \in J$ let $B_j = \bigcup_{k \in \omega} B_{j,k}$ with each $B_{j,k} \in \bigcup_{\xi' < \xi} \mathbf{\Pi}^0_{\xi'}$. Using Lemma 1.1.1 and modifying each $\langle B_{j,k} \mid k \in \omega \rangle$ by inserting $\emptyset$ where necessary, we may assume that each $B_{j,k} \in \mathbf{\Pi}^0_{\eta_k}$. For $k \in \omega$, let $A_k = \bigcup_{j \in J} B_{j,k}$. Now $\{D_j \mid j \in J\}$ witnesses that each A_k is the fully open-separated union of $\{B_{j,k} \mid j \in J\}$, so our induction hypothesis gives that each $A_k \in \mathbf{\Pi}^0_{\eta_k}$. Thus $A = \bigcup_{k \in \omega} A_k$ belongs to $\mathbf{\Sigma}^0_\xi$.

Now assume that each $B_j \in \mathbf{\Pi}^0_\xi$. For each $j \in J$, $D_j \setminus B_j \in \mathbf{\Sigma}^0_\xi$. (1) therefore gives that $\neg A = \bigcup_{j \in J}(D_j \setminus B_j) \in \mathbf{\Sigma}^0_\xi$. Hence $A \in \mathbf{\Pi}^0_\xi$, and we have proved (2) for ξ. $\square$

Theorem 1.4.21. (ZC$^-$+ Σ_1 Replacement) *For all T,* $\mathbf{Diff}^*(\mathbf{\Pi}^0_\xi) = \mathbf{\Delta}^0_{\xi+1}$.

Proof. We first show that $\mathbf{Diff}^*(\mathbf{\Pi}^0_\xi) \subseteq \mathbf{\Delta}^0_{\xi+1}$. Let $\langle A_\beta \mid \beta < \alpha\rangle$ and $f : T \to \alpha$ witness that $A \in \mathbf{Diff}^*(\mathbf{\Pi}^0_\xi)$. Assume for definiteness that α is even. It will also be convenient to assume that $(\forall p \in T)(\ell\mathrm{h}(p)$ even $\leftrightarrow f(p)$ even$)$. Since by Theorem 1.4.2 we may certainly assume that $\alpha \geq 2$, we can modify f if necessary to make this assumption hold.

For $n \in \omega$ let

$$C_n \ = \ \{x \in \lceil T\rceil \mid x \in A_{f(x\restriction n)}\}.$$

Each C_n is the fully open-separated union of

$$\{A_{f(p)} \cap \lceil T_p\rceil \mid p \in T \wedge \ell\mathrm{h}(p) = n\}.$$

By Lemma 1.4.20, each $C_n \in \mathbf{\Pi}^0_\xi$. Since α is even,

$$\begin{aligned} A \ &= \ \bigcup_{\substack{n\in\omega\\ n \text{ odd}}} (\neg C_n \cap \bigcap_{m\in\omega} (C_m \cup \{x \mid f(x\restriction m) \geq f(x\restriction n)\})) \\ &= \ \bigcup_{n\in\omega} \neg C_n \ \cap \ \bigcap_{\substack{n\in\omega\\ n \text{ even}}} \ (C_n \cup \bigcup_{m\in\omega} (\neg C_m \cap \{x \mid f(x\restriction m) < f(x\restriction n)\}))\end{aligned}$$

For each n and m, $\{x \mid f(x\restriction m) \geq f(x\restriction n)\}$ is clopen, so we can show as in the first part of the proof of Theorem 1.4.2 that $A \in \mathbf{\Delta}^0_{\xi+1}$.

Now let $A \in \mathbf{\Delta}^0_{\xi+1}$. Repeating the second part of the proof of Theorem 1.4.2, we get $\langle A_\alpha \mid \alpha \leq 2\gamma\rangle$, with $\gamma < |T|^+$, such that each $A_\alpha \in \mathbf{\Pi}^0_\xi$ and such that

$$(\forall x \in \lceil T\rceil)(x \in A \ \leftrightarrow \ \mu\beta(x \notin A_\beta \vee \beta = \alpha) \text{ is odd}).$$

Moreover, in the notation of the proof of Theorem 1.4.2,

$$\begin{aligned}&(\forall x \in \lceil T\rceil)(\forall \alpha < \gamma)(x \notin A_{2\alpha} \cap A_{2\alpha+1} \ \to \\ &\quad (\exists n \in \omega)(\exists t \in \mathcal{S}_{II}(S^n))(\langle x\restriction n, t\rangle \in U_n \ \wedge \ \mathrm{ord}(x\restriction n, t) = \alpha)).\end{aligned}$$

Let $\langle t_i \mid i \in \omega\rangle$ be an enumeration of $\bigcup_{n\in\omega} \mathcal{S}_{II}(S^n)$ with the property that each t_i belongs to $\mathcal{S}_{II}(S^n)$ for some $n \leq i$. Define $f : T \to 2\gamma$ by

$$f(p) \ = \ \begin{cases} 2\,\mathrm{ord}(p\restriction n, t_i) & \text{if } \ell\mathrm{h}(p) = 2i \ \wedge \ \langle p\restriction n, t_i\rangle \in U_n; \\ 2\,\mathrm{ord}(p\restriction n, t_i) + 1 & \text{if } \ell\mathrm{h}(p) = 2i+1 \ \wedge \ \langle p\restriction n, t_i\rangle \in U_n; \\ 0 & \text{otherwise.}\end{cases}$$

Clearly f satisfies (b) in the definition of $((2\gamma+1)\text{-}\mathbf{\Pi}^0_\xi)^*$. $\square$

Theorem 1.4.22. (Rec(ZC$^-$+ Σ_1 Replacement)) *All* $\mathbf{Diff}^*(\mathbf{\Pi}^0_3)$ *games are determined.*

Proof. Let $\langle A_\alpha \mid \alpha < \gamma\rangle$ and $f : T \to \gamma$ witness that $A \in \mathbf{Diff}^*(\mathbf{\Pi}^0_3)$. In the proof of Theorem 1.4.10—and so in the proof of Theorem 1.4.19—the countability of T was used only in the proof of Lemma 1.4.14 and in the final argument after the proof of that lemma. We can adapt the proof of Lemma 1.4.14 to the present situation as follows: For $p \in T$ and $\ell\text{h}(p) = 2^k 3^m$, let

$$\begin{aligned} \beta_p &= \begin{cases} f(p \restriction k) & \text{if } f(p\restriction k) < r(n) \ \wedge \ f(p\restriction k) \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases} \\ m_p &= m. \end{aligned}$$

Repeat the proof of Lemma 1.4.14, except replace, in the definition of U, the pairs $\langle \beta_j, m_j\rangle$ by $\langle \beta_{q\restriction j}, m_{q\restriction j}\rangle$. A similar change will adapt the final argument in the proof of Theorem 1.4.10 to the present situation. □

Corollary 1.4.23. (Rec(ZC$^-$+ Σ_1 Replacement)) *All* $\mathbf{\Delta}^0_4$ *games are determined.*

Theorem 1.4.16 holds for the generalized difference hierarchy, with essentially the same proof as for the original theorem.

Theorem 1.4.24. *Let $(M;E)$ be an ω-model of ZC$^-$+ Σ_1 Replacement. Assume that T is, in $(M;E)$, a game tree. Assume that T belongs to the wellfounded part of $(M;E)$. Assume that γ is an ordinal of $(M;E)$ and that γ belongs to the wellfounded part of $(M;E)$. Then*

$$(M;E) \models \text{“All } (\gamma\text{-}\mathbf{\Pi}^0_3)^* \text{ games are determined.”}$$

Corollary 1.4.25. *Every wellfounded model of ZC$^-$+ Σ_1 Replacement satisfies “All Δ^0_4 games are determined.”*

Exercise 1.4.1. ZFC$^-$ is ZFC without the Axiom of Power Set. In other words, ZFC$^-$ is ZC$^-$ + Replacement. Show that the determinacy of all Σ^0_4 games in countable trees is not provable in ZFC$^-$. This is a refinement, due to the author, of a theorem of [Friedman, 1971]. Friedman's theorem has Σ^0_5 is instead of Σ^0_4.

Hint. Let β_0 be the least ordinal number β such that $L_\beta \models \mathrm{ZC}^- + \Sigma_1$ Replacement. Show that β_0 is the least β such that $L_\beta \models Z$ and that it is the least β such that $L_\beta \models \mathrm{ZFC}^-$.

Show that there is no $a \subseteq \omega$ such that $a \in L_{\beta_0+1} \setminus L_{\beta_0}$, and show that β_0 is the least ordinal with this property. Show that every set in L_{β_0} is definable in L_{β_0}.

The plan is to prove that L_{β_0} does not satisfy the determinacy of all Σ_4^0 games in ${}^{<\omega}\omega$. This is to be done by defining a Σ_4^0 game G in ${}^{<\omega}\omega$ such that G is a win for I but the set of Gödel numbers of sentences true in L_{β_0} is recursive uniformly in any winning strategy for I for G.

For a model $(M; E)$, $\mathrm{WFP}(M; E)$, the *wellfounded part* of $(M; E)$, is the union of all subsets N of M such that

(a) $(\forall x)(\forall y)(xEy \in N \to x \in N)$;

(b) the restriction of E to N is wellfounded.

It is easy to show the restriction of E to $\mathrm{WFP}(M; E)$ is wellfounded, so that $\mathrm{WFP}(M; E)$ is the largest subset N of M satisfying (a) and (b).

It will be convenient, in the exercises that follow and those for §2.3, to use "ω-model" to mean a model $(M; E)$ such that $\omega \in \mathrm{WFP}(M; E)$ and the restriction of E to $\mathrm{WFP}(M; E)$ is the membership relation.

If $\mathbf{S}$ is a complete theory in the language of set theory and $\mathbf{S}$ extends some weak fragment of ZFC and $\mathbf{S} \vdash V = L$, then there is a canonical model of $\mathbf{S}$, the *term model*. The model consists of equivalence classes of formulas $\varphi(v)$. Formulas $\varphi(v)$ and $\psi(v)$ are equivalent if $\mathbf{S} \vdash$ "The $<_L$-least v such that either $\varphi(v)$ or else $v = 0$ and $(\forall v')\neg\varphi(v')$ is identical with the $<_L$-least v such that either $\psi(v)$ or else $v = 0$ and $(\forall v')\neg\psi(v')$." The interpretation of $\in$ is defined in the obvious way. Note that every element of the term model of $\mathbf{S}$ is definable in the model.

Consider the following game G in ${}^{<\omega}\omega$: Gödel numbering all the sentences of the language of set theory, we define, for each play x,

$$\begin{aligned} \mathbf{S}_{\mathrm{I}}(x) &= \{\varphi \mid x(2\,\#(\varphi)) = 1\} \\ \mathbf{S}_{\mathrm{II}}(x) &= \{\varphi \mid x(2\,\#(\varphi)+1) = 1\} \end{aligned}$$

If $\mathbf{S}_{\mathrm{I}}(x)$ is not the set of sentences true in an ω-model of ZFC^- + "$V = L_{\beta_0}$," then I loses. Otherwise II loses unless $\mathbf{S}_{\mathrm{II}}(x)$ is also the set of sentences true in an ω-model of ZFC^- + "$V = L_{\beta_0}$." If neither player has lost for this reason, then the term models of $\mathbf{S}_{\mathrm{I}}(x)$ and $\mathbf{S}_{\mathrm{II}}(x)$ are isomorphic to ω-models. Let $\mathfrak{M}_{\mathrm{I}}$ and $\mathfrak{M}_{\mathrm{II}}$ be such ω-models. Player I wins just in case one of the following holds:

(1) The model $\mathfrak{M}_{\mathrm{I}}$ is isomorphic to a initial segment of $\mathfrak{M}_{\mathrm{II}}$.

(2) There is an ordinal a of $\mathfrak{M}_{\mathrm{I}}$ such that $L_a^{\mathfrak{M}_{\mathrm{I}}}$ is isomorphic to an initial segment of $\mathfrak{M}_{\mathrm{II}}$ but $L_{a+1}^{\mathfrak{M}_{\mathrm{I}}}$ is not.

By *an ordinal of* $\mathfrak{M}_{\mathrm{I}}$ we mean an a such that $\mathfrak{M}_{\mathrm{I}} \models$ "a is an ordinal number." By an *initial segment of* $\mathfrak{M}_{\mathrm{II}}$ we mean $\bigcup_{b\in X} L_b^{\mathfrak{M}_{\mathrm{II}}}$, where X is a (not necessarily proper) initial segment of the ordinals of $\mathfrak{M}_{\mathrm{II}}$. It is not required that X be the initial segment of any ordinal of $\mathfrak{M}_{\mathrm{II}}$.

G is a win for I, who can simply play as $\mathbf{S}_{\mathrm{I}}(x)$ the set of sentences true in L_{β_0}. But show that, as long as II simply copies I's moves, I can maintain a winning position *only* by following this strategy. Thus the set of Gödel numbers of sentences true in L_{β_0} is recursive uniformly in any winning strategy for I for G. It is fairly easy to see that this set of Gödel numbers does not belong to L_{β_0}. Thus no winning strategy for G belongs to L_{β_0}. By absoluteness, $L_{\beta_0} \models$ "G is not determined."

Show that $G = G(A; {}^{<\omega}\omega)$ with $A \in \Sigma_4^0$. There are two main points. First, there is a fixed Π_1^1 formula $\varphi(X,Y)$ of second order arithmetic such that, for any ω-model $\mathfrak{N}$ of (a weak fragment of ZFC +) $V = L$, for any ordinal a of $\mathfrak{N}$, and for any subset b of ω belonging to $L_{a+1}^{\mathfrak{N}} \setminus L_a^{\mathfrak{N}}$,

(i) $L^{\mathfrak{N}} \cap \mathcal{P}(\omega) \models (\exists Y)\varphi(b,Y)$;

(ii) for all $c \in L_{a+1}^{\mathfrak{N}} \cap \mathcal{P}(\omega)$, $L^{\mathfrak{N}} \cap \mathcal{P}(\omega) \models \varphi(b,c)$ if and only if c codes a model $(\omega; E)$ isomorphic to $L_a^{\mathfrak{N}}$.

It follows that, for ω-models $\mathfrak{N}$ of $V = L$ + "β_0 does not exist," $\mathfrak{N} \cap \mathcal{P}(\omega)$ determines the isomorphism type of $\mathfrak{N}$. This implies, for example, that (1) holds just in case the subsets of ω of $\mathfrak{M}_{\mathrm{I}}$ are the same as those of some initial segment of $\mathfrak{M}_{\mathrm{II}}$ in the sense of the L-hierarchy of $\mathfrak{M}_{\mathrm{II}}$. The second main point is that, for formulas $\varphi(v)$ and $\psi(v)$ whose equivalence classes correspond to subsets of ω in $\mathfrak{M}_{\mathrm{I}}$ and $\mathfrak{M}_{\mathrm{II}}$ respectively, the condition that these subsets of ω are the same is a Π_1^0 condition. This enables one to show that (1) is a Π_3^0 condition. Similarly, the condition (2) is Σ_4^0.

Remark. In [Montalban and Shore, 2012], the authors prove a sharper result than that of Exercise 1.4.1:

$$\mathrm{ZFC}^- \nvdash (\forall n \in \omega)\text{All } n\text{-}\mathbf{\Pi}_3^0 \text{ games are determined.}$$

See the remark after the hint for Exercise 1.4.2 for discussion of a still stronger result of Montalban and Shore.

Exercise 1.4.2. Work in ZC$^-$+ Σ_1 Replacement. Assume that all Σ^0_4 games are determined. Prove that there is an ordinal β such that $L_\beta \models$ ZFC$^-$; i.e., prove that β_0 exists. Like the result of of Exercise 1.4.1, this is a refinement by the author of a result of [Friedman, 1971], replacing Σ^0_5 by Σ^0_4. One of its consequences is that the consistency of ZFC$^-$ can be proved in ZC$^-$+ Σ_1 Replacement + "all Σ^0_4 games are determined."

Hint. Show (in ZC$^-$+ Σ_1 Replacement) that there are arbitrarily large admissible ordinals, i.e., ordinals α such that $L_\alpha \models$ Kripke–Platek set theory, KP.

Let T be the theory KP + $V = L$ + "β_0 does not exist."

Show that, for any ω-model $\mathfrak{M}$ of KP, WFP($\mathfrak{M}$) $\models$ KP. This implies, in particular, that if $\mathfrak{M}$ is any ω-model of T and β is the least ordinal not belonging to WFP($\mathfrak{M}$), then β is admissible .

In order to derive a contradiction, assume that β_0 does not exist.

Let Y be the set of ordinals α such that $L_\alpha \models T$ and every member of L_α is definable in L_α. Prove that Y is unbounded in the ordinals. Do do so, assume to the contrary that $\gamma = \sup(Y)$. Let α be the least admissible greater than γ. Let X be the elementary submodel of L_α consisting of the elements of L_α definable in L_α. The ordinal γ belongs to X. The non-existence of β_0 implies that there is some subset a of ω such that $a \in L_{\gamma+1} \setminus L_\gamma$. The $<_L$-least such a belongs to X. Deduce that $X = L_\alpha$. This gives the contradiction that $\alpha \in Y$.

Consider the following game G' in ${}^{<\omega}\omega$: For $x \in {}^\omega\omega$, let $\mathbf{S}_{\mathrm{I}}(x)$ and $\mathbf{S}_{\mathrm{II}}(x)$ be defined as in Exercise 1.4.1. If $\mathbf{S}_{\mathrm{I}}(x)$ is not the set of sentences true in an ω-model of T then I loses. Otherwise II loses unless $\mathbf{S}_{\mathrm{II}}(x)$ is also the set of sentences true in an ω-model of T. If neither player has lost for this reason, then let $\mathfrak{M}_{\mathrm{I}}$ and $\mathfrak{M}_{\mathrm{II}}$ be ω-models isomorphic to the term models of $\mathbf{S}_{\mathrm{I}}(x)$ and $\mathbf{S}_{\mathrm{II}}(x)$ respectively. Player I wins just in case one of the following holds:

(1) The model $\mathfrak{M}_{\mathrm{I}}$ is isomorphic to an initial segment of $\mathfrak{M}_{\mathrm{II}}$ not of the form $L_b^{\mathfrak{M}_{\mathrm{II}}}$ for b an ordinal of $\mathfrak{M}_{\mathrm{II}}$.

(2) There is an ordinal a of $\mathfrak{M}_{\mathrm{I}}$ such that $L_a^{\mathfrak{M}_{\mathrm{I}}}$ is isomorphic to an initial segment of $\mathfrak{M}_{\mathrm{II}}$ but $L_{a+1}^{\mathfrak{M}_{\mathrm{I}}}$ is not.

Prove that G' is not a win for I. To do so, assume that σ is a winning strategy for I. By absoluteness, you may assume that $\sigma \in L$. Let II play against σ the set of sentences true in some $L_\alpha \in Y$ such that $\sigma \in L_\alpha$. It is easily seen that II's part of the resulting play x belongs to $L_{\alpha+2}$. It follows that x belongs to $L_{\alpha+2}$. Thus $\mathbf{S}_{\mathrm{I}}(x) \in L_{\alpha+2}$, and so $\mathcal{P}(\omega) \cap \mathfrak{M}_{\mathrm{I}} \subseteq L_{\alpha+2}$. Let β be the least ordinal not belonging to $\mathrm{WFP}(\mathfrak{M}_{\mathrm{I}})$. The wellfoundedness of $\mathfrak{M}_{\mathrm{II}}$ implies that (1) can hold only if $\mathfrak{M}_{\mathrm{I}} \cong \mathfrak{M}_{\mathrm{II}}$, i.e., if $\mathfrak{M}_{\mathrm{I}} = L_\alpha$. But this is impossible, for σ belongs to L_α, and II is simply copying as long as I is playing the set of sentences true in L_α. Therefore (2) holds. This can happen only if $\beta > \alpha$. But β is admissible, so $\beta > \alpha + 2$. Therefore $L_{\alpha+3} \subseteq \mathfrak{M}_{\mathrm{I}}$. Because β_0 does not exist, there is an $a \subseteq \omega$ such that $a \in L_{\alpha+3} \setminus L_{\alpha+2}$. This gives the contradiction that a both belongs and does not belong to $\mathfrak{M}_{\mathrm{I}}$.

Now prove that the game G' is not a win for II. For this assume that $\tau \in L$ is a winning strategy for II. Let I play against τ the set of sentences true in some $L_\alpha \in Y$ such that $\tau \in L_\alpha$. It follows that $\mathcal{P}(\omega) \cap \mathfrak{M}_{\mathrm{II}} \subseteq L_{\alpha+2}$. Let β be the least ordinal not belonging to $\mathrm{WFP}(\mathfrak{M}_{\mathrm{II}})$. Since (1) fails, it is impossible that $\beta = \alpha$. Since (2) fails, it is also impossible that $\beta < \alpha$. Thus $\beta > \alpha$. By an argument like that of the last paragraph, this gives a contradiction.

Derive a contradiction by showing that G' is Σ^0_4 and therefore, by hypothesis, determined.

Remark. In [Montalban and Shore,], a strengthening of the result of Exercise 1.4.2 is proved. In [Montalban and Shore,], they improve this result by showing that the consistency ZFC$^-$ is implied by the statement that, for all $n \in \omega$, all n-$\mathbf{\Pi}^0_3$ games are determined. (The latter statement is what they call ω-$\mathbf{\Pi}^0_3$ determinacy.) This follows by an ultraproduct construction from their theorem that, for all $n \geq 1$, KP $\vdash \mathbf{\Pi}^0_{n+2}$ Determinacy $\rightarrow$ "There is a wellfounded model of KP + $\mathbf{\Delta}^0_{n+2}$ Comprehension." The theorem is proved using games similar to—but more complex than—the game used in the proof of Exercise 1.4.2.

Exercise 1.4.3. Let $\mathcal{D}$ be the set of all degrees of unsolvability, i.e., of all Turing degrees. A *cone* of Turing degrees is the set of all degrees greater than or equal to some fixed degree, which is called the *vertex* of the cone. For classes $\mathbf{\Gamma}$ of sets, $\mathbf{\Gamma}$ *Turing determinacy* is the assertion that, for all $\mathcal{A} \subseteq \mathcal{D}$ such that $\bigcup \mathcal{A} \in \mathbf{\Gamma}$, either $\mathcal{A}$ or $\mathcal{D} \setminus \mathcal{A}$ contains a cone.

Work in ZC$^-$+ Σ_1 Replacement. Assume Σ^0_5 Turing determinacy and prove that β_0 exists. This is another strengthening by the author of a result of [Friedman, 1971], with Σ^0_5 replacing the Σ^0_6 of [Friedman, 1971].

Hint. For elements y and z of ${}^\omega\omega$, let $y \oplus z \in {}^\omega\omega$ be such that $(y \oplus z)(2n) = y(n)$ and $(y \oplus z)(2n+1) = z(n)$ for all $n \in \omega$. Let B be the set of all $\langle y, z\rangle$ such that I wins the play $y \oplus z$ of the game G' of Exercise 1.4.2. Let

$$\mathcal{A} = \{\mathbf{a} \in \mathcal{D} \mid (\exists z \in {}^\omega\omega)(\mathbf{d}(z) \leq \mathbf{a} \ \wedge \ (\forall y \in {}^\omega\omega)(\mathbf{d}(y) \leq \mathbf{a} \to \langle y, z\rangle \notin B))\}.$$

Note that $\bigcup \mathcal{A} \in \Sigma^0_5$.

Assume that β_0 does not exist. Let $\mathbf{b} \in \mathcal{D} \cap L$. Show that the cone with vertex $\mathbf{b}$ is not contained in $\mathcal{A}$. To do so, let α belong to the set Y defined in the hint to Exercise 1.4.2 and let $\mathbf{b} \in L_\alpha$. Let $\mathbf{S}$ be the set of sentences true in L_α. Let y be the characteristic function of the set of Gödel numbers of members of $\mathbf{S}$. Let $\mathbf{a} = \mathbf{d}(y)$. Show that

$$(\forall z)(\mathbf{d}(z) \leq \mathbf{a} \ \to \ \langle y, z\rangle \in B).$$

(This is more than what is needed for $\mathbf{a} \notin \mathcal{A}$.) Similarly show that the cone with vertex $\mathbf{b}$ is not disjoint from $\mathcal{A}$.

Exercise 1.4.4. Work in $\mathrm{ZC}^- + \Sigma_1$ Replacement. Let $\mathcal{D}$ be as in Exercise 1.4.3.

(a) Let $\mathcal{A} \subseteq \mathcal{D}$. Prove that $G(\bigcup \mathcal{A}; {}^{<\omega}\omega)$ is determined if and only if either $\mathcal{A}$ or $\mathcal{D} \setminus \mathcal{A}$ contains a cone. This observation is from [Martin, 1968]. It implies that, for all classes $\mathbf{\Gamma}$, $\mathbf{\Gamma}$ Turing determinacy follows from the hypothesis that all $\mathbf{\Gamma}$ games are determined.

(b) Let α be a countable ordinal. Assume that all $\mathbf{\Sigma}^0_\alpha$ games in ${}^{<\omega}\omega$ are determined. Let $A \subseteq {}^\omega\omega$ with $A \in \mathbf{\Sigma}^0_{\alpha+1}$. Show that if A has members of arbitrarily large degree then A meets each member of some cone of degrees. This result appears in [Harrington and Kechris, 1975]; the authors report that it was independently proved by Ramez Sami.

(c) Prove that, for every countable ordinal α, $\mathbf{\Delta}^0_{\alpha+2}$ Turing determinacy follows from the determinacy of all $\mathbf{\Sigma}^0_\alpha$ games in ${}^{<\omega}\omega$. This consequence of (b) is due to the author.

(d) Prove $\mathbf{\Delta}^0_5$ Turing determinacy. (By Exercise 1.4.3, this is an optimal result.)

Hint. For (b), let $A \in \mathbf{\Sigma}^0_{\alpha+1}$ and let B_n, $n \in \omega$, be $\mathbf{\Pi}^0_\alpha$ sets such that be such that $A = \bigcup_{n\in\omega} B_n$. Let $B = \{\langle n\rangle^\frown x \mid x \in B_n\}$. Consider the game G in which I plays x, II plays y, and I wins if $x \in B$ and $\mathbf{d}(y) \leq \mathbf{d}(x)$. Show that G is a win for I and use this to show that A meets all sufficiently large degrees.

For (c), use Theorem 1.4.2.

Remark. The result of Exercise 1.4.1 and those of parts (c) and (d) of Exercise 1.4.4 were proved in 1974, and the proofs were circulated in unpublished manuscripts. The same is true of the theorem that Σ^0_5 Turing determinacy is false L_{β_0}. The result of Exercise 1.4.2 was proved somewhat later, and that of Exercise 1.4.3 was noticed only in 1995.

Exercise 1.4.5. There are various ways to generalize Friedman-style results on Borel games to the case of uncountable trees. Here is one generalization of Exercise 1.4.1.

Let ρ be any ordinal. Let $\beta(\rho)$ be the least ordinal $\beta > \rho$ such that $L_\beta \models \text{ZFC}^-$. Prove that $L_{\beta(\rho)}$ does not satisfy the determinacy of all $\mathbf{\Sigma}^0_4$ games in ${}^{<\omega}\rho$.

Hint. Relativize to arbitrary elements of ${}^\omega\omega$ the argument of the hint for Exercise 1.4.1. Now collapse ρ by forcing.

Exercise 1.4.6. Let ZFCRec$^-$ be the theory Rec(ZC$^-$+ Σ_1 Replacement) introduced on page 47. Let ZFCRec$^-$ be the result of adding, to the axioms of Rec(ZC$^-$+ Σ_1 Replacement), the Replacement Schema for all formulas of the language $\mathcal{L}_{\text{rec}}$. Show that the determinacy of all Σ^0_4 games in countable trees is not provable in ZFCRec$^-$.

Hint. Replace the L_α hierarchy by the L^{rec}_α hierarchy, where $L^{\text{rec}}_{\alpha+1}$ is the set of subsets of L^{rec}_α defined over L^{rec}_α from parameters in L^{rec}_α by formulas of $\mathcal{L}_{\text{rec}}$. It is easy to see that $L^{\text{rec}}_\lambda = L_\lambda$ for all limit ordinals λ.

Let β^{rec}_0 be the least ordinal β such that L^{rec}_β satisfies ZFCRec$^-$. Show that $L^{\text{rec}}_{\omega_1}$ is such an ordinal. Show that Condensation holds for the L^{rec}_α hierarchy as it does for the L_α hierarchy, and so β^{rec}_0 is a countable ordinal. Show also that β^{rec}_0 is the least ordinal β such that no $a \subseteq \omega$ belongs to $L^{\text{rec}}_{\beta+1} \setminus L^{\text{rec}}_\beta$.

Continue to imitate the proof for Exercise 1.4.1.

Chapter 2

General Borel Games

In this chapter we introduce the technical concept of a *covering* of a game tree, and we use this concept to prove the determinacy of all Borel games and—in uncountable trees—the determinacy of all games in a larger class.

Borel determinacy is proved in §2.1. In countable trees, the Borel sets are the same as the the $\mathbf{\Delta}^1_1$ sets (to be defined in §2.2). In general, however, $\mathbf{\Delta}^1_1$ is a larger class, the class of what we will call *quasi-Borel* sets. In §2.2 we prove this and also prove that all quasi-Borel games are determined. §2.1 and §2.2 depend only on §1.1 and §1.2 (and not on the rest of Chapter 1).

Readers interested only in main results may confine themselves to §2.1 (though §2.2 should present no extra difficulties).

In §2.3 we work again in the weak set theory of §§1.3–1.4. We use the proofs of §2.1 and the results of §1.4 to get $\mathbf{\Sigma}^0_\alpha$ determinacy with the minimal possible amount of Power Set and Replacement (allowed by refinements—given in the exercises—of results of Harvey Friedman).

In §2.4 we consider a class of infinite games of imperfect information called *Blackwell games* after David Blackwell, who initiated their study. We introduce the basic theory of imperfect information games, and then we prove the determinacy of Borel Blackwell games by showing that it follows from ordinary Borel deteminacy. This is done by proving a general theorem reducing the the determinacy of Blackwell games of any reasonably closed class to the determinacy of ordinary games of that class. Thus all our determinacy results in later chapters will imply corresponding determinacy results for Blackwell games.

2.1 Borel Determinacy

Almost all the determinacy results in the remainder of this book will be proved by the technique of *auxiliary games*: To prove $G(A;T)$ determined, we will associate with $G(A;T)$ another game $G(A^*;T^*)$. This auxiliary game we will know to be determined. Moreover the two games will be so related that the determinacy of $G(A;T)$ will follow from that of $G(A^*;T^*)$. In a sense we have already seen this technique. To prove Theorem 1.3.1, for example, we made use of the closed games $G(C;T)$ occurring in the proof of Lemma 1.3.2. Such games were used also in proving Theorems 1.3.3, 1.4.10, and 1.4.22. The auxiliary game technique as we will use it later differs from these examples in two important ways: (1) The determinacy of the given game $G(A;T)$ will be reduced to the determinacy of a *single* game $G(A^*;T^*)$. (2) T^* will be *larger* than T, whereas the auxiliary game trees in the earlier examples were all subtrees of the given T. Indeed the results of Friedman [1971] show that (for, e.g., $T = {}^{<\omega}\omega$) some use of existence principles for sets larger than T is necessary to prove the determinacy of Borel games in T. (See Exercises 1.4.1–1.4.5 and Exercises 2.3.2–2.3.12.)

In using the auxiliary game technique, one can think of moves in the auxiliary tree as being moves in T together with extra components. In later chapters the extra components will be elements of measure spaces. Winning strategies for the main game will be derived from winning strategies for the auxiliary game by *integration*. In this chapter the extra components of moves in the auxiliary tree will be, in the basic case, (a) subtrees of T and (b) decisions about whether the element of $\lceil T \rceil$ being produced will belong to certain subsets of $\lceil T \rceil$. Exercise 2.1.2 illustrates this technique, reproving Theorem 1.3.1 with the help of an auxiliary game. However, components of the form (b) do not appear in this example. In more general cases, auxiliary trees will come from iterations of the process that gives the basic case.

Remark. The first proof of $\mathbf{\Sigma}^0_4$ determinacy, that in [Paris, 1972], used an auxiliary game technique modeled on the one we will present in Chapter 4. James Baumgartner had earlier found, adapting the method of Chapter 4, a new proof of $\mathbf{\Sigma}^0_3$ determinacy,

Our proof of Borel determinacy will be like that in [Martin, 1985] in that we will prove inductively that all Borel sets have a certain property, the property of being reducible in a certain way to a clopen set of plays in a different tree. The determinacy of a set with this property will follow easily from the determinacy of a set related to the clopen set. In the details there we will be several differences between the proof in [Martin, 1985] and the proof as we will present it below. Our presentation will be similar to that in [Hurkens, 1993]. This similarity is partly coincidental and partly by choice. When the first draft of this section was written around 1990, it was influenced by an idea of Moschovakis (found in the proof of Theorem 6F.1 of [Moschovakis, 1980]). Moschovakis' idea eliminates from the original proof of Borel determinacy (the proof in [Martin, 1975]) part of its use of quasistrategies and subsidiary games. In writing the present chapter, the author wished to go further: (a) to combine Moschovakis' idea with the purely inductive proof in [Martin, 1985] and (b) to eliminate from the proof every vestige of the use of quasistrategies. To accomplish these aims, the author introduced *game trees with taboos*, game trees in which each terminal position is automatically lost for one player or the other—is *taboo* for one player or the other—independently of the payoff set. (In the first draft of the section, non-taboo terminal positions were also permitted.) Hurkens, who explicitly had aim (a), produced a proof that has essentially all the ingredients in the author's draft (which Hurkens had not seen). Hurkens' proof introduces one additional idea, an idea that both simplifies and helps motivate the main construction of the proof. Although the author had in his possession a copy of [Hurkens, 1993], he learned about this idea only indirectly, in conversation with Marco Vervoort. Afterwards he actually consulted [Hurkens, 1993] and discovered the similiarities between Hurkens' proof and his own. Hurkens' additional idea seemed too valuable to omit, so the author has revised his draft to incorporate that idea (and to make some other modifications). In the course of giving the proof, we will explain Hurkens' idea and we will comment on relations between the two proofs.

A *game tree with taboos* is a triple $\mathbf{T} = \langle T, \mathcal{T}_{\mathrm{I}}, \mathcal{T}_{\mathrm{II}} \rangle$, where

(1) T is a game tree;

(2) $\mathcal{T}_{\mathrm{I}}$ and $\mathcal{T}_{\mathrm{II}}$ are disjoint sets of terminal positions in T;

(3) every terminal position in T belongs to $\mathcal{T}_{\mathrm{I}}$ or to $\mathcal{T}_{\mathrm{II}}$.

Recall that terminal positions in T are members of T that are also finite plays in T. Infinite plays are not positions, and so are not terminal positions.

Convention. We always use boldface letters, perhaps with other markings, for game trees with taboos. For the underlying game trees, we use the corresponding italic lightface letters, with the same markings; for the other two components, we use the corresponding calligraphic letters, with the same markings and with subscripts "I" and "II." For example, $\tilde{\mathbf{T}}^i$ will be $\langle \tilde{T}^i, \tilde{\mathcal{T}}^i_{\mathrm{I}}, \tilde{\mathcal{T}}^i_{\mathrm{II}} \rangle$.

If $\mathbf{T}$ is a game tree with taboos, then *positions*, *moves*, *plays*, *strategies*, etc. in $\mathbf{T}$ are just positions, moves, plays, strategies, etc. in T. If $p \in T$, then $\mathbf{T}_p$ is the game tree with taboos $\langle T_p, \mathcal{T}_{\mathrm{I}} \cap T_p, \mathcal{T}_{\mathrm{II}} \cap T_p \rangle$.

For any game tree T, we let $[T]$ be the set of all infinite plays in T. Note that $[T]$ is a closed subset of $\lceil T \rceil$.

Let $\mathbf{T}$ be a game tree with taboos. Plays belonging to $\mathcal{T}_{\mathrm{I}}$ are *taboo* for I in $\mathbf{T}$, and plays belonging to $\mathcal{T}_{\mathrm{II}}$ are *taboo* for II in $\mathbf{T}$. Hence $[T]$ is the set of all plays that are not taboo for either player in $\mathbf{T}$, i.e., that are not *taboo* in $\mathbf{T}$. For $A \subseteq [T]$, we define the game $G(A; \mathbf{T})$ as follows: A finite play of $G(A; \mathbf{T})$ is lost by the player for whom it is taboo. A play $x \in [T]$ is won by I if and only if $x \in A$. Thus $G(A; \mathbf{T})$ is the same game as $G((A \cup \mathcal{T}_{\mathrm{II}}) \setminus \mathcal{T}_{\mathrm{I}}; T)$. The notions, for $G(A; \mathbf{T})$, of *winning strategy* and being *determined* are the same as those for $G((A \cup \mathcal{T}_{\mathrm{II}}) \setminus \mathcal{T}_{\mathrm{I}}; T)$.

Remark. [Hurkens, 1993] does not have game trees with taboos, but it has a device that does the same work. It has a move function of the sort we discussed on page 2. The move function is defined even in terminal positions, and whichever player has the impossible task of moving in a terminal position loses the that play of the game.

We could have omitted clause (3) from the definition of game trees with taboos, i.e., we could have permitted the existence of finite non-taboo plays. Indeed, this would have been the more natural definition, since we permitted finite plays throughout Chapter 1. The reason why we include clause (3) is that without it many of our definitions and proofs would have been more complicated, since we would have had to worry about whether any given finite play was taboo or not.

Remark. While Hurkens' use of a move function does all the work done by game trees with taboos, it would not in a straightforward way do the work of game trees with taboos in the more liberal sense just discussed.

It is important to make sure that proving determinacy results only for game trees with taboos in our restricted sense involves no loss of generality. First note that, for ordinary game trees (without taboos), nothing is lost by considering only trees without finite plays. To see this, let T be a game tree.

Consider the tree

$$T' = T \cup \{p^\frown \langle \underbrace{0, \ldots, 0}_{n} \rangle \mid n \in \omega \wedge p \text{ is terminal in } T\}.$$

The tree T' has no terminal postions. The obvious bijection $f : \lceil T' \rceil \to \lceil T \rceil$ is a homeomorphism such that, for each $A \subseteq \lceil T \rceil$, $G(A; T)$ is determined if and only if $G(f^{-1}(A); T')$ is determined. Similarly, let $\mathbf{T}$ be an game tree with taboos in the unrestricted sense (possibly not satifying clause (3)). Set

$$T' = T \cup \{p^\frown \langle \underbrace{0, \ldots, 0}_{n} \rangle \mid n \in \omega \wedge p \text{ is terminal in } T \text{ and not taboo in } \mathbf{T}\}.$$

Let $\mathbf{T}' = \langle T', \mathcal{T}_{\mathrm{I}}, \mathcal{T}_{\mathrm{II}} \rangle$. Then $\mathbf{T}'$ is a game tree with taboos. Furthermore, the obvious homeomorphism $f : \lceil T' \rceil \to \lceil T \rceil$ restricts to a homeomorphism (in the sense of the definition below, adapted to allow for game trees with taboos in the unrestricted sense) from $[T']$ to the set of all non-taboo plays in $\mathbf{T}$. Moreover, for any set A of non-taboo plays in $\mathbf{T}$, $G(A; \mathbf{T})$ is determined if and only if $G(f^{-1}(A); \mathbf{T}')$ is determined (under the obvious definition).

We give $[T]$ the relative topology: A subset A of $[T]$ is open just in case there is an open $B \subseteq \lceil T \rceil$ such that $A = B \cap [T]$. We will construe our topological definitions as making sense even in the case $[T]$ is empty, so that the unique subset $\emptyset$ of $[T]$ is open, Borel, etc. The following easy lemma will allow us usually not to worry about the distinction between the Borel hierarchy on $[T]$ and that on $\lceil T \rceil$.

Lemma 2.1.1. *Let $\mathbf{T}$ be a game tree with taboos. For all ordinals $\alpha \geq 1$ and all subsets A of $[T]$, A belongs to $\mathbf{\Pi}^0_\alpha$ as a subset of $[T]$ if and only if A belongs to $\mathbf{\Pi}^0_\alpha$ as a subset of $\lceil T \rceil$. For all ordinals $\alpha > 1$ and all subsets A of $[T]$, A belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $[T]$ if and only if A belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $\lceil T \rceil$.*

Proof. We prove the lemma by induction on $\alpha \geq 1$.

By definition of the relative topology, every subset of $[T]$ closed as a subset of $\lceil T \rceil$ is closed as a subset of $[T]$. Since $[T]$ is closed as a subset of $\lceil T \rceil$, every subset of $[T]$ closed as a subset of $[T]$ is closed as a subset of $\lceil T \rceil$.

Let $\alpha > 1$ and assume that the lemma holds for all $\beta < \alpha$. The fact that any subset of $[T]$ belongs to $\mathbf{\Sigma}^0_\alpha$ as subset of $[T]$ if and only if it belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $\lceil T \rceil$ follows directly from the definition of $\mathbf{\Sigma}^0_\alpha$ and our induction hypothesis. Suppose that $A \in \mathbf{\Pi}^0_\alpha$ as a subset of $[T]$. Thus $[T] \setminus A \in \mathbf{\Sigma}^0_\alpha$ as a subset of $[T]$ and so also as a subset of $\lceil T \rceil$. $\lceil T \rceil \setminus [T] \in \mathbf{\Sigma}^0_1 \subseteq$ (by Lemma 1.1.1) $\mathbf{\Sigma}^0_\alpha$. By Lemma 1.1.1 again, $\lceil T \rceil \setminus A = ([T] \setminus A) \cup (\lceil T \rceil \setminus [T])$ belongs to $\mathbf{\Sigma}^0_\alpha$. By the definition of $\mathbf{\Pi}^0_\alpha$, $A \in \mathbf{\Pi}^0_\alpha$ as a subset of $\lceil T \rceil$. Suppose now that $A \subseteq [T]$ and that $A \in \mathbf{\Pi}^0_\alpha$ as a subset of $\lceil T \rceil$. Thus $\lceil T \rceil \setminus A \in \mathbf{\Sigma}^0_\alpha$. By Lemma 1.1.1, the closed set $[T]$ belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $\lceil T \rceil$. By Lemma 1.1.1 again, $[T] \setminus A = (\lceil T \rceil \setminus A) \cap [T]$ belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $\lceil T \rceil$. Thus $[T] \setminus A \in \mathbf{\Sigma}^0_\alpha$ as a subset of $[T]$, and so $A \in \mathbf{\Pi}^0_\alpha$ as a subset of $[T]$. □

There is another way to characterize the topology on $[T]$. Note that $[T] = \lceil \bar{T} \rceil$, where $\bar{T} = \{p \in T \mid (\exists x \supseteq p)\, x \in [T]\}$. If $[T]$ is nonempty, then $\bar{T}$ is a game tree, and our topology for $[T]$ is the same as the topology it has as $\lceil \bar{T} \rceil$. Thus Lemma 1.1.1 holds for the Borel hierarchy on $[T]$. (One can also see this using Lemma 2.1.1.)

Let us now show that determinacy for games in game trees with taboos is level by level equivalent to determinacy for games in ordinary game trees. By the remark above, determinacy in ordinary game trees is equivalent level by level to determinacy in ordinary game trees that have no terminal positions, so we need only consider the latter. In one direction, note that any game tree without terminal nodes can be considered a game tree with taboos by setting $\mathcal{T}_{\mathrm{I}} = \mathcal{T}_{\mathrm{II}} = \emptyset$. In the other direction, let $\mathbf{T}$ be a game tree with taboos. If $G(\lceil T \rceil \setminus \mathcal{T}_{\mathrm{I}}; T)$ is a win for II, then all games in $\mathbf{T}$ are wins for II. Assume otherwise and let R be I's non-losing quasistrategy for $G(\lceil T \rceil \setminus \mathcal{T}_{\mathrm{I}}; T)$. If $G(\mathcal{T}_{\mathrm{II}}; R)$ is a win for I, then all games in $\mathbf{T}$ are wins for I. Assume otherwise and let S be II's non-losing quasistrategy for $G(\mathcal{T}_{\mathrm{II}}; R)$. The game subtree S of T satisfies $\lceil S \rceil \subseteq [T]$. Moreoever, for any $A \subseteq [T]$, the games $G(A \cap \lceil S \rceil; S)$ and $G(A; \mathbf{T})$ are completely equivalent; in particular, the latter is determined if the former is. Finally, we have that $A \cap \lceil S \rceil$ is as simple topologically as A. One consequence of this is that our previous determinacy results hold also for game trees with taboos:

Lemma 2.1.2. *Theorems 1.2.4, 1.3.1, 1.3.3, 1.4.10, and 1.4.22 and Corollaries 1.2.3, 1.4.15, and 1.4.23, hold for games in game trees with taboos.*

Proof. The argument given in the paragraph preceding the statement of the lemma goes through in ZC$^-$ + Σ_1 Replacement. Thus the Theorems listed in the statement of the lemma holds for games in trees with taboos. Corollary 1.2.3 follows from Theorem 1.2.4. To see that Corollaries 1.4.15

and 1.4.23 follow, it is suffices to show that Theorems 1.4.2 and 1.4.21 hold in each $[T]$. This in turn follows from the original Theorems 1.4.2 and 1.4.21 for $\lceil \bar{T} \rceil$, where $\bar{T}$ is as above. □

Remark. Since games in $\mathbf{T}$ are equivalent to games in the S defined above, we could avoid dealing with game trees with taboos by replacing each $\mathbf{T}$ with the corresponding S. In a sense, that is what is done in [Martin, 1975] and [Martin, 1985]. Here, however, we are interested in avoiding the nuisance of quasistrategies, and so we put up with the nuisance of taboos.

If $\tilde{\mathbf{T}}$ and $\mathbf{T}$ are game trees with taboos, we write $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$ to mean that

(i) $\pi : \tilde{T} \to T$;

(ii) $\tilde{p} \subseteq \tilde{q} \to \pi(\tilde{p}) \subseteq \pi(\tilde{q})$ for all $\tilde{p}$ and $\tilde{q}$ belonging to $\tilde{T}$;

(iii) $\ell\mathrm{h}(\pi(\tilde{p})) = \ell\mathrm{h}(\tilde{p})$ for all $\tilde{p} \in \tilde{T}$.

(iv) $\pi(\tilde{p}) \in \mathcal{T}_{\mathrm{I}} \to \tilde{p} \in \tilde{\mathcal{T}}_{\mathrm{I}}$ for all $\tilde{p} \in \tilde{T}$;

(v) $\pi(\tilde{p}) \in \mathcal{T}_{\mathrm{II}} \to \tilde{p} \in \tilde{\mathcal{T}}_{\mathrm{II}}$ for all $\tilde{p} \in \tilde{T}$;

Note that it is allowed that $\tilde{p}$ be terminal in $\tilde{T}$ (and so taboo in $\tilde{\mathbf{T}}$) even though $\pi(\tilde{p})$ is not terminal in T.

Let $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$. If $\tilde{x}$ is a play in $\tilde{T}$, then clause (ii) implies that $\bigcup_{\tilde{p} \subseteq \tilde{x}} \pi(\tilde{p})$ is either a position or a play in T. If $\tilde{x}$ is finite, then $\bigcup_{\tilde{p} \subseteq \tilde{x}} \pi(\tilde{p}) = \pi(\tilde{x})$. Thus we can extend π to a function, which we also denote by "π," from $\tilde{T} \cup \lceil \tilde{T} \rceil$ to $T \cup \lceil T \rceil$. By clause (iii), $\ell\mathrm{h}(\pi(\tilde{x})) = \ell\mathrm{h}(\tilde{x})$ for all plays $\tilde{x}$, where we recall that $\ell\mathrm{h}(\tilde{x}) = \omega$ if x is infinite. If $\tilde{x}$ is an infinite play in $\tilde{T}$, then $\pi(\tilde{x})$ is an infinite play in T. Thus π induces a function

$$\boldsymbol{\pi} : [\tilde{T}] \to [T].$$

The function $\boldsymbol{\pi}$ is continuous and satisfies a "Lipschitz condition," i.e. $\boldsymbol{\pi}(\tilde{x}) \restriction n$ depends only on $\tilde{x} \restriction n$.

If $\tilde{\mathbf{T}}$ and $\mathbf{T}$ are game trees with taboos, we write $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$ to mean that

(i) $\phi : \mathcal{S}(\tilde{T}) \to \mathcal{S}(T)$;

(ii) each $\phi(\tilde{\sigma})$ is a strategy for the same player as is $\tilde{\sigma}$;

(iii) for each $n \in \omega$, the restriction of $\phi(\tilde{\sigma})$ to positions of length $< n$ depends only on the restriction of $\tilde{\sigma}$ to positions of length $< n$.

If T is a game tree and $k \in \omega$, let

$$_kT = \{p \in T \mid \ell\mathrm{h}(p) \leq k\}.$$

By clause (iii) of the definition, we can think of a ϕ such that $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$ as acting on $\bigcup_{k\in\omega} \mathcal{S}(_k\tilde{T})$ so that, for each k, $\phi \restriction \mathcal{S}(_k\tilde{T}) : \mathcal{S}(_k\tilde{T}) \to \mathcal{S}(_kT)$.

We are now ready to give the main technical definition of this chapter. If $\mathbf{T}$ is a game tree with taboos, then a *covering* of $\mathbf{T}$ is a quadruple $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ such that

(a) $\tilde{\mathbf{T}}$ is a game tree with taboos;

(b) $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$;

(c) $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$;

(d) $\Psi : \{\langle \tilde{\sigma}, x\rangle \mid \tilde{\sigma} \in \mathcal{S}(\tilde{T}) \wedge x \in \lceil T \rceil \wedge x \text{ is consistent with } \phi(\tilde{\sigma})\} \to \lceil \tilde{T} \rceil$, and, for all $\langle \tilde{\sigma}, x\rangle \in \mathrm{domain}\,(\Psi)$,

  (i) $\Psi(\tilde{\sigma}, x)$ is consistent with $\tilde{\sigma}$;

  (ii) $\pi(\Psi(\tilde{\sigma}, x)) \subseteq x$;

  (iii) either $\pi(\Psi(\tilde{\sigma}, x)) = x$ or $\Psi(\tilde{\sigma}, x)$ is taboo for the player for whom $\tilde{\sigma}$ is a strategy.

With regard to clause (d)(iii), note that $\pi(\Psi(\tilde{\sigma}, x)) = x$ implies $\ell\mathrm{h}(\Psi(\tilde{\sigma}, x)) = \ell\mathrm{h}(x)$; and this in turn implies that $\Psi(\tilde{\sigma}, x)$ and x are both finite or both infinite. Note also that if both are finite then, by clauses (iv) and (v) of the definition of $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$, both are taboo for the same player.

Remarks:

(a) A variant definition, and one that has some advantages which we will point out later, would replace the quadruple $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ by the triple $\langle \tilde{\mathbf{T}}, \pi, \phi, \rangle$ and replace clause (d) by

(d′) if $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ and x is consistent with $\tilde{\sigma}$, then there is an $\tilde{x} \in \lceil \tilde{T} \rceil$ such that

  (i) $\tilde{x}$ is consistent with $\tilde{\sigma}$;

  (ii) $\pi(\tilde{x}) \subseteq x$;

  (iii) either $\pi(\tilde{x}) = x$ or $\tilde{x}$ is taboo for the player for whom $\tilde{\sigma}$ is a strategy.

(b) Although the fact will not be directly used by us, the π of a covering is a surjection. Indeed, every play in T is in the range of the extended π. (Exercise 2.1.4). For an example and an almost-example of coverings, see Exercises 2.1.3 and 2.1.5.

We say that a covering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ *unravels* a subset A of $[T]$ if the preimage $\boldsymbol{\pi}^{-1}(A)$ is a clopen subset of $[\tilde{T}]$.

We prove at once the basic lemma connecting coverings and unraveling with determinacy:

Lemma 2.1.3. *Let* $\mathbf{T}$ *be a game tree with taboos. If there is a covering of* $\mathbf{T}$ *that unravels* $A \subseteq [T]$*, then* $G(A; \mathbf{T})$ *is determined.*

Proof. Let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a covering of $\mathbf{T}$ that unravels $A \subseteq [T]$. By Lemma 2.1.2 (as applied to Corollary 1.2.3), $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ is determined. Let us call the player for whom $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ is a win the *good player* and let us call the other player the *bad player.* Let $\tilde{\sigma}$ be a winning strategy for the good player for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$. We show that $\phi(\tilde{\sigma})$ is a winning strategy for the good player for $G(A; \mathbf{T})$. Let x be a play in T consistent with $\phi(\tilde{\sigma})$. We must prove that x is a win for the good player in $G(A; \mathbf{T})$. We may assume that x is not taboo for the bad player.

It is enough to show that $\pi(\Psi(\tilde{\sigma}, x)) = x$ and that $\Psi(\tilde{\sigma}, x)$ is infinite. If this is true then, since $\boldsymbol{\pi} = \pi \restriction [\tilde{T}]$ and $\boldsymbol{\pi} : [\tilde{T}] \to [T]$,

$$\Psi(\tilde{\sigma}, x) \in \boldsymbol{\pi}^{-1}(A) \leftrightarrow \pi(\Psi(\tilde{\sigma}, x)) \in A \leftrightarrow x \in A.$$

Because $\Psi(\tilde{\sigma}, x)$ is a win for the good player in $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$, it follows that x is a win for the good player in $G(A; \mathbf{T})$.

By clause (d)(i) in the definition of a covering, $\Psi(\tilde{\sigma}, x)$ is a play in $\tilde{T}$ that is consistent with $\tilde{\sigma}$. Since $\tilde{\sigma}$ is a winning strategy, $\Psi(\tilde{\sigma}, x)$ cannot be taboo for the good player. Thus clause (d)(iii) gives that $\pi(\Psi(\tilde{\sigma}, x)) = x$. By the observations after the definition of a covering, x and $\Psi(\tilde{\sigma}, x)$ are both finite or both taboo for the same player. They cannot both be taboo for the same player, for $\Psi(\tilde{\sigma}, x)$ is not taboo for the good player, and we are assuming that x is not taboo for the bad player. □

In the proof of Lemma 2.1.3, the fact that $\boldsymbol{\pi}^{-1}(A)$ is clopen was used only to get that $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ is determined. Thus we have the following generalization of that Lemma.

Lemma 2.1.4. *Let* $\mathbf{T}$ *be a game tree with taboos, and let* $A \subseteq [T]$*. If there is a covering* $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ *of* $\mathbf{T}$ *such that* $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ *is determined, then* $G(A; \mathbf{T})$ *is determined.*

Borel determinacy will be proved if we can show that every Borel set is unraveled by a covering. To do this, we need to do two things: (i) We must show that every open set can be unraveled. (ii) We must find some operations on coverings corresponding to the operations that generate the Borel sets from the open sets. (i) is the heart of the proof. We begin with the more routine (ii).

Let $\mathbf{T}$ be a game tree with taboos and let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a covering of $\mathbf{T}$. For $k \in \omega$, $\mathcal{C}$ is a *k-covering* of $\mathbf{T}$ if

(i) ${}_k\tilde{T} = {}_kT$, ${}_k\tilde{T} \cap \tilde{\mathcal{T}}_{\mathrm{I}} = {}_kT \cap \mathcal{T}_{\mathrm{I}}$, and ${}_k\tilde{T} \cap \tilde{\mathcal{T}}_{\mathrm{II}} = {}_kT \cap \mathcal{T}_{\mathrm{II}}$;

(ii) $\pi \restriction {}_k\tilde{T}$ is the identity;

(iii) $\phi \restriction \mathcal{S}({}_k\tilde{T})$ is the identity.

Suppose that $\mathcal{C}_1 = \langle \mathbf{T}_1, \pi_1, \phi_1, \Psi_1 \rangle$ is a covering of $\mathbf{T}_0$ and that $\mathcal{C}_2 = \langle \mathbf{T}_2, \pi_2, \phi_2, \Psi_2 \rangle$ is a covering of $\mathbf{T}_1$. We define the *composition* $\mathcal{C}_1 \circ \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ to be

$$\langle \mathbf{T}_2, \pi_1 \circ \pi_2, \phi_1 \circ \phi_2, \Psi \rangle,$$

where $\Psi(\sigma, x) = \Psi_2(\sigma, \Psi_1(\phi_2(\sigma), x))$. We omit the routine proof of the following lemma.

Lemma 2.1.5. *The composition of coverings is a covering. For natural numbers* k_1 *and* k_2*, the composition of a* k_1*-covering and a* k_2*-covering is a* $\min\{k_1, k_2\}$*-covering.*

The next lemma gives us a sufficient condition that the limit of a sequence of k-coverings exist and be a k-covering. It is for constructing such limits that the concept of k-covering was introduced.

Lemma 2.1.6. *Let* $k \in \omega$*, let* $\mathbf{T}_i$*,* $i \in \omega$*, be game trees with taboos, and let* $\langle k_{j,i}, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \mid i \leq j \in \omega \rangle$ *be such that*

(1) if $i \leq j \in \omega$ *then* $\mathcal{C}_{j,i} = \langle \mathbf{T}_j, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \rangle$ *is a* $k_{j,i}$*-covering of* $\mathbf{T}_i$*;*

(2) if $i_1 \leq i_2 \leq i_3 \in \omega$ *then* $\mathcal{C}_{i_3,i_1} = \mathcal{C}_{i_2,i_1} \circ \mathcal{C}_{i_3,i_2}$*;*

(3) $\inf_{i \leq j \in \omega} k_{j,i} \geq k$*;*

(4) $\underline{\lim}_{j\in\omega}\inf_{j'\geq j} k_{j',j} = \infty$*; i.e., for all* $n \in \omega$ *there is an* $i \in \omega$ *such that* $k_{j',j} \geq n$ *for all* $j' \geq j \geq i$*.*

Then there is a $\mathbf{T}_\infty$ *with* $|T_\infty| \leq \sum_{i\in\omega} |T_i|$ *and there is a system*

$$\langle \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \mid i \in \omega \rangle$$

such that each $\mathcal{C}_{\infty,i} = \langle \mathbf{T}_\infty, \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \rangle$ *is a k-covering of* $\mathbf{T}_i$ *and such that, for* $i \leq j \in \omega$*,* $\mathcal{C}_{\infty,i} = \mathcal{C}_{j,i} \circ \mathcal{C}_{\infty,j}$*.*

Proof. The idea is that, because of (4), what is in essence the inverse limit exists. For $n \in \omega$, let i_n be the least number i such that, for all $j' \geq j \geq i$, $k_{j',j} \geq n$. Thus ${}_nT_j$, ${}_nT_j \cap (\mathcal{T}_j)_{\mathrm{I}}$, and ${}_nT_j \cap (\mathcal{T}_j)_{\mathrm{II}}$, are independent of j for $j \geq i_n$. For any finite sequence p, let

$$\begin{aligned} p \in T_\infty &\;\leftrightarrow\; p \in T_{i_{\ell\mathrm{h}(p)}}; \\ p \in (\mathcal{T}_\infty)_{\mathrm{I}} &\;\leftrightarrow\; p \in (\mathcal{T}_{i_{\ell\mathrm{h}(p)}})_{\mathrm{I}}; \\ p \in (\mathcal{T}_\infty)_{\mathrm{II}} &\;\leftrightarrow\; p \in (\mathcal{T}_{i_{\ell\mathrm{h}(p)}})_{\mathrm{II}}. \end{aligned}$$

Clearly $\mathbf{T}_\infty$ is a game tree with taboos and $|T_\infty| \leq \sum_{i\in\omega} |T_i|$. Since (3) gives that $i_n = 0$ for $n \leq k$, we have that ${}_kT_\infty = {}_kT_j$, ${}_kT_\infty \cap (\mathcal{T}_\infty)_{\mathrm{I}} = {}_kT_j \cap (\mathcal{T}_j)_{\mathrm{I}}$, and ${}_kT_\infty \cap (\mathcal{T}_\infty)_{\mathrm{II}} = {}_kT_j \cap (\mathcal{T}_j)_{\mathrm{II}}$ for each j, as required by clause (i) of the definition of a k-covering.

For $p \in {}_nT_\infty$, we let

$$\pi_{\infty,j}(p) = \begin{cases} p & \text{if } j \geq i_n; \\ \pi_{i_n,j}(p) & \text{if } j < i_n. \end{cases}$$

It is routine to check that each $\pi_{\infty,j}$ is well-defined, that $\pi_{\infty,j} : \mathbf{T}_\infty \Rightarrow \mathbf{T}_j$, and that $\pi_{\infty,j} = \pi_{j',j} \circ \pi_{\infty,j'}$ whenever $j \leq j' \in \omega$. Clearly $\pi_{\infty,j} \restriction {}_nT_\infty$ is the identity for each $j \geq i_n$, and so the fact that $i_n = 0$ for $n \leq k$ guarantees that every $\pi_{\infty,j} \restriction {}_kT_\infty$ is the identity, as required by clause (ii) of the definition of a k-covering.

Similarly, for $\sigma \in \mathcal{S}({}_nT_\infty)$, we let

$$\phi_{\infty,j}(\sigma) = \begin{cases} \sigma & \text{if } j \geq i_n; \\ \phi_{i_n,j}(\sigma) & \text{if } j < i_n. \end{cases}$$

We omit the verifications that each $\phi_{\infty,j}$ is well-defined, that each $\phi_{\infty,j} : \mathbf{T}_\infty \stackrel{\mathcal{S}}{\Rightarrow} \mathbf{T}_j$, and that $\phi_{\infty,j} = \phi_{j',j} \circ \phi_{\infty,j'}$ for all $j \leq j' \in \omega$. Since $\phi_{\infty,j} \restriction \mathcal{S}({}_nT_\infty)$

is the identity whenever $j \geq i_n$, the fact that $i_n = 0$ for $n \leq k$ guarantees that clause (iii) of the definition of a k-covering holds.

It remains to define the $\Psi^{j,\infty}$ and to verify clause (d) in the definition of a covering.

First note that we always have $(\Psi^{j,j'}(\sigma,x))\restriction k_{j',j} = x\restriction k_{j',j}$; for $(\Psi^{j,j'}(\sigma,x))\restriction k_{j',j} = \pi_{j',j}((\Psi^{j,j'}(\sigma,x)) \restriction k_{j',j}) \subseteq x \restriction k_{j',j}$, and $(\Psi^{j,j'}(\sigma,x)) \restriction k_{j',j} \subsetneq x \restriction k_{j',j}$ is impossible. Let $j \in \omega$ and let $\sigma \in \mathcal{S}(T_\infty)$. For $x \in \lceil T_j \rceil$ and x consistent with $\phi_{\infty,j}(\sigma)$, we can set

$$\Psi^{j,\infty}(\sigma,x) \;=\; \lim_{j'\to\infty}\Psi^{j,j'}(\phi_{\infty,j'}(\sigma),x),$$

since, for each $n \in \omega$,

$$\lim_{j'\to\infty}((\Psi^{j,j'}(\phi_{\infty,j'}(\sigma),x)) \restriction n) \;=\; \begin{cases} x \restriction n & \text{if } j \geq i_n; \\ (\Psi^{j,i_n}(\phi_{\infty,i_n}(\sigma),x)) \restriction n & \text{if } j < i_n. \end{cases}$$

If some $(\Psi^{j,\infty}(\sigma,x))\restriction n$ is not consistent with σ, then, for any j' such that $j \leq j'$ and $i_n \leq j'$, the same position $(\Psi^{j,j'}(\phi_{\infty,j'}(\sigma),x))\restriction n$ is not consistent with $\phi_{\infty,j'}(\sigma)$, which agrees with σ on positions of length $< n$. This contradicts property (d)(i) of the covering $\mathcal{C}_{j',j}$, so and property (d)(i) is verified for $\mathcal{C}_{\infty,j}$. For (d)(ii) and (d)(iii), note that we have, for each $n \in \omega$, for each j' such that $j \leq j'$ and $j' \geq i_n$, for each $\sigma \in \mathcal{S}(T_\infty)$, and for each $x \in \lceil T_j \rceil$ consistent with $\phi_{\infty,j}(\sigma)$, that

$$(\pi_{\infty,j}(\Psi^{j,\infty}(\sigma,x))) \restriction n \;=\; (\pi_{j',j}(\Psi^{j,j'}(\phi_{\infty,j'}(\sigma),x))) \restriction n.$$

Property (d)(ii) for $\mathcal{C}_{\infty,j}$ thus follows from property (d)(ii) for $\mathcal{C}_{j',j}$. Moreover, since $j' \geq i_n$ implies that $(\Psi^{j,\infty}(\sigma,x)) \restriction n = (\Psi^{j,j'}(\phi_{\infty,j'}(\sigma),x)) \restriction n$, property (d)(iii) for $\mathcal{C}_{\infty,j}$ also follows from property (d)(iii) for $\mathcal{C}_{j',j}$. We omit the verification that $\Psi^{j,\infty}(\sigma,x) = \Psi^{j',\infty}(\sigma,\Psi^{j,j'}(\phi_{\infty,j'}(\sigma),x))$ for all $j \leq j'$ and all $\langle\sigma,x\rangle$ in domain $(\Psi^{j,\infty})$. □

Remark. One advantage of adopting the alternative definition of covering considered in remark (a) on page 66 would be that the construction of the proof of Lemma 2.1.6 would literally be the construction of the inverse limit of the given system of coverings.

Lemma 2.1.7. *Let* $\mathbf{T}$ *be a game tree with taboos. If* $A \subseteq [T]$ *is open or closed and* $k \in \omega$*, then there is a* k*-covering of* $\mathbf{T}$ *that unravels* A*.*

Proof. Since any covering that unravels a set also unravels its complement, it is enough to prove that every closed subset of $[T]$ is, for each $k \in \omega$, unraveled by some k-covering of $\mathbf{T}$. Let then $A \subseteq [T]$ be closed. Recall that A is also closed as a subset of $\lceil T \rceil$. Let $k \in \omega$ and, since every $(k+1)$-covering is also a k-covering, assume without loss of generality that k is even.

We will define $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ and show that $\mathcal{C}$ is a k-covering and that $\mathcal{C}$ unravels A.

We begin with $\tilde{\mathbf{T}}$. Because we have to make $\mathcal{C}$ a k-covering, we let ${}_k\tilde{T} = {}_kT$, ${}_k\tilde{T} \cap \tilde{\mathcal{T}}_{\mathrm{I}} = {}_kT \cap \mathcal{T}_{\mathrm{I}}$, and ${}_k\tilde{T} \cap \tilde{\mathcal{T}}_{\mathrm{II}} = {}_kT \cap \mathcal{T}_{\mathrm{II}}$. All moves in $\tilde{T}$ will be moves in T, except for move k and move $k+1$. Each of these two moves will consist of a move in T together with one or two extra components.

To describe move k, let $p \in \tilde{T}$ with $\ell\mathrm{h}(p) = k$. Thus $p \in T$ also. If p is terminal in T—and so taboo in $\mathbf{T}$—then p is taboo in $\tilde{\mathbf{T}}$—and so terminal in $\tilde{T}$; and hence there is no move k. Assume therefore that p is not terminal in T. Since k is even, it is I's turn to move at p. We stipulate that I's move at p in $\tilde{T}$ must be of the form

$$\langle a, X \rangle,$$

where a is a move legal in T at p and X is a subset of the set Z of all $q \in T$ satisfying the following conditions:

(i) $p^\frown\langle a \rangle \subsetneq q$.

(ii) q is not terminal in T.

(iii) $[T_q] \cap A = \emptyset$.

(iv) $(\forall r)(p^\frown\langle a \rangle \subsetneq r \subsetneq q \rightarrow [T_r] \cap A \neq \emptyset)$.

Remark. Here is how to think of the move X. Suppose that the players are considering playing some game $G(B; \mathbf{T})$. Player I is asserting that he can win $G(B; \mathbf{T}_q)$ for every $q \in X$ and is conceding that II can win $G(B; \mathbf{T}_q)$ for every $q \in Z \setminus X$. If $x \in [T]_{p^\frown\langle a \rangle}$ then $x \notin A$ if and only if x extends some $q \in Z$. I is proposing that if and when a position $q \in Z$ is reached the play be terminated immediately, with I declared the winner if $q \in X$ and II declared the winner otherwise. In proposing this, I is proposing that the players should play out an infinite play only when that play belongs to A. The idea of having I play subsets of Z, rather than quasistrategies for I in $T_{p^\frown\langle a \rangle}$ or subtrees of $T_{p^\frown\langle a \rangle}$ is the idea of Hurkens mentioned on page 61.

If $p^\frown\langle a \rangle$ is taboo in $\mathbf{T}$, then we must let $p^\frown\langle\langle a, X \rangle\rangle$ be taboo for the same player in $\tilde{\mathbf{T}}$. Suppose that $p^\frown\langle a \rangle$ is not taboo in $\mathbf{T}$, and so is not terminal, in T. Then we make $p^\frown\langle\langle a, X\rangle\rangle$ not terminal in $\tilde{T}$. We allow II, in principle, two options for move $k+1$ in $\tilde{T}$, though the second option is available only if $X \neq \emptyset$.

Option (1). II may *accept* X. If II accepts X, then II's move in $\tilde{T}$ at $p^\frown\langle\langle a, X\rangle\rangle$ must be of the form

$$\langle 1, b\rangle,$$

where b is a legal move for II in T at $p^\frown\langle a\rangle$. We stipulate that the positions in $\tilde{T}$ that extend the resulting position $p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle$ are an initial seqment of the finite sequences of the form

$$p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle^\frown s$$

with $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s \in T$. A position of this form is to be terminal in $\tilde{T}$ if and only if one of the following holds.

(i) $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$ is terminal in T.

(ii) $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s \in Z$.

If (i) holds, then we make $p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle^\frown s$ is taboo in $\tilde{\mathbf{T}}$ for the player for whom $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$ is taboo in $\mathbf{T}$. If (ii) holds, then we let $p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle$ be taboo in $\tilde{\mathbf{T}}$ for II if $p^\frown\langle a\rangle^\frown\langle b\rangle \in X$ and for I otherwise. Note that (i) and (ii) cannot both hold, and note that either might hold for $s = \emptyset$.

Option (2). II may *challenge* X. If II challenges X, then II's move in $\tilde{T}$ at $p^\frown\langle\langle a, X\rangle\rangle$ must be of the form

$$\langle 2, r, b\rangle,$$

where $r \in X$ and $b = r(k+1)$ (so that $p^\frown\langle a\rangle^\frown\langle b\rangle \in T_r$). The positions in $\tilde{T}$ that extend $p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 2, r, b\rangle\rangle$ are to be precisely those of the form

$$p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 2, r, b\rangle\rangle^\frown s$$

with $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s \in T_r$. Such a position in $\tilde{T}$ is taboo for a player in $\tilde{\mathbf{T}}$ if and only if $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$ is taboo for that player in $\tilde{\mathbf{T}}$.)

Remark. Here is the way to think about II's two options. If II accepts X, then II accepts the proposal of I that was described in the remark on page 71. If II challenges X and makes the move $\langle 2, r, b\rangle$, then II is denying

I's contention that I can win the game $G(B; \mathbf{T}_r)$. The players then play that game to decide who is right. (Remember, of course, that the set B is entirely imaginary. We imagine it only to motivate the definition of $\tilde{\mathbf{T}}$.)

The definition of π is the obvious one:

$$(\pi(\tilde{p}))(i) = \begin{cases} \tilde{p}(i) & \text{if } i \neq k \text{ and } i \neq k+1; \\ a & \text{if } i = k \text{ and } \tilde{p}(k) = \langle a, X\rangle; \\ b & \text{if } i = k+1 \text{ and } \tilde{p}(k+1) = \langle 1, b\rangle; \\ b & \text{if } i = k+1 \text{ and } \tilde{p}(k+1) = \langle 2, r, b\rangle. \end{cases}$$

In other words, $\pi(\tilde{p})$ is obtained from $\tilde{p}$ by deleting the components X, 1, 2, and r that occur in $\tilde{p}$.

Before defining the rest of our covering, let us pause to verify that $\boldsymbol{\pi}^{-1}(A)$ is clopen, so that our covering will unravel A. If a $\tilde{x}$ is an infinite play in $\tilde{T}$ in which II accepts I's X, then no position in $\pi(\tilde{x})$ belongs to the associated Z. Hence $\lceil T \rceil_q \cap A \neq \emptyset$ for all $q \subseteq \pi(\tilde{x})$. Since A is closed, $\pi(\tilde{x}) \in A$. If $\tilde{x}$ is any play in $\tilde{T}$ of length $> k+1$ in which II challenges I's X, then $\pi(\tilde{x}) \notin A$, for $\pi(\tilde{x})$ must extend the r played by II at move $k+1$, and this r belongs to X and so to Z. Define $\tilde{A} \subseteq \lceil \tilde{T} \rceil$ by stipulating, for $\tilde{x} \in \lceil \tilde{T} \rceil$, that

$$\tilde{x} \in \tilde{A} \leftrightarrow (\ell\mathrm{h}(\tilde{x}) > k+1 \wedge \text{II accepts I's } X).$$

Clearly $\tilde{A}$ is clopen. Moreover $\tilde{A} \cap [\tilde{T}] = \boldsymbol{\pi}^{-1}(A)$, as required for unraveling.

Next we define ϕ and Ψ simultaneously. It will be clear from the definitions that clauses (c) and (d) in the definition of a covering and clause (iii) in the definition of a k-covering are satisfied.

First let $\tilde{\sigma} \in \mathcal{S}_{\mathrm{I}}(\tilde{T})$. Here is the idea: The strategy $\tilde{\sigma}$ supplies us with values of $(\phi(\tilde{\sigma}))(p)$ for $\ell\mathrm{h}(p) \leq k$. Furthermore $\tilde{\sigma}$ supplies us with an X, and thus we have a clear choice for $\Psi(\tilde{\sigma}, x) \restriction k+1$. As long as no position is reached that belongs to X, we get subsequent values of $\phi(\tilde{\sigma})$ from values of $\tilde{\sigma}$ gotten by assuming that II accepts X. If no position belonging to X is ever reached, then this assumption gives us $\Psi(\tilde{\sigma}, x)$ also. Suppose we reach a position $r \in X$. If we were to define $\Psi(\tilde{\sigma}, x)$ using the assumption that II accepts X, then we would make $\Psi(\tilde{\sigma}, x)$ taboo for II, in violation of clause (d)(iii) in the definition of a covering. But we can avoid such a violation, for $\langle 2, r, r(k+1)\rangle$ is a legal move $k+1$ in $\tilde{T}$ in the position $\Psi(\tilde{\sigma}, x) \restriction k+1$. We get subsequent values of $\phi(\tilde{\sigma})$, and we get $\Psi(\tilde{\sigma}, x)$, by assuming that this move is made.

Here are the formal details. We describe $\phi(\tilde{\sigma}) = \sigma$ by describing an arbitrary play x consistent with σ. We thus omit the definition of $\sigma(p)$ for p

inconsistent with σ. Such values can be assigned arbitrarily, except for the easily met constraints from clause (iii) in the definition of $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$ and clause (iii) in the definition of a k-covering.

At each position $p \subseteq x$, either we will have a guess $\psi(p)$ for $\Psi(\tilde{\sigma}, x) \restriction \ell\mathrm{h}(p)$ or else there will be a $q \subsetneq p$ such that $\psi(q)$ is taboo for I in $\tilde{\mathbf{T}}$ and we will have already set $\Psi(\tilde{\sigma}, x) = \psi(q)$. Each $\psi(p)$ will be such that $\psi(p) \in \tilde{T}$, $\psi(p)$ is consistent with $\tilde{\sigma}$, and $\pi(\psi(p)) = p$. At most once during the construction we will contradict our previous guesses: for at most one $p \subseteq x$, $\psi(p)$ will be defined but will not be an extension of the $\psi(p \restriction i)$ for $i < \ell\mathrm{h}(p)$.

We will arrange that $\psi(p)$ is taboo for II in $\tilde{\mathbf{T}}$ only if p is taboo for II in $\mathbf{T}$. If we reach a p such that $\psi(p)$ is terminal in $\tilde{T}$, then we set $\Psi(\tilde{\sigma}, x) = \psi(p)$. In such a case, if p is not terminal then we define σ on extensions of p to agree with some fixed (independent of x) strategy σ_p in T_p.

To begin, we let σ agree with $\tilde{\sigma}$ and $\psi(p) = p$ until (if ever) we have reached a position p of length k. At this point we still let $\psi(p) = p$. If p is not terminal and $\tilde{\sigma}(p) = \langle a, X\rangle$, then set $\sigma(p) = a$ and $\psi(p^\frown\langle a\rangle) = p^\frown\langle\langle a, X\rangle\rangle$. If the position $p^\frown\langle a\rangle$ is not terminal, let b be II's next move.

As long as no position is reached that belongs to X, we proceed as follows. For positions $q = p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$, let $\tilde{q} = p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle^\frown s$. If $\tilde{q} \in \tilde{T}$, then let $\psi(q) = \tilde{q}$ and, if $\tilde{q}$ is non-terminal and of even length, let $\sigma(q) = \tilde{\sigma}(\tilde{q})$. If there is a last $q \subseteq x$ such that the associated $\tilde{q}$ belongs to $\tilde{T}$, then there are two possibilities for this last q.

(a) q is terminal. Then $q = x$ and we let $\Psi(\tilde{\sigma}, x) = \psi(q)$.

(b) $q \in Z \setminus X$. Then $\psi(q)$ is taboo for I and we let $\Psi(\tilde{\sigma}, x) = \psi(q)$.

If there is no last q such that the associated $\tilde{q} \in \tilde{T}$, then the play x is infinite. In this case we set $\Psi(\tilde{\sigma}, x) = \bigcup_{q\subseteq x} \psi(q)$.

Suppose that there is a position $r \subseteq x$ that belongs to X. For some s, $r = p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$. We let $\tilde{r} = p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 2, r, b\rangle\rangle^\frown s$. Note that $\tilde{r}$ is a legal postion in $\tilde{T}$. Note also that $\tilde{r}^\frown t \in \tilde{T}$ for any t such that $r^\frown t \in T$. For positions $r^\frown t$, we set $\psi(r^\frown t) = \tilde{r}^\frown t$ and, for $r^\frown t$ of even length and not terminal, we let $\sigma(r^\frown t) = \tilde{\sigma}(\tilde{r}^\frown t)$. If the play x is infinite, we let $\Psi(\tilde{\sigma}, x) = \bigcup_{n\geq \ell\mathrm{h}(r)} \psi(x \restriction n)$.

Next let $\tilde{\tau} \in \mathcal{S}_{\mathrm{II}}(\tilde{T})$. Here is the idea: When we reach a position $p^\frown\langle a\rangle$ in T of length $k + 1$, there is a subset Y of the Z associated with $p^\frown\langle a\rangle$ such that

(i) τ calls for II to accept Y;

(ii) for any $r \in Z \setminus Y$, there is an $X \subseteq Z$ such that $\tau(p^\frown\langle\langle a, X\rangle\rangle) = \langle 2, r, r(k+1)\rangle$.

As long as no position is reached that belongs to $Z \setminus Y$, we get subsequent values of $\phi(\tilde{\tau})$ from valuses of $\tilde{\tau}$ gotten by assuming that I plays $\langle a, Y\rangle$. If no position belonging to $Z \setminus Y$ is ever reached, then this assumption gives us $\Psi(\tilde{\tau}, x)$ also. Suppose we reach a position $r \in Z \setminus Y$. If we were to define $\Psi(\tilde{\tau}, x)$ using the assumption that I plays $\langle a, Y\rangle$, then we would make $\Psi(\tilde{\tau}, x)$ taboo for I, in violation of clause (d)(iii) in the definition of a covering. We can avoid such a violation by using property (ii) of Y. If X is as given by (ii), then we get subsequent values of $\phi(\tilde{\tau})$, and we get $\Psi(\tilde{\tau}, x)$, by assuming that the moves $\langle a, X\rangle$ and $\langle 2, r, r(k+1)\rangle$ are made.

Now we give the formal details. As in the preceding case, we describe $\phi(\tilde{\tau}) = \tau$ by describing an arbitrary play x consistent with τ. At each position $p \subseteq x$, either we will have a guess $\psi(p)$ for $\Psi(\tilde{\tau}, x) \restriction \ell\text{h}(p)$ or else there will be $q \subsetneq p$ such that $\psi(q)$ is taboo for II in $\tilde{\mathbf{T}}$ and we will have already set $\Psi(\tilde{\tau}, x) = \psi(q)$. Each $\psi(p)$ will be such that $\psi(p) \in \tilde{T}$, $\psi(p)$ is consistent with $\tilde{\tau}$, and $\pi(\psi(p)) = p$. As before, there will be at most one $p \subseteq x$ such that $\psi(p)$ is defined but is not an extension of the $\psi(p \restriction i)$ for $i < \ell\text{h}(p)$.

We will arrange that $\psi(p)$ is not taboo for I in $\tilde{\mathbf{T}}$ unless p is taboo for I in $\mathbf{T}$. If we reach a p such that $\psi(p)$ is terminal, then we set $\Psi(\tilde{\tau}, x) = \psi(p)$. We use the same method as we used before for σ to define τ on extensions of p when $\psi(p)$ is terminal in $\tilde{T}$ but p is not terminal in T.

To begin, we let τ agree with $\tilde{\tau}$ and $\psi(p) = p$ until (if ever) we have reached a position p of length k. For this p also, we let $\psi(p) = p$. If p is not terminal, let a be I's move at p. Let Z be the set associated with $p^\frown\langle a\rangle$, the set of which the second component of move k must be a subset. Let

$$Y = \{r \in Z \mid (\forall X \subseteq Z)\, \tilde{\tau}(p^\frown\langle\langle a, X\rangle\rangle) \neq \langle 2, r, r(k+1)\rangle\}.$$

The move $\langle a, Y\rangle$ is legal for I in $\tilde{T}$ at p, and so we can let $\psi(p^\frown\langle a\rangle) = p^\frown\langle\langle a, Y\rangle\rangle$. Assume that $p^\frown\langle a\rangle$ is not terminal in T. Then $p^\frown\langle\langle a, Y\rangle\rangle$ is not terminal in $\tilde{T}$.

It is obvious from the definition of Y that Y has property (ii) above. Let us show that Y has property (i), i.e., that $\tilde{\tau}$ cannot call for II to challenge Y at $p^\frown\langle\langle a, Y\rangle\rangle$. Assume the contrary and let $\tilde{\tau}(p^\frown\langle\langle a, Y\rangle\rangle) = \langle 2, r, r(k+1)\rangle$. By the definition of Y, we have that $r \notin Y$. But challenging Y requires that $r \in Y$, so we have a contradiction.

Thus $\tilde{\tau}(p^\frown\langle\langle a, Y\rangle\rangle) = \langle 1, b\rangle$ for some b with $p^\frown\langle a\rangle^\frown\langle b\rangle \in T$. We let $\tau(p^\frown\langle a\rangle) = b$.

As long as no position is reached that belongs to $Z \setminus Y$, we proceed as follows. For positions $q = p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$, let $\tilde{q} = p^\frown\langle\langle a, Y\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle^\frown s$. If $\tilde{q} \in \tilde{T}$, then let $\psi(q) = \tilde{q}$ and, if $\tilde{q}$ is non-terminal and of odd length, let $\tau(q) = \tilde{\tau}(\tilde{q})$. If there is a last $q \subseteq x$ such that the associated $\tilde{q}$ belongs to $\tilde{T}$, then there are two possibilities for this last q.

(a) q is terminal. Then $q = x$ and we let $\Psi(\tilde{\tau}, x) = \psi(q)$.

(b) $q \in Y$. Then $\psi(q)$ is taboo for II and we let $\Psi(\tilde{\tau}, x) = \psi(q)$.

If there is no last q such that the associated $\tilde{q} \in \tilde{T}$, then the play x is infinite. In this case we set $\Psi(\tilde{\tau}, x) = \bigcup_{q \subseteq x} \psi(q)$.

Suppose that there is a position $r \subseteq x$ that belongs to $Z \setminus Y$. By property (ii) of Y, let $X \subseteq Z$ be such that $\tilde{\tau}(p^\frown\langle\langle a, X\rangle\rangle) = \langle 2, r, r(k+1)\rangle$. For some s, $r = p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$. We let $\tilde{r} = p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 2, r, b\rangle\rangle^\frown s$. Note that $\tilde{r}$ is a legal postion in $\tilde{T}$. Note also that $\tilde{r}^\frown t \in \tilde{T}$ for any t such that $r^\frown t \in T$. For positions $r^\frown t$, we set $\psi(r^\frown t) = \tilde{r}^\frown t$ and, for $r^\frown t$ of odd length and not terminal, we let $\tau(r^\frown t) = \tilde{\tau}(\tilde{r}^\frown t)$. If the play x is infinite, we let $\Psi(\tilde{\tau}, x) = \bigcup_{n \geq \ell\mathrm{h}(r)} \psi(x \restriction n)$. $\square$

Theorem 2.1.8. ([Martin, 1985]) *Let* $\mathbf{T}$ *be a game tree with taboos. If* $A \subseteq [T]$ *is Borel and* $k \in \omega$*, then there is a* k*-covering of* $\mathbf{T}$ *that unravels* A*.*

Proof. By induction on countable ordinals $\alpha \geq 1$, we prove

$(\dagger)_\alpha$ *For all* $\mathbf{T}$*, for all* $A \subseteq [T]$ *such that* $A \in \mathbf{\Sigma}^0_\alpha$*, and for all* $k \in \omega$*, there is a* k*-covering of* $\mathbf{T}$ *that unravels* A*.*

$(\dagger)_1$ is equivalent with Lemma 2.1.7. Assume then that $\alpha > 1$ and that $(\dagger)_\beta$ holds for all β with $1 \leq \beta < \alpha$. Let $k \in \omega$ and let $A \subseteq [T]$ with $A \in \mathbf{\Sigma}^0_\alpha$. By the definition of $\mathbf{\Sigma}^0_\alpha$, there are B_n, $n \in \omega$, such that each B_n belongs to $\mathbf{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$ and such that $A = \bigcup_{n \in \omega} B_n$.

Let $\mathbf{T}_0 = \mathbf{T}$. By induction on $j' \in \omega$, we define $\mathbf{T}_{j'}$ and

$$\mathcal{C}_{j',j} = \langle \mathbf{T}_{j'}, \pi_{j',j}, \phi_{j',j}, \Psi^{j,j'} \rangle$$

for $j \leq j'$ such that $\mathcal{C}_{j',i} = \mathcal{C}_{j,i} \circ \mathcal{C}_{j',j}$ for all $i \leq j \leq j'$. We do this in such a way that each $\mathcal{C}_{j',j}$ is a $(k+j)$-covering of $\mathbf{T}_j$ and $\mathcal{C}_{j',0}$ unravels B_j for each

$j \le j'$. Note that $\mathcal{C}_{j',j'}$ must be the trivial covering, with $\pi_{j',j'}$ and $\phi_{j',j'}$ the identities and $\Psi^{j',j'}(\sigma, x) = x$ for all σ and x.

Suppose that we have defined $\mathbf{T}_{j'}$ and the $\mathcal{C}_{j',j}$ for all $j' \le n$. By the continuity of $\boldsymbol{\pi}_{n,0}$, we have that $\boldsymbol{\pi}_{n,0}{}^{-1}(B_n) \in \mathbf{\Pi}^0_{\beta_n}$. By $(\dagger)_{\beta_n}$, let $\mathcal{C}_n = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a $(k+n)$-covering of $\mathbf{T}_n$ that unravels $\boldsymbol{\pi}_{n,0}{}^{-1}([T] \setminus B_n)$ and so unravels $\boldsymbol{\pi}_{n,0}{}^{-1}(B_n)$. Let $\mathbf{T}_{n+1} = \tilde{\mathbf{T}}$. For $j \le n$, let $\mathcal{C}_{n+1,j} = \mathcal{C}_{n,j} \circ \mathcal{C}_n$; let $\mathcal{C}_{n+1,n+1}$ be the trivial covering. The required properties of the $\mathcal{C}_{n+1,j}$ follow directly from Lemma 2.1.5 and the continuity of the $\boldsymbol{\pi}_{n,j}$.

If we let $k_{j,i} = k + i$, then the hypotheses of Lemma 2.1.6 hold. Let $\mathbf{T}_\infty$ and, for $i \in \omega$, $\mathcal{C}_{\infty,i} = \langle \mathbf{T}_\infty, \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \rangle$ be given by that lemma. For each n, $\boldsymbol{\pi}_{\infty,0}{}^{-1}(B_n)$ is clopen, by the continuity of $\boldsymbol{\pi}_{\infty,n+1}$. Thus $\boldsymbol{\pi}_{\infty,0}{}^{-1}(A)$ is open. By Lemma 2.1.7, let $\tilde{\mathcal{C}}$ be a k-covering of $\mathbf{T}_\infty$ that unravels $\boldsymbol{\pi}_{\infty,0}{}^{-1}(A)$. $\mathcal{C}_{\infty,0} \circ \tilde{\mathcal{C}}$ is a k-covering of T that unravels A. $\square$

Theorem 2.1.9. ([Martin, 1975]) *All Borel games are determined.*

Proof. The theorem follows immediately from Lemma 2.1.3 and Theorem 2.1.8. $\square$

Exercise 2.1.1. Consider the following two strengthenings of AD.

(1) $\mathrm{AD}_{\mathbb{R}}$, the assertion that all games in ${}^{<\omega}({}^{\omega}\omega)$ are determined;

(2) $\mathrm{AD}(\omega^2)$, the assertion that all games of length ω^2 with moves in ω are determined.

Prove that $\mathrm{AD}_{\mathcal{R}}$ and $\mathrm{AD}(\omega_2)$ are equivalent.

Hint. In the non-trivial direction, consider a game of length ω in which I's individual moves are strategies for games in ${}^{<\omega}\omega$ and II's moves are plays consistent with these strategies.

Remarks:

(a) This result was proved independently by Andreas Blass and Jan Mycielski. (See [Blass, 1975].) Until the author learned of it in 1974, his and others' attempts to prove Borel determinacy involved auxiliary games with individual moves that were ordinal numbers. (See [Paris, 1972] for a partial success.) The Blass–Mycielski proof suggested trying games with individual moves that were strategies (or something similar). In [Martin, 1975] and [Martin, 1985], there are individual moves that are quasistrategies. In the version of the proof we have just presented, however, the quasistrategies have disappeared.

(b) Oddly enough, the determinacy of all games of countable length, with real or natural number moves, follows from $\mathrm{AD}_{\mathbb{R}}$. This fact is a consequence of a theorem independently proved by Hugh Woodin and the author, together with another theorem of Woodin. See [Martin, 2015].

Exercise 2.1.2. Let $A \subseteq \lceil T \rceil$ and suppose that $A = \bigcup_{i\in\omega} A_i$, with each A_i closed. Consider the following game $G^* = G(A^*; T^*)$. I begins by picking a strategy σ_0 for I in T. II then chooses a position $p_0 \in T$ consistent with σ_0. If the position in T^* is not terminal (as defined below), I next picks a strategy σ_1 for I in T_{p_0}; II picks $p_1 \in T_{p_0}$ consistent with σ_1 such that $p_1 \supseteq p_0$; etc. If some $\lceil T_{p_i} \rceil$ is not disjoint from A_i, then the position just after II has picked p_i is terminal. This is the only way terminal positions in G^* arise. A play of G^* is a win for I if and only if the play is finite. Prove using G^* and Theorem 1.2.4 that $G(A;T)$ is determined.

Exercise 2.1.3. Modify the T^* of Exercise 2.1.2 to get a covering of $\mathbf{T} = \langle T, \emptyset, \emptyset \rangle$ that unravels the A of Exercise 2.1.2.

Exercise 2.1.4. Let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a covering of $\mathbf{T}$. Show that the extended $\pi : \tilde{T} \cup \lceil \tilde{T} \rceil \to T \cup \lceil T \rceil$ is a surjection.

Hint. Let $x \in \lceil T \rceil$. Consider the game in $\tilde{T}$ that I wins unless someone makes a Move $\tilde{p}$ such that $\pi(\tilde{p}) \not\subseteq x$ and I is the first player to do so. Prove that this game is a win for I. Prove that the analogous game with the roles of the players reversed is a win for II.

Exercise 2.1.5. Work in ZF and assume AD. Let $T = {}^{<\omega}\omega$. Let games in $\tilde{T}$ be played as follows:

$$\begin{array}{lccccccc} \mathrm{I} & \langle \sigma, n_0 \rangle & & n_2 & & n_4 & & \ldots \\ \mathrm{II} & & \langle x, n_1 \rangle & & n_3 & & n_5 & & \ldots \end{array}$$

Here σ must be a strategy for I in T with $\sigma(\emptyset) = n_0$, and x must be a play in T consistent with σ. Each n_i must be $x(i)$. (Thus only σ and x matter.) Use $\tilde{T}$ to get a $(\tilde{\mathbf{T}}, \pi, \phi, \Psi)$ that fails to be a covering of $\mathbf{T} = \langle T, \emptyset, \emptyset \rangle$ unraveling every subset of ${}^{\omega}\omega$ only in that ϕ is not single-valued.

In [Martin, 1985] it is asserted that a certain uniformization hypothesis permits one to get a single-valued ϕ. The hypothesis is mistated in

[Martin, 1985], but the intended one does not work. In [Neeman, 2000] it is shown that every $\mathbf{\Pi}^1_1$ subset of ${}^\omega\omega$ can be unraveled by a covering of $\langle {}^{<\omega}\omega, \emptyset, \emptyset\rangle$.

Exercise 2.1.6. Under the hypotheses of Lemma 2.1.6, let $\mathbf{T}_\infty$ and $\langle \mathcal{C}_{\infty,i} \mid i \in \omega\rangle$ be the tree and sequence of coverings constructed in the proof of that lemma. Suppose that $\mathbf{T}'$ and $\langle \mathcal{C}'_{\infty,i} \mid i \in \omega\rangle$ are such that each $\mathcal{C}'_{\infty,i}$ is a k-covering of $\mathbf{T}_i$ with first component $\mathbf{T}'$ and such that, for $i \leq j \in \omega$, $\mathcal{C}'_{\infty,i} = \mathcal{C}'_{j,i} \circ \mathcal{C}'_{\infty,j}$. Show that there are π',ϕ', and Ψ' such that $\mathcal{C}' = \langle \mathbf{T}', \pi', \phi', \Psi'\rangle$ is a k-covering of $\mathbf{T}_\infty$ and, for each $i \in \omega$, $\mathcal{C}'_{\infty,i} = \mathcal{C}_{\infty,i} \circ \mathcal{C}'$.

2.2 Uncountable Trees

The Souslin Theorem (see Theorem 2E.2 of [Moschovakis, 1980]) asserts that, in countable trees, the Borel sets are the same as the $\mathbf{\Delta}^1_1$ sets (which will be defined below). For uncountable trees, the $\mathbf{\Delta}^1_1$ sets form a larger class than the Borel sets. In this section, we will define the class of *quasi-Borel* sets. We will prove that the quasi-Borel subsets of $\lceil T\rceil$ are the same as the $\mathbf{\Delta}^1_1$ subsets of $\lceil T\rceil$ for every T. This is the special case for spaces of the form $\lceil T\rceil$ of a theorem of R.W. Hansell ([Hansell, 1973a] and [Hansell, 1973b].) We will prove general $\mathbf{\Delta}^1_1$ determinacy by proving that all quasi-Borel games are determined. This determinacy result is from [Martin, 1990].

Remark. In [Martin, 1990], the author credited the concept of quasi-Borel sets to himself. After the publication of [Martin, 1990], Alberto Marcone pointed out to the author that the concept had been introduced by R.W. Hansell in Hansell [1972]. In [Hansell, 1973a] and [Hansell, 1973b], what we call quasi-Borel sets were called *extended Borel* sets. In [Martin, 1990] the author also wrongly credited to himself the fact that the quasi-Borel sets are the same as the $\mathbf{\Delta}^1_1$ sets.

The definition of the quasi-Borel sets is like that of the Borel sets, except that an additional operation, besides those of forming countable unions and complements, is required to generate them from the open sets. Our proof of quasi-Borel determinacy will be a minor modification of our proof of Borel determinacy, with extra lemmas to take care of the extra operation. This result is relevant even for games in countable trees: we will use it later §5.2 in getting as strong a determinacy result as possible from the assumption that a measurable cardinal exists.

We begin by defining the quasi-Borel sets and studying their properties. To do so we must define the extra operation needed to generate them. Let T be a game tree. If A and B_j, $j \in J$, are all subsets of $\lceil T \rceil$, then A comes from $\{B_j \mid j \in J\}$ by the operation of *open-separated union*, or, equivalently, A is the *open-separated union* of $\{B_j \mid j \in J\}$, if

(a) $A = \bigcup_{j \in J} B_j$;

(b) there are disjoint open sets D_j, $j \in J$, such that $B_j \subseteq D_j$ for each $j \in J$.

A set is *quasi-Borel* if it belongs to the smallest class containing the open sets and closed under countable unions, open-separated unions, and complements.

There is no clearly best way to define a quasi-Borel hierarchy. The one in [Martin, 1990] is different from the one we are about to give here.

Recall from §1.4 that A is the *fully open-separated union* of $\{B_j \mid j \in J\}$ if some $\{D_j \mid j \in J\}$ witnessing that A is the open-separated union of $\{B_j \mid j \in J\}$ satisfies $\bigcup_{j \in J} D_j = \lceil T \rceil$. Recall also that if $\{D_j \mid j \in J\}$ witnesses that A is the fully open-separated union of $\{B_j \mid j \in J\}$, then each D_j is clopen.

If α is a limit ordinal, then $\mathrm{cf}(\alpha)$, the *cofinality* of α, is the least ordinal ρ such that some $f : \rho \to \alpha$ has unbounded range. The cofinality of α is always a regular cardinal $\leq \alpha$.

We define the *quasi-Borel hierarchy* (of subsets of $\lceil T \rceil$) as follows:

(1) $\boldsymbol{\Sigma}^*_1$ is the class of open sets;

(2) $\boldsymbol{\Pi}^*_\alpha$ is the set of complements of members of $\boldsymbol{\Sigma}^*_\alpha$;

(3) if $\alpha > 1$ is a successor ordinal or if α is a limit ordinal and $\mathrm{cf}(\alpha) = \omega$, then $\boldsymbol{\Sigma}^*_\alpha$ is the set of countable unions of members of $\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^*_\beta$;

(4) if α is a limit ordinal and $\mathrm{cf}(\alpha) \neq \omega$ (so $\mathrm{cf}(\alpha)$ is uncountable), then $\boldsymbol{\Sigma}^*_\alpha$ is the set of all fully open-separated unions of members of $\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^*_\beta$.

The following lemma gives some properties of the quasi-Borel hierarchy and of quasi-Borel sets. The important ones for us are (1)(f) and (2). The latter is non-trivial, because our definition of the $\boldsymbol{\Sigma}^*_\alpha$ uses fully open-separated unions.

Lemma 2.2.1. *(1) The following assertions hold for every $\alpha \geq 1$:*

(a) $(\forall \beta > \alpha)\, \mathbf{\Sigma}^*_\alpha \cup \mathbf{\Pi}^*_\alpha \subseteq \mathbf{\Delta}^*_\beta$.

(b) If α *is a limit ordinal of uncountable cofinality, then* $\mathbf{\Sigma}^*_\alpha = \mathbf{\Pi}^*_\alpha$*, i.e.* $\mathbf{\Sigma}^*_\alpha$ *is closed under complements.*

(c) If α *is a successor ordinal or a limit ordinal of cofinality* ω*, then* $\mathbf{\Sigma}^*_\alpha$ *is closed under countable unions.*

(d) If α *is a successor ordinal or a limit ordinal of cofinality* ω*, then* $\mathbf{\Pi}^*_\alpha$ *is closed under countable intersections.*

(e) $\mathbf{\Sigma}^*_\alpha$ *is closed under finite unions and finite intersections.*

(f) If α *is a limit ordinal of uncountable cofinality and* A *is any subset of* $\lceil T \rceil$*, then* A *belongs to* $\mathbf{\Sigma}^*_\alpha$ *if and only if there is a set* $\mathcal{D} \subseteq T$ *and there are* B_d*,* $d \in \mathcal{D}$*, such that* $\bigcup_{d\in\mathcal{D}} \lceil T_d \rceil = \lceil T \rceil$*, such that any two elements of* $\mathcal{D}$ *are incomparable with respect to* $\subseteq$*, such that each* $B_d \subseteq \lceil T_d \rceil$*, such that each* $B_d \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$*, and such that* $A = \bigcup_{d\in\mathcal{D}} B_d$*. In other words, we can replace "fully open-separated unions" by "fully basic-open-separated unions" in clause (4) of the definition of the quasi-Borel hierarchy.*

(2) If T *is infinite, then a subset of* $\lceil T \rceil$ *is quasi-Borel if and only if it belongs to* $\bigcup_{\alpha<|T|^+} \mathbf{\Sigma}^*_\alpha$*. Thus, in particular, the Borel sets and the quasi-Borel sets are the same for countable* T*.*

Proof. (1)(a). If $A \in \mathbf{\Pi}^*_\alpha$, then $A = \bigcup\{A\}$. Since this trivial union is both countable and fully open-separated, it follows that $A \in \mathbf{\Sigma}^*_\beta$ for every $\beta > \alpha$. Thus $\mathbf{\Pi}^*_\alpha \subseteq \mathbf{\Sigma}^*_\beta$ whenever $1 \le \alpha < \beta$. From this we have also that $\mathbf{\Sigma}^*_\alpha \subseteq \mathbf{\Pi}^*_\beta$ whenever $1 \le \alpha < \beta$. If $1 \le \alpha < \beta$ and $\beta > \alpha + 1$, then we have that

$$\mathbf{\Sigma}^*_\alpha \subseteq \mathbf{\Pi}^*_{\alpha+1} \subseteq \mathbf{\Sigma}^*_\beta.$$

It remains then only to show that $\mathbf{\Sigma}^*_\alpha \subseteq \mathbf{\Sigma}^*_{\alpha+1}$ for all $\alpha \ge 1$. The proof of (1)(a) of Lemma 1.1.1 showed that that every element of $\mathbf{\Sigma}^*_1$ $(= \mathbf{\Sigma}^0_1)$ is a countable union of $\mathbf{\Pi}^*_1$ sets, and so that $\mathbf{\Sigma}^*_1 \subseteq \mathbf{\Sigma}^*_2$. If α is a successor or has cofinality ω, then it is immediate from clause (3) of the definition that $\mathbf{\Sigma}^*_\alpha \subseteq \mathbf{\Sigma}^*_{\alpha+1}$. If α is a limit ordinal of uncountable cofinality, then it follows from (1)(b)—which is proved below using only the part of (1)(a) already proved—that $\mathbf{\Sigma}^*_\alpha = \mathbf{\Pi}^*_\alpha \subseteq \mathbf{\Sigma}^*_{\alpha+1}$.

(1)(b). Let α be a limit ordinal of uncountable cofinality and suppose that $\{D_j \mid j \in J\}$ witnesses that A is the fully open-separated union of

$\{B_j \mid j \in J\}$, where each $B_j \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$. Then $\{D_j \mid j \in J\}$ also witnesses that $\neg A$ is the fully open-separated union of $\{D_j \setminus B_j \mid j \in J\}$. Moreover each $(D_j \setminus B_j) \in \bigcup_{\beta<\alpha} \mathbf{\Sigma}^*_\beta \subseteq \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_{\beta+1} \subseteq \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$.

(1)(c) is immediate as in Lemma 1.1.1, and (1)(d) follows directly from (1)(c).

(1)(e). We prove by induction on α that $\mathbf{\Sigma}^*_\alpha$ is closed under finite intersections. (1)(e) then follows by (1)(b) and (1)(c). The case $\alpha = 1$ is immediate. For α a successor or a limit of cofinality ω, we argue as in the proof of part (1)(b) of Lemma 1.1.1, except that the last step of the argument now comes by our induction hypothesis. Assume then that α is a limit of uncountable cofinality and suppose that, for each $i < n \in \omega$, $\{D^i_j \mid j \in J_i\}$ witnesses that A^i is the fully open-separated union of $\{B^i_j \mid j \in J_i\}$, with each $B^i_j \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$. Then

$$\{\bigcap_{i<n} D^i_{s(i)} \mid s \in \prod_{i<n} J_i\}$$

witnesses that $\bigcap_{i<n} A^i$ is the fully open-separated union of

$$\{\bigcap_{i<n} B^i_{s(i)} \mid s \in \prod_{i<n} J_i\}.$$

(1)(a) and our induction hypothesis give that each $\bigcap_{i<n} B^i_{s(i)} \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$.

(1)(f). The "if" direction is trivial, so we prove only the other direction. Let α be a limit ordinal of uncountable cofinality. Let $\{D'_j \mid j \in J\}$ witness that A is the fully open-separated union of $\{B'_j \mid j \in J\}$, where each $B'_j \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$. For each $j \in J$, let $\mathcal{D}_j$ be the set of all $p \in T$ such that $\lceil T_p \rceil \subseteq D'_j$ but $(\forall q \subsetneq p)\, \lceil T_q \rceil \not\subseteq D'_j$. Clearly $\bigcup_{p\in\mathcal{D}_j} \lceil T_p \rceil \subseteq D'_j$. To see that the reverse inclusion also holds, suppose that $x \in D'_j$. Since D'_j is open, there is an $n \in \omega$ such that $\lceil T_{x\restriction n} \rceil \subseteq D'_j$. For the least such n, $x \restriction n \in \mathcal{D}_j$. Let $\mathcal{D} = \bigcup_{j\in J} \mathcal{D}_j$. By the definition of the $\mathcal{D}_j$ and by the disjointness of the D'_j, any two elements of $\mathcal{D}$ are incomparable with respect to $\subseteq$. Furthermore, $\bigcup_{d\in\mathcal{D}} \lceil T_d \rceil = \bigcup_{j\in J} \lceil D'_j \rceil = \lceil T \rceil$. For $j \in J$ and $d \in \mathcal{D}_j$, let $B_d = B'_j \cap \lceil T_d \rceil$. Then

$$A = \bigcup_{j\in J} B'_j = \bigcup_{j\in J} \bigcup_{d\in\mathcal{D}_j} B_d = \bigcup_{d\in\mathcal{D}} B_d.$$

Each B_d is the intersection of a clopen set with a member of $\bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$, and so (1)(e) gives that each $B_d \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$.

(2). We may suppose that T is infinite, since otherwise every subset of $\lceil T \rceil$ is clopen and so belongs to $\mathbf{\Sigma}^*_1$. As in the proof of Lemma 1.1.1, we get that $\bigcup_{\alpha<|T|^+} \mathbf{\Sigma}^*_\alpha$ is a class containing the open sets and closed under countable unions and complements. To see that this class is closed under open-separated unions as well, and so that every quasi-Borel set belongs to it, suppose that $\{D'_j \mid j \in J\}$ witnesses that A is the open-separated union of $\{B'_j \mid j \in J\} \subseteq \bigcup_{\alpha<|T|^+} \mathbf{\Sigma}^*_\alpha$. We may assume that J is uncountable, since otherwise $A \in \bigcup_{\alpha<|T|^+} \mathbf{\Sigma}^*_\alpha$, by closure under countable unions. For each $j \in J$, define $\mathcal{D}_j$ as in the proof above of (1)(f). Similarly define $\mathcal{D}$ and B_d, $d \in \mathcal{D}$. We have that any two elements of $\mathcal{D}$ are incomparable with respect to $\subseteq$, that $A = \bigcup_{d\in\mathcal{D}} B_d$, and that each $B_d \in \bigcup_{\alpha<|T|^+} \mathbf{\Sigma}^*_\alpha$. Since $\mathcal{D} \subseteq T$ and $\mathrm{cf}\,(|T|^+) > |T| \geq |\mathcal{D}| \geq |J| > \aleph_0$, there is limit ordinal $\alpha < |T|^+$ such that $\mathrm{cf}\,(\alpha) > \omega$ and such that each B_d belongs to $\bigcup_{\beta<\alpha} \mathbf{\Sigma}^*_\beta$ and so to $\bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$. Our problem is that we may not have that $\bigcup_{d\in\mathcal{D}} \lceil T_d \rceil = \lceil T \rceil$. To deal with this problem, let

$$\mathcal{D}^n = \{d \in \mathcal{D} \mid \ell\mathrm{h}(d) = n\}.$$

Let $A^n = \bigcup_{d\in\mathcal{D}^n} B_d$. Let $\mathcal{D}^n_+ = \{p \in T \mid \ell\mathrm{h}(p) = n\}$. For each n, $\{\lceil T_d \rceil \mid d \in \mathcal{D}^n_+\}$ witnesses that A^n is the fully open-separated union of $\{B_d \mid d \in \mathcal{D}^n\} = \{B_d \mid d \in \mathcal{D}^n\} \cup \emptyset$. Hence each $A^n \in \mathbf{\Sigma}^*_\alpha$. Since $A = \bigcup_{n\in\omega} A^n$, we get that $A \in \mathbf{\Sigma}^*_{\alpha+1}$.

The fact that every member of $\bigcup_{\alpha<|T|^+} \mathbf{\Sigma}^*_\alpha$ is quasi-Borel is proved by an easy induction on α. $\square$

Remarks:

(a) In general, the quasi-Borel sets form a larger class than the Borel sets. For example, let $T = \{\langle\alpha\rangle^\frown p \mid p \in {}^{<\omega}\omega \wedge \alpha < \omega_1\}$. For $\alpha < \omega_1$, let $B_\alpha \subseteq {}^\omega\omega$ with $B_\alpha \in \mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha$. Let $A = \{\langle\alpha\rangle^\frown y \mid y \in B_\alpha\}$. A is quasi-Borel but not Borel. See Exercise 2.2.1.

(b) Parts (1)(f) and (2) of Lemma 2.2.1 shows that, in the definition of *quasi-Borel*, we could replace "open-separated union" by "(fully) basic-open-separated union." What if we made replacements in the other direction, broadening rather than narrowing the class of allowable separating sets? Unfortunately, this would trivialize the concept: All points in $\lceil T \rceil$ are closed, so every subset of $\lceil T \rceil$ is a closed-separated union of closed sets.

(c) If $\mathbf{T}$ is a game tree with taboos and $[T]$ is nonempty, then, as we remarked in §2.1, the topological space $[T]$ is the same as the space $\lceil \bar{T} \rceil$, where $\bar{T}$ is the set of all $p \in T$ such that some infinite play in T extends

p. Thus Lemma 2.2.1 applies to $[T]$. In addition, we have the following generalization of Lemma 2.1.1.

Lemma 2.2.2. *Let* $\mathbf{T}$ *be a game tree with taboos. For all ordinals* α*, a subset* A *of* $[T]$ *belongs to* $\mathbf{\Pi}^*_\alpha$ *if and only if* $A \in \mathbf{\Pi}^*_\alpha$ *as a subset of* $\lceil T \rceil$*. For all ordinals* $\alpha > 1$*, a subset* A *of* $[T]$ *belongs to* $\mathbf{\Sigma}^*_\alpha$ *if and only if* $A \in \mathbf{\Sigma}^*_\alpha$ *as a subset of* $\lceil T \rceil$*.*

Proof. We prove the lemma by induction on α. The cases other than that of α a limit ordinal of uncountable cofinality are handled as in the proof of Lemma 2.1.1. Assume then that α is a limit ordinal and that $\mathrm{cf}(\alpha) > \omega$. Suppose first that $A \subseteq [T]$ belongs to $\mathbf{\Sigma}^*_\alpha$ as a subset of $\lceil T \rceil$. Let $\{D_j \mid j \in J\}$ witness that A is the fully open-separated union of $\{B_j \mid j \in J\}$, with each $B_j \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$ as a subset of $\lceil T \rceil$ and so, by induction, as a subset of $[T]$. Then $\{D_j \cap [T] \mid j \in J\}$ witnesses for the space $[T]$ that A is the fully open-separated union of $\{B_j \mid j \in J\}$. Thus $A \in \mathbf{\Sigma}^*_\alpha$ as a subset of $[T]$. Now suppose that $A \in \mathbf{\Sigma}^*_\alpha$ as a subset of $[T]$. Let $\{D_j \mid j \in J\}$ witness that A is the fully open-separated union of $\{B_j \mid j \in J\}$, with each $B_j \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta$. For each $j \in J$ let D'_j be open in $\lceil T \rceil$ with $D_j = D'_j \cap [T]$. Let $J' = J \cup \{j'\}$, where $j' \notin J$, and let $D'_{j'} = \lceil T \rceil \setminus \bigcup_{j \in J} D'_j$. Since all members of $D'_{j'}$ are finite, $D'_{j'}$ is open. Let $B_{j'} = \emptyset$. Then $\{D_j \mid j \in J'\}$ witnesses that A is the fully open-separated union of $\{B_j \mid j \in J'\}$. Hence $A \in \mathbf{\Sigma}^*_\alpha$ as a subset of $\lceil T \rceil$. □

If T is a game tree and $A \subseteq \lceil T \rceil$, then $A \in \mathbf{\Sigma^1_1}$ if and only if there is a closed $C \subseteq \lceil T \rceil \times {}^\omega\omega$ $(= \lceil T \rceil \times \lceil {}^{<\omega}\omega \rceil)$ such that

$$(\forall x \in \lceil T \rceil)(x \in A \leftrightarrow (\exists y \in {}^\omega\omega)\, \langle x, y \rangle \in C).$$

If $A \subseteq \lceil T \rceil$ then $A \in \mathbf{\Pi^1_1}$ if and only if $\lceil T \rceil \setminus A \in \mathbf{\Sigma}^1_1$. We let $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$. (In Part 1 of Rogers *et al.* [1980], elements of $\mathbf{\Sigma}^1_1$ are called *Souslin-$\mathcal{F}$* sets.)

The following theorem generalizes the Souslin Theorem.

Theorem 2.2.3. ([Hansell, 1973a] and [Hansell, 1973b]) *For every game tree* T*, the class of quasi-Borel sets coincides with* $\mathbf{\Delta}^1_1$*.*

Proof. In the proof of Lemma 1.1.1, we showed that every open set is a countable union of clopen sets: If A is open then $A = \bigcup_{n \in \omega} A_n$, where

$$A_n = \bigcup \{ \lceil T_p \rceil \mid p \in T \wedge \ell\mathrm{h}(p) = n \wedge \lceil T_p \rceil \subseteq A \}.$$

Thus the quasi-Borel sets form the smallest class that contains the clopen sets and is closed under complements, countable unions, and open-separated unions. To prove that all quasi-Borel sets belong to $\mathbf{\Delta}^1_1$, it then suffices to show (a) that every clopen set belongs $\mathbf{\Sigma}^1_1$ (and so that every clopen set belongs to $\mathbf{\Pi}^1_1$) and (b) that both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ are closed under (i) countable unions and (ii) open-separated unions.

(a). If A is clopen (or even just closed), then let $C = A \times {}^\omega\omega$. C witnesses that $A \in \mathbf{\Sigma}^1_1$.

(b)(i). Suppose that, for each $n \in \omega$, C_n witnesses that $A_n \in \mathbf{\Sigma}^1_1$. Hence each C_n is closed, and $A_n = \{x \mid (\exists y \in {}^\omega\omega)\, \langle x, y\rangle \in C_n\}$. Let C be defined by

$$\langle x, \langle n\rangle^\frown y\rangle \in C \;\leftrightarrow\; \langle x, y\rangle \in C_n,$$

where $(\langle n\rangle^\frown y)(0) = n$ and, for each i, $(\langle n\rangle^\frown y)(i+1) = y(i)$. It is easy to see that C witnesses that $\bigcup_{n\in\omega} A_n \in \mathbf{\Sigma}^1_1$.

Now suppose that, for each $n \in \omega$, C_n witnesses that $\neg A_n \in \mathbf{\Sigma}^1_1$. For $y \in {}^\omega\omega$ and $n \in \omega$, let $(y)_n \in {}^\omega\omega$ be defined by

$$(y)_n(k) \;=\; y({p_n}^k),$$

where $\langle p_n \mid n \in \omega\rangle$ is the sequence of all prime numbers in increasing order. Let

$$\langle x, y\rangle \in C \;\leftrightarrow\; (\forall n \in \omega)\, \langle x, (y)_n\rangle \in C_n.$$

C witnesses that $\bigcap_{n\in\omega} \neg A_n \in \mathbf{\Sigma}^1_1$, and so that $\bigcup_{n\in\omega} A_n \in \mathbf{\Pi}^1_1$.

For (b)(ii), first assume that $\{D_j \mid j \in J\}$ witnesses that A is the open-separated union of $\{B_j \mid j \in J\}$ with each $B_j \in \mathbf{\Sigma}^1_1$. Let $B_j = \{x \mid (\exists y \in {}^\omega\omega)\, \langle x, y\rangle \in C_j\}$, with each C_j closed. Let

$$\langle x, y\rangle \in C \;\leftrightarrow\; (\forall j \in J)(x \in D_j \;\rightarrow\; \langle x, y\rangle \in C_j).$$

C witnesses that $A \cup (\lceil T\rceil \setminus \bigcup_{j\in J} D_j) \in \mathbf{\Sigma}^1_1$. (a) and (b)(i) imply that $A \in \mathbf{\Sigma}^1_1$. (We could also have applied parts (1)(f) and (2) of Lemma 2.2.1 to get our $\{D_j \mid j \in J\}$ such that $\bigcup_{j\in J} D_j = \lceil T\rceil$.)

Next assume that $\{D_j \mid j \in J\}$ witnesses that A is the open-separated union of $\{B_j \mid j \in J\}$, with each $B_j \in \mathbf{\Pi}^1_1$. Let $\lceil T\rceil \setminus B_j = \{x \mid (\exists y \in {}^\omega\omega)\, \langle x, y\rangle \in C_j\}$, with each C_j closed. Let

$$\langle x, y\rangle \in C \;\leftrightarrow\; (\forall j \in J)(x \in D_j \;\rightarrow\; \langle x, y\rangle \in C_j).$$

C witnesses that $A \in \mathbf{\Pi}^1_1$.

For the other half of the theorem we repeat the proof of the result of [Lusin, 1927] (Theorem 2E.1 of [Moschovakis, 1980] and §35 III of [Kuratowski, 1958]), for the case of countable T, that any two disjoint $\mathbf{\Sigma}^1_1$ sets can be separated by a Borel set; we just replace "Borel" by "quasi-Borel." Let $A = \{x \mid (\exists y \in {}^\omega\omega)\, \langle x, y\rangle \in C\}$, with C closed, and let $A' = \{x \mid (\exists y \in {}^\omega\omega)\, \langle x, y\rangle \in C'\}$, with C' closed. Assume that A and A' are not separated by any quasi-Borel set, i.e. assume that there is no quasi-Borel B such that $A \subseteq B$ and $A' \cap B = \emptyset$. We will prove that $A \cap A' \neq \emptyset$. For $q \in T$, $r \in {}^{<\omega}\omega$, and $r' \in {}^{<\omega}\omega$, let

$$\begin{aligned} A_{q,r} &= \lceil T_q \rceil \cap \{x \mid (\exists y \in {}^\omega\omega)(r \subseteq y \wedge \langle x, y\rangle \in C)\} \\ A'_{q,r'} &= \lceil T_q \rceil \cap \{x \mid (\exists y \in {}^\omega\omega)(r' \subseteq y \wedge \langle x, y\rangle \in C')\} \end{aligned}$$

Assume inductively that $n \in \omega$ and that we have defined q_n, r_n, and r'_n, all of length n, such that A_{q_n,r_n} and A'_{q_n,r'_n} are not separated by any quasi-Borel set. First note that there are k and k' such that $A_{q_n,r_n{}^\frown\langle k\rangle}$ and $A'_{q_n,r'_n{}^\frown\langle k'\rangle}$ are not separated by any quasi-Borel set, since if sets $B_{k,k'}$, k, $k' \in \omega$, contradict this then

$$\bigcup_{k\in\omega} \bigcap_{k'\in\omega} B_{k,k'}$$

separates A_{q_n,r_n} and A_{q_n,r'_n}. Choose such k and k' and let r_{n+1} and r'_{n+1}, be $r_n{}^\frown\langle k\rangle$ and $r'_n{}^\frown\langle k'\rangle$ respectively. Now $A_{q_n,r_{n+1}}$ is the open-separated union of $\{A_{s,r_{n+1}} \mid q_n \subseteq s \wedge \ell\mathrm{h}(s) = \ell\mathrm{h}(q_n)+1\}$ and $A'_{q_n,r_{n+1}}$ is the open-separated union of $\{A'_{s,r'_{n+1}} \mid q_n \subseteq s \wedge \ell\mathrm{h}(s) = \ell\mathrm{h}(q_n)+1\}$. If for each $s \supseteq q_n$ with $\ell\mathrm{h}(s) = \ell\mathrm{h}(q_n)+1$ there were a quasi-Borel set B_s separating $A_{s,r_{n+1}}$ and $A'_{s,r'_{n+1}}$, then $\bigcup_s (B_s \cap \lceil T_s \rceil)$ would be a quasi-Borel set separating $A_{q_n,r_{n+1}}$ and $A'_{q_n,r'_{n+1}}$. Thus we can let q_{n+1} be some $s \supseteq q_n$ with $\ell\mathrm{h}(s) = \ell\mathrm{h}(q_n)+1$ such that no quasi-Borel set separates $A_{s,r_{n+1}}$ and $A'_{s,r'_{n+1}}$. This completes the induction step. Now let $x = \bigcup_n q_n$, let $y = \bigcup_n r_n$, and let $y' = \bigcup_n r'_n$. Then y and y' witness that $x \in A \cap B$. □

Remarks:

(a) Of course, Hansell's theorem is not about spaces of the form $\lceil T \rceil$ but about a wider class that includes these spaces.

(b) The separation theorem, which is what the second half of our proof of Theorem 2.2.3 actually proves, is in [Hansell, 1973a] and [Hansell, 1973b].

(c) If $\mathbf{T}$ is a game tree with taboos, it is easy to see that any subset A of $[T]$ belongs to $\mathbf{\Sigma}^1_1$ as a subset of $[T]$ if and only if it belongs to $\mathbf{\Sigma}^1_1$ as a subset of $\lceil T \rceil$. By the first part of Theorem 2.2.3 and by the closure properties of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ demonstrated in the proof, this also holds for $\mathbf{\Pi}^1_1$ and $\mathbf{\Delta}^1_1$.

Our proof of quasi-Borel determinacy will parallel that of Borel determinacy. For the analogue of Lemma 2.1.8, we will prove the analogue $(\dagger)^*_\alpha$ of $(\dagger)_\alpha$, for all ordinals α. For α of uncountable cofinality, where $\mathbf{\Sigma}^*_\alpha$ is gotten by the new operation of open-separated union, we will need an additional method of combining coverings.

If we are considering $\mathbf{T}$, a game tree with taboos, and S a subtree of T or $S = \emptyset$, let us denote by $\mathbf{S}$ the triple $\langle S, \mathcal{T}_{\mathrm{I}} \cap S, \mathcal{T}_{\mathrm{II}} \cap S \rangle$. If S is a game subtree of T, then $\mathbf{S}$ is a game tree with taboos.

Suppose that $\mathbf{T}$ is a game tree with taboos and that $p \in T$. Let

$$_{(p)}T \;=\; \{q \in T \mid \neg(p \subsetneq q)\}.$$

Let $\mathbf{T}$ be a game tree with taboos and let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a covering of $\mathbf{T}$. For $p \in T$, $\mathcal{C}$ is a (p)-*covering* of $\mathbf{T}$ if

(i) $_{(p)}\tilde{\mathbf{T}} = {}_{(p)}\mathbf{T}$;

(ii) $\pi \restriction {}_{(p)}\tilde{T}$ is the identity;

(iii) $(\phi(\tilde{\sigma}))(q) = \tilde{\sigma}(q)$, for all $\tilde{\sigma}$ and all $q \not\supseteq p$;

(iv) $\phi(\tilde{\sigma}) \restriction \{q \in T \mid q \supseteq p\}$ depends only on $\tilde{\sigma} \restriction \{\tilde{q} \in \tilde{T} \mid \tilde{q} \supseteq p\}$.

Lemma 2.2.4. *Let* $\mathbf{T}$ *be a game tree with taboos and let* $p \in T$. *Every* $\ell\mathrm{h}(p)$-*covering of* $\mathbf{T}_p$ *induces a unique* (p)-*covering of* $\mathbf{T}$*; i.e., if* $\langle \tilde{\mathbf{T}}', \pi', \phi', \Psi' \rangle$ *is a* $\ell\mathrm{h}(p)$-*covering of* $\mathbf{T}_p$*, then there is a unique* (p)-*covering* $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ *of* $\mathbf{T}$ *such that* $\tilde{\mathbf{T}}_p = \tilde{\mathbf{T}}'$, $\pi \restriction \tilde{T}' = \pi'$, $\phi(\tilde{\sigma}) \restriction \{q{\in}T \mid q \supseteq p\} = (\phi'(\tilde{\sigma}')) \restriction \{q{\in}T \mid q \supseteq p\}$ *for all* $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$, *and* $\Psi(\tilde{\sigma}, x) = \Psi'(\tilde{\sigma}', x)$ *for all* $\langle \tilde{\sigma}, x \rangle \in \mathit{domain}(\Psi)$ *with* $p \subseteq x$, *where in the last two clauses* $\tilde{\sigma}'$ *is the unique element of* $\mathcal{S}(\tilde{T}')$ *agreeing with* $\tilde{\sigma}$ *on* $\{q \in \tilde{T} \mid q \supseteq p\}$.

Proof. The proof is quite routine, so we verify only the uniqueness of Ψ. Suppose that $x \in \lceil {}_{(p)}T \rceil$ is consistent with $\phi(\tilde{\sigma})$, with $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$. We show that we must have $\Psi(\tilde{\sigma}, x) = x$. First note that $\Psi(\tilde{\sigma}, x) \in \lceil {}_{(p)}\tilde{T} \rceil$, since otherwise clause (d)(ii) of the definition of a covering and (ii) above imply that $x \supseteq \pi(\Psi(\tilde{\sigma}, x)) \supsetneq p$. Hence (d)(ii) and (ii) give that $x \supseteq \pi(\Psi(\tilde{\sigma}, x)) = \Psi(\tilde{\sigma}, x)$. But then (i) implies that $\Psi(\tilde{\sigma}, x) = x$. $\square$

Remark. It is also true that every (p)-covering of $\mathbf{T}$ induces a unique $\ell\mathrm{h}(p)$-covering of $\mathbf{T}_p$. (See Exercise 2.2.3.) Thus a (p)-covering of $\mathbf{T}$ is essentially the same thing as a $\ell\mathrm{h}(p)$-covering of $\mathbf{T}_p$.

Lemma 2.2.5. *Let* $\mathbf{T}$ *be a game tree with taboos and let* $\mathcal{D} \subseteq T$ *be such that* $\bigcup_{d\in\mathcal{D}}\lceil T_d\rceil = \lceil T\rceil$ *and such that any two distinct elements of* $\mathcal{D}$ *are incomparable with respect to* $\subseteq$*. Suppose that* $k \in \omega$ *and, for each* $d \in \mathcal{D}$*, that* $\mathcal{C}_d = \langle \mathbf{T}^d, \pi_d, \phi_d, \Psi_d\rangle$ *is both a* k*-covering and a* (d)*-covering of* $\mathbf{T}$*. Then there is a* k*-covering* $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ *of* $\mathbf{T}$ *and, for each* $d \in \mathcal{D}$*, there are* $\tilde{\pi}_d$*,* $\tilde{\phi}_d$*, and* $\tilde{\Psi}_d$ *such that*

(i) $\tilde{\mathcal{C}}_d = \langle \tilde{\mathbf{T}}, \tilde{\pi}_d, \tilde{\phi}_d, \tilde{\Psi}_d\rangle$ *is a* k*-covering of* T^d*;*

(ii) $\mathcal{C} = \mathcal{C}_d \circ \tilde{\mathcal{C}}_d$*.*

Proof. We get $\tilde{\mathbf{T}}$ from $\mathbf{T}$ by replacing each $\mathbf{T}_d$, $d \in \mathcal{D}$, by $\mathbf{T}^d{}_d$:

$$\begin{aligned}
\tilde{p} \in \tilde{T} \quad &\leftrightarrow \quad \begin{cases} (\exists d \in \mathcal{D})(d \subseteq \tilde{p} \,\wedge\, \tilde{p} \in T^d) \\ \text{or } \tilde{p} \in T \,\wedge\, (\forall d \in \mathcal{D})\, d \not\subseteq \tilde{p}; \end{cases}\\
\tilde{p} \in \tilde{\mathcal{T}}_{\mathrm{I}} \quad &\leftrightarrow \quad \begin{cases} (\exists d \in \mathcal{D})(d \subseteq \tilde{p} \,\wedge\, \tilde{p} \in \mathcal{T}^d_{\mathrm{I}}) \\ \text{or } \tilde{p} \in \mathcal{T}_{\mathrm{I}} \,\wedge\, (\forall d \in \mathcal{D})\, d \not\subseteq \tilde{p}; \end{cases}\\
\tilde{p} \in \tilde{\mathcal{T}}_{\mathrm{II}} \quad &\leftrightarrow \quad \begin{cases} (\exists d \in \mathcal{D})(d \subseteq \tilde{p} \,\wedge\, \tilde{p} \in \mathcal{T}^d_{\mathrm{II}}) \\ \text{or } \tilde{p} \in \mathcal{T}_{\mathrm{II}} \,\wedge\, (\forall d \in \mathcal{D})\, d \not\subseteq \tilde{p}. \end{cases}
\end{aligned}$$

In the notation introduced on page 87, clause (i) in the definition of a k-covering says that ${}_k\tilde{\mathbf{T}} = {}_k\mathbf{T}$. That this is true follows from the fact that the $\mathcal{C}_d$ are k-coverings.

We define π and $\tilde{\pi}_d$, for $d \in \mathcal{D}$, by

$$\begin{aligned}
\pi(\tilde{p}) \quad &= \quad \begin{cases} \pi_d(\tilde{p}) & \text{if } d \in \mathcal{D} \,\wedge\, d \subseteq \tilde{p}; \\ \tilde{p} & \text{if } (\forall d \in \mathcal{D})\, d \not\subseteq \tilde{p}; \end{cases}\\
\tilde{\pi}_d(\tilde{p}) \quad &= \quad \begin{cases} \pi_{d'}(\tilde{p}) & \text{if } d' \in (\mathcal{D} \setminus \{d\}) \wedge d' \subseteq \tilde{p}; \\ \tilde{p} & \text{if } (\forall d' \in (\mathcal{D} \setminus \{d\}))\, d' \not\subseteq \tilde{p}. \end{cases}
\end{aligned}$$

It is easy to check that $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$ and that each $\tilde{\pi}_d : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}^d$. The fact that π and the $\tilde{\pi}_d$ are the identity on ${}_k\tilde{T} = {}_kT$ follows from the fact that the $\mathcal{C}_d$ are k-coverings. To verify that $\pi = \pi_d \circ \tilde{\pi}_d$ for each $d \in \mathcal{D}$, let $\tilde{p} \in \tilde{T}$ and $d \in \mathcal{D}$. If $(\forall d' \in \mathcal{D})\, d' \not\subseteq \tilde{p}$, then $\pi(\tilde{p}) = \pi_d(\tilde{\pi}_d(\tilde{p})) = \tilde{p}$. So assume that $d' \in \mathcal{D}$ and $d' \subseteq \tilde{p}$. By definition, $\pi(\tilde{p}) = \pi_{d'}(\tilde{p})$. Assume first that $d' = d$. By definition, $\tilde{\pi}_d(\tilde{p}) = \tilde{p}$. Thus $\pi_d(\tilde{\pi}_d(\tilde{p})) = \pi_d(\tilde{p}) = \pi(\tilde{p})$. Assume now that $d' \neq d$. We have that $\tilde{\pi}_d(\tilde{p}) = \pi_{d'}(\tilde{p})$ and, since $\pi_{d'}(\tilde{p}) \supseteq d' \neq d$, that $\pi_d(\pi_{d'}(\tilde{p})) = \pi_{d'}(\tilde{p})$. Hence $\pi_d(\tilde{\pi}_d(\tilde{p})) = \pi_{d'}(\tilde{p}) = \pi(\tilde{p})$.

For $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ and $d \in \mathcal{D}$, let $\tilde{\sigma}_d$ be any element of $\mathcal{S}(T^d)$ that agrees with $\tilde{\sigma}$ on $\{q \in \tilde{T} \mid q \supseteq d\}$. Clause (iv) in the definition of a (p)-covering guarantees that the following definitions of ϕ and $\tilde{\phi}_d$, for $d \in \mathcal{D}$, are independent of the choices of the $\tilde{\sigma}_d$.

$$
\begin{aligned}
(\phi(\tilde{\sigma}))(p) &= \begin{cases} (\phi_d(\tilde{\sigma}_d))(p) & \text{if } d \in \mathcal{D} \wedge d \subseteq p; \\ \tilde{\sigma}(p) & \text{if } (\forall d \in \mathcal{D})\, d \not\subseteq p; \end{cases} \\
(\tilde{\phi}_d(\tilde{\sigma}))(p) &= \begin{cases} (\phi_{d'}(\tilde{\sigma}_{d'}))(p) & \text{if } d' \in (\mathcal{D} \setminus \{d\}) \wedge d' \subseteq p; \\ \tilde{\sigma}(p) & \text{if } (\forall d' \in (\mathcal{D} \setminus \{d\}))\, d' \not\subseteq p. \end{cases}
\end{aligned}
$$

It is easy to verify that $\phi : \tilde{\mathbf{T}} \stackrel{\mathcal{S}}{\Rightarrow} \mathbf{T}$. The fact that ϕ and the $\tilde{\phi}_d$ are the identity on ${}_k\mathcal{S}(\tilde{T})$ follows from the fact that the $\mathcal{C}_d$ are k-coverings. The proof that $\phi_d \circ \tilde{\phi}_d = \phi$ for every $d \in \mathcal{D}$ is like the proof above that $\pi_d \circ \tilde{\pi}_d = \pi$, and we omit it.

We define Ψ and $\tilde{\Psi}_d$, for $d \in \mathcal{D}$, as follows:

$$
\begin{aligned}
\Psi(\tilde{\sigma}, x) &= \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x), \quad \text{where } d \in \mathcal{D} \wedge d \subseteq x; \\
\tilde{\Psi}_d(\tilde{\sigma}, x) &= \begin{cases} \Psi_{d'}(\tilde{\phi}_{d'}(\tilde{\sigma}), x) & \text{if } d' \in \mathcal{D} \setminus \{d\} \wedge d' \subseteq x; \\ x & \text{if } d \subseteq x. \end{cases}
\end{aligned}
$$

Let us check clause (d) in the definition of a covering for $\mathcal{C}$. (Clause (d) for the $\tilde{\mathcal{C}}_d$ has a similar proof.) Let $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ and let $x \in \lceil \tilde{T} \rceil$ be consistent with $\tilde{\sigma}$. Let $d \in \mathcal{D}$ be such that $d \subseteq x$. Since x is consistent with $\phi(\tilde{\sigma}) = \phi_d(\tilde{\phi}_d(\tilde{\sigma}))$, it follows that $\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)$ is consistent with $\tilde{\phi}_d(\tilde{\sigma})$. Now $\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \supseteq d$, since otherwise we would have $x \supseteq \pi_d(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) = \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \not\supseteq d$. But $\tilde{\sigma}$ and $\tilde{\phi}_d(\tilde{\sigma})$ agree on $\tilde{T}_d$, so $\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)$ is consistent with $\tilde{\sigma}$. The fact that $\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \supseteq d$ implies that $\pi(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) = \pi_d(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x))$, and so we have that $\pi(\Psi(\tilde{\sigma}, x)) = \pi(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) = \pi_d(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) \subseteq x$. For clause (d)(iii), suppose for definiteness that $\tilde{\sigma}$ is a strategy for I. If $\Psi(\tilde{\sigma}, x)$ is not taboo for I in $\tilde{\mathbf{T}}$, then this same play $\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)$ is not taboo for I in $\mathbf{T}^d$. By clause (d)(iii) for $\mathcal{C}_d$, $\pi_d(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) = x$. Thus $\pi(\Psi(\tilde{\sigma}, x)) = x$.

Finally, we must verify that $\Psi(\tilde{\sigma}, x) = \tilde{\Psi}_d(\tilde{\sigma}, \Psi_d(\phi_d(\tilde{\sigma}), x))$. If $d \subseteq x$, then $\Psi(\tilde{\sigma}, x) = \Psi_d(\phi_d(\tilde{\sigma}), x) =$ (since $\Psi_d(\phi_d(\tilde{\sigma}), x) \supseteq d$) $\tilde{\Psi}_d(\tilde{\sigma}, \Psi_d(\phi_d(\tilde{\sigma}), x)$. If $x \supseteq d' \neq d$, then $\Psi(\tilde{\sigma}, x) = \Psi_{d'}(\phi_{d'}(\tilde{\sigma}), x) = \tilde{\Psi}_d(\tilde{\sigma}, x) =$ (since $\mathcal{C}_d$ is a d-covering) $\tilde{\Psi}_d(\tilde{\sigma}, \Psi_d(\phi_d(\tilde{\sigma}), x)$. $\square$

Remark. Lemma 2.2.5 is the basic new step in the proof of quasi-Borel determinacy. Its proof turns on the fact that the non-trivial parts of the $\mathcal{C}_d$ are separated, and so these coverings can be combined without interference. We have given most of the details of the proof, but the proof really should be obvious. The significance of the lemma is that all subsets of $[T]$ unraveled by any of the given coverings are simultaneously unraveled by $\mathcal{C}$.

Theorem 2.2.6. ([Martin, 1990]) *Let* $\mathbf{T}$ *be a game tree with taboos. If* A *is a quasi-Borel subset of* $[T]$ *and* $k \in \omega$, *then there is a* k*-covering of* $\mathbf{T}$ *that unravels* A.

Proof. By induction of ordinals $\alpha \geq 1$, we prove

$(\dagger)^*_\alpha$ *For all* $\mathbf{T}$, *for all* $A \subseteq [T]$ *such that* $A \in \mathbf{\Sigma}^*_\alpha$, *and for all* $k \in \omega$, *there is a* k*-covering of* $\mathbf{T}$ *that unravels* A.

Since $\mathbf{\Sigma}^*_1 = \mathbf{\Sigma}^0_1$, $(\dagger)^*_1$ is equivalent with Lemma 2.1.7. Assume then that $\alpha > 1$ and that $(\dagger)_\beta$ holds for all β with $1 \leq \beta < \alpha$. If α is a successor ordinal or if $\mathrm{cf}\,(\alpha) = \omega$, then the proof of Lemma 2.1.8 gives $(\dagger)^*_\alpha$. We may then assume that α has uncountable cofinality. Let $k \in \omega$ and let $A \subseteq [T]$ with $A \in \mathbf{\Sigma}^*_\alpha$. By Lemma 2.2.2 and part (1)(f) of Lemma 2.2.1 there is a set $\mathcal{D} \subseteq T$ such that $\bigcup_{d \in \mathcal{D}} \lceil T_d \rceil = \lceil T \rceil$ and any two elements of $\mathcal{D}$ are incomparable with respect to $\subseteq$, and there are B_d, $d \in \mathcal{D}$, such that each $B_d \subseteq \lceil \mathbf{T}_d \rceil$, such that each $B_d \in \bigcup_{\beta < \alpha} \mathbf{\Pi}^*_\beta$, and such that $A = \bigcup_{d \in \mathcal{D}} B_d$. By our induction hypothesis, for each $d \in \mathcal{D}$ there is a $\max\{k, \ell\mathrm{h}(d)\}$-covering $\mathcal{C}'_d$ of $\mathbf{T}_d$ that unravels B_d. For $d \in \mathcal{D}$, let $\mathcal{C}_d = \langle \mathbf{T}^d, \pi_d, \phi_d, \Psi'_d \rangle$ be the (d)-covering of $\mathbf{T}$ given by Lemma 2.2.4. Each $\mathcal{C}_d$ is a k-covering and unravels B_d. Let $\mathcal{C}$ and $\tilde{\mathcal{C}}_d$, $d \in \mathcal{D}$, be given by Lemma 2.2.5. Since $\mathcal{C} = \mathcal{C}_d \circ \tilde{\mathcal{C}}_d$ for each $d \in \mathcal{D}$, it follows that $\mathcal{C}$ unravels each B_d. Thus $\boldsymbol{\pi}^{-1}(A)$ is open. Let $\hat{\mathcal{C}}$ be a k-covering of $\tilde{\mathbf{T}}$ that unravels $\boldsymbol{\pi}^{-1}(A)$. Then $\mathcal{C} \circ \hat{\mathcal{C}}$ is a k-covering of $\mathbf{T}$ that unravels A. □

Theorem 2.2.7. ([Martin, 1990]) *All quasi-Borel games are determined.*

Proof. The theorem follows immediately from Lemma 2.1.3 and Theorem 2.2.6 □

Theorem 2.2.8. *All* $\mathbf{\Delta}^1_1$ *games are determined.*

Proof. The theorem follows immediately from Theorem 2.2.3 and Theorem 2.2.7 □

Exercise 2.2.1. Show that the set A defined in Remark (a) following the proof of Lemma 2.2.1 is quasi-Borel but not Borel.

Exercise 2.2.2. Prove the remark following the proof of Theorem 2.2.3.

Exercise 2.2.3. Let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a (p)-covering of $\mathbf{T}$. Show that there is a unique $\ell\mathrm{h}(p)$-covering $\langle \tilde{\mathbf{T}}', \pi', \phi', \Psi' \rangle$ of $\mathbf{T}_p$ such that the conditions of Lemma 2.2.4 are met.

2.3 Optimal Hypotheses

Results of [Friedman, 1971] show that more and more of the strength of the Power Set and Replacement Axioms is needed to prove $\mathbf{\Sigma}^0_\alpha$ determinacy for larger and larger countable α. (See Exercises 2.3.2–2.3.5.) Our aim in this section is to show that $\mathbf{\Sigma}^0_\alpha$ determinacy follows from essentially the weakest Power Set and Replacement assumptions permitted by slight refinements of Friedman's theorems. Throughout the section, we work again in ZC$^-$+ Σ_1 Replacement.

Note first that the proof of Lemma 2.1.6 goes through in our weak set theory, provided we take the given $i \mapsto \mathbf{T}_i$ to be a genuine function (i.e. a set) rather than just a (class) operation.

The proofs of Lemmas 2.1.3 and 2.1.4 also go through in the weak set theory. Here are some results that come from combining Lemma 2.1.4 with facts proved in Chapter 1.

Lemma 2.3.1. (ZC$^-$+ Σ_1 Replacement) *Let $\mathbf{T}$ be a game tree with taboos and let $A \subseteq [T]$. If there is a covering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ such that $\boldsymbol{\pi}^{-1}(A) \in \mathbf{\Sigma}^0_3$, then $G(A; \mathbf{T})$ is determined.*

Proof. This follows from Lemmas 2.1.4 and Corollary 1.3.4. □

Using Theorem 1.4.9, the Montalban-Shore theorem, we can get a stronger result, at least in countable trees:

Lemma 2.3.2. *For all $k \in \omega$,* ZC$^-$+ Σ_1 Replacement $\vdash$ *" For all countable game trees with taboos $\mathbf{T}$, and for all $A \subseteq [T]$, if there is a covering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ such that $\boldsymbol{\pi}^{-1}(A) \in k\text{-}\mathbf{\Pi}^0_3$, then $G(A; \mathbf{T})$ is determined."*

If we strengthen ZC$^-$+ Σ_1 Replacement to Rec(ZC$^-$+ Σ_1 Replacement), then Corollary 1.4.23 lets us can strengthen the conclusion to $\mathbf{\Delta}^0_4$.

Lemma 2.3.3. (Rec(ZC$^-$+ Σ_1 Replacement)) *Let* $\mathbf{T}$ *be a game tree with taboos and let* $A \subseteq [T]$*. If there is a covering* $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ *of* $\mathbf{T}$ *such that* $\boldsymbol{\pi}^{-1}(A) \in \mathbf{\Delta}^0_4$*, then* $G(A; \mathbf{T})$ *is determined.*

We will mainly use the first of the three lemmas just stated, but we will occasionally mention the consequences of the others.

From Lemma 2.1.6 and the proof of Lemma 2.1.7 we can extract the following fact.

Lemma 2.3.4. (ZC$^-$ + Σ_1 Replacement) *Let* $\mathbf{T}$ *be a game tree with taboos. If* $k \in \omega$*, if* $\mathcal{A}$ *is a countable set of open or closed subsets of* $[T]$*, and if* $\mathcal{P}(T)$ *(the power set of* T*) exists, then there is a* k*-covering* $\mathcal{C}$ *of* $\mathbf{T}$ *that unravels every member of* $\mathcal{A}$ *and is such that if* T *is infinite then*

$$|\tilde{T}| \leq |\mathcal{P}(T)|.$$

Proof. The proof of Lemma 2.1.7 gives an operation

$$\langle \mathbf{T}, A, k \rangle \mapsto \mathcal{C}(\mathbf{T}, A, k),$$

defined on triples consisting of (1) a game tree with taboos $\mathbf{T}$ such that the power set of T exists, (2) a closed subset A of $[T]$, (3) and an even $k \in \omega$. Let $\tilde{\mathbf{T}}(\mathbf{T}, A, k)$ be the first component of $\mathcal{C}(\mathbf{T}, A, k)$ and let $\pi(\mathbf{T}, A, k)$ be its second component. The main properties of this operation are the following, where we suppress $(\mathbf{T}, A, k)$:

(i) $\mathcal{C}$ is a k-covering of $\mathbf{T}$ that unravels A;

(ii) if T is infinite, then $|\tilde{T}| \leq |\mathcal{P}(T)|$;

(iii) if $\tilde{p} \in \tilde{T}$ and $\ell\mathrm{h}(\tilde{p}) \geq k + 2$, then every move in $\tilde{T}$ at $\tilde{p}$ is a move in T at $\pi(\tilde{p})$.

(i) and (iii) are clear. (ii) holds because the two extra components (other than the numbers 1 and 2) of moves in $\tilde{T}$ are subsets or members of T. Because we are working in the weak set theory, it will simplify matters if we change $\tilde{\mathbf{T}}$ so that we have

(iv) if T is infinite, if $\tilde{p} \in \tilde{T}$, and if $\ell\mathrm{h}(\tilde{p}) \in \{k, k+1\}$, then every move in $\tilde{T}$ at $\tilde{p}$ is a subset of $T_{\pi(\tilde{p})}$.

For example, we can make move k be a subset of $T_{\pi(\tilde{p})} = T_p$ by having I play $\{p^\frown\langle a\rangle\} \cup X$ instead of $\langle a, X\rangle$. We leave it to the reader the problem of finding an appropriate modification of the rules for move $k+1$.

Assume that $\mathcal{P}(T)$ exists and, without loss of generality, assume that T is infinite. Let k and $\mathcal{A}$ be as in the statement of the lemma. We may assume that k is even, and we may assume that all members of $\mathcal{A}$ are closed. Let then $\mathcal{A} = \{A_i \mid i \in \omega\}$, with each A_i closed. Let $\mathbf{T}_0 = \mathbf{T}$. Inductively define $\mathcal{C}_i = \langle \mathbf{T}_{i+1}, \pi_{i+1}, \phi_{i+1}, \Psi_{i+1}\rangle$ by

$$\mathcal{C}_i \;=\; \mathcal{C}(\mathbf{T}_i, (\pi_1 \circ \cdots \circ \pi_i)^{-1}(A_i), k+2i).$$

For $i < j \in \omega$, let

$$\mathcal{C}_{j,i} \;=\; \mathcal{C}_{i+1} \circ \cdots \circ \mathcal{C}_j.$$

For $j \in \omega$, let $\mathcal{C}_{j,j}$ be the trivial covering of $\mathbf{T}_i$. It follows by induction using (iii) and (iv) that, for all $i \in \omega$, (iii) holds with "T_i" replacing "$\tilde{T}$" and "$k+2i$" replacing "$k+2$," and

$$(\forall \tilde{p} \in T_i)(\forall m < \ell\mathrm{h}(\tilde{p}))(k \le m < k+2i \;\rightarrow\; \tilde{p}(m) \in \mathcal{P}(T)).$$

Thus we can set $k_{j,i} = k + 2i$, and the hypotheses of Lemma 2.1.6 will be satisfied. Applying Lemma 2.1.6, we get, in particular, a covering $\mathcal{C}_{\infty,0} = \langle \mathbf{T}_\infty, \pi_{\infty,0}, \phi_{\infty,0}, \Psi^{0,\infty}\rangle$, a $(k+2i)$-covering of $\mathbf{T}$ that unravels all the A_i and is such that $|T_\infty| \le \sum_{i\in\omega} |T_i| \le |{}^\omega\mathcal{P}(T)| = |\mathcal{P}(T)|$. $\Box$.

We next prove a standard fact about Borel sets that will be useful in deriving a fact related to Theorem 2.1.8 as Lemma 2.3.4 is related to Lemma 2.1.7. Let us call a set $\mathcal{A}$ of Borel subsets of $[T]$ *self-sufficient* if, whenever $\beta > 1$ and $A \in \mathcal{A} \cap (\mathbf{\Sigma}^0_\beta \setminus \bigcup_{\gamma<\beta} \mathbf{\Sigma}^0_\gamma)$, there are A_i, $i \in \omega$, with each $A_i \in \mathcal{A} \cap \bigcup_{\gamma<\beta} \mathbf{\Sigma}^0_\gamma$ and with $A = \bigcup_{i\in\omega} \neg A_i$.

Lemma 2.3.5. (ZC$^-$+ Σ_1 Replacement) *Every countable set of Borel subsets of $[T]$ can be extended to a countable, self-sufficient set.*

Proof. For every countable set $\mathcal{A}$ of Borel sets, there is a countable ordinal α such that $\mathcal{A} \subseteq \mathbf{\Sigma}^0_\alpha$. Thus we may assume inductively that $\mathcal{A}$ is a countable subset of $\mathbf{\Sigma}^0_\alpha$ with α countable and ≥ 1, and that for each $\beta < \alpha$ every countable subset of $\mathbf{\Sigma}^0_\beta$ can be extended to a countable self-sufficient set. The case $\alpha = 1$ is trivial, so assume that $\alpha > 1$. For each $A \in \mathcal{A} \setminus \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$, let $\langle B_{i,A} \mid i \in \omega \rangle$ be such that each $B_{i,A} \in \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$ and $A = \bigcup_{i\in\omega} \neg B_{i,A}$. By induction, for each $\beta < \alpha$ let $\mathcal{B}_\beta$ be a countable self-sufficient set extending $\{B_{i,A} \mid i \in \omega \wedge A \in \mathcal{A} \setminus \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta \wedge B_{i,A} \in \mathbf{\Sigma}^0_\beta\} \cup \{A \in \mathcal{A} \mid A \in \mathbf{\Sigma}^0_\beta\}$. Let

$$\mathcal{B} = \mathcal{A} \cup \bigcup_{\beta<\alpha} \mathcal{B}_\beta.$$

It is easy to see that $\mathcal{B}$ is self-sufficient. $\square$

For sets X and ordinals α, we define $\mathcal{P}^\alpha(X)$ inductively as follows:

$$\begin{aligned}
\mathcal{P}^0(X) &= X; \\
\mathcal{P}^{\alpha+1}(X) &= \mathcal{P}(\mathcal{P}^\alpha(X)) \cup \mathcal{P}^\alpha(X); \\
\mathcal{P}^\lambda(X) &= \bigcup_{\beta<\lambda} \mathcal{P}^\beta(X) \text{ for } \beta \text{ a limit ordinal.}
\end{aligned}$$

Of course, it does not follow in our weak set theory that $\mathcal{P}^\alpha(X)$ always exists, even for $\alpha = 1$.

Lemma 2.3.6. (ZC$^-$+ Σ_1 Replacement) *Let α be any countable ordinal ≥ 1. Let*

$$\alpha^* = \begin{cases} \alpha - 1 & \textit{if } \alpha \textit{ is finite;} \\ \alpha & \textit{if } \alpha \textit{ is infinite.} \end{cases}$$

Let $k \in \omega$, and let $\mathbf{T}$ be a game tree with taboos. Let $\mathcal{A}$ be a be countable set of subsets of $[T]$ such that $\mathcal{A} \subseteq \bigcup_{1\leq\beta<\alpha} \mathbf{\Sigma}^0_\beta$. If $\mathcal{P}^{\alpha^*}(T)$ exists then there is a k-covering $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ such that $\mathcal{C}$ unravels every member of $\mathcal{A}$ and such that if T is infinite then $|\tilde{T}| \leq |\mathcal{P}^{\alpha^*}(T)|$.

Proof. We prove the lemma by induction on α. The case $\alpha = 1$ is trivial. Let $\alpha > 1$ and assume that the lemma holds for all non-zero ordinals smaller than α. Fix $\mathbf{T}$ and assume that $\mathcal{P}^{\alpha^*}(T)$ exists. Let $k \in \omega$ and let $\mathcal{A}$ be a countable set of subsets of $[T]$ such that $\mathcal{A} \subseteq \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$. By Lemma 2.3.5, we may assume that $\mathcal{A}$ is self-sufficient. Clearly we may assume that T is infinite.

First suppose that $\alpha = \beta + 1$ for some β. If $\beta > 1$, then by induction let $\mathcal{C}' = \langle \mathbf{T}', \pi', \psi', \phi' \rangle$ be a k-covering of $\mathbf{T}$ that unravels every member of

$\mathcal{A} \cap \bigcup_{\gamma<\beta} \mathbf{\Sigma}^0_\gamma$ and is such that $|T'| \le |\mathcal{P}^{\beta^*}(T)|$. If $\beta = 1$, let $\mathcal{C}'$ be the trivial covering, with $\mathbf{T}' = \mathbf{T}$, etc. Note that in this case $|T'| = |T| = |\mathcal{P}^0(T)| = |\mathcal{P}^{\beta^*}(T)|$.

Let $A \in \mathcal{A} \backslash \bigcup_{\gamma<\beta} \mathbf{\Sigma}^0_\gamma$. If $\beta > 1$ then, since $A \in \mathbf{\Sigma}^0_\beta$ and $\mathcal{A}$ is self-sufficient, there are A_i, $i \in \omega$, such that $A = \bigcup_{i\in\omega} \neg A_i$ and each $A_i \in \mathcal{A} \cap \bigcup_{\gamma<\beta} \mathbf{\Sigma}^0_\gamma$. Hence $\boldsymbol{\pi}'^{-1}(A) = \bigcup_{i\in\omega} \boldsymbol{\pi}'^{-1}(\neg A_i)$, and therefore $\boldsymbol{\pi}'^{-1}(A)$ is open. If $\beta = 1$ then $\boldsymbol{\pi}'^{-1}(A) = A$, which is $\mathbf{\Sigma}^0_1$, i.e. open.

Since $\mathcal{P}^{\alpha^*}(T) = \mathcal{P}(\mathcal{P}^{\beta^*}(T)) \cup \mathcal{P}^{\beta^*}(T)$, we have the existence of $\mathcal{P}(T')$. Applying Lemma 2.3.4 to $\mathbf{T}'$ and $\mathcal{A}' = \{\pi'^{-1}(A) \mid A \in \mathcal{A}\}$, we get a covering $\hat{\mathcal{C}} = \langle \hat{\mathbf{T}}, \hat{\pi}, \hat{\phi}, \hat{\Psi} \rangle$ of $\mathbf{T}'$ that unravels every member of $\mathcal{A}'$ and satisfies

$$|\hat{T}| \le |\mathcal{P}(T')| \le |\mathcal{P}^{\alpha^*}(T)|.$$

Let $\mathcal{C} = \mathcal{C}' \circ \hat{\mathcal{C}}$.

Now suppose that α is a limit ordinal. Let $\langle \beta_n \mid n \in \omega \rangle$ be an increasing sequence of ordinals $< \alpha$ such that $\sup_{n\in\omega}\beta_n = \alpha$ and such that $\beta_0 = 1$.

Inductively we define $\mathbf{T}_j$, $j \in \omega$, and $\mathcal{C}_{j,i} = \langle \mathbf{T}_j, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \rangle$, $i \le j \in \omega$, so that

(i) the hypotheses of Lemma 2.1.6 are satisfied with $k_{j,i} = k + i$.

(ii) $\mathbf{T}_0 = \mathbf{T}$;

(iii) for all $n \in \omega$, every move in T_n belongs to $\mathcal{P}^{\beta_n^*}(T)$;

(iv) for all $n \in \omega$, $\mathcal{C}_{n,0}$ unravels every element of $\mathcal{A}_n = \mathcal{A} \cap \bigcup_{\gamma<\beta_n} \mathbf{\Sigma}^0_{\gamma_n}$.

Let $\mathcal{A}^m_n = \{\boldsymbol{\pi}_{m,0}{}^{-1}(A) \mid A \in \mathcal{A}_n\}$. Clause (iv) says that each $\mathcal{A}^n_n$ is a set of clopen sets.

Assume that the $\mathbf{T}_j$ and the $\mathcal{C}_{j,i}$ are defined for $i \le j \le n$ and have the stated properties. (This is trivial for $n = 0$.)

Let γ be such that $\beta_n + \gamma^* = \beta_{n+1}$. Now $\mathcal{A}^n_{n+1}$ is readily seen to be self-sufficient, and $\mathcal{A}^n_{n+1} \cap \{\boldsymbol{\pi}_{n,0}{}^{-1}(A) \mid A \in \bigcup_{\delta<\beta_n} \mathbf{\Sigma}^0_\delta\} = \mathcal{A}^n_n$, a set of clopen sets. It follows by an easy inductive argument that $\mathcal{A}^n_{n+1} \in \bigcup_{\delta<\gamma} \mathbf{\Sigma}^0_\delta$. Now $\mathcal{P}^{\gamma^*}(\mathcal{P}^{\beta_n^*}(T)) = \mathcal{P}^{\beta_{n+1}^*}(T)$. It follows by (iii) that $\mathcal{P}^{\gamma^*}(T_n)$ exists. Since $\mathcal{A}^n_{n+1}$ is a countable subset of $\lceil T_n \rceil \cap \bigcup_{\delta<\gamma} \mathbf{\Sigma}^0_\delta$ and $\gamma \le \beta_{n+1} < \alpha$, it follows by induction that there is a $(k+n)$-covering $\mathcal{C}' = \langle \mathbf{T}', \pi', \phi', \Psi' \rangle$ of $\mathbf{T}_n$ that unravels every member of $\mathcal{A}^n_{n+1}$ and satisfies $|T'| \le |\mathcal{P}^{\gamma^*}(T_n)| \le |\mathcal{P}^{\beta_{n+1}^*}(T)|$. Modifying $\mathbf{T}'$ to make clause (iii) hold, we get our $\mathcal{C}_{n+1,n}$; we get the $\mathcal{C}_{n+1,j}$ for $j < n$ by composition.

Lemma 2.1.6 yields, in particular, a k-covering

$$\mathcal{C}_\infty = \langle \mathbf{T}_\infty, \pi_{\infty,0}, \phi_{\infty,0}, \Psi^{0,\infty}\rangle$$

of T that unravels every member of $\mathcal{A}$ and satisfies $|T_\infty| \leq |\mathcal{P}^\alpha(T)|$. Since $\alpha^* = \alpha$, we can let $\mathcal{C} = \mathcal{C}_{\infty,0}$. □

Theorem 2.3.7. (ZC$^-$+ Σ_1 Replacement) *Let* $\mathbf{T}$ *be a game tree with taboos.*

(a) If $n \in \omega$ *and* $\mathcal{P}^n(T)$ *exists, then all* $\mathbf{\Sigma}^0_{n+3}$ *games in* T *are determined.*

(b) If α *is an infinite countable ordinal and* $\mathcal{P}^\alpha(T)$ *exists, then all* $\mathbf{\Sigma}^0_{\alpha+2}$ *games in* $\mathbf{T}$ *are determined.*

Proof. (a) Assume that $\mathcal{P}^n(T)$ exists and let $A \subseteq [T]$ with $A \in \mathbf{\Sigma}^0_{n+3}$. By Lemma 2.3.5, let $\mathcal{B}$ be countable and self-sufficient with $A \in \mathcal{B}$. By Lemma 2.3.6, let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be a covering of $\mathbf{T}$ that unravels every member of $\mathcal{B} \cap \mathbf{\Sigma}^0_n$. We have that $\boldsymbol{\pi}^{-1}(A) \in \mathbf{\Sigma}^0_3$. By Lemma 2.3.1, $G(A; \mathbf{T})$ is determined.

The proof of (b) is similar to that of (a), and we omit it. □

Corollary 2.3.8. (ZC$^-$+ Σ_1 Replacement) *If* $\mathcal{P}^\alpha(T)$ *exists for every countable* α*, then all Borel games in* $\mathbf{T}$ *are determined.*

Corollary 2.3.9. (ZC + Σ_1 Replacement) *For all* $n \in \omega$*, every* $\mathbf{\Sigma}^0_n$ *game is determined.*

Proof. Since Zermelo Set Theory (ZC) gives the existence of $\mathcal{P}^n(T)$ for every $n \in \omega$, the Corollary follows by Theorem 2.3.7. □

Remarks:

(i) "Σ_1 Replacement" can be dropped from the statements of Corollaries 2.3.8 and 2.3.9. Though we cannot then use (von Neumann) ordinal numbers, we can replace them by wellordered sets, making use of Zermelo's theorem that every set can be wellordered. If we restrict ourselves to, say, the tree ${}^{<\omega}\omega$, then "+ Σ_1 Replacement" can be dropped from the statements of all our other results as well, though—since we don't have in general the existence of cartesian products—we must exercise some care in *formulating* these results.

(ii) For countable trees and for any fixed $k \in \omega$, "$\mathbf{\Sigma}^0_{n+3}$" can be replaced in Theorem 2.3.7 by "k-$\mathbf{\Pi}^0_3$," and "$\mathbf{\Sigma}^0_{\alpha+2}$" can be replaced there by "k-$\mathbf{\Pi}^0_{\alpha+2}$." This follows by Lemma 2.3.2.

(iii) If "ZC$^-$+ Σ_1 Replacement" is replaced by in Theorem 2.3.7 by "Rec(ZC$^-$+ Σ_1 Replacement)," then "$\mathbf{\Sigma}^0_{n+3}$" and "$\mathbf{\Sigma}^0_{\alpha+2}$" can be replaced by "$\mathbf{\Delta}^0_{n+4}$" and "$\mathbf{\Delta}^0_{\alpha+3}$" respectively. This follows by Lemma 2.3.3.

For elements x of ${}^\omega\omega$ and countable ordinals α, let β^x_α be the least ordinal β such that $L_\beta[x] \models$ ZC$^-$+ Σ_1 Replacement + "$\mathcal{P}^\alpha(\omega)$ exists" (provided, of course, that such a β exists).

Theorem 2.3.10. *(*ZC$^-$+ Σ_1 Replacement)

(a) If $n \in \omega$ and β^x_n exists for every $x \in {}^\omega\omega$, then all $\mathbf{\Delta}^0_{n+4}$ games in ${}^{<\omega}\omega$ are determined.

(b) If α is an infinite countable ordinal and β^x_α exists for every $x \in {}^\omega\omega$, then all $\mathbf{\Delta}^0_{\alpha+3}$ games in ${}^{<\omega}\omega$ are determined.

Proof. (a) Assume that β^x_n exists for every $x \in {}^\omega\omega$. Let A be a $\mathbf{\Delta}^0_{n+4}$ subset of ${}^\omega\omega$. Let x be such that A is Δ^0_{n+4} in x. Fix a definition witnessing that A is Δ^0_{n+4} in x.

During this paragraph, we work in $L_{\beta^x_n}[x]$ and we write "A" for the set satisfying in $L_{\beta^x_n}[x]$ our chosen definition of A. By Lemma 2.3.5, let $\mathcal{B}$ be countable and self-sufficient with $A \in \mathcal{B}$. By 2.3.6, let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a covering of $\mathbf{T}$ that unravels every member of $\mathcal{B} \cap \mathbf{\Sigma}^0_n$. The set $\boldsymbol{\pi}^{-1}(A)$ belongs to $\mathbf{\Delta}^0_4$. By Theorem 1.4.2 (which holds in $L_{\beta^x_n}[x]$), $\boldsymbol{\pi}^{-1}(A) \in \mathbf{Diff}(\mathbf{\Pi}^0_3)$.

We have just shown that $L_{\beta^x_n}[x]$ satisfies "$\boldsymbol{\pi}^{-1}(A) \in \mathbf{Diff}(\mathbf{\Pi}^0_3)$." By Theorem 1.4.16, $L_{\beta^x_n}[x] \models$ "All $\mathbf{Diff}(\mathbf{\Pi}^0_3$ games in $\tilde{\mathbf{T}}$ are determined." Thus $L_{\beta^x_n}[x] \models$ "$G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ is determined." By Lemma 2.1.4, $L_{\beta^x_n}[x] \models$ "$G(A; {}^{<\omega}\omega)$ is determined." By absoluteness, $G(A; {}^{<\omega}\omega))$ is determined.

The proof of (b) is similar. $\square$

As in the exercises at the end of §1.4, in the exercises below we will use "ω-model" to mean a model $(M; E)$ such that $\omega \in \mathrm{WFP}(M; E)$ and the restriction of E to $\mathrm{WFP}(M; E)$ is the membership relation.

Exercise 2.3.1. This exercise extends the results of the the present section to the quasi-Borel hierarchy introduced in §2.2.

Work in ZC$^-$+ Σ_1 Replacement. Let α be any infinite ordinal. Prove that, for every game tree with taboos $\mathbf{T}$, for every countable set $\mathcal{A}$ of subsets of $[T]$ such that $\mathcal{A} \subseteq \bigcup_{1 \le \beta < \alpha} \mathbf{\Sigma}^*_\beta$, and for every $k \in \omega$, if $\mathcal{P}^\alpha(T)$ exists then there is a k-covering $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ such that $\mathcal{C}$ unravels every member of $\mathcal{A}$ and such that if T is infinite then $|\tilde{T}| \le |\mathcal{P}^\alpha(T)|$.

Deduce that, if $\mathbf{T}$ is a game tree with taboos and $\mathcal{P}^\alpha(T)$ exists, then all $\mathbf{\Sigma}^*_{\alpha+2}$ games in $\mathbf{T}$ are determined.

Hint. Adapt the proof of Lemma 2.3.6, using the proof of Lemma 2.2.5 to handle the case that α is the successor of an ordinal of uncountable cofinality.

Exercise 2.3.2. This exercise and the four that follow it are, like Exercises 1.4.1 and 1.4.2, refinements by the author of results of [Friedman, 1971].

Show that, for each ordinal $\alpha < \omega_1$, there is a model $(M; \in)$ of ZC$^-$+ Σ_1 Replacement such that M is a transitive set, $\alpha \in M$, and $(M; \in) \models$ "$\mathcal{P}^\alpha(\omega)$ exists," and (a) $\mathbf{\Sigma}^0_{\alpha+4}$ determinacy for games in ${}^{<\omega}\omega$ fails in $(M; \in)$ if α is finite and (b) $\mathbf{\Sigma}^0_{\alpha+3}$ determinacy for games in ${}^{<\omega}\omega$ fails for α infinite.

Hint. Proceed as with Exercise 1.4.1, except—for $\alpha < {\omega_1}^L$—replace β_0 by β_α, where β_α is the least ordinal number β such that $L_\beta \models$ ZC$^-$+ Σ_1 Replacement + "$\mathcal{P}^\alpha(\omega)$ exists." A key fact about β_α for $\alpha < {\omega_1}^L$ is that it is the least ordinal β such that there is no $a \subseteq \mathcal{P}^\alpha(\omega)$ such that $a \in L_{\beta_\alpha+1} \setminus L_{\beta_\alpha}$. For $\alpha \geq {\omega_1}^L$, first generically add an $f : \omega \to \alpha$ and then define β_α using $L_\beta[f]$ instead of L_β.

Exercise 2.3.3. Work in ZC$^-$+ Σ_1 Replacement. Let α be a small enough countable ordinal that α is definable and the lightface class Σ^0_α makes sense (e.g., let $\alpha < \omega_1^{\mathrm{CK}}$). Assume that (a) all $\Sigma^0_{\alpha+4}$ games in ${}^{<\omega}\omega$ are determined if α is finite and that all $\Sigma^0_{\alpha+3}$ games in ${}^{<\omega}\omega$ are determined if α is infinite. Prove that β_α exists. It follows that the consistency of ZFC$^-$ + "$\mathcal{P}^\alpha(\omega)$ exists" can be proved in ZC$^-$+ Σ_1 Replacement + "either all $\Sigma^0_{\alpha+4}$ games are determined or α is infinite and all $\Sigma^0_{\alpha+3}$ games are determined."

Hint. Combine the hints to Exercises 1.4.2 and 2.3.2.

Exercise 2.3.4. Show that, for each limit ordinal $\lambda < \omega_1$, there is a model $(M; \in)$ of ZC$^-$+ Σ_1 Replacement such that M is a transitive set, $(\forall \alpha < \lambda)\, \mathcal{P}^\alpha(\omega) \cap M \in M$, and $\mathbf{\Sigma}^0_{\lambda+1}$ determinacy for games in countable trees fails in $(M; \in)$. Your $(M; \in)$ should also be a model of the Power Set Axiom, and so you can deduce that the determinacy of all $\mathbf{\Sigma}^0_{\omega+1}$ games in ${}^{<\omega}\omega$ is not provable in ZC + Σ_1 Replacement.

Exercise 2.3.5. Work in ZC$^-$+ Σ_1 Replacement. Let λ be a small enough countable limit ordinal that λ is definable and the Σ^0_λ makes sense. Assume that all $\Sigma^0_{\lambda+1}$ games in ${}^{<\omega}\omega$ are determined. Let β_λ be the least ordinal γ, if one exists, such that $L_\gamma \models$ ZC$^-$+ Σ_1 Replacement + "$(\forall\alpha < \lambda)\,\mathcal{P}^\alpha(\omega)$ exists." Prove that β_λ exists.

In particular, this means that in ZC$^-$+ Σ_1 Replacement the determinacy of all $\Sigma^0_{\omega+1}$ games in ${}^{<\omega}\omega$ implies the consistency of ZC.

Exercise 2.3.6. Work in ZC$^-$+ Σ_1 Replacement. Prove the following generalization of the result of Exercise 1.4.3. Let α be a countable ordinal and assume that $\mathbf{\Sigma}^0_{\alpha+5}$ Turing determinacy holds if α is finite and that $\mathbf{\Sigma}^0_{\alpha+4}$ Turing determinacy holds if α is infinite. Show that β^x_α exists for every $x \in {}^\omega\omega$.

Hint. Combine the hints for Exercises 2.3.3 and 1.4.3. In the game, require that the models satisfy $V = L[x]$ instead of $V = L$.

Exercise 2.3.7. Work in ZC$^-$+ Σ_1 Replacement. Show that, for every countable ordinal α, $\mathbf{\Sigma}^0_{\alpha+5}$ Turing determinacy implies the determinacy of all $\mathbf{\Delta}^0_{\alpha+4}$ games in ${}^{<\omega}\omega$. Note that a consequence of this is that Borel Turing determinacy implies the determinacy of all Borel games in ${}^{<\omega}\omega$.

Hint. Use Theorem 2.3.10.

Exercise 2.3.8. This and the following four exercises give a a proof due to Ramez Sami of a non-level-by-level form of of Friedman's result on the strength of Borel determinacy. Sami's proof has more in common with the proof in [Friedman, 1971] than with the proof sketched in our hints to earlier exercises. In particular, Friedman and Sami use Turing degrees in similar ways. But Sami's proof has ingredients not in either Friedman's proofs or ours, principally the result of the present exercise.

In order that the result of this exercise will be applicable to Exercise 2.3.12, work in the theory ZFC$^-$. Assume that all Δ^1_1 games in ${}^{<\omega}\omega$ are determined. Let $A \in \Sigma^1_1$. Show that at least one of the following holds.

(1) There is a strategy σ for I such that, if $x \in \Delta^1_1(\sigma)$ is any play consistent with σ, then $x \in A$.

(2) There is a winning strategy for II for $G(A; {}^{<\omega}\omega)$.

Hint. Assume that (1) fails and show that (1) fails for some $B \supseteq A$ with $B \in \Delta^1_1$. Then use Borel determinacy.

Exercise 2.3.9. Once again work in ZFC$^-$. Assume, say, that $\nu < \omega_1^{\mathrm{CK}}$. Let T_ν be the theory KP + $V = L$ + "there is no ordinal α such that $L_\alpha \models$ (ZFC$^-$ + $\aleph_\nu$ exists)."

Let $\mathcal{M}$ and $\mathcal{N}$ be ω-models of T_ν. Let d be an ordinal of M and let e_1 and e_2 be ordinals of $\mathcal{N}$. Suppose that $f_1 : L_d^{\mathcal{M}} \cong L_{e_1}^{\mathcal{N}}$ and $f_2 : L_d^{\mathcal{M}} \cong L_{e_2}^{\mathcal{N}}$. Prove that $f_1 = f_2$.

Hint. It is enough to show that f_1 and f_2 agree on the ordinals of $\mathcal{M}$ that are less than d. Show that the order type of the infinite cardinals of $L_d^{\mathcal{M}}$ is $\leq \nu + 1$. Prove by induction on infinite cardinals b of $\mathcal{M}$ that $f_1(a) = f_2(a)$ for every ordinal a of $\mathcal{M}$ such that $|a|^{\mathcal{M}} \leq b$.

Exercise 2.3.10. Once again work in the theory ZFC$^-$. Let ν and T_ν be as in Exercise 2.3.9. Assume $V = L$ and assume that there is no ordinal α such that $L_\alpha \models$ ZFC$^-$ + "$\aleph_\nu$ exists." Prove that the set of Turing degrees of complete extensions of T_ν with wellfounded term models is unbounded.

Hint. Under the assumptions it is enough to prove that there are arbitrarily large countable ordinals α such that $L_\alpha \models$ KP and every member of L_α is definable in L_α. See Exercise 1.4.2.

Exercise 2.3.11. Once again work in the theory ZFC$^-$. Let ν and T_ν be as in the preceding two exercises. Say that the term model $\mathcal{M}$ of a complete extension S of T_ν is *pseudo-wellfounded* if every non-empty subset of the universe of $\mathcal{M}$ that is Δ^1_1 in S has an element that is minimal with respect to $\in_{\mathcal{M}}$. Let $\mathcal{S}_\nu$ be the set of all complete extensions S of T_ν whose term models are pseudo-wellfounded. Note that $\mathcal{S}_\nu$ is Σ^1_1. Prove that two distinct members of $\mathcal{S}_\nu$ cannot be Δ^1_1 in one another.

Hint. Assume this is false and let $\mathcal{M}$ and $\mathcal{N}$ be the term models of theories witnessing its falsity. For d and e ordinals of $\mathcal{M}$ and $\mathcal{N}$ respectively, say that $d \sim e$ if $L_d^{\mathcal{M}} \cong L_e^{\mathcal{N}}$. Use Exercise 2.3.9 to prove that $\sim$ is Δ^1_1 in $\mathcal{M}$ and in $\mathcal{N}$. Deduce that either $\mathcal{M} \cong L_e^{\mathcal{N}}$ for some ordinal e of $\mathcal{N}$ or else $\mathcal{N} \cong L_d^{\mathcal{M}}$ for some ordinal d of $\mathcal{M}$. Get a contradiction as in the analogous parts of the proofs for Exercises 1.4.2, 2.3.3, and 2.3.5.

Exercise 2.3.12. Work again in ZFC$^-$. Let ν be as in the preceding three exercises. Prove that the determinacy of all Δ^1_1 games in ${}^{<\omega}\omega$ implies that there is an ordinal α such that $L_\alpha \models$ ZFC$^-$ + "$\aleph_\nu$ exists."

Hint. By absoluteness, you may assume $V = L$. Assume that what you are trying to prove is false. Use Exercises 2.3.8 and 2.3.10 to prove that there is a Turing degree $\mathbf{d}$ such that, for any $\mathbf{d}' \geq \mathbf{d}$ and Δ^1_1 in $\mathbf{d}$, there is a member of $\mathcal{S}_\nu$ of degree $\mathbf{d}'$. Use Exercise 2.3.11 to obtain a contradiction.

Exercise 2.3.13. If A and B are subsets of ${}^{\omega}\omega$, we say that A is *Wadge reducible* to be, or $A \leq_w B$, if there is a continuous $f : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $A = f^{-1}B$. If $A \leq_w B$ then, in a strong sense, A is at least as simple a set as B. Prove that, for any Borel A and B, either $A \leq_w B$ or $B \leq_w \neg A$.

Hint. Say that A is *Lipschitz reducible* to B, or $A \leq_\ell B$, if there is a winning strategy τ for II for the game $G_w(A, B; {}^{<\omega}\omega) = G(C; {}^{<\omega}\omega)$, where $\langle a_i \mid i \in \omega\rangle \in C$ if and only if

$$\langle a_{2i} \mid i \in \omega\rangle \in A \;\leftrightarrow\; \langle a_{2i+1} \mid i \in \omega\rangle \in B.$$

Note that $A \leq_\ell B \to A \leq_w B$. Use Borel determinacy.

Remarks:

(a) This fundamental result (announced in Wadge [1972]) was proved by William Wadge in about 1967. Wadge used determinacy as a hypothesis. Borel determinacy had not been proved at the time. In the papers [Louveau and Saint-Raymond, 1987] and [Louveau and Saint-Raymond, 1988], it is shown that the result, unlike Borel determinacy, can be proved in, say, ZC$^-$. They do this by proving in the weak theory that the *Wadge game* $G(A, B; {}^{<\omega}\omega)$ is determined for Borel sets A and B.

(b) Clearly Wadge's result still holds if we replace ${}^{\omega}\omega$ by $\lceil T\rceil$, where T is any game tree, and if we replace "Borel" by "quasi-Borel." Clearly also, determinacy hypotheses for larger classes imply Wadge's result for larger classes. For example, the determinacy of all projective games in ${}^{<\omega}\omega$ implies that Wadge's result holds for all projective subsets of ${}^{\omega}\omega$. (See Chapter 8 for the definition of "projective.") Moreover AD implies that Wadge's result holds for all subsets of ${}^{\omega}\omega$.

Exercise 2.3.14. This exercise show that Wadge reducibility stratifies the Borel subsets of ${}^{\omega}\omega$ into a wellordered hierarchy.

If A and B are subsets of ${}^{\omega}\omega$, say that $A \sim_w B$ if both $A \leq_w B$ and $B \leq_w A$. Similarly define $A \sim_\ell B$. Exercise 2.3.13 and the argument of the hint show that $\leq_w$ and $\leq_\ell$ give linear orderings of the equivalence classes with respect to $\sim_w$ and $\sim_\ell$ respectively of Borel sets, except that the classes of a set and its complement may be incomparable. Show that these linear orderings are well-orderings.

Hint. Let A be such that every continuous preimage of A in ${}^{\omega}2$ is measurable with respect to the product of the measure on $\{0, 1\}$ that gives each point measure 1/2. Show that for any $\langle A_i \mid i \in \omega\rangle$ such that $A_0 = A$, there is an $i \in \omega$ such that I does not have winning strategies for both $G_w(A_i, A_{i+1}; {}^{<\omega}\omega)$ and $G_w(A_i, \neg A_{i+1}; {}^{<\omega}\omega)$. To do this, assume for a contradiction that winning strategies σ_0^i for I for $G_w(A_i, A_{i+1}; {}^{<\omega}\omega)$ and σ_1^i for I for $G_w(A_i, \neg A_{i+1}; {}^{<\omega}\omega)$ exist. For $z \in {}^{\omega}2$, we get a sequence $\langle x_z^i \mid i \in \omega\rangle$ of elements of ${}^{\omega}\omega$ by letting I follow the strategies $\sigma_{z(i)}^i$ to produce simultaneously the plays $x_z^i(0), x_z^{i+1}(0), x_z^i(1), x_z^{i+1}(1), \ldots$. Consider the probability that $x_i^z \in A_i$. Show that the 0–1 law is contradicted.

Remarks:

(a) This result is due to Martin, but the basic idea of using the 0–1 law was introduced by Leonard Monk, who proved a partial result.

(b) Remark (b) for Exercise 2.3.13 applies to the result of this exercise as well.

2.4 Blackwell Games

In [Blackwell, 1969], David Blackwell introduced a class of infinite games of imperfect information. These games differ in one basic way from the ones we have been studying. Instead of taking turns moving, the players make their nth moves *simultaneously*. The fact that moves are made simultaneously rules out, even for games where each player makes only a single move, determinacy theorems of the kind we have been studying. To make determinacy theorems possible, the strategies we have considered heretofore have to be replaced by *mixed strategies*, strategies that involve randomization. Moreover determinacy has to be defined in terms of *having a value*.

Another other difference between Blackwell's games and the ones we have been studying is a restriction: in each position, each player can have only finitely many legal moves. We will explain the reason for this restriction later, and we will also weaken the restriction. A third difference from our perfect information games is that we will allow *payoff functions*, not just payoff sets. Payoff functions make sense for perfect information games also, but their introduction adds little to the theory or applications of such games.

The concepts (mixed strategies, values of games, and payoff functions) we have just mentioned and will shortly explain in detail are among the basic concepts of ordinary game theory. What [Blackwell, 1969] proposes is that these concepts be studied in the context of games of infinite length.

From the fundamental von Neumann Minimax Theorem of [von Neumann, 1928], it follows that all Blackwell games of finite length are determined. [Blackwell, 1969] proves that all $\mathbf{\Sigma}^0_2$ ($\mathbf{F}_\sigma$) Blackwell games with payoff *sets* are determined. [Orkin, 1972] extends this result to Boolean combinations of $\mathbf{F}_\sigma$'s. Determinacy for what we will consider the general class of $\mathbf{\Sigma}^0_2$ Blackwell games ("general" in that payoff functions are allowed) is proved in [Maitra and Sudderth, 1992]. [Vervoort, 1996] betters Blackwell's result by a whole level of the Borel hierarchy, proving the determinacy of all $\mathbf{\Sigma}^0_3$ ($\mathbf{G}_{\delta\sigma}$) Blackwell games with payoff sets.

In this section, we present a result from [Martin, 1998] showing that the determinacy of any given Blackwell game is implied by the determinacy of perfect information games of roughly the same complexity. This result yields, in particular, a level-by-level reduction of Borel Blackwell determinacy to ordinary Borel determinacy. Blackwell conjectured in [Blackwell, 1969] that all Borel Blackwell games are determined, and so his conjecture is confirmed. (Blackwell did not, of course, use the word "Blackwell.")

We now turn to the formal introduction of Blackwell games. A game tree T is a *Blackwell game tree* if

(a) the members of T are finite sequences of ordered pairs;

(b) if $p \in T$ is non-terminal and has length i, then there are non-empty sets X_p and Y_p such that

  (i) at least one of X_p and Y_p is finite;

  (ii) the length $i+1$ extensions of p that belong to T are precisely the $p^\frown\langle\langle a, b\rangle\rangle$ with $a \in X_p$ and $b \in Y_p$.

If T is a Blackwell game tree, then *Blackwell games* in T are played as follows.

$$\begin{array}{lllll} \mathrm{I} & a_0 & a_1 & a_2 & \dots \\ \mathrm{II} & b_0 & b_1 & b_2 & \dots \end{array}$$

In other words, for each i the moves a_i and b_i are made simultaneously. It is required that all positions $\langle\langle a_i, b_i\rangle \mid i < n\rangle$ belong to T.

Remark. If clause (b) is relaxed to allow both players to have countably infinitely many moves, then determinacy fails even for games in which each player makes only a single move. See Exercise 2.4.1.

Let T be a Blackwell game tree. A *mixed strategy* for player I or II in T is a function σ that assigns to each position in T a discrete probability distribution on the set of legal moves for that player in the position. To see what we mean by this, assume for definiteness that σ is a strategy for I. Let X_p be as in the definition of Blackwell game trees. Then, for each $p \in T$,

(i) $\sigma(p) : X_p \to [0,1]$;

(ii) $\sum_{a \in X_p} (\sigma(p))(a) = 1$.

Note that (i) and (ii) imply that the set of $a \in X_p$ such that $(\sigma(p))(a) > 0$ is countable. We are not, then, considering the more general case in which the sum of (ii) is replaced by an integral.

Let T be a Blackwell game tree. Let σ and τ be mixed strategies for I and II respectively in T. The strategies σ and τ give, in the following manner, a probability measure $\mu_{\sigma,\tau}$ on the set of all plays in T. If $p = \langle\langle a_i, b_i\rangle \mid i < n\rangle$ is a position in T then set

$$\mu_{\sigma,\tau}(\lceil T_p \rceil) = \prod_{i<n} (\sigma(p \restriction i))(a_i) \cdot (\tau(p \restriction i))(b_i).$$

By a standard construction and argument, there is a unique probability measure defined on the Borel subsets of $\lceil T \rceil$ that has the specified values on the $\lceil T_p \rceil$. The $\mu_{\sigma,\tau}$*-measurable* sets are defined in the usual way, as the set of all $A \subseteq \lceil T \rceil$ such that A the symmetric difference of A and some Borel set is contained in a Borel set B with $\mu_{\sigma,\tau}(B) = 0$. A function $f : \lceil T \rceil \to \mathbb{R}$ is $\mu_{\sigma,\tau}$*-measurable* if the f-preimage of each open set is $\mu_{\sigma,\tau}$-measurable.

A *payoff function* for a game tree T is a function f from the set of all plays in T into a bounded subset of the real numbers. For each Blackwell game tree T and each payoff function f for T, there is a *Blackwell game* which we call $\Gamma(f;T)$.

If $\Gamma(f;T)$ is a Blackwell game and f is $\mu_{\sigma,\tau}$-measurable, then set

$$E_{\sigma,\tau}(f) = \int f \, d\mu_{\sigma,\tau} \, .$$

For for Blackwell games $\Gamma(f,T)$ with arbitrary payoff functions f, set

$$E^-_{\sigma,\tau}(f) = \sup\{E_{\sigma,\tau}(g) \mid g \text{ is Borel measurable } \wedge \ (\forall x)\, g(x) \leq f(x)\}\,;$$
$$E^+_{\sigma,\tau}(f) = \inf\{E_{\sigma,\tau}(g) \mid g \text{ is Borel measurable } \wedge \ (\forall x)\, g(x) \geq f(x)\}\,.$$

If f is $\mu_{\sigma,\tau}$-measurable, then $E^-_{\sigma,\tau}(f) = E^+_{\sigma,\tau}(f) = E_{\sigma,\tau}(f)$.

Let $\Gamma(f;T)$ be a Blackwell game. If σ is a mixed strategy for I in T then the *value* of σ in $\Gamma(f;T)$ is

$$\inf\{E^-_{\sigma,\tau}(f) \mid \tau \text{ is a mixed strategy for II}\}\,.$$

If τ is a mixed strategy for II in T then the *value* of τ in $\Gamma(f;T)$ is

$$\sup\{E^+_{\sigma,\tau}(f) \mid \sigma \text{ is a mixed strategy for I}\}\,.$$

Let $\mathrm{val}_\downarrow(\Gamma(f;T))$ be the supremum over all mixed strategies σ for I in T of the value of σ in $\Gamma(f;T)$ and let $\mathrm{val}^\uparrow(\Gamma(f;T))$ be the infinum over all mixed strategies τ for II in T of the value of τ in $\Gamma(f)$. The game $\Gamma(f;T)$ is *determined* if

$$\mathrm{val}_\downarrow(\Gamma(f;T)) = \mathrm{val}^\uparrow(\Gamma(f;T))\,.$$

If $\Gamma(f;T)$ is determined, then

$$\text{the } \textit{value} \text{ of } \Gamma(f;T) = \mathrm{val}(\Gamma(f;T)) = \mathrm{val}_\downarrow(\Gamma(f;T)) = \mathrm{val}^\uparrow(\Gamma(f;T))\,.$$

Remarks:

(a) We are using the term "Blackwell games" to cover a rather wide class. It might be more accurate to reserve the term "Blackwell games" for infinite length games. Moreover Blackwell considered only measurable payoff functions. The definitions for the non-measurable case are from [Vervoort, 1996].

Somewhat artificially, we say that a Blackwell game $\Gamma(f;T)$ is *open*, *closed*, *Borel*, etc., if for all rationals y the set of all plays x such that $y \leq f(x)$ is open, closed, Borel, etc. Note that a Blackwell game is Borel just in case its payoff function is Borel measurable, i.e., just in case the f-preimage of each Borel set is Borel.

Suppose that T is a Blackwell game tree and that A is a set of plays in T. Let $\chi(A)$ be the *characteristic function* of A, the function f such that $f(x) = 1$ for $x \in A$ and $f(x) = 0$ for $x \notin A$. According to the definition of the preceding paragraph, $\Gamma(\chi(A);T)$ is open, closed, Borel, etc., just in case A is open, closed, Borel, etc.

Say that a mixed strategy σ for I for a Blackwell game $\Gamma(f;T)$ is an *optimal strategy* if the value of σ in $\Gamma(f;T)$ is $\mathrm{val}_\downarrow(\Gamma(f;T))$. Similarly say that a mixed strategy τ for II is an *optimal strategy* if the value of τ is $\mathrm{val}^\uparrow(\Gamma(f;T))$. We say that $\Gamma(f;T)$ is *determined in optimal strategies* if $\Gamma(f;T)$ is determined and each player has an optimal strategy. Note that this means that each player has a mixed strategy whose value is the value of $\Gamma(f;T)$.

The basic form of the von Neumann Minimax Theorem of [von Neumann, 1928] is as follows.

Theorem 2.4.1 (Minimax Theorem) *Let T be a Blackwell game tree in which all plays have length 1 and for which both $X_\emptyset$ and $Y_\emptyset$ are finite. Then all Blackwell games in T are determined in optimal strategies.*

A proof of this theorem may be found in [Vervoort, 2000] (and, of course, in [von Neumann, 1928] and in many other places).

Corollary 2.4.2. *Let T be a Blackwell game tree in which all plays have length $\leq$ some fixed natural number n. Assume that all the sets X_p and Y_p associated with T are finite. Then all Blackwell games in T are determined in optimal strategies.*

Proof. We proceed by induction on n. The case $n = 0$ is trivial. Let $n \geq 0$ and assume that the corollary holds for n. Let T be a Blackwell game tree in which all plays have length $\leq n + 1$. For each non-terminal position p in T of length n, let

$$T^p = \{\emptyset\} \cup \{\langle w\rangle \mid p^\frown\langle w\rangle \in T\}.$$

Let f^p be the payoff function for T^p given by

$$f^p(\langle w\rangle) = f(p^\frown\langle w\rangle).$$

By the theorem, each $\Gamma(f^p;T^p)$ is determined in optimal strategies.

Let

$$T' = \{p \in T \mid \ell\mathrm{h}(p) \leq n\}.$$

Define a payoff function f' for T' by

$$f'(p) = \begin{cases} f(p) & \text{if } p \text{ is terminal in } T; \\ \mathrm{val}(\Gamma(f^p;T^p)) & \text{otherwise.} \end{cases}$$

By our induction hypothesis, $\Gamma(f';T')$ is determined in optimal strategies.

It is easy to see that one gets optimal strategies witnessing that the corollary holds for $\Gamma(f;T)$ by combining in the obvious way optimal strategies for $\Gamma(f';T')$ with optimal strategies for the $\Gamma(f^p;T^p)$. $\square$

Corollary 2.4.3. *Let T be a Blackwell game tree in which all plays have length $\leq$ some fixed natural number n. Then all Blackwell games in T are determined.*

Proof. First consider the case of a T in which all plays have length 1. By symmetry, we may assume that $X_\emptyset$ is finite. For any finite subset u of $Y_\emptyset$, let

$$T^u = \{\emptyset\} \cup \{\langle\langle a, b\rangle\rangle \mid a \in X_\emptyset \wedge b \in u\}$$

and let $f^u = f \restriction \lceil T^u \rceil$. By the theorem, for each u let σ^u and τ^u be optimal strategies for I and II respectively for $\Gamma(f^u; T^u)$. Let

$$v = \inf_u \mathrm{val}(\Gamma(f^u; T^u)).$$

It is easy to see, using the τ^u, that $\mathrm{val}^\uparrow(\Gamma(f; T)) \leq v$. To finish the proof, we show that $\mathrm{val}_\downarrow(\Gamma(f; T)) \geq v$. Let $\varepsilon > 0$. We will show that $\mathrm{val}_\downarrow(\Gamma(f; T)) \geq v - \varepsilon$.

Since the range of a payoff function is required to be bounded, let $s \in \mathbb{R}$ with $s > |f(x)|$ for all plays x in T.

First we prove that there are r_a, $a \in X_\emptyset$, such that each $r_a \in [0, 1]$, such that $\sum_{a \in X_\emptyset} r_a = 1$, and such that for every finite $u \subseteq Y_\emptyset$ there is a finite $u' \subseteq Y_\emptyset$ such that, for all $a \in X_\emptyset$,

$$u \subseteq u' \;\wedge\; \left|(\sigma^{u'}(\emptyset))(a) - r_a\right| \leq \frac{\varepsilon}{2s|X_\emptyset|}.$$

Let $X_\emptyset = \{a_1, \ldots a_k\}$. Let $0 \leq j < k$ and assume that we have $\bar{r}_{a_i}$ for $i < j$ such that

$$(\forall u)(\exists u' \supseteq u')(\forall i < j) \left|(\sigma^{u'}(\emptyset))(a) - \bar{r}_a\right| \leq \frac{\varepsilon}{4s|X_\emptyset|}.$$

If there were no $\bar{r}_{a_j}$ that made this hold with "$j + 1$" replacing "j," then we could generate an infinite sequence of elements of $[0, 1]$ any two of which would differ by more than $\frac{\varepsilon}{4s|X_\emptyset|}$, and there can be no such sequence. Since

$$\left|1 - \sum_{a \in X_\emptyset} \bar{r}_a\right| \;\leq\; |X_\emptyset| \frac{\varepsilon}{4s|X_\emptyset|},$$

we can get our r_a by adding to or subtracting from each $\bar{r}_a$ some number $\leq \frac{\varepsilon}{4s|X_\emptyset|}$.

Define a mixed strategy σ for I in T by

$$(\sigma(\emptyset))(a) = r_a.$$

We show that the value of σ in $\Gamma(f;T)$ is $\geq v - \varepsilon$. Let τ be any mixed strategy for II in T. There is a non-empty finite subset u of $Y_\emptyset$ such that

$$\sum_{b \notin u} (\tau(\emptyset))(b) < \frac{\varepsilon}{2s}.$$

Let $u' \supseteq u$ be given by the property of the r_a. Let $b_0 \in u'$. Let τ' be the mixed strategy for II in $T^{u'}$ given by

$$\tau'(b) = \begin{cases} \tau(b) & \text{if } b \neq b_0; \\ \tau(b) + \sum_{b \notin u'} (\tau(\emptyset))(b) & \text{if } b = b_0. \end{cases}$$

We have that

$$\begin{aligned} E_{\sigma^{u'},\tau'}(f^{u'}) - E_{\sigma,\tau}(f) &\leq \frac{\varepsilon}{2s} s + \sum_{a \in X_\emptyset} \frac{\varepsilon}{2s|X_\emptyset|} s \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since

$$E_{\sigma^{u'},\tau'}(f^{u'}) \geq \mathrm{val}(\Gamma(f^{u'}; T^{u'})) \geq v,$$

this completes the proof of the determinacy of $\Gamma(f;T)$.

The corollary can now be proved by an induction similar to the proof of Corollary 2.4.2. □

Remark. For simplicity's sake, in proving determinacy theorems we will work mainly with Blackwell game trees that have no terminal nodes and with payoff funtions whose range is a subset of $[0,1]$. Arguments like those given in §2.1 for the perfect information case show that Blackwell determinacy is level-by-level equivalent to Blackwell determinacy in trees without terminal nodes. It is easy to see that restricting to payoff functions bounded to $[0,1]$ similarly makes no difference for determinacy.

Announcement. Until further notice, let T be a Blackwell game tree with no terminal positions and let f be a payoff function for T such that $0 \leq f(x) \leq 1$ for every $x \in [T]$.

For each $v \in (0,1]$, we define a perfect information game G_v.

Play in the game G_v is as follows:

$$\begin{array}{lccccccccc} \mathrm{I} & h_0 & & h_1 & & h_2 & & \ldots & \\ \mathrm{II} & & p_1 & & p_2 & & p_3 & & \ldots \end{array}$$

Set $p_0 = \emptyset$, the starting position in T. For $i \geq 1$, p_i must a position in T of length i. It is required that $p_0 \subseteq p_1 \subseteq \ldots$. For each i, h_i must be a function into $[0,1]$ from the set of positions in T that are length $i+1$ extensions of p_i. Let $v_0 = v$ and for $i \geq 0$ let

$$v_{i+1} = h_i(p_{i+1}).$$

For each i, let T^i be the Blackwell game tree in which the players start at p_i and simultaneously make one move legal in T. I's move h_i is a payoff function for T^i. By Corollary 2.4.3, the game $\Gamma(h_i; T^i)$ is determined. The final requirement on h_i is that

$$\mathrm{val}(\Gamma(h_i; T^i)) \geq v_i.$$

Note that I always has a legal move that fulfills this requirement. For example, I may set $h_i(q) = 1$ for all q. The final requirement on p_{i+1} is that

$$v_{i+1} > 0\,.$$

By induction on i, we can see that II always has a legal move that fulfills this requirement; for if $h_i(q) = 0$ for all q, then $v_i \leq \mathrm{val}(\Gamma(h_i; T^i)) = 0$.

For each position p^* in G_v, let $\pi(p^*)$ be the union of all the moves made by II in arriving at p^*. (If $\ell\mathrm{h}(p^*) \leq 1$, then $\pi(p^*) = \emptyset$; otherwise $\pi(p^*)$ is the last move made by II.) For any play x^* of G_v, let $\pi(x^*) = \bigcup_i \pi(x^* \restriction i)$, i.e., let $\pi(x^*)$ be the play of Γ extending all the p_i. A play x^* is a win for I if and only if

$$\limsup{}_i\, v_i \leq f(\pi(x^*))\,.$$

One way to think of the game G_v is to imagine that player I is trying to show that $\mathrm{val}(\Gamma(f;T)) \geq v$. This account takes I to be asserting, at the point when p_i has been chosen, that $\mathrm{val}(\Gamma(f; T_{p_i})) \geq v_i$. To substantiate this assertion, I chooses the $h_i(q)$. If $\mathrm{val}(\Gamma(f; T_q)) \geq h_i(q)$ for each q, then the fact that $\mathrm{val}(\Gamma(h_i; T^i)) \geq v_i$ shows that I's assertion is correct. Player II is therefore required to choose some q as p_{i+1}, thus asking I to show that $\mathrm{val}(\Gamma(f; T_q) \geq h_i(q)$.

Remark. The motivation just given for G_v might suggest that G_v is a win for I if and only if $\mathrm{val}_\downarrow(\Gamma(f;T)) \geq v$. But the "if" part of this statement is not true in general, even in when both games are determined. Suppose that the X_p and Y_p for T are all $\{0,1\}$. Suppose also that $f(x) = 0$ if 1 is never played or if the two players first play 1 simultaneously and that $f(x) = 1$ otherwise. This game has value 1, but II has a winning strategy for G_1. ([Vervoort, 1996] uses this $\Gamma(f;T)$ to illustrate a different, though related, point. There are other choices for T and f such that I has a winning strategy for $G_{\mathrm{val}(\Gamma(f;T))}$.) The "only if" part of the statement *is* true, as is the anlogous assertion about II. Both will be proved below.

Theorem 2.4.4. *If* I *has a winning strategy for* G_v*, then* $\mathrm{val}_\downarrow(\Gamma(f;T)) \geq v$.

Proof. Suppose that σ^* is a winning strategy for I for G_v. Let $\delta > 0$. We will prove that $\mathrm{val}(\Gamma(f;T)) \geq v - \delta$.

We simultaneously define

(i) a mixed strategy σ for I in T;

(ii) the notion of an *acceptable* position in T;

(iii) for each acceptable position p in T, a position $\psi(p)$ in G_v such that $\ell\mathrm{h}(\psi(p)) = 2\ell\mathrm{h}(p) + 1$, $\psi(p)$ is consistent with σ^*, and $\pi(\psi(p)) = p$.

The function $p \mapsto \psi(p)$ will satisfy the condition

$$p \subseteq q \to \psi(p) \subseteq \psi(q)\,.$$

Thus for each play x in T that contains only acceptable positions, we will have a play $\Psi(x) = \bigcup_{p \subseteq x} \psi(p)$ of G_v such that $\Psi(x)$ is consistent with σ^* and $\pi(\Psi(x)) = x$.

The starting position $\emptyset$ is acceptable. Every position extending an unacceptable position is unacceptable.

For unacceptable positions p, define $\sigma(p)$ in an arbitrary fashion.

Let $\psi(\emptyset) = \langle h_0 \rangle$, where h_0 is given by σ^*.

Suppose inductively that we are given an acceptable p of length i and that either (a) $i = 0$ or else (b) $i > 0$ and we have defined

$$\psi(p) = \langle h_0, \ldots, p_i, h_i \rangle\,,$$

a position in G_v consistent with σ^* and with $p_i = p$. Let v_i and T^i be the v_i and the T_i associated with $\psi(p)$.

For positions q of length $i+1$ that extend p, define q to be acceptable if and only if $h_i(q) > 0$.

Because $\mathrm{val}(\Gamma(h_i; T^i)) \geq v_i$, there is a mixed strategy for I in T^i whose value in $\Gamma(h_i; T^i)$ is $\geq v_i - \delta/2^{i+1}$. Let $\sigma(p)$ be the probability distribution given by such a mixed strategy. Given any acceptable q of length $i+1$ that extends p, set $\psi(q) = \psi(p)^\frown\langle p_{i+1}, h_{i+1}\rangle$, where $p_{i+1} = q$ and where h_{i+1} is given by σ^*.

For acceptable positions p in T, let h^p be the last move made in reaching the position $\psi(p)$, i.e., let h^p be the $h_{\ell\mathrm{h}(p)}$ of $\psi(p)$. For acceptable p, also let T^p be the $T^{\ell\mathrm{h}(p)}$ of $\psi(p)$ and let v^p be the $v_{\ell\mathrm{h}(p)}$ of $\psi(p)$. For unacceptable p, let $v^p = 0$.

Lemma 2.4.5. *Let τ be a mixed strategy for* II *in T and let $\mu = \mu_{\sigma,\tau}$. Let $p \in T$ with $\ell\mathrm{h}(p) = i$. Then*

$$v^p \mu([T_p]) \leq \sum_{\substack{p \subseteq q \\ \ell\mathrm{h}(q) = i+1}} (v^q + \delta/2^{i+1})\mu([T_q]).$$

Proof of Lemma. Since $v^p = 0$ for unacceptable p, we may assume that p is acceptable. Because $\sigma(p)$ is the probability distribution of a mixed strategy in T^p whose value in $\Gamma(h^p; T^p)$ is $\geq v^p - \delta/2^{i+1}$, we have that

$$v^p - \delta/2^{i+1} \leq \sum_{\substack{p \subseteq q \\ \ell\mathrm{h}(q) = i+1}} h^p(q)\frac{\mu([T_q])}{\mu([T_p])}.$$

Using the facts that $h^p(q) = v^q$ and that $\mu([T_p]) = \sum_q \mu([T_q])$, we get the desired inequality. $\square$

For plays x in T, set

$$g(x) = \limsup{}_i\, v^{x\restriction i}.$$

Note that g is Borel measurable and that $\mathrm{range}(g) \subseteq [0,1]$. Note also that $g(x) \leq f(x)$ for every play x in T. This is trivially true for those x such that $g(x) = 0$. For any other x, $\Psi(x)$ is a play consistent with the winning strategy σ^*, and so $g(x) \leq f(\pi(\Psi(x))) = f(x)$.

Lemma 2.4.6. *For any mixed strategy τ for* II *in T, $E_{\sigma,\tau}(g) \geq v - \delta$.*

Proof of Lemma. Given τ, let $\mu = \mu_{\sigma,\tau}$. Assume that $E_{\sigma,\tau}(g) < v - \delta$. Thus $\int g\, d\mu < v - \delta$. Let $\varepsilon > 0$ be such that $\int g\, d\mu < v - \delta - \varepsilon$. Then $\int (1-g)\, d\mu > 1 - v + \delta + \varepsilon$. There is a closed set C such that g is continuous on C and $\int_C (1-g)\, d\mu > 1 - v + \delta + \varepsilon$. (See Kechris [1994], Theorem 17.12.)

We will define a play x in T such that, for all i, $x \restriction i$ is acceptable and

$$\int_{C \cap [T_{x\restriction i}]} (1-g)\, d\mu > (1 - v^{x\restriction i} + \delta/2^i + \varepsilon)\, \mu([T_{x\restriction i}])\,.$$

Suppose that $x\restriction i$ has been defined so that $x\restriction i$ is acceptable and the inequality just stated holds. If there is an acceptable q of length $i+1$ that extends $x \restriction i$ and is such that $\int_{C\cap[T_q]}(1-g)\, d\mu > (1 - v^q + \delta/2^{i+1} + \varepsilon)\, \mu([T_q])$, then let $x \restriction i+1$ be such a q. Assume, in order to derive a contradiction, that the inequality

$$\int_{C\cap[T_q]} (1-g)\, d\mu \;\le\; (1 - v^q + \delta/2^{i+1} + \varepsilon)\, \mu(T_q])$$

holds for every acceptable q of length $i+1$ that extends $x \restriction i$. This inequality holds also for unacceptable q, since for them $v^q = 0$. Thus

$$\begin{aligned}
\int_{C\cap[T_{x\restriction i}]} (1-g)\, d\mu &= \sum_q \int_{C\cap[T_q]} (1-g)\, d\mu \\
&\le \sum_q (1 - v^q + \delta/2^{i+1} + \varepsilon)\, \mu(T_q]) \\
&\le (1 - v^{x\restriction i} + \delta/2^i + \varepsilon)\, \mu([T_{x\restriction i}])\,,
\end{aligned}$$

where the last inequality is by Lemma 2.4.5. This contradicts our induction hypothesis for i.

We next observe that for any i there is a $y_i \in C \cap [T_{x\restriction i}]$ such that

$$g(y_i) < v^{x\restriction i} - \delta/2^i - \varepsilon\,.$$

Assume to the contrary that $g(y) \ge v^{x\restriction i} - \delta/2^i - \varepsilon$ for every $y \in C \cap [T_{x\restriction i}]$. Then

$$\int_{C\cap[T_{x\restriction i}]} (1-g)\, d\mu \;\le\; (1 - v^{x\restriction i} + \delta/2^i + \varepsilon)\, \mu([T_{x\restriction i}])\,,$$

contradicting what we have just proved by induction about our play x.

Since $x = \lim_i y_i$, we have that $x \in C$ and so that $g(x) = \lim_i g(y_i)$. Let j be such that

$$(\forall i \geq j)\, |g(x) - g(y_i)| < \varepsilon/2\,.$$

Then

$$(\forall i \geq j)\, g(x) < g(y_i) + \varepsilon/2 < v^{x \restriction i} - \delta/2^i - \varepsilon/2\,.$$

Therefore

$$g(x) \leq \limsup{}_i\, v^{x \restriction i} - \varepsilon/2 = g(x) - \varepsilon/2\,.$$

This contradiction completes the proof of the lemma. □

Lemma 2.4.7. *The value of σ in $\Gamma(f;T)$ is $\geq v - \delta$.*

Proof of Lemma. By the fact that $g(x) \leq f(x)$ for all x, the value of σ in $\Gamma(f;T)$ is $\geq$ the value of σ in $\Gamma(g;T)$, which is $\geq v - \delta$ by Lemma 2.4.6. □

Since δ was an arbitrary positive real number, the theorem is proved. □

Theorem 2.4.8. *If* II *has a winning strategy for G_v, then* $\mathrm{val}_{\downarrow}(\Gamma(f;T)) \leq v$.

Proof. Suppose that τ^* is a winning strategy for II for G_v. Let $\delta > 0$. We will prove that $\mathrm{val}(\Gamma(f;T)) \leq v + \delta$.

We simultaneously define

(i) a mixed strategy τ for II in T;

(ii) the notion of an *acceptable* position in T;

(iii) for each acceptable position p in T, a function u_p into $[0,1]$ from the set of all q extending p such that $\ell\mathrm{h}(q) = \ell\mathrm{h}(p) + 1$;

(iv) for each acceptable position p in T, a position $\psi(p)$ in G_v such that $\ell\mathrm{h}(\psi(p)) = 2\ell\mathrm{h}(p)$, $\psi(p)$ is consistent with τ^*, and $\pi(\psi(p)) = p$.

The function $p \mapsto \psi(p)$ will satisfy the condition

$$p \subseteq q \rightarrow \psi(p) \subseteq \psi(q)\,.$$

Thus for each play x in T that contains only acceptable positions, we will have a play $\Psi(x) = \bigcup_{p \subseteq x} \psi(p)$ of G_v such that $\Psi(x)$ is consistent with τ^* and $\pi(\Psi(x)) = x$.

The starting position $\emptyset$ is acceptable. Every position extending an unacceptable position is unacceptable.

For unacceptable positions p, define $\tau(p)$ in an arbitrary fashion.

Let $\psi(\emptyset) = \emptyset$.

Suppose inductively that we are given an acceptable p of length i and that either (a) $i = 0$ or else (b) $i > 0$ and we have defined

$$\psi(p) = \langle h_0, \ldots, p_i \rangle ,$$

a position in G_v consistent with τ^* and with $p_i = p$. Let v_i and T^i be the v_i and the T_i associated with $\psi(p)$.

For positions q of length $i+1$ that extend p, define q to be acceptable if and only if there is a legal move h for I in G_v at $\psi(p)$ such that $\tau^*(\psi(p)^\frown\langle h \rangle) = q$.

For acceptable q, set

$$u_p(q) = \inf\{h(q) \mid h \text{ is legal in } G_v \text{ at } \psi(p) \ \wedge \tau^*(\psi(p)^\frown\langle h \rangle) = q\} .$$

For unacceptable q, set $u_p(q) = 1$.

Lemma 2.4.9. $\mathrm{val}(\Gamma(u_p; T^i)) \leq v_i$.

Proof of Lemma. Assume that $\mathrm{val}(\Gamma(u_p; T^i)) > v_i$. Let $\varepsilon > 0$ be such that $\mathrm{val}(\Gamma(u_p; T^i)) \geq v_i + \varepsilon$. Define a function h, with the same domain as u_p, by

$$h(q) = \begin{cases} u_p(q) - \varepsilon & \text{if } u_p(q) > \varepsilon; \\ 0 & \text{if } u_p(q) \leq \varepsilon. \end{cases}$$

Then $\mathrm{val}(\Gamma(h; T^i)) \geq \mathrm{val}(\Gamma(u_p; T^i)) - \varepsilon \geq v_i$, and therefore h is a is a legal move for I at the position $\psi(p)$. Hence there is some q such that $\tau^*(\psi(p)^\frown\langle h \rangle) = q$. If $u_p(q) \leq \varepsilon$ then $h(q) = 0$, and so q is not a legal move. If $u_p(q) > \varepsilon$ then $h(q) < u_p(q)$, and this contradicts the definition of $u_p(q)$. □

Let $\tau(p)$ be the probability distribution given by some mixed strategy for II in T^i whose value in $\Gamma(u_p; T^i)$ is $\leq v_i + \delta/2^{i+2}$.

For each acceptable q of length $i + 1$ and extending p, we define $\psi(q)$ as follows. Pick a legal move h_i for I at $\psi(p)$ such that $h_i(q) \leq u_p(q) + \delta/2^{i+2}$ and such that $\tau^*(\psi(p)^\frown\langle h_i \rangle) = q$. Set $\psi(q) = \psi(p)^\frown\langle h_i, p_{i+1} \rangle$, where $p_{i+1} = q$.

For acceptable positions p in T with $\ell\mathrm{h}(p) > 0$, let h^p be the next to last move made in reaching the position $\psi(p)$, i.e., let h^p be the $h_{\ell\mathrm{h}(p)-1}$ of $\psi(p)$. For acceptable $p \in T$ of any length, let T^p be the $T^{\ell\mathrm{h}(p)}$ of $\psi(p)$ and let v^p be the $v_{\ell\mathrm{h}(p)}$ of $\psi(p)$. If p is unacceptable, set $v^p = 1$.

Lemma 2.4.10. *Let σ be a mixed strategy for* I *in T and let $\mu = \mu_{\sigma,\tau}$. Let $p \in T$ with $\ell\mathrm{h}(p) = i$. Then*

$$v^p \mu([T_p]) \geq \sum_{\substack{p \subseteq q \\ \ell\mathrm{h}(q) = i+1}} (v^q - \delta/2^{i+1})\mu([T_q]).$$

Proof of Lemma. Since $v^p = 1$ for unacceptable p, we may assume that p is acceptable. Because $\tau(p)$ is the probability distribution of a mixed strategy in T^p whose value in $\Gamma(u_p; T^p)$ is $\leq v^p + \delta/2^{i+2}$, we have that

$$v^p + \delta/2^{i+2} \geq \sum_{\substack{p \subseteq q \\ \ell\mathrm{h}(q) = i+1}} u_p(q)\frac{\mu([T_q])}{\mu([T_p])}.$$

Since $\mu([T_p]) = \sum_q \mu([T_q])$, we get that

$$v^p \mu([T_p]) \geq \sum_{\substack{p \subseteq q \\ \ell\mathrm{h}(q) = i+1}} (u_p(q) - \delta/2^{i+2})\mu([T_q]).$$

For acceptable q, $v^q = h^q(q) \leq u_p(q) + \delta/2^{i+2}$. For unacceptable q, $v^q = 1 = u_p(q)$. In either case, $v^q - \delta/2^{i+2} \leq u_p(q)$, and this gives us the desired inequality. $\square$

For plays x in T, set

$$g(x) = \limsup{}_i\, v^{x \restriction i}.$$

Note that g is Borel measurable and that range$(g) \subseteq [0, 1]$. Note also that $g(x) \geq f(x)$ for every play x in T. This is trivially true for those x such that $g(x) = 1$. For any other x, $\Psi(x)$ is a play consistent with the winning strategy τ^*, and so $g(x) \leq f(\pi(\Psi(x))) = f(x)$.

Lemma 2.4.11. *For any strategy σ for* I *for Γ,*

$$E_{\sigma,\tau}(g) \leq v + \delta\,.$$

Proof of Lemma. Given σ, let $\mu = \mu_{\sigma,\tau}$. Assume that $E_{\sigma,\tau}(g) > v + \delta$. Let $\varepsilon > 0$ be such that $E_{\sigma,\tau}(g) > v + \delta + \varepsilon$. Then $\int g\, d\mu > v + \delta + \varepsilon$. Let C be a closed set such that g is continuous on C and such that $\int_C g\, d\mu > v + \delta + \varepsilon$.

We will define a play x in T such that, for all i, $x \restriction i$ is acceptable and

$$\int_{C\cap[T_{x\restriction i}]} g\,d\mu > (v^{x\restriction i} + \delta/2^i + \varepsilon)\,\mu([T_{x\restriction i}])\,.$$

Suppose that $x\restriction i$ has been defined so that $x\restriction i$ is acceptable and the inequality just stated holds. If there is an acceptable q such that $\int_{C\cap[T_q]} g\,d\mu > (v^q + \delta/2^{i+1} + \varepsilon)\,\mu([T_q])$, then let $x \restriction i+1$ be such a q. If, for every acceptable q,

$$\int_{C\cap[T_q]} g\,d\mu \;\leq\; (v^q + \delta/2^{i+1} + \varepsilon)\,\mu([T_q])\,,$$

then

$$\begin{aligned}\int_{C\cap[T_{x\restriction i}]} g\,d\mu &= \sum_q \int_{C\cap[T_q]} g\,d\mu \\ &\leq \sum_q (v^q + \delta/2^{i+1} + \varepsilon)\,\mu([T_q]) \\ &\leq (v_i^{x\restriction i} + \delta/2^i + \varepsilon)\,\mu([T_{x\restriction i}])\,,\end{aligned}$$

where the last inequality is by Lemma 2.4.10. This contradicts our induction hypothesis for i.

We next observe that for any i there is a $y_i \in C \cap [T_{x\restriction i}]$ such that

$$g(y_i) > v^{x\restriction i} + \delta/2^i + \varepsilon\,.$$

Assume to the contrary that $g(y) \leq v^{x\restriction i} + \delta/2^i + \varepsilon$ for every $y \in C \cap [T_{x\restriction i}]$. Then

$$\int_{C\cap[T_{x\restriction i}]} g\,d\mu \leq (v^{x\restriction i} + \delta/2^i + \varepsilon)\,\mu([T_{x\restriction i}])\,,$$

contradicting what we have just proved by induction about our play x.

Since $x = \lim_i y_i$, we have that $x \in C$ and so that $g(x) = \lim_i g(y_i)$. Let j be such that

$$(\forall i \geq j)\,|g(x) - g(y_i)| < \varepsilon/2\,.$$

Then

$$(\forall i \geq j)\,g(x) > g(y_i) - \varepsilon/2 > v_i^{x\restriction i} + \delta/2^i + \varepsilon/2\,.$$

Therefore

$$g(x) \geq \limsup{}_i v_i^{x\restriction i} + \varepsilon/2 = g(x) + \varepsilon/2\,.$$

This contradiction completes the proof of the lemma. $\square$

Lemma 2.4.12. *The value of τ in $\Gamma(f;T)$ is $\le v+\delta$.*

Proof of Lemma. By the fact that $g(x) \geq f(x)$ for all x, the value of τ in $\Gamma(f;T)$ is $\le$ the value of τ in $\Gamma(g;T)$, which is $\le v+\delta$ by Lemma 2.4.11. □

Since δ was an arbitrary positive real number, the theorem is proved. □

Theorem 2.4.13. *If G_v is determined for every $v \in (0,1]$, then $\Gamma(f;T)$ is determined.*

Proof. Assume that all the G_v are determined. Let w be the least upper bound of all the numbers v such that I has a winning strategy for G_v. By Lemma 2.4.4, $\mathrm{val}_{\downarrow}(\Gamma(f;T)) \geq w$. By Lemma 2.4.8, $\mathrm{val}^{\uparrow}(\Gamma(f;T)) \leq w$. Thus $\mathrm{val}(\Gamma(f;T)) = w$. □

Since the games G_v are of the form $G(A^*;T^*)$, Theorem 2.4.13 allows us to show that Blackwell determinacy for any given class follows from ordinary determinacy for a related class. But Theorem 2.4.13 does not yield optimal results when $|T| < 2^{\aleph_0}$. This is because the tree T^* for G_v has size $\geq 2^{\aleph_0}$ for all non-trivial trees T. This problem is easily remedied, however, as we now explain.

For $v \in (0,1]$, let $\bar{G}_v$ differ from G_v only in an additional requirement that all values $h_i(q)$ be rational. It is easy to check that our proofs go through unchanged for $\bar{G}_v$ in place of G_v. We state this formally as the following theorem.

Theorem 2.4.14. *If $\bar{G}_v$ is determined for every real (indeed, for every rational) $v \in (0,1]$, then $\Gamma(f;T)$ is determined.*

If f is the characteristic function of a set, then there is a modification of the games G_v that gives our results with simpler proofs.

Announcement. Until further notice, let A be a subset of $[T]$.

For $v \in (0,1]$ let G'_v be played exactly as is G_v, but let a play x^* of G'_v be a win for I if and only if $\pi(x^*) \in A$.

Theorem 2.4.15. *If I has a winning strategy for G'_v, then $\mathrm{val}_{\downarrow}(\Gamma(\chi(A);T)) \geq v$.*

Suppose that σ^* is a winning strategy for I for G_v. Let $\delta > 0$. We will prove that $\mathrm{val}(\Gamma(\chi(f); T)) \geq v - \delta$. Define σ, *acceptable* positions, and ψ, exactly as in the proof of Theorem 2.4.4. Define h^p, T^p, and v^p as before.

Lemma 2.4.5 holds as before.

Lemma 2.4.16. *Let τ be a mixed strategy for* II *in T and let $\mu = \mu_{\sigma,\tau}$. For each $i \in \omega$,*

$$v \leq \sum_{\ell\mathrm{h}(p)\,=\,i} (v^p + \delta(1 - 1/2^i)\mu([T_p]).$$

Let C_1 be the closed set of all plays of Γ that contain only acceptable positions. Since $x = \pi(\psi(x))$ for $x \in C_1$, $C_1 \subseteq A$.

Lemma 2.4.17. *For any strategy τ for* II *for Γ, $\mu_{\sigma,\tau}(C_1) \geq v$.*

Proof. Given τ, assume that $\mu_{\sigma,\tau}(C_1) < v$. It follows that there is a closed set C disjoint from C_1 such that $\mu_{\sigma,\tau}(C) > 1 - v$. By a construction like that in the proof of Lemma 2.4.6, there is an $x \in C_1$ such that, for all i, $\mu_{\sigma,\tau}(C \cap [T_{x\restriction i}]) > 1 - v_i^{x\restriction i}$. But this is a contradiction, for such an x must belong to $C_1 \cap C$. □

Lemma 2.4.18. *The value of σ in $\Gamma(\chi(A))$ is $\geq v$.*

Proof. The lemma follows from Lemma 2.4.17.

Here is a direct proof of the lemma. For each i, consider the game Γ^i which is played in the same way as Γ except that play terminates when the position p has length i. For plays p of Γ^i, let

$$h^i(p) = \begin{cases} v_i^p & \text{if } p \text{ is acceptable;} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove by induction on i that the value of the appropriate fragment σ^i of σ in $\Gamma^i(h^i)$ is $\geq v$. Thus the value of σ^i in $\Gamma^i(\chi(C_1^i))$ is $\geq v$, where C_1^i is the set of all acceptable plays of Γ^i. Thus the value of σ in $\Gamma(\chi(C_1))$ is $\geq v$. □

Dropping our assumption about σ^*, we get the following.

Lemma 2.4.19. *If* I *has a winning strategy for G'_v, then* $\mathrm{val}_\downarrow(\Gamma(\chi(A))) \geq v$.

Now assume that τ^* is a winning strategy for II for G'_v. Let $\delta > 0$. Define τ, *acceptable* positions, and ψ, exactly as in §1.

Let C_2 be the closed set of all plays of Γ that contain only acceptable positions. Since $x = \pi(\psi(x))$ for $x \in C_2$, $C_2 \cap A = \emptyset$.

Lemma 2.4.20. *For any strategy σ for* I *for Γ, $\mu_{\sigma,\tau}(C_2) \geq 1 - v - \delta$.*

Lemma 2.4.21. *The value of τ in $\Gamma(\chi(A))$ is $\leq v + \delta$.*

Proof. The lemma follows from Lemma 2.4.20. There is also a direct proof of the lemma, analogous to the direct proof of Lemma 2.4.18. □

Dropping our assumption about τ^*, we get the following.

Lemma 2.4.22. *If* II *has a winning strategy for G'_v, then* $\mathrm{val}^{\uparrow}(\Gamma(\chi(A))) \leq v$.

Theorem 2.4.23. *If G'_v is determined for every $v \in (0, 1]$, then $\Gamma(\chi(A))$ is determined.*

We next indicate how to extend our results to stochastic games. In doing so we are reporting an observation of Maitra and Sudderth.

Stochastic games $\tilde{\Gamma}$ are played like Blackwell games, except that each pair of moves of I and II is followed by a move of a third player, whom we will call Nature. We will restrict ourselves to the case that Nature has a countable set of legal moves in each postion in which she must make a move. Payoff functions for $\tilde{\Gamma}$ are functions of the entire play, including Nature's moves. The analogue of $\Gamma(f)$ is $\tilde{\Gamma}(\tilde{f}, \rho)$ where $\tilde{f}$ is a payoff function and ρ is a mixed strategy for Nature. If σ and τ are strategies for I and II respectively, then σ, τ, and ρ give a probability measure $\mu_{\sigma,\tau,\rho}$ on the set of plays of $\tilde{\Gamma}$. Using this measure, we define $E_{\sigma,\tau,\rho}(\tilde{f})$, $E^-_{\sigma,\tau,\rho}(\tilde{f})$, $E^+_{\sigma,\tau,\rho}(\tilde{f})$, $\mathrm{val}_{\downarrow}(\tilde{\Gamma}(\tilde{f}, \rho))$, $\mathrm{val}^{\uparrow}(\tilde{\Gamma}(\tilde{f}, \rho))$, *determinacy* of $\tilde{\Gamma}(\tilde{f}, \rho)$, and the *value* of $\tilde{\Gamma}(\tilde{f}, \rho)$ in the obvious way.

Fix $\tilde{\Gamma}$ with no terminal positions. Fix $\tilde{f}$ and ρ. For $v \in (0, 1]$, let $\tilde{G}_v$ be the perfect information game played as follows. Set $p_0 = \emptyset$. I's moves are functions $h_0, h_1, \ldots$ and II's moves are positions $p_1, p_2, \ldots$ in $\tilde{\Gamma}$. For each i, h_i is a function into $[0, 1]$ from the set of all length $2i + 2$ extensions of p_i, which has length $2i$. Let $v_0 = v$ and for $i \geq 0$ let $v_{i+1} = h_i(p_{i+1})$. For each i, let $\tilde{\Delta}_i$ be the game in which, starting at p_i, the two players and then Nature make legal moves in $\tilde{\Gamma}$. The final requirement on h_i is that

$$\mathrm{val}(\tilde{\Delta}_i(h_i, \rho_{p_i})) \geq v_i,$$

where ρ_{p_i} is the strategy for Nature for $\tilde{\Delta}_i$ that is given by ρ. The final requirement on p_{i+1} is that $v_{i+1} > 0$.

For positions p^* in $\tilde{G}_v$, define $\pi(p^*)$, a position in $\tilde{G}$ of length $2\ell\mathrm{h}(p^*)$, in the obvous way. Also call π the function induced by π from plays of $\tilde{G}_v$ to plays of $\tilde{\Gamma}$. A play x^* is a win for I if and only if $\limsup_i v_i \leq \tilde{f}(\pi(x^*))$.

The constructions, lemmas, and proofs of the earlier part of this section adapt in obvious ways to $\tilde{G}_v$ and $\tilde{\Gamma}(\tilde{f}, \rho)$. (The first draft of our paper had a slightly different definition of the function h on page 114. Maitra and Sudderth remarked that the current definition, unlike the original one, would work for stochastic games.) Thus we get the following theorem.

Theorem 2.4.24. *If $\tilde{G}_v$ is determined for every $v \in (0, 1]$, then $\tilde{\Gamma}(\tilde{f}, \rho)$ is determined.*

For more details, see [Maitra and Sudderth, 1993]. There Maitra and Sudderth adapt our proof to demonstrate the determinacy of a wider class of stochastic games. They work in the context of finitely additive probabililty measures, removing the restrictions that I and II choose their moves from finite sets and that Nature's moves are chosen from countable sets.

The proof of Theorem 2.4.23 gives the following stronger result.

Theorem 2.4.25. *Assume that all G'_v are determined. Then*

$$\mathrm{val}(\Gamma(\chi(A))) = \sup\{\mathrm{val}(\Gamma(\chi(C))) \mid C \text{ closed and } C \subseteq A\}.$$

Proof. Let $v < \mathrm{val}(\Gamma(\chi(A)))$. Let σ^* be a winning strategy for I for G'_v. Let σ be the strategy defined from σ^* as above. Let C be the set C_1 defined just before the statement of Lemma 2.4.17. The proofs of Lemma 2.4.18 both show that the value of σ in $\Gamma(\chi(C))$ is $\geq v$. □

Remarks:

(a) For $\mathbf{F}_{\sigma\delta}$ sets A, Vervoort in [Vervoort, 1996] directly proves a strengthening of the conclusion of Theorem 2.4.25. He conjectures that the conclusion of Theorem 2.4.25 holds for all Borel sets A. Since the hypothesis of Theorem 2.4.25 holds for Borel A, his conjecture is confirmed.

(b) Applying Theorem 2.4.25 to the complement of A, we see that the theorem's hypothesis implies that

$$\mathrm{val}(\Gamma(\chi(A))) = \inf\{\mathrm{val}(\Gamma(\chi(B))) \mid B \text{ open and } B \supseteq A\}\,.$$

One can also get this conclusion directly from the proofs of Lemma 2.4.21.

(d) For Borel sets A, Maitra, Purves, and Sudderth [Maitra et al., 1991] show that the determinacy of $\Gamma(\chi(A))$ implies the conclusion of Theorem 2.4.25. As mentioned in (a) above, their result follows *a fortiori* from Theorem 2.4.25 and the determinacy of all Borel perfect information games. It is not true that for arbitrary A the determinacy of $\Gamma(\chi(A))$ implies the conclusion of Theorem 2.4.25. For a counterexample, see page 126 below. The last paragraph of of the paper also discusses issues related to the theorem of [Maitra et al., 1991].

Let $\bar{G}'_v$ be like G'_v except that all $h_i(q)$ must be rational.

Theorem 2.4.26. *If $\bar{G}'_v$ is determined for every rational $v \in (0,1]$, then $\Gamma(f)$ is determined.*

Theorem 2.4.27. *Assume that $\bar{G}'_v$ is determined for every rational $v \in (0,1]$. Then*

$$\mathrm{val}(\Gamma(\chi(A))) = \sup\{\mathrm{val}(\Gamma(\chi(C))) \mid C \text{ closed } \;\wedge\; C \subseteq A\}\,.$$

Combining the proof of Theorem 2.4.23 with a proof of Vervoort [Vervoort, 1996] that Blackwell determinacy implies that all sets are Lebesgue measurable, one gets [Martin, 2003] on eliminating the Blackwell game, a new way to deduce Lebesgue measurability from the determinacy of perfect information games.

For convenenience, we think of Lebesgue measure as being the coin-flipping measure on $2^{\mathbb{N}}$.

Until the end of the statement of Theorem 2.4.29, let $B \subseteq 2^{\mathbb{N}}$.

Let H_v be played as follows:

$$\begin{array}{lccccccccc} \mathrm{I} & h_0 & & h_1 & & h_2 & & \ldots & \\ \mathrm{II} & & p_1 & & p_2 & & p_3 & & \ldots \end{array}$$

Set $p_0 = \emptyset$. For $i \geq 1$, p_i must a sequence of 0's and 1's of length i. It is required that $p_0 \subseteq p_1 \subseteq \ldots$. For each i, h_i must be a function into $[0,1]$ from $\{p_i{}^\frown\langle 0\rangle, p_i{}^\frown\langle 1\rangle\}$. Let $v_0 = v$ and for $i \geq 0$ let

$$v_{i+1} = h_i(p_{i+1}).$$

The final requirement on h_i is that

$$\frac{1}{2}h_i(p_i{}^\frown\langle 0\rangle) + \frac{1}{2}h_i(p_i{}^\frown\langle 1\rangle) \;\geq\; v_i.$$

The final requirement on p_{i+1} is that $v_{i+1} > 0$.

For any play x^* of H_v, let $\pi(x^*)$ be the member of $2^{\mathbb{N}}$ extending all the p_i. The play x^* is a win for I if and only if $\pi(x^*) \in B$.

Theorem 2.4.28. *If H_v is determined for every v, then B is Lebesgue measurable.*

Proof. Analogues of Lemmas 2.4.19 and 2.4.22 give that the inner measure of B is $\geq v$ if I has an winning strategy for H_v and that the outer measure of B is $\leq v$ if II has a winning strategy for H_v. □

Let $\bar{H}_v$ be like H_v except that all $h_i(q)$ must be rational.

Theorem 2.4.29. *If $\bar{H}_v$ is determined for every rational $v \in (0,1]$, then B is Lebesgue measurable.*

Our definition of Blackwell games requires that each player has only finitely many legal moves in each position. We could relax this requirement by demanding only that, in each position, each player has only countably many legal moves and at least one of the players has only finitely many legal moves. All our determinacy proofs would still work for this more general concept. Some proofs would change in a very minor way, because 1-move games would no longer have optimal strategies. Of course, one could generalize further by allowing positions in which one or the other player makes a move alone, from a countable set of possibilities.

We have thus far dealt only with Blackwell games $\Gamma(f)$ such that all plays of Γ are infinite and such that the range of f is a subset of $[0,1]$ (though we made no real use of the former hypothesis). It is clear that our proofs work with only trivial modifications for general Blackwell games. We will therefore cite the theorems we have proved as if they were their generalizations.

Theorem 2.4.30. *All Borel Blackwell games are determined.*

Proof. For Borel measurable f, the games G_v and $\bar{G}_v$ have Borel payoff sets. By [Martin, 1975] or [Martin, 1985], Borel games of perfect information are determined. □

Theorem 2.4.31. *All Borel stochastic games (of the kind above) are determined.*

Proof. This follows from Theorem 2.4.24 and Borel perfect information determinacy. □

As we said earlier, it was Maitra and Sudderth who noticed that our methods yield Theorem 2.4.31, and in [Maitra and Sudderth, 1993] they prove a more general version of it.

Borel perfect information determinacy for the case of countable game trees can be stated in, for example, formal second order number theory. The same is true of Borel Blackwell determinacy. Results of Friedman [Friedman, 1971] show, in a technical sense, that Borel perfect information determinacy cannot be proved without appealing to uncountably many uncountable cardinal numbers. Indeed, for every new level of the Borel hierarchy beyond the third level, a new cardinal number is needed. Thus it is of interest that the proof of Theorem 2.4.30 goes through in second order number theory and that the proof is "local," i.e., Blackwell determinacy for a given Borel level needs only perfect information Borel determinacy for the same level.

Theorem 2.4.32. *Work in formal second order number theory. Let α be a countable ordinal. Assume that, for countable game trees, every $\mathbf{\Pi}^0_\alpha$ perfect information game is determined. Then every $\mathbf{\Pi}^0_\alpha$ Blackwell game is determined. (For what is we mean by a "$\mathbf{\Pi}^0_\alpha$ Blackwell game," see page 105.)*

Proof. If $\Gamma(f)$ is $\mathbf{\Pi}^0_\alpha$, then the games $\bar{G}_v$ are $\mathbf{\Pi}^0_\alpha$ as long as $\alpha > 2$. □

Remarks:

(a) For all $\alpha \geq 1$, the games G'_v and $\bar{G}'_v$ are $\mathbf{\Pi}^0_\alpha$ if the set A is $\mathbf{\Pi}^0_\alpha$.

(b) Theorem 2.4.32 holds for the stochastic games defined in §1, since the proof of Theorem 2.4.31 is local in the same way as the proof of Theorem 2.4.30.

Going beyond the Borel sets, we can derive from the results of §1 and §2 local results for pretty much any natural classes. Here are just two examples. Projective perfect information determinacy for countable game trees implies projective Blackwell determinacy. For each positive integer n, $\mathbf{\Sigma}^1_n$ perfect information determinacy implies $\mathbf{\Sigma}^1_n$ Blackwell determinacy.

As we have already said, the determinacy of many classes of perfect information games has been deduced from so-called large cardinal axioms. With the aid of our theorem, we get corresponding determinacy results for Blackwell games. For example, for all $n \geq 0$, $\mathbf{\Sigma}^1_{n+1}$ Blackwell determinacy follows from the existence of n Woodin cardinals with a measurable cardinal above them.

Vervoort in [Vervoort, 1996] introduces the *Axiom of Determinacy for Blackwell Games* (*AD-Bl*), the assertion that all Blackwell Games are determined. He shows that AD-Bl, like AD, contradicts the Axiom of Choice. He deduces from AD-Bl an important known consequence of AD: that all sets of reals are Lebesgue measurable.

Itay Neeman pointed out to us that there are several variants of AD-Bl that are not obviously equivalent to one another. One could restrict to games of the form $\Gamma(\chi(A))$. Whether or not one did this, one could require that each player has exactly 2 legal moves in each position, or one could replace 2 by some other number n. In the opposite direction, one could allow that in each position one of the players has countably infinitely many legal moves. We know of no simple argument that any two of the possible versions of AD-Bl are equivalent. Nonetheless, it can be shown that they are all equivalent. The games $\bar{G}_v$ of §2 can easily be turned into equivalent games in which only two legal moves are available to each player in each position. Our proofs adapt to show that the mixed strategy determinacy of these games is enough to yield the determinacy of the given game $\Gamma(f)$.

[Vervoort, 1996] asks whether either of AD and AD-Bl implies the other. Our results obviously give an implication in one direction.

Theorem 2.4.33. *Work in ZF without the Axiom of Choice. AD implies AD-Bl.*

What about the converse? Also, do forms of Blackwell determinacy consistent with the Axiom of Choice imply the corresponding forms of perfect information determinacy?

The main results on these questions are in [Martin et al., 2003]. Examples of pointclasses Γ for which this paper proves that Blackwell determinacy implies determinacy are $\mathbf{\Delta}^1_{2n}$ for $n \in \omega$, the class of projective sets, the class of sets in $L(\mathbb{R})$. The fact that Blackwell determinacy for sets in $L(\mathbb{R})$ implies determinacy for sets in $L(\mathbb{R})$ implies that AD is consistent if AD-Bl is consistent.

Here is a rough sketch of how such theorems are proved. By a *perfect information game*, let us mean a game in ${}^{<\omega}2$ played as in all sections of the book prior to the present one. To show that Γ Blackwell determinacy implies Γ determinacy, it is enough to show that if all perfect information Γ games are determined in the sense of mixed strategies, then all perfect information Γ games are determined in the sense of pure strategies. A theorem of Vervoort shows that every perfect information game determined in mixed strategies has value 0 or 1 and is determined in optimal strategies. In other words, either player I has a strategy whose value is 1 or player II has a strategy whose value is 0. In §6E of [Moschovakis, 1980], a method is introduced for using perfect information determinacy to get optimal (in a different sense from the present one) pure winning strategies for Γ games for certain pointclasses Γ. It turns out that the method applies when the given determinacy is not in pure strategies but just in mixed strategies, provided that the games have value 0 or 1. Even though the input is weakened to such mixed strategies, the output is still pure strategies. This yields Lemma 4.1 of [Martin et al., 2003]:

Let Γ be a weakly scaled adequate pointclass. Let Δ be the intersection of Γ and its dual. Then Δ perfect information Blackwell determinacy implies Δ determinacy.

Here *adequate* means closed under recursive substitutions, $\vee$, $\wedge$, and bounded number quantifcation, and Γ is *weakly scaled* if every set in Γ has a scale such that each of the norms is a Γ norm. The mentioned results for the projective hierarchy are proved using by bootstrapping, using Lemma 4.1 and Moschovakis' method for getting scales for $\boldsymbol{\Sigma}^1_{2n+2}$ from $\boldsymbol{\Delta}^1_{2n}$ determinacy. The case of $L(\mathbb{R})$ is handled using a result of [Kechris and Woodin, 1983].

Here is a more indirect method that sometimes works for getting determinacy from Blackwell determinacy. Many of the proofs of consequences of perfect information determinacy still work if the existence of mixed strategies replaces that of pure strategies. Among the consequences of perfect information determinacy is the existence of good inner models for various large cardinal axioms. Many of the proofs of perfect information determinacy from large cardinal axioms need as hypotheses only the existence of good inner models of the large cardinal axioms. In this way one often gets the equivalence of forms of determinacy and the existence of good inner models of large cardinal axioms. These facts provide a method for proving perfect information determinacy from Blackwell determinacy. A sample theorem that can be proved in this way is the following. $\mathbf{\Sigma}^1_1$ Blackwell determinacy—even just for games of the form $\Gamma(\chi(A))$—implies $\mathbf{\Sigma}^1_1$ perfect information determinacy.

What about trying to show directly that Blackwell determinacy implies determinacy? The most direct way to proceed would be to show that any countable-tree perfect information game that is determined in the sense of mixed strategies is determined in the sense of pure strategies. Unfortunately, this is false, as the following example of Greg Hjorth shows. Let A be any uncountable subset of $2^{\mathbb{N}}$ such that $\mu(A) = 0$ for every atomless Borel probability measure μ. (For example, let the members of A code wellorderings, exactly one of the order type of each countable ordinal.) Consider the game $G^*(A)$ defined on page 149 of [Kechris, 1994]. Player II has a mixed strategy whose value in $G^*(A)$ is 0: in each position, assign $1/2$ to each of the two legal moves. But II has no winning pure strategy. (See part (ii) of Theorem 21.1 of [Kechris, 1994].) This counterexample does not destroy all branches of the direct route. For example, Vervoort's theorem lets one assume that all sets are Lebesgue measurable, and this rules out counterexamples of the kind described in parentheses above. Moreover, although mixed strategy determinacy for a perfect information game does not imply pure strategy determinacy, there are useful constraints on the values of perfect information games. We have been able to prove that the upper or lower value (in the mixed strategy sense) of a perfect information game in our sense (i.e., one whose winning condition is given by a set of plays) is either 0 or 1.

Theorems 2.4.25 and 2.4.27 and the related result for general payoff functions give a strong version of Blackwell determinacy. It is easy to see that this strong version implies, level by level, perfect information determinacy. Thus another route to our goal would be to show that Blackwell determinacy implies strong Blackwell determinacy. As we mentioned earlier, Maitra, Purves, and Sudderth [Maitra et al., 1991] have shown that, for Borel A, the determinacy of $\Gamma(\chi(A))$ implies the strong determinacy of $\Gamma(\chi(A))$. Hjorth's example given in the preceding paragraph shows that, under Choice, this is not true for arbitrary A. Nevertheless, their proof may still be relevant. That proof uses the fact that Borel sets are universally measurable. The proof of the Lebesgue measurability result of [Vervoort, 1996] shows that the universal measurability of a set follows from the Blackwell determinacy of sets of about the same complexity. The additional fact about Borel sets used in the proof of [Maitra et al., 1991] is their universal capacitability. This does not generalize to all sets under AD-Bl, for there exist even $\mathbf{\Pi}^1_1$ sets that are not universally capacitable. But to prove that AD-Bl implies strong

AD-Bl it would be enough to prove from AD-Bl that all sets are capacitable with respect to the capacities of [Maitra et al., 1991]. See Section 30 and Exercises 36.22 and 39.14 of [Kechris, 1994] for some of the capacitability consequences of perfect information determinacy, consequences that are due independently to Busch, Shochat, and Mycielski.

Exercise 2.4.1. Let T be the game tree in which every play has length 1 and for which $X_\emptyset = Y_\emptyset = \omega$. Blackwell games in T are thus played by each player's choosing a natural number. Let A be the set of plays in T such that I's number is $\geq$ II's number. Prove that $\mathrm{val}_\downarrow(\Gamma(A;T)) = 0$ and $\mathrm{val}^\uparrow(\Gamma(A;T)) = 1$.

Exercise 2.4.2. Verify that the proofs of Theorems 2.4.4 and 2.4.8, with trivial changes, still work if we replace "lim sup" by "lim inf" in stating the winning condition for G_v.

Exercise 2.4.3. (c) Prove the following strengthening of Theorem 2.4.13: If G_v is determined for every $v \in (0,1]$, then $\mathrm{val}(\Gamma(f;T))$ is the supremum of the $\mathrm{val}(\Gamma(g;T))$ for functions g such that $(\forall x \in [T])\, g(x) \leq f(x)$ and g is the lim sup of a function defined on positions in T.

This is the analogue of Theorem 2.4.25 in the context of the G_v (instead of the G'_v).

Chapter 3

Measurable Cardinals

The results of Chapter 2 exhaust the determinacy that can be proved in ZFC, at least if we are talking of proving the determinacy of all games in natural topological or definability classes. The next natural class after $\mathbf{\Delta}^1_1$ is $\mathbf{\Pi}^1_1$ or its dual $\mathbf{\Sigma}^1_1$. (See page 84 for the definitions of these classes.) By what is essentially a result of [Davis, 1964], the determinacy of all $\mathbf{\Pi}^1_1$ games is not provable in ZFC. (See Exercise 4.1.1.) The rest of the determinacy results in this book will be proved with the help of large cardinal axioms. In the next chapter, we will prove the determinacy of $\mathbf{\Pi}^1_1$ games in an arbitrary tree T from the assumption that a *measurable cardinal* exists.

The purpose of this chapter is to introduce measurable cardinals and related notions and to prove the basic facts about them. In §3.1 we give the definition of measurable cardinals and establish the facts about them that we need in order to give the proof of $\mathbf{\Pi}^1_1$ determinacy in §4.1. Sections 4.1 and 4.2 may be read independently of the rest of Chapter 3. In §3.2 we introduce ultrapowers and prove a characterization of measurable cardinals in terms of elementary embeddings. The elementary embedding definition of measurable cardinals is important not only because we will later make use of it but also because it is the elementary embedding versions of large cardinal axioms that (1) support most arguments for their plausibility and (2) allow one to see that the known large cardinal axioms are arranged in a coherent hierarchy. In §3.3 we extend the concepts and results of §3.2 to iterated ultrapowers and iterations of elementary embeddings. These notions will be used in §4.3 and throughout the later chapters. As always in this book, our aim is to prove determinacy results from the weakest possible large cardinal axioms. In §4.4. we show that $\mathbf{\Pi}^1_1$ determinacy for games in countable trees can be proved from a consequence of measurable cardinals, the existence of *sharps* of elements of ${}^{\omega}\omega$. §3.4 is devoted to an exposition of constructible sets, relative constructibility, and sharps. In §3.5 we study canonical inner models for ZFC + "there is a measurable cardinal." We also study related models. These models will be used in the last three sections of Chapter 5.

Measurable cardinals were introduced in [Ulam, 1930]. Most of the material in this chapter dates, however, from the 1960's, when there was a major revival in the study of large cardinals.

3.1 Basic Properties

For any nonempty set A, a *filter* on A is a set $\mathcal{F}$ of subsets of A such that

(a) $A \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;

(b) $(\forall X \in \mathcal{F})(\forall Y \in \mathcal{F})\, X \cap Y \in \mathcal{F}$;

(c) $(\forall X \in \mathcal{F})(\forall Y \subseteq A)(X \subseteq Y \to Y \in \mathcal{F})$.

An *ultrafilter* on a set A is a filter $\mathcal{U}$ on A such that

(d) $(\forall X \subseteq A)(X \in \mathcal{U} \ \vee\ A \setminus X \in \mathcal{U})$.

A filter $\mathcal{F}$ on A is *principal* if there is a $Y \subseteq A$ such that $\mathcal{F} = \{X \subseteq A \mid Y \subseteq X\}$.

Lemma 3.1.1. *An ultrafilter $\mathcal{U}$ on A is principal if and only if there is an $a \in A$ such that $\{a\} \in \mathcal{U}$.*

Proof. Let $\mathcal{U}$ be an ultrafilter on A. If $a \in A$ and $\{a\} \in \mathcal{U}$, then clause (c) in the definition of a filter implies that $\mathcal{U} \supseteq \{X \subseteq A \mid a \in X\}$, and clauses (a) and (b) then imply that $\mathcal{U} = \{X \subseteq A \mid a \in X\}$. Suppose that $Y \subseteq A$ and that $\mathcal{U} = \{X \subseteq A \mid Y \subseteq X\}$. By (a), Y is nonempty. Let $a \in Y$. By (d), one of $\{a\}$ and $A \setminus \{a\}$ belongs to $\mathcal{U}$ and so is a superset of Y. This is possible only if $Y = \{a\}$. $\square$

If κ is a cardinal number, a filter $\mathcal{F}$ is *κ-complete* if every intersection of fewer than κ elements of $\mathcal{F}$ belongs to $\mathcal{F}$. Clause (b) in the definition of a filter is thus equivalent with the assertion that $\mathcal{F}$ is $\aleph_0$-complete. A filter $\mathcal{F}$ is *countably complete* if $\mathcal{F}$ is closed under countable intersections. Note that countable completeness is equivalent with $\aleph_1$-completeness, not with $\aleph_0$-completeness.

A cardinal number κ is *measurable* if $\kappa > \aleph_0$ and there is a κ-complete, non-principal ultrafilter on κ. (Recall that a cardinal number κ is identical with the set of all ordinals α such that the cardinal number $|\alpha|$ of α—i.e., of the set of predecessors of α—is smaller than κ. Thus a cardinal number κ is a set and $|\kappa| = \kappa$.)

The study of measurable cardinals arose out of [Banach, 1930] and especially [Ulam, 1930]. These papers dealt with the question of whether there can be a countably additive real-valued measure defined on all subsets of the unit interval and giving singletons measure 0 and the whole unit interval positive measure. An ultrafilter on A is essentially the same as a finitely additive $\{0,1\}$-measure (a finitely additive measure taking only the values 0 and 1) defined on the whole power set $\mathcal{P}(A)$ of A and giving the empty set measure 0 and A measure 1: If μ is such a measure, let $\mathcal{U} = \{X \subseteq A \mid \mu(X) = 1\}$. If $\mathcal{U}$ is an ultrafilter on A, let $\mu : \mathcal{P}(A) \to \{0,1\}$ be given by $\mu(X) = 1 \leftrightarrow X \in \mathcal{U}$. Thus the definition of a measurable cardinal can be given in terms of measures, and this fact explains the name. If we left out the conventional constraint that a measurable cardinal must be uncountable, then $\aleph_0$ would qualify as a measurable cardinal.

If a filter $\mathcal{F}$ is non-principal, then $\bigcap \mathcal{F} \notin \mathcal{F}$. We may then define the *completeness* of a non-principal filter $\mathcal{F}$ to be the least cardinal κ such that some intersection of κ elements of $\mathcal{F}$ does not belong to $\mathcal{F}$. It is not hard to show that every non-principal filter on A has a completeness $\leq |A|$. For ultrafilters this is immediate from Lemma 3.1.1 and clause (a) in the definition of a filter. By clause (b), the completeness of a filter is at least $\aleph_0$.

The following lemma gives a very useful method of constructing new filters and ultrafilters from old ones.

Lemma 3.1.2. *Let A and B be sets, let $\mathcal{F}$ be a filter on A, and let $f : A \to B$. Let $\mathcal{G}$ be the set of all $X \subseteq B$ such that $f^{-1}(X) \in \mathcal{F}$. Then*

(1) $\mathcal{G}$ is a filter on B;

(2) if $\mathcal{F}$ is an ultrafilter then so is $\mathcal{G}$;

(3) the completeness of $\mathcal{G}$ is is at least as large as the completeness of $\mathcal{F}$, where we think of the completeness of a principal filter as Ord.

(4) $\mathcal{G}$ is a principal ultrafilter if and only if $(\exists b \in B)\, f^{-1}(\{b\}) \in \mathcal{F}$.

Proof. (1) Since $f^{-1}(B) = A$ and $f^{-1}(\emptyset) = \emptyset$, the fact that $\mathcal{F}$ satisfies clause (a) in the definition of a filter implies that $\mathcal{G}$ satisfies (a). For clause (b), we note that $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$. Hence clause (b) for $\mathcal{F}$ implies clause (b) for $\mathcal{G}$. Similarly clause (c) for $\mathcal{G}$ follows from clause (c) for $\mathcal{F}$, because if $X \subseteq Y$ then $f^{-1}(X) \subseteq f^{-1}(Y)$.

(2) Assume that $\mathcal{F}$ is an ultrafilter. Since $f^{-1}(B \setminus X) = A \setminus f^{-1}(X)$, property (d) for $\mathcal{F}$ implies property (d) for $\mathcal{G}$.

(3) Let κ be a cardinal number. Since

$$f^{-1}(\bigcap_{\alpha<\kappa} X_\alpha) = \bigcap_{\alpha<\kappa} f^{-1}(X_\alpha),$$

it follows that $\mathcal{G}$ is closed under intersections of size κ if $\mathcal{F}$ is closed under intersections of size κ.

(4) This is an immediate consequence of Lemma 3.1.1 and the definition of $\mathcal{G}$. □

We will see later that the existence of measurable cardinals cannot be demonstrated in ZFC. The following lemma shows that their existence is equivalent with the existence of a countably complete, non-principal ultrafilter on some set.

Lemma 3.1.3. *If $\mathcal{U}$ is a countably complete, non-principal ultrafilter on A and κ is the completeness of $\mathcal{U}$, then κ is a measurable cardinal.*

Proof. Let $\kappa > \aleph_0$ be the completeness of a non-principal ultrafilter $\mathcal{U}$ on A. Let $\{X_\alpha \mid \alpha < \kappa\}$ witness that the completeness of $\mathcal{U}$ is no greater than κ. Thus each $X_\alpha \in \mathcal{U}$ but $\bigcap_{\alpha<\kappa} X_\alpha \notin \mathcal{U}$. Define $f : A \to \kappa$ by

$$f(a) = \begin{cases} \mu\gamma(a \notin X_\gamma) & \text{if } a \notin \bigcap_{\gamma<\kappa} X_\gamma; \\ 0 & \text{otherwise.} \end{cases}$$

Here, as usual, "μ" means "the least." Let $\mathcal{V} = \{X \subseteq \kappa \mid f^{-1}(X) \in \mathcal{U}\}$. By Lemma 3.1.2, we get that $\mathcal{V}$ is a κ-complete ultrafilter on κ. For each non-zero $\alpha < \kappa$, $f^{-1}(\{\alpha\})$ is disjoint from X_α, and $f^{-1}(\{0\})$ is disjoint from $X_0 \setminus \bigcap_{\gamma<\kappa} X_\gamma$. Thus no $f^{-1}(\{\alpha\})$ belongs to $\mathcal{U}$, and so clause (4) of Lemma 3.1.2 implies that $\mathcal{V}$ is non-principal. □

Corollary 3.1.4. *If there is a cardinal κ such that there is a countably complete, non-principal ultrafilter on κ, then there is a measurable cardinal $\leq \kappa$.*

In the next section, we will present techniques that give easy proofs that measurable cardinals are very large. Even without these techniques, it is not hard to show (Exercises 3.1.1 and 3.1.2) that every measurable cardinal is *inaccessible*, i.e. regular and a strong limit. An infinite cardinal κ is *regular* if there is no ordinal $\lambda < \kappa$ such that some $f : \lambda \to \kappa$ has unbounded range; equivalently, κ is regular if $\mathrm{cf}(\kappa) = \kappa$. An infinite cardinal κ is a *strong limit* if whenever λ is a cardinal $< \kappa$ then $2^\lambda < \kappa$.

An ultrafilter $\mathcal{U}$ on an infinite cardinal κ is *normal* if, for all functions $f : \kappa \to \kappa$, if $\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in \mathcal{U}$ then there is a $\beta < \kappa$ such that $\{\alpha < \kappa \mid f(\alpha) = \beta\} \in \mathcal{U}$.

Let κ be an infinite cardinal. No non-principal ultrafilter on κ can be closed under all intersections of κ-many sets. If $\langle X_\beta \mid \beta < \kappa\rangle$ is a sequence of subsets of κ, then the *diagonal intersection* $\Delta_{\beta<\kappa} X_\beta$ is defined by

$$\Delta_{\beta<\kappa} X_\beta = \{\alpha < \kappa \mid (\forall \beta < \alpha)\, \alpha \in X_\beta\}.$$

Lemma 3.1.5. (Dana Scott; see [Keisler and Tarski, 1964]) *If $\mathcal{U}$ is an ultrafilter on an infinite cardinal κ, then $\mathcal{U}$ is normal if and only if $\mathcal{U}$ is closed under diagonal intersections.*

Proof. Let $\mathcal{U}$ be an ultrafilter on κ, with κ infinite. Assume first that $\mathcal{U}$ is normal. Let $\langle X_\beta \mid \beta < \kappa\rangle$ be such that each $X_\beta \in \mathcal{U}$. Suppose that $\Delta_{\beta<\kappa} X_\beta \notin \mathcal{U}$. Define $f : \kappa \to \kappa$ by

$$f(\alpha) = \begin{cases} \mu\beta(\beta < \alpha \wedge \alpha \notin X_\beta) & \text{if } (\exists \beta < \alpha)\, \alpha \notin X_\beta; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Delta_{\beta<\kappa} X_\beta \notin \mathcal{U}$, the set of α for which the first clause in the definition of f applies is a set in $\mathcal{U}$, i.e. $\{\alpha \mid f(\alpha) < \alpha \wedge \alpha \notin X_{f(\alpha)}\} \in \mathcal{U}$. By normality, let $\beta < \kappa$ be such that $\{\alpha < \kappa \mid f(\alpha) = \beta\} \in \mathcal{U}$. But then $\{\alpha < \kappa \mid \alpha \notin X_\beta\} \in \mathcal{U}$, contrary to assumption.

Now assume that $\mathcal{U}$ is closed under diagonal intersections. Let $f : \kappa \to \kappa$ be such that $\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in \mathcal{U}$. For $\beta < \kappa$, let $X_\beta = \{\alpha < \kappa \mid f(\alpha) \neq \beta\}$. We have that $\Delta_{\beta<\kappa} X_\beta \notin \mathcal{U}$. From closure under diagonal intersections, we get a $\beta < \kappa$ such that $X_\beta \notin \mathcal{U}$. But then $\{\alpha < \kappa \mid f(\alpha) = \beta\} \in \mathcal{U}$. □

An ultrafilter $\mathcal{U}$ on an infinite cardinal κ is *uniform* if every element of $\mathcal{U}$ has size κ; it is *weakly uniform* if for each $\delta < \kappa$ the set of all $\alpha < \kappa$ such that $\delta \leq \alpha$ belongs to $\mathcal{U}$.

Lemma 3.1.6. *Let $\mathcal{U}$ be a normal ultrafilter on an infinite cardinal κ. Then the following are equivalent:*

(a) $\mathcal{U}$ is κ-complete and non-principal.

(b) $\mathcal{U}$ is uniform.

(c) $\mathcal{U}$ is weakly uniform.

Proof. That (a) implies (b) and that (b) implies (c) follow directly from the definitions, and these implications do not depend on the hypothesis of normality.

To show that (c) implies (a) assume that $\mathcal{U}$ is weakly uniform. Obviously $\mathcal{U}$ is non-principal. For κ-completeness, let $\delta < \kappa$ and let $\langle X_\gamma \mid \gamma < \delta \rangle$ be a sequence of members of $\mathcal{U}$. For $\gamma \geq \delta$, let $X_\gamma = \kappa$. By Lemma 3.1.5, $\Delta_{\gamma<\kappa} X_\gamma \in \mathcal{U}$. Since $\mathcal{U}$ is uniform, $\{\alpha < \kappa \mid \delta \leq \alpha\} \in \mathcal{U}$. But then

$$\bigcap_{\gamma<\delta} X_\gamma \supseteq ((\kappa \setminus \delta) \cap \Delta_{\gamma<\kappa} X_\gamma) \in \mathcal{U}.$$

□

Lemma 3.1.7. (Dana Scott; see [Keisler and Tarski, 1964]) *If κ is a measurable cardinal, then there is a uniform normal ultrafilter on κ.*

Proof. Let κ be a measurable cardinal. Let $\mathcal{U}$ be a non-principal, κ-complete ultrafilter on κ.

We first show that there is an $f : \kappa \to \kappa$ such that

(i) f is not constant on any member of $\mathcal{U}$; i.e., $(\forall \beta < \kappa)\{\alpha < \kappa \mid f(\alpha) = \beta\} \notin \mathcal{U}$;

(ii) for every $g : \kappa \to \kappa$, if g is not constant on any member of $\mathcal{U}$, then $\{\alpha < \kappa \mid f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$.

Assume that no f satisfying (i) and (ii) exists. Let f_0 be the identity function on κ. Since $\mathcal{U}$ is non-principal, f_0 satisfies (i). Assume inductively that $f_0, f_1, \ldots, f_n$ all satisfy (i) and that $(\forall i < n)\{\alpha < \kappa \mid f_{i+1}(\alpha) < f_i(\alpha)\} \in \mathcal{U}$. By assumption, f_n does not satisfy (ii). Let f_{n+1} be a g witnessing this fact. Our induction hypothesis thus holds for $n+1$. Since $\mathcal{U}$ is κ-complete and κ is uncountable, we have that $\mathcal{U}$ is countably complete. Hence

$$\bigcap_{n\in\omega} \{\alpha < \kappa \mid f_{n+1}(\alpha) < f_n(\alpha)\} \in \mathcal{U}.$$

By clause (a) in the definition of a filter, no element of $\mathcal{U}$ can be empty. Let then $\alpha \in \bigcap_{n\in\omega}\{\alpha < \kappa \mid f_{n+1}(\alpha) < f_n(\alpha)\}$. We have that

$$f_0(\alpha) > f_1(\alpha) > f_2(\alpha) > \cdots,$$

contradicting the fact that κ is wellordered by $<$.

Let $f : \kappa \to \kappa$ satisfy (i) and (ii). Define an ultrafilter $\mathcal{V} \subseteq \mathcal{P}(\kappa)$ by

$$X \in \mathcal{V} \leftrightarrow f^{-1}(X) \in \mathcal{U}.$$

Lemma 3.1.2 and (i) give that $\mathcal{V}$ is a non-principal ultrafilter on κ. To prove the normality of $\mathcal{V}$, suppose that $\{\alpha < \kappa \mid h(\alpha) < \alpha\} \in \mathcal{V}$. We must show that $(\exists\beta < \kappa)\,\{\alpha < \kappa \mid h(\alpha) = \beta\} \in \mathcal{V}$. Define $g : \kappa \to \kappa$ by

$$g(\alpha) = h(f(\alpha)).$$

Since $\{\alpha < \kappa \mid h(\alpha) < \alpha\} \in \mathcal{V}$, it follows from the definition of $\mathcal{V}$ that $\{\alpha < \kappa \mid h(f(\alpha)) < f(\alpha)\} \in \mathcal{U}$. Hence $\{\alpha < \kappa \mid g(\alpha) < f(\alpha)\} \in \mathcal{U}$. But f satisfies (ii); so g cannot satisfy (i). Thus we get a $\beta < \kappa$ such that $\{\alpha < \kappa \mid g(\alpha) = \beta\} \in \mathcal{U}$. By the definitions of g and $\mathcal{V}$, this implies that $\{\alpha < \kappa \mid h(\alpha) = \beta\} \in \mathcal{V}$. □

The proof of Lemma 3.1.7 uses the uncountability of measurable cardinals. This use is necessary: no non-principal ultrafilter on ω is normal. (See Exercise 3.1.3.)

For any set z and any cardinal number λ, $[z]^\lambda$ is the set of all subsets w of z such that $|w| = \lambda$. One reason that normal ultrafilters are useful is the following result of Frederick Rowbottom, which shows that a κ-complete normal ultrafilter on κ generates κ-complete ultrafilters on $[\kappa]^n$ for all $n \in \omega$.

Lemma 3.1.8. ([Rowbottom, 1964]) *Let $n \in \omega$ and let $\mathcal{U}$ be a normal ultrafilter on a cardinal κ. If $Z \subseteq [\kappa]^n$, there is an $X \in \mathcal{U}$ such that either $[X]^n \subseteq Z$ or $[X]^n \cap Z = \emptyset$.*

Proof. We prove the lemma by induction on n.

The case $n = 0$ is trivial, since $[\kappa]^0$ has only one member, $\emptyset$.

Assume that the lemma holds for $n \geq 0$. Let $Z \subseteq [\kappa]^{n+1}$. For each $\beta < \kappa$, let

$$Z_\beta = \{u \in [\kappa \setminus \{\beta\}]^n \mid \{\beta\} \cup u \in Z\}.$$

By our induction hypothesis, we have for each β an $X_\beta \in \mathcal{U}$ such that either $[X_\beta]^n \subseteq Z_\beta$ or $[X_\beta]^n \cap Z_\beta = \emptyset$. Let Y be $\{\beta \mid [X_\beta]^n \subseteq Z_\beta\}$ if that set belongs to $\mathcal{U}$ and $\{\beta \mid [X_\beta]^n \cap Z_\beta = \emptyset\}$ otherwise. Let

$$X = Y \cap \Delta_{\beta<\kappa} X_\beta.$$

By Lemma 3.1.5, we have that $X \in \mathcal{U}$. Assume first that $Y = \{\beta \mid [X_\beta]^n \subseteq Z_\beta\}$. Let $t \in [X]^{n+1}$. Let β be the least element of t. Let $u = t \setminus \{\beta\}$. We have that $u \in [X]^n \subseteq [\Delta_{\eta<\kappa} X_\eta]^n$, so $u \in [X_\beta]^n \subseteq Z_\beta$. By the definitions of t and Z_β, we get that $t \in Z$. If we now assume that $Y = \{\beta \mid [X_\beta]^n \cap Z_\beta = \emptyset\}$, then a similar argument shows that no $t \in [X]^{n+1}$ belongs to Z. □

Suppose that $\mathcal{U}$ is a uniform normal ultrafilter on a cardinal κ. For $n \in \omega$, we define the *Rowbottom ultrafilter* $\mathcal{U}^{[n]}$ on $[\kappa]^n$ by

$$Z \in \mathcal{U}^{[n]} \leftrightarrow (\exists X \in \mathcal{U})\, [X]^n \subseteq Z.$$

The fact that $\mathcal{U}$ is κ-complete and non-principal implies that $\mathcal{U}^{[n]}$ is a κ-complete filter. (We need that $\mathcal{U}$ is non-principal in order to show that $\emptyset \notin \mathcal{U}^{[n]}$.) Lemma 3.1.8 implies that $\mathcal{U}^{[n]}$ is an ultrafilter. It is clear that $\mathcal{U}^{[n]}$ is non-principal if $n > 0$. The Rowbottom ultrafilter on $[\kappa]^n$ is essentially the same as the *iterated product ultrafilter* on κ^n. (See Exercise 3.1.7.)

Exercise 3.1.1. Prove that every measurable cardinal is regular. (Ulam [1930].)

Exercise 3.1.2. Prove that every measurable cardinal is a strong limit. (This is due to Alfred Tarski and Ulam independently. See [Ulam, 1930].)

Hint. Assume that $\lambda < \kappa$ and that $\mathcal{U}$ is a κ-complete ultrafilter on a subset of ${}^\lambda 2$. Prove that $\mathcal{U}$ is principal.

Exercise 3.1.3. Show that every normal ultrafilter on ω is principal.

Exercise 3.1.4. Let $\mathcal{U}$ be a normal ultrafilter on κ. Let $f : \kappa \to \kappa$ be such that $(\forall \alpha < \kappa)\, f^{-1}(\{\alpha\}) \notin \mathcal{U}$. Prove that there is a set belonging to $\mathcal{U}$ on which f is one-one.

Exercise 3.1.5. If $\mathcal{F}$ and $\mathcal{G}$ are filters on A and B respectively, then $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* ($\mathcal{F} \cong \mathcal{G}$) if there is a bijection $f : A \to B$ such that $\mathcal{G} = \{X \subseteq B \mid f^{-1}(X) \in \mathcal{F}\}$.

Prove that a κ-complete non-principal ultrafilter $\mathcal{U}$ on a measurable cardinal κ is isomorphic to a normal ultrafilter if and only if $\mathcal{U}$ satisfies Lemma 3.1.8, i.e. if and only if

$$(\forall n \in \omega)(\forall Z \subseteq [\kappa]^n)(\exists X \in \mathcal{U})([X]^n \subseteq Z \vee [X]^n \cap Z = \emptyset).$$

(This result is probably due to Dana Scott.)

Hint. For the non-trivial direction, consider a function f satisfying (i) and (ii) in the proof of Lemma 3.1.7.

Exercise 3.1.6. Prove that not every κ-complete non-principal ultrafilter on a measurable cardinal κ is isomorphic to a normal ultrafilter.

Hint. Prove that the function $f : [\kappa]^2 \to \kappa$ given by $f(u) = \min(u)$ is not one-one on any set belonging to $\mathcal{U}^{[2]}$.

Exercise 3.1.7. If $\mathcal{U}$ is a filter on a set A, then for the iterated product filter $\mathcal{U}^n$ on ${}^nA$ is defined by letting $\mathcal{U}^0$ be the unique filter on ${}^0A = \{\emptyset\}$ and inductively setting

$$W \in \mathcal{U}^{n+1} \leftrightarrow \{a \in A \mid \{s \in {}^nA \mid \langle a\rangle^\frown s \in W\} \in \mathcal{U}^n\} \in \mathcal{U}.$$

Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal κ and, for $n \in \omega$, let the injection $g : [\kappa]^n \to {}^n\kappa$ be given by letting each $g(u)$ enumerate u in increasing order. Prove that, for all $Z \in [\kappa]^n$, $Z \in \mathcal{U}^{[n]}$ if and only if $g(Z) \in \mathcal{U}^n$.

3.2 Ultrapowers and Elementary Embeddings

We now have developed enough of the theory of measurable cardinals to prove the main theorem of Chapter 4: that the determinacy of all $\mathbf{\Pi}^1_1$ games in a tree T follows from the existence of a measurable cardinal larger than $|T|$. Section 4.1, which contains the proof of this theorem, and also Section 4.2 can be read without reading the rest of Chapter 3.

For the determinacy proofs of the later chapters, however, we need a further technical tool: the ultrapower construction.

Convention. Except where we explicitly state otherwise, we mean by a *model* a model for the language of set theory: a model $\mathcal{M} = (M; E)$, where M is a nonempty set and E is a binary relation in M (a subset of $M \times M$).

Let $\mathcal{U}$ be an ultrafilter on a set A. Let $\mathcal{M} = (M; E)$ be a model. The ultrafilter $\mathcal{U}$ gives rise to an equivalence relation $\sim_{\mathcal{U},M}$ on ${}^A M$: If $f : A \to M$ and $g : A \to M$, then

$$f \sim_{\mathcal{U},M} g \leftrightarrow \{a \in A \mid f(a) = g(a)\} \in \mathcal{U}.$$

We will sometimes suppress the subscript $\mathcal{U}, M$ and write simply $\sim$ when there is no ambiguity about $\mathcal{U}$ or M.

For $f \in {}^A M$, we write $[\![f]\!]_{\mathcal{U},M}$ or just $[\![f]\!]$ for the equivalence class of f with respect to $\sim_{\mathcal{U},M}$. We denote the set of all the equivalence classes by ${}^A M/\mathcal{U}$.

We define a binary relation $E_{\mathcal{U},M}$ in ${}^A M/\mathcal{U}$ as follows:

$$[\![f]\!] \, E_{\mathcal{U},M} \, [\![g]\!] \leftrightarrow \{a \in A \mid f(a) \, E \, g(a)\} \in \mathcal{U}.$$

It is easy to see that $E_{\mathcal{U},M}$ is well-defined.

The *ultrapower* of $\mathcal{M}$ with respect to $\mathcal{U}$ is the model

$$\prod_{\mathcal{U}} \mathcal{M} = ({}^A M/\mathcal{U}; E_{\mathcal{U},M}).$$

Note that $\prod_{\mathcal{U}} \mathcal{M}$ is, like $\mathcal{M}$, a model for the language of set theory.

Remark. We have defined $\prod_{\mathcal{U}} \mathcal{M}$ only for models of one particular similarity type, but the definition can easily be extended to arbitrary models. $\prod_{\mathcal{U}} \mathcal{M}$ is always a model of the same similarity type as $\mathcal{M}$. The proof of Theorem 3.2.1 below works also for ultrapowers in this more general sense.

Ultrapowers were introduced in [Łoś, 1955], where the following theorem essentially appears.

Theorem 3.2.1. ([Łoś, 1955]) *Let $\mathcal{U}$ be an ultrafilter on A and let $\mathcal{M} = (M; E)$ be a model. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $f_1, \ldots, f_n$ be elements of ${}^A M$. Then*

$$\prod_{\mathcal{U}} \mathcal{M} \models \varphi[[\![f_1]\!], \ldots, [\![f_n]\!]] \leftrightarrow \{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)]\} \in \mathcal{U}.$$

Proof. We may assume that the only connectives in φ are $\wedge$ and $\neg$ and that the only quantifier in φ is $\exists$. We prove the theorem by induction on the complexity of the formula φ.

For φ atomic, the theorem holds of φ by the definitions of $\sim_{\mathcal{U},M}$ and $E_{\mathcal{U},M}$.

If φ is $\psi \wedge \chi$, then

$$\begin{array}{l} \prod_{\mathcal{U}} \mathcal{M} \models \varphi[[\![f_1]\!], \ldots, [\![f_n]\!]] \ \leftrightarrow \\ \prod_{\mathcal{U}} \mathcal{M} \models \psi[[\![f_1]\!], \ldots, [\![f_n]\!]] \wedge \prod_{\mathcal{U}} \mathcal{M} \models \chi[[\![f_1]\!], \ldots, [\![f_n]\!]] \ \leftrightarrow \\ \left(\begin{array}{l} \{a \in A \mid \mathcal{M} \models \psi[f_1(a), \ldots, f_n(a)]\} \in \mathcal{U} \wedge \\ \{a \in A \mid \mathcal{M} \models \chi[f_1(a), \ldots, f_n(a)]\} \in \mathcal{U} \end{array} \right) \leftrightarrow \\ \{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)]\} \in \mathcal{U}. \end{array}$$

Here we have used induction to get the second equivalence, and we have used clauses (b) and (c) in the definition of a filter to get the last equivalence.

If φ is $\neg\psi$, then

$$\begin{array}{l} \prod_{\mathcal{U}} \mathcal{M} \models \varphi[[\![f_1]\!], \ldots, [\![f_n]\!]] \ \leftrightarrow \\ \neg \left(\prod_{\mathcal{U}} \mathcal{M} \models \psi[[\![f_1]\!], \ldots, [\![f_n]\!]]\right) \ \leftrightarrow \\ \{a \in A \mid \mathcal{M} \models \psi[f_1(a), \ldots, f_n(a)]\} \notin \mathcal{U} \ \leftrightarrow \\ \{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)]\} \in \mathcal{U}. \end{array}$$

The last line follows from the preceding line by clause (d) in the definition of an ultrafilter. This is the only place in the proof where we use the fact that $\mathcal{U}$ is an ultrafilter rather than just a filter.

If φ is $(\exists v_0)\psi$, then

$$\begin{array}{l} \prod_{\mathcal{U}} \mathcal{M} \models \varphi[[\![f_1]\!], \ldots, [\![f_n]\!]] \ \leftrightarrow \\ (\exists f_0 \in {}^{A}M) \prod_{\mathcal{U}} \mathcal{M} \models \psi[[\![f_0]\!], [\![f_1]\!], \ldots, [\![f_n]\!]] \ \leftrightarrow \\ (\exists f_0 \in {}^{A}M) (\{a \in A \mid \mathcal{M} \models \psi[f_0(a), f_1(a), \ldots, f_n(a)]\} \in \mathcal{U}) \ \leftrightarrow \\ \{a \in A \mid (\exists b \in M) \mathcal{M} \models \psi[b, f_1(a), \ldots, f_n(a)]\} \in \mathcal{U} \ \leftrightarrow \\ \{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)]\} \in \mathcal{U}. \end{array}$$

Note that the Axiom of Choice is used to deduce that the fourth line implies the third. □

If $\mathcal{M} = (M; E)$ and $\mathcal{N} = (N; F)$ are models, an *elementary embedding* of $\mathcal{M}$ into $\mathcal{N}$ is a function $j : M \to N$ such that, for any formula $\varphi(v_1, \ldots, v_n)$ of the language of set theory and for any n-tuple $\langle b_1, \ldots, b_n \rangle$ of elements of M,

$$\mathcal{M} \models \varphi[b_1, \ldots, b_n] \leftrightarrow \mathcal{N} \models \varphi[j(b_1), \ldots, j(b_n)].$$

We write $j : \mathcal{M} \prec \mathcal{N}$ to mean that j is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

Let $\mathcal{M} = (M; E)$ be a model and let $\mathcal{U}$ be an ultrafilter on a set A. We define $j : M \to {}^A M/\mathcal{U}$ by

$$j(b) = [\![c_b]\!],$$

where $c_b : A \to M$ is the constant function with value b.

Corollary 3.2.2. $j : \mathcal{M} \prec \prod_{\mathcal{U}} \mathcal{M}$.

Proof. For any $\varphi(v_1, \ldots, v_n)$ and $\langle b_1, \ldots, b_n \rangle$, we have by Theorem 3.2.1 that

$$\prod_{\mathcal{U}} \mathcal{M} \models \varphi[[\![c_{b_1}]\!], \ldots, [\![c_{b_n}]\!]] \leftrightarrow \{a \in A \mid \mathcal{M} \models \varphi[c_{b_1}(a), \ldots, c_{b_n}(a)]\} \in \mathcal{U}.$$

But the left-hand side is equivalent with $\prod_{\mathcal{U}} \mathcal{M} \models \varphi[j(b_1), \ldots, j(b_n)]$, and the right-hand side just says that $\{a \in A \mid \mathcal{M} \models \varphi[b_1, \ldots, b_n]\} \in \mathcal{U}$, i.e. that $\mathcal{M} \models \varphi[b_1, \ldots, b_n]$. $\square$

A a model $(M; E)$ is *wellfounded* if the relation E is wellfounded, i.e. if every nonempty subset of M has an E-minimal element. This is equivalent (using Choice) with the non-existence of an infinite sequence $\langle b_i \mid i \in \omega \rangle$ such that $b_{i+1} \mathrel{E} b_i$ for each $i \in \omega$. In [Keisler, 1962b] ultrapowers of wellfounded structures were first used to get results about measurable cardinals. The next lemma is fundamental for the method.

Lemma 3.2.3. *Let $\mathcal{U}$ be a countably complete ultrafilter on the set A and and let $\mathcal{M} = (M; E)$ be a wellfounded model. Then $\prod_{\mathcal{U}} \mathcal{M}$ is also wellfounded.*

Proof. Suppose that $\langle f_i \mid i \in \omega \rangle$ is a counterexample to the wellfoundedness of $\prod_{\mathcal{U}} \mathcal{M}$; that is suppose that

$$\cdots E_{\mathcal{U},M} [\![f_2]\!] \, E_{\mathcal{U},M} [\![f_1]\!] \, E_{\mathcal{U},M} [\![f_0]\!].$$

By the definition of $E_{\mathcal{U},M}$,

$$(\forall i \in \omega)\{a \in A \mid f_{i+1}(a) \mathrel{E} f_i(a)\} \in \mathcal{U}.$$

By the countable completeness of $\mathcal{U}$, $\bigcap_{i \in \omega} \{a \in A \mid f_{i+1}(a) \mathrel{E} f_i(a)\} \in \mathcal{U}$. Let a belong to this set. Then

$$\cdots E \, f_2(a) \, E \, f_1(a) \, E \, f_0(a),$$

contrary to the wellfoundedness of $\mathcal{M}$. $\square$

We mainly want to apply the ultrapower construction to the case that $\mathcal{M}$ is a model of ZFC and that the relation E is the restriction of the membership relation to M, i.e. $\mathcal{M} = (M; \in \cap (M \times M))$. For simplicity we will write $(M; \in)$ instead of $\mathcal{M} = (M; \in \cap (M \times M))$. The Axiom of Foundation asserts that such models are always wellfounded. The following lemma of Andrzej Mostowski implies that wellfounded models of ZFC are all isomorphic to such models.

Recall that a set x is transitive if every member of a member of x belongs to x.

Lemma 3.2.4. ([Mostowski, 1949]) *Let $(M; E)$ be a wellfounded model of the Axiom of Extensionality. Then there is a unique transitive set N such that $(M; E) \cong (N; \in)$, and the isomorphism $\pi : (M; E) \cong (N; \in)$ is unique.*

Proof. We define $\pi(x)$ by transfinite recursion on the wellfounded relation E:

$$\pi(x) = \{\pi(y) \mid y \, E \, x\}.$$

Note that this must be $\pi(x)$ if π is to be an isomorphism. Let, as we must, $N = \{\pi(x) \mid x \in M\}$. It is immediate that N is transitive. It is immediate from the definition that $(\forall x \in M)(\forall y \in M)(y \, E \, x \to \pi(y) \in \pi(x))$. If π is one-one, then it also follows that $(\forall x \in M)(\forall y \in M)(\pi(y) \in \pi(x) \to y \, E \, x)$, and so that π is an isomorphism. We prove by induction on E that for every $x \in M$ there is no $x' \in M$ such that $x' \neq x$ and $\pi(x') = \pi(x)$. Assume then that $x' \neq x$. Since $(M; E)$ satisfies Extensionality, there is a $z \in M$ that bears E to exactly one of x' and x. Assume for definiteness that $z \, E \, x'$ but that not $z \, E \, x$; the other case is similar. Then by induction there is no $w \, E \, x$ such that $\pi(z) = \pi(w)$. But this means that $\pi(z) \in \pi(x') \setminus \pi(x)$ and so that $\pi(x') \neq \pi(x)$. $\square$

Suppose that $(M; \in)$ is a model and that $\mathcal{U}$ is a countably complete ultrafilter on a set A. Let $j : (M; \in) \prec \prod_{\mathcal{U}}(M; \in)$ be the canonical elementary embedding as defined on page 140. By Lemma 3.2.3, $\prod_{\mathcal{U}}(M; \in)$ is wellfounded. Let $\pi : \prod_{\mathcal{U}}(M; \in) \cong (N; \in)$ be given by Lemma 3.2.4. We have then that

$$\pi \circ j : (M; \in) \prec (N; \in).$$

We want to study such embeddings arising from a uniform normal $\mathcal{U}$ on a measurable cardinal κ. However, we want to replace M by a proper class, in particular by the set-theoretic universe V, and consequently to replace N also by a proper class. For this we must first check that the results we have derived so far hold for ultrapowers of proper class models.

Since we are officially working in ZFC, we can't literally talk about proper classes. What we mean by a *class* is something of the form

$$\{x \mid \varphi(x, y_1, \ldots, y_n)\},$$

where $y_1, \ldots, y_n$ are sets and φ is a formula of the language of (ZFC) set theory. Hence each class is determined by a formula and a finite sequence of sets. We cannot in our language make *general* statements about classes; thus most of the theorems in the rest of this section should be construed as theorem *schemata*. See pages 23–24 of [Kunen, 1980] for a discussion of this. If the reader prefers to construe our talk of classes literally, he can mostly take us to be working in von Neumann–Bernays–Gödel set theory.

Warning. We will be very casual in dealing with proper classes. The advantage of doing so is that ideas are less likely to be obscured by technical details. The disadvantage is that it will sometimes be a non-trivial problem for the careful reader to see how our discussion could be formalized in ZFC or even in von Neumann–Bernays–Gödel set theory.

As is usual, we will identify a non-proper class with the corresponding set.

Except where we explicitly state otherwise, we mean by a *class model* something of the form $(M; E)$ where M is a nonempty class and E is a subclass of $M \times M$. We haven't actually indicated what kind of set-theoretic object an ordinary model is (we haven't used ordered pair notation $\langle M, E\rangle$), so we may blithely keep the same ambiguity as to what specific object a class model is.

Let $\mathcal{M} = (M; E)$ be a class model and let $\mathcal{U}$ be an ultrafilter on a set A. We want to define an ultrapower as in the set model case. As before, for $f : A \to M$ and $g : A \to M$ we can let $f \sim_{\mathcal{U},M} g$ just in case $\{a \in A \mid f(a) = g(a)\} \in \mathcal{U}$. The first problem comes when we try to define the equivalence class $[\![f]\!]_{\mathcal{U},M}$. The genuine equivalence class, $\{g \mid f \sim_{\mathcal{U},M} g\}$, is a proper class if M is a proper class, unless $|A| = 1$. Since all our classes are to be classes of sets, using these classes for the $[\![f]\!]_{\mathcal{U},M}$ would render us unable to define ${}^{A}M/\mathcal{U}$. We could try picking a representative from each equivalence class, but this would require a global form of the Axiom of Choice, and we do not want to make such an assumption. A satisfactory solution comes from [Scott, 1955], and we will make use of it below.

First let us recall the cumulative hierarchy of sets. Inductively we define for each ordinal α a set V_α:

(i) $V_0 = \emptyset$;

(ii) $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$;

(iii) $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ if λ is a limit ordinal.

It is easy to show by induction that each V_α is transitive and so that the definition would be unaffected if we changed clause (ii) to set $V_{\alpha+1} = \mathcal{P}(V_\alpha)$. The Axiom of Foundation implies that every set belongs to the class $V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$. (See III, §4 of [Kunen, 1980].) Thus we can define the *rank* of any set x by

$$\mathrm{rank}\,(x) = \mu\alpha\, x \in V_{\alpha+1}.$$

Let us say that $f \sim_{\mathcal{U}} g$ if $f \sim_{\mathcal{U},V} g$, i.e. if f and g are functions with domain A and $\{a \in A \mid f(a) = g(a)\} \in \mathcal{U}$. Following [Scott, 1961] we define $[\![f]\!]_{\mathcal{U}}$ for $f : A \to V$ to be the set of all g of minimal rank such that $f \sim_{\mathcal{U}} g$, i.e.

$$[\![f]\!]_{\mathcal{U}} = \{g \mid f \sim_{\mathcal{U}} g \wedge (\forall h)(f \sim_{\mathcal{U}} h \to \mathrm{rank}\,(g) \le \mathrm{rank}\,(h))\}.$$

With this definition, $[\![f]\!]_{\mathcal{U}} \subseteq V_\alpha$ for some $\alpha \le \mathrm{rank}\,(f) + 1$. Thus $[\![f]\!]_{\mathcal{U}}$ is a set. When there is no ambiguity, we may write "$[\![f]\!]$" for "$[\![f]\!]_{\mathcal{U}}$."

Remark. We have chosen, since Scott's trick makes it possible, to use "equivalence classes" $[\![f]\!]_{\mathcal{U}}$ that are independent of M.

Continuing with our class model $\mathcal{M} = (M; E)$ and our ultrafilter $\mathcal{U}$ on the set A, we denote as in the set model case the class of all $[\![f]\!]_{\mathcal{U}}$ for $f : A \to M$ by ${}^A M/\mathcal{U}$. The class ${}^A M/\mathcal{U}$ is a proper class if (and only if) M is a proper class. Also as in the set model case we let $[\![f]\!]\, E_{\mathcal{U}}\, [\![g]\!]$ hold if and only if $\{a \in A \mid f(a) \in g(a)\} \in \mathcal{U}$ and we let $\prod_{\mathcal{U}} \mathcal{M}$ be the class model $({}^A M/\mathcal{U}; E_{\mathcal{U}})$.

Remark. Since a set model is also a class model, there is an ambiguity in our definitions of ${}^A M/\mathcal{U}$ and $\prod_{\mathcal{U}} \mathcal{M}$ when applied to set models. Let us officially adopt the new definition in all cases, though nothing important will turn on this.

Theorem 3.2.5. *Let $\mathcal{U}$ be an ultrafilter on A and let $\mathcal{M} = (M; E)$ be a class model. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $f_1, \ldots, f_n$ be elements of ${}^A M$. Then*

$$\prod_{\mathcal{U}} \mathcal{M} \models \varphi[[\![f_1]\!], \ldots, [\![f_n]\!]] \leftrightarrow \{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)]\} \in \mathcal{U}.$$

The proof of Theorem 3.2.5 is just like that of Theorem 3.2.1.

Remark. Theorem 3.2.5 is a theorem schema both because it is about arbitrary class models and because it is about arbitrary formulas. Since we cannot in ZFC talk in general about the satisfaction relation even for a fixed class model, φ as well as $\mathcal{M}$ must be treated schematically. Thus we are not proving a fixed sentence by induction but rather are inductively showing how to prove all sentences of a certain form.

We define elementary embeddings for class models just as we defined them for set models. We can define a function $j : M \to {}^A M/\mathcal{U}$ just as on page 140. The proof of elementarity of j in the set case works in the class case.

Corollary 3.2.6. $j : \mathcal{M} \prec \prod_{\mathcal{U}} \mathcal{M}$.

Wellfoundedness is defined for class models as for set models. The proof of Lemma 3.2.3 also gives the following lemma.

Lemma 3.2.7. *Let $\mathcal{U}$ be a countably complete ultrafilter on the set A and let $\mathcal{M}$ be a wellfounded class model. Then $\prod_{\mathcal{U}} \mathcal{M}$ is also wellfounded.*

Mostowski's Lemma (Lemma 3.2.4) does not hold in general for class models. The point is that wellfounded class models can be longer than the ordinals, and so need not be isomorphic to class models $(N; \in)$. (See Exercise 3.2.1.) To rule out this possibility we define (more or less following [Kunen, 1980]) a class model $(M; E)$ to be *set-like* if for all $x \in M$ the class $\{y \in M \mid y \, E \, x\}$ is a set.

Lemma 3.2.8. *If $(M; E)$ is a wellfounded set-like model of the Axiom of Extensionality, then there is a unique transitive class N such that $(M; E) \cong (N; \in)$, and the isomorphism $\pi : (M; E) \cong (N; \in)$ is unique.*

The proof of 3.2.8 is just like that of 3.2.4. The assumption that $(M;E)$ is set-like justifies the inductive definition of π, in particular it guarantees that $\pi(x) = \{\pi(y) \mid y\,E\,x\}$ is a set.

Remark. Since the N and the π of Lemma 3.2.8 are proper classes, a word is in order about the significance of the "there is" in the statement of the lemma. The point is that our proof defines N and π from M and E, and hence we show how to construct formulas determining N and π from formulas determining M and E.

The following lemma guarantees that the class models we are interested in are set-like.

Lemma 3.2.9. *Let $\mathcal{M} = (M;\in)$. Let $\mathcal{U}$ be an ultrafilter on a set A. Then $\prod_{\mathcal{U}} \mathcal{M}$ is set-like.*

Proof. Let $f : A \to M$. We must prove that $\{[\![g]\!] \in {}^{A}M/\mathcal{U} \mid [\![g]\!] \in_{\mathcal{U}} [\![f]\!]\}$ is a set. Let $g : A \to M$ be such that $[\![g]\!] \in_{\mathcal{U}} [\![f]\!]$. By definition, this means that $\{a \in A \mid g(a) \in f(a)\} \in \mathcal{U}$. Define $g' : A \to M$ by

$$g'(a) = \begin{cases} g(a) & \text{if } g(a) \in f(a); \\ f(a) & \text{otherwise.} \end{cases}$$

Clearly $g' \sim g$ and $\mathrm{rank}\,(g') \leq \mathrm{rank}\,(f)$. By the definition of $[\![g]\!]$, it follows that every member of $[\![g]\!]$ has rank no greater than $\mathrm{rank}\,(f)$. Let $\alpha = \mathrm{rank}\,(f)$. We have shown that whenever $[\![g]\!] \in_{\mathcal{U}} [\![f]\!]$ then $[\![g]\!] \subseteq V_{\alpha+1}$ and so $[\![g]\!] \in V_{\alpha+2}$. But then $\{[\![g]\!] \mid [\![g]\!] \in_{\mathcal{U}} [\![f]\!]\} \subseteq V_{\alpha+2}$ and is therefore a set. $\square$

Let $\mathcal{U}$ be a countably complete ultrafilter on a set A. Let $i'_{\mathcal{U}}$ be the embedding $j : (V;\in) \prec ({}^{A}V/\mathcal{U};\in_{\mathcal{U}})$ defined on page 140. By Lemmas 3.2.7 and 3.2.9, $({}^{A}V/\mathcal{U};\in_{\mathcal{U}})$ is a wellfounded set-like class model. Let $\pi_{\mathcal{U}} : ({}^{A}V/\mathcal{U};\in_{\mathcal{U}}) \cong (N;\in)$ be given by Lemma 3.2.8. Note that $(N;\in)$ is a class model of ZFC, since V is such a model. By $\mathrm{Ult}(V;\mathcal{U})$ we mean the class N. We denote by $i_{\mathcal{U}}$ the embedding $\pi \circ i'_{\mathcal{U}}$.

Convention. We will often attribute properties of models $(M;\in)$ to the corresponding sets M. Thus we will say, e.g. that $V \models \mathrm{ZFC}$ and that, for the $i_{\mathcal{U}}$ and N of the last paragraph, that $i_{\mathcal{U}} : V \prec N$.

If M and N are classes, if $h : M \to N$, and if there is an ordinal $\alpha \in M$ such that $h(\alpha) \neq \alpha$, then we let $\mathrm{crit}(h)$ be the least such α and we call it the *critical point* of h. We will mainly use this terminology when $h : M \prec N$.

Lemma 3.2.10. *Let $\mathcal{U}$ be a countably complete ultrafilter on a set A. If $\mathcal{U}$ is principal, then* $\mathrm{Ult}(V;\mathcal{U}) = V$ *and $i_{\mathcal{U}}$ is identity. If $\mathcal{U}$ is non-principal, then $i_{\mathcal{U}}$ is the identity on V_κ, where κ is the completeness of $\mathcal{U}$, but $i_{\mathcal{U}}$ is not the identity and* $\kappa = \mathrm{crit}(i_{\mathcal{U}})$.

Proof. Let us construe the completeness of $\mathcal{U}$ to be Ord if $\mathcal{U}$ is principal, since then $\mathcal{U}$ is closed under arbitrary intersections. (But $\mathcal{U}$ is closed under intersections of size Ord, so this convention is not completely natural.) Let κ be the completeness of $\mathcal{U}$.

We first show that $i_{\mathcal{U}}$ is the identity on κ. To do this we prove by induction that $i_{\mathcal{U}}(\alpha) = \alpha$ for all $\alpha < \kappa$. Suppose then that $\alpha < \kappa$ and that $i_{\mathcal{U}}(\beta) = \beta$ for all $\beta < \alpha$. For each $\beta < \alpha$, $i_{\mathcal{U}}(\beta) \in i_{\mathcal{U}}(\alpha)$, by the elementarity of $i_{\mathcal{U}}$. Suppose that $\pi(\llbracket g \rrbracket) \in i_{\mathcal{U}}(\alpha)$, where $\pi = \pi_{\mathcal{U}} : \prod_{\mathcal{U}}(V;\in) \cong (\mathrm{Ult}(V;\mathcal{U});\in)$. Then $\llbracket g \rrbracket \in_{\mathcal{U}} \llbracket c_\alpha \rrbracket$, and so $\{a \in A \mid g(a) \in \alpha\} \in \mathcal{U}$. But $\alpha < \kappa$ and so the κ-completeness of $\mathcal{U}$ implies that there is a $\beta < \alpha$ such that $\{a \in A \mid g(a) = \beta\} \in \mathcal{U}$. But then $g \sim c_\beta$, and so

$$\pi(\llbracket g \rrbracket) = \pi(\llbracket c_\beta \rrbracket) = i_{\mathcal{U}}(\beta) = \beta.$$

This completes the inductive proof that $i_{\mathcal{U}}$ is the identity on κ.

Next we prove by induction on $\alpha < \kappa$ that $i_{\mathcal{U}}$ is the identity on V_α. The only non-trivial case of the induction is that of successor α. Assume then that $\alpha = \beta + 1$ and $i_{\mathcal{U}} \restriction V_\beta$ is the identity. Let $x \in V_\alpha$. By the elementarity of $i_{\mathcal{U}}$,

$$i_{\mathcal{U}}(x) \in V_{i_{\mathcal{U}}(\alpha)} = V_\alpha.$$

Thus every member of $i_{\mathcal{U}}(x)$ belongs to V_β. If $y \in V_\beta$, then the elementarity of $i_{\mathcal{U}}$ and our induction hypothesis give that

$$y \in x \;\leftrightarrow\; i_{\mathcal{U}}(y) \in i_{\mathcal{U}}(x) \;\leftrightarrow\; y \in i_{\mathcal{U}}(x).$$

We have shown that $i_{\mathcal{U}}(x)$ and x have the same members and hence that $i_{\mathcal{U}}(x) = x$.

It only remains to prove that if $\mathcal{U}$ is non-principal then $i_{\mathcal{U}}(\kappa) \neq \kappa$. Assume that $\mathcal{U}$ is non-principal. Let $\langle X_\alpha \mid \alpha < \kappa \rangle$ be such that each $X_\alpha \in \mathcal{U}$ but $\bigcap_{\alpha<\kappa} X_\alpha \notin \mathcal{U}$. Let $f : A \to V$ be given by

$$f(a) = \begin{cases} \mu\alpha\; a \notin X_\alpha & \text{if } a \notin \bigcap_{\alpha<\kappa} X_\alpha; \\ 0 & \text{if } a \in \bigcap_{\alpha<\kappa} X_\alpha. \end{cases}$$

Now, for each $\alpha < \kappa$, $c_\alpha(a) = \alpha < f(a)$ for every $a \in \bigcap_{\beta<\alpha} X_\beta \setminus \bigcap_{\alpha<\kappa} X_\alpha$. Since $\bigcap_{\beta<\alpha} X_\beta \in \mathcal{U}$ and $\bigcap_{\alpha<\kappa} X_\alpha \notin \mathcal{U}$, we have that $[\![c_\alpha]\!] < [\![f]\!]$ for each $\alpha < \kappa$. But then

$$\alpha = i_{\mathcal{U}}(\alpha) = \pi([\![c_\alpha]\!]) < \pi([\![f]\!]),$$

for each $\alpha < \kappa$. Thus $\pi([\![f]\!])$ is an ordinal $\geq \kappa$. But we also have that $f(a) < \kappa$ for every $a \in A$. Hence $[\![f]\!] < [\![c_\kappa]\!]$, and so

$$\kappa \leq \pi([\![f]\!]) < \pi([\![c_\kappa]\!]) = i_{\mathcal{U}}(\kappa). \qquad \square$$

Remark. Note that the proof that $i_{\mathcal{U}}$ is the identity on V_κ used only that $i_{\mathcal{U}} : V \prec M$ for some M and that $i_{\mathcal{U}}$ is the identity on κ. Thus any $j : V \prec M$ is the identity on $V_{\mathrm{crit}(j)}$.

If M is a class model of ZFC (or a large enough fragment of ZFC), then by V^M_α we mean the αth stage of the rank hierarchy as defined in M. If M is transitive, this is just $V_\alpha \cap M$. If $\mathcal{U}$ is a non-principal ultrafilter, then Lemma 3.2.10 shows that $\mathrm{Ult}(V;\mathcal{U})$ and V agree to the completeness κ of $\mathcal{U}$, i.e. $V^M_\kappa = V_\kappa$. The following lemma shows that they agree one level further.

Lemma 3.2.11. *Let κ be the completeness of a non-principal ultrafilter $\mathcal{U}$ on a set A. Then $V^{\mathrm{Ult}(V;\mathcal{U})}_{\kappa+1} = V_{\kappa+1}$. Indeed, ${}^\kappa(\mathrm{Ult}(V;\mathcal{U})) \subseteq \mathrm{Ult}(V;\mathcal{U})$.*

Proof. The second assertion actually implies the first; for, by Lemma 3.1.3 and either Exercises 3.1.1 and 3.1.2 or Lemma 3.2.15, the cardinal κ is inaccessible and so $|V_\kappa| = \kappa$. But the first assertion has a simpler proof, so we give that proof separately: Let $x \in V_{\kappa+1}$. Thus $x \subseteq V_\kappa$. If $y \in V_\kappa$, then

$$y \in x \;\leftrightarrow\; i_{\mathcal{U}}(y) \in i_{\mathcal{U}}(x) \;\leftrightarrow\; y \in i_{\mathcal{U}}(x).$$

Thus $i_{\mathcal{U}}(x) \cap V_\kappa = x$. Since V_κ belongs to the transitive $\mathrm{Ult}(V;\mathcal{U})$, it follows that $x \in \mathrm{Ult}(V;\mathcal{U})$.

For the second assertion, let $h : \kappa \to \mathrm{Ult}(V;\mathcal{U})$. For each $\alpha < \kappa$, let $h(\alpha) = \pi([\![f_\alpha]\!])$, with $\pi = \pi_{\mathcal{U}}$. Let $g : A \to {}^\kappa V$ be given by

$$(g(a))(\alpha) = f_\alpha(a).$$

Now $\pi([\![g]\!]) : i_{\mathcal{U}}(\kappa) \to \mathrm{Ult}(V;\mathcal{U})$ and for $\alpha < \kappa$ we have that $(\pi([\![g]\!]))(\alpha) = (\pi([\![g]\!]))(i_{\mathcal{U}}(\alpha)) = (\pi([\![g]\!]))(\pi([\![c_\alpha]\!])) = \pi([\![f_\alpha]\!]) = h(\alpha)$. Thus $h = \pi([\![g]\!]) \restriction \kappa \in \mathrm{Ult}(V;\mathcal{U})$. $\square$

Theorem 3.2.12. ([Scott, 1961], [Keisler, 1962a]) *Let κ be an ordinal number. The following are equivalent:*

(a) κ is a measurable cardinal.

(b) There are a transitive class M and an embedding $j : V \prec M$ such that $\mathrm{crit}(j) = \kappa$.

(c) There is transitive set N and an embedding $k : V_{\kappa+1} \prec N$ such that $\mathrm{crit}(k) = \kappa$.

Proof. (a) $\Rightarrow$ (b): Assume (a) and let $\mathcal{U}$ be a κ-complete non-principal ultrafilter on κ. Since the completeness of $\mathcal{U}$ cannot be greater than $|\kappa| = \kappa$, we can apply Lemma 3.2.10 with $A = \kappa$. Thus (b) holds with $M = \mathrm{Ult}(V;\mathcal{U})$ and $j = i_{\mathcal{U}}$.

(b) $\Rightarrow$ (c): Assume that M and j witness (b). Then $N = V_{j(\kappa)+1} \cap M$ and $k = j \restriction V_{\kappa+1}$ witness (c).

(c) $\Rightarrow$ (a): Assume that N and k witness (c). Let

$$\mathcal{U} = \{X \subseteq \kappa \mid \kappa \in k(X)\}.$$

Since κ is the critical point of k, $\kappa < k(\kappa)$, i.e. $\kappa \in k(\kappa)$. By the elementarity of k, we have that $k(\emptyset) = \emptyset$ and so that $\kappa \notin k(\emptyset)$. Thus $\mathcal{U}$ satisfies clause (a) in the definition of a filter. The elementarity of k also gives that $k(X \cap Y) = k(X) \cap k(Y)$, that $X \subseteq Y \to k(X) \subseteq k(Y)$, and that $k(\kappa \setminus X) = k(\kappa) \setminus k(X)$; therefore $\mathcal{U}$ satisfies clauses (b), (c), and (d) in the definition of an ultrafilter. To verify the κ-completeness of $\mathcal{U}$, let $\delta < \kappa$ and let $X = \langle X_\gamma \mid \gamma < \delta \rangle$ be a sequence of elements of $\mathcal{U}$. The elementarity of k and the fact that $\delta < \mathrm{crit}(k)$ yield that

$$k(\bigcap_{\gamma<\delta} X_\gamma) = \bigcap_{\gamma<k(\delta)} (k(X))_\gamma = \bigcap_{\gamma<\delta} k(X_\gamma).$$

But $\kappa \in \bigcap_{\gamma<\delta} k(X_\gamma)$, so $\bigcap_{\gamma<\delta} X_\gamma \in \mathcal{U}$. $\mathcal{U}$ is non-principal, since for $\alpha < \kappa$ we have that $\kappa \notin \{\alpha\} = k(\{\alpha\})$. $\square$

Remark. We included (c) in the statement of Theorem 3.2.12 to show that (b), which involves proper classes, has an equivalent version that involves only sets. Obviously (b) and (c) are also equivalent to each of the intermediate propositions gotten by replacing $V_{\kappa+1}$ in (c) by $V_{\kappa+\alpha}$ for ordinals $\alpha > 1$.

It is of interest that the ultrafilter $\mathcal{U}$ defined in the proof of (a) from (c) is actually normal:

Lemma 3.2.13. *Let $j : V \prec M$ with M transitive or let $j : V_{\kappa+\alpha} \prec N$ with N transitive and $\alpha \geq 1$. Assume that $\kappa = \operatorname{crit}(j)$. Let $\mathcal{U} = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. Then $\mathcal{U}$ is a normal ultrafilter on κ.*

Proof. Suppose that $\langle X_\beta \mid \beta < \kappa\rangle$ is a sequence of elements of $\mathcal{U}$. Then $\kappa \in j(X_\beta)$ for each $\beta < \kappa$. Thus κ belongs to the diagonal intersection of $j(\langle X_\beta \mid \beta < \kappa\rangle)$. Thus $\Delta_{\beta<\kappa} X_\beta \in \mathcal{U}$. □

Many large cardinal properties of a cardinal κ can be expressed in the form:

> There is an elementary embedding $j : V \prec M$ with M transitive, with $\operatorname{crit}(j) = \kappa$, and with M like V in respect R.

For the property of being measurable, nothing like the last clause appears in (b) of Theorem 3.2.12. But such a clause could be added, as Lemma 3.2.11 shows. Thus we could strengthen (b) by adding "and with $V_{\kappa+1} \subseteq M$" or even "and with ${}^\kappa M \subseteq M$." In fact, the proof of the first assertion of Lemma 3.2.11 uses nothing special about $i_{\mathcal{U}}$ and $\operatorname{Ult}(V;\mathcal{U})$, so we have:

Lemma 3.2.14. *If $j : V \prec M$ with M transitive and $\operatorname{crit}(j) = \kappa$, then $V_{\kappa+1} \subseteq M$.*

The proof of the second part of Lemma 3.2.11 (that ${}^\kappa(\operatorname{Ult}(V;\mathcal{U})) \subseteq \operatorname{Ult}(V;\mathcal{U})$, or—as we will say—that $\operatorname{Ult}(V;\mathcal{U})$ is κ-*closed*) depended on specific properties of $\operatorname{Ult}(V;\mathcal{U})$. Indeed the analogue of Lemma 3.2.14 fails: If κ is a measurable cardinal, then there is an embedding $j : V \prec M$ with M transitive and $\operatorname{crit}(j) = \kappa$ such that M is not even countably closed. (See Exercise 3.3.2.)

Various properties of measurable cardinals can be proved rather easily from the elementary embedding version of measurability. For example, the fact that there is a normal ultrafilter on each measurable cardinal (Lemma 3.1.7) follows from Lemma 3.2.13. Another example is the following lemma. The original proof of its first assertion is in [Ulam, 1930]. (See Exercises 3.1.1 and 3.1.2.) The original proof of the second assertion is in [Hanf, 1964] and [Tarski, 1962]. The first part of the proof below is essentially from [Keisler, 1962b].

Lemma 3.2.15. *Let κ be a measurable cardinal. Then κ is inaccessible. In fact κ is the κth inaccessible cardinal.*

Proof. Let $j : V \prec M$ with M transitive and $\operatorname{crit}(j) = \kappa$.

To prove that κ is regular, let $\delta < \kappa$ and let $f : \delta \to \kappa$. Elementarity gives that $j(f) : j(\delta) \to j(\kappa)$ and so that $j(f) : \delta \to j(\kappa)$. Moreover for all $\gamma < \delta$ we have that $(j(f))(\gamma) = (j(f))(j(\gamma)) = j(f(\gamma)) = f(\gamma)$ (since $f(\gamma) < \kappa$). Hence $j(f) = f$. But then the range of $j(f)$ is not unbounded in $j(\kappa)$, since it is bounded by $\kappa < j(\kappa)$. By elementarity, the range of f is bounded in κ.

To show κ is a strong limit cardinal, let $\delta < \kappa$. If $x \subseteq \delta$, then $j(x) = x$. Moreover $j(\mathcal{P}(\delta)) = \mathcal{P}(\delta)$. Let $\lambda = |\mathcal{P}(\delta)|$. Let $h : \mathcal{P}(\delta) \to \lambda$ be a bijection. Then $j(h) : \mathcal{P}(\delta) \to j(\lambda)$ is a bijection. For each $x \in \mathcal{P}(\delta)$, $(j(h))(x) = (j(h))(j(x)) = j(h(x))$. But then every ordinal smaller than $j(\lambda)$ belongs to the range of j. Since $\kappa \notin \operatorname{range}(j)$, it follows that $j(\lambda) \leq \kappa$ and so that $\lambda < \kappa$. Thus we have shown that $2^\delta < \kappa$.

Now κ is inaccessible in M, since any witness that κ is not inaccessible in M would also be a witness that κ is not inaccessible in V. If $\alpha < \kappa$, then $M \models (\exists\beta)(\alpha < \beta < j(\kappa) \wedge \beta \text{ is inaccessible})$. (Take κ for β.) Hence in $V \models (\exists\beta)(\alpha < \beta < \kappa \wedge \beta \text{ is inaccessible})$. We have shown that the there are unboundedly many inaccessible cardinals smaller than κ. Since κ is regular, this means that there are κ inaccessible cardinals smaller than κ. □

Exercise 3.2.7 is another example of this sort.

Suppose that we start with a uniform normal ultrafilter $\mathcal{U}$ on a measurable cardinal κ, that we form the elementary embedding $i_{\mathcal{U}}$, and that we then construct a normal measure $\mathcal{V}$ on κ from $i_{\mathcal{U}}$ by letting $\mathcal{V} = \{X \subseteq \kappa \mid \kappa \in i_{\mathcal{U}}(X)\}$. Then $\mathcal{V} = \mathcal{U}$ (Exercise 3.2.3). On the other hand, if we start with $j : V \prec M$ with M transitive and $\operatorname{crit}(j) = \kappa$ and if we then form $\mathcal{V} = \{X \subseteq \kappa \mid \kappa \in j(X)\}$, it need not be true that $i_{\mathcal{V}} = j$. (See Exercise 3.2.4.)

Exercise 3.2.1. Let $M = \mathrm{Ord}$. Define a relation E in M by

$$\alpha \mathrel{E} \beta \leftrightarrow \begin{cases} (\alpha < \beta \text{ and } \alpha \text{ and } \beta \text{ are even}) \vee \\ (\alpha < \beta \text{ and } \alpha \text{ and } \beta \text{ are odd}) \vee \\ (\alpha \text{ is even and } \beta \text{ is odd}) \end{cases}$$

Prove that there is no class N such that $(M; E) \cong (N; \in)$.

Exercise 3.2.2. Let $\mathcal{U}$ be a κ-complete, non-principal ultrafilter on a measurable cardinal κ. Let $\mathrm{id} : \kappa \to \kappa$ be the identity. Prove that $\mathcal{U}$ is normal if and only if $\pi_{\mathcal{U}}([\![\mathrm{id}]\!]_{\mathcal{U}}) = \kappa$.

Exercise 3.2.3. Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal κ. Let $\mathcal{V} = \{X \subseteq \kappa \mid \kappa \in i_{\mathcal{U}}(X)\}$. Show that $\mathcal{V} = \mathcal{U}$.

Exercise 3.2.4. Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal κ. Let $\mathcal{V} = \{X \subseteq \kappa \mid \kappa \in i_{\mathcal{U}^{[2]}}(X)\}$, where $\mathcal{U}^{[2]}$ is the Rowbottom measure defined on page 136. Prove that $\mathcal{V} = \mathcal{U}$ and that $i_{\mathcal{V}}(\kappa) < i_{\mathcal{U}^{[2]}}(\kappa)$.

Exercise 3.2.5. Let $\mathcal{U}$ be a uniform normal ultrafilter on κ. Let $f : \kappa \to V$. Show that

$$(i_{\mathcal{U}}(f))(\kappa) = \pi_{\mathcal{U}}([\![f]\!]_{\mathcal{U}}).$$

Hint. Use Exercise 3.2.2.

Exercise 3.2.6. (a) Let $j : V \prec M$ with M transitive and $\mathrm{crit}(j) = \kappa$. Let $\mathcal{U} = \{x \subseteq \kappa \mid \kappa \in j(X)\}$. Prove that there is a unique $k : \mathrm{Ult}(V;\mathcal{U}) \prec M$ such that $k \circ i_{\mathcal{U}} = j$ and $k \restriction \kappa + 1$ is the identity.

(b) Show that if κ is a measurable cardinal then there is a $j : V \prec M$ with M transitive and $\mathrm{crit}(j) = \kappa$ such that, with $\mathcal{U}$ as in (a), there is more than one $k : \mathrm{Ult}(V;\mathcal{U}) \prec M$ with $k \circ i_{\mathcal{U}} = j$.

Hint. For (a), define k by setting $k(\pi_{\mathcal{U}}([\![f]\!]_{\mathcal{U}})) = (j(f))(\kappa)$. For (b), let $j = i_{\mathcal{U}^{[2]}}$ and let $k' = i_{\mathcal{U}} \restriction \mathrm{Ult}(V;\mathcal{U})$.

Exercise 3.2.7. If λ is an ordinal number, then a subset C of λ is *closed in* λ if it is closed in the order topology or, equivalently, if $\alpha \in C$ whenever $\alpha < \lambda$ is a limit ordinal and C is unbounded in α. If λ is a limit ordinal, then a subset X of λ is *stationary in* λ if X meets every closed, unbounded subset of λ. Note that the only stationary subsets of an ordinal λ of cofinality ω are the complements in λ of bounded sets. A cardinal κ is *Mahlo* if κ is a strong limit cardinal and the set of all regular $\alpha < \kappa$ is stationary in κ. Clearly every Mahlo cardinal has uncountable cofinality. If $\mathrm{cf}(\kappa) > \omega$ and $f : \delta \to \kappa$ witnesses that κ is not regular, then the set C of limit points of $\mathrm{range}(f)$ which are greater than δ witnesses that κ is not Mahlo. Thus every Mahlo cardinal is inaccessible. Prove that every Mahlo cardinal κ is the κth inaccessible cardinal. Prove that every measurable cardinal κ is the κth Mahlo cardinal. This last result is a consequence of theorems in [Tarski, 1962] and [Hanf, 1964].

3.3 Iterated Ultrapowers

In this section we show how to iterate the ultrapower construction to get a sequence

$$V = M_0 \stackrel{j_0}{\to} M_1 \stackrel{j_1}{\to} M_2 \stackrel{j_2}{\to} \dots$$

of class models and elementary embeddings. This system will be used in §4.3 and in determinacy proofs in Chapter 5. We also show how to extend the sequence into the transfinite. The machinery of iterated ultrapowers was introduced by Haim Gaifman and used by him to obtain the results of [Gaifman, 1964]. The machinery, in generalized form, is presented in [Gaifman, 1974].

If $\mathcal{U}$ is a countably complete ultrafilter on A, then

$$\mathrm{Ult}(V;\mathcal{U}) \models \mathrm{ZFC} + \chi[i_{\mathcal{U}}(\mathcal{U}), i_{\mathcal{U}}(A)],$$

where $\chi(v_1, v_2)$ says "v_1 is a countably complete ultrafilter on v_2." Thus we can, within $\mathrm{Ult}(V;\mathcal{U})$, form the ultrapower of the universe of sets with respect to $i_{\mathcal{U}}(\mathcal{U})$. The elements of this ultrapower are the "equivalence classes" of functions $f : i_{\mathcal{U}}(A) \to \mathrm{Ult}(V;\mathcal{U})$ with $f \in \mathrm{Ult}(V;\mathcal{U})$; in other words, each element of the ultrapower is, for some such f, the set of all $g : i_{\mathcal{U}}(A) \to \mathrm{Ult}(V;\mathcal{U})$ of minimal rank such that $g \in \mathrm{Ult}(V;\mathcal{U})$ and $\{a \in i_{\mathcal{U}}(A) \mid f(a) = g(a)\} \in i_{\mathcal{U}}(\mathcal{U})$. We denote these classes by

$$[\![f]\!]^{\mathrm{Ult}(V;\mathcal{U})}_{i_{\mathcal{U}}(\mathcal{U})}.$$

Note that this need not be a true ultrapower (in the full universe V), since (1) $i_{\mathcal{U}}(\mathcal{U})$ may not be (and, unless $\mathcal{U}$ is principal, is in fact not) an ultrafilter in V, and (2) we are using only functions in $\mathrm{Ult}(V;\mathcal{U})$. The class model $\mathrm{Ult}(V;\mathcal{U})$ satisfies the formula saying that this ultrapower is wellfounded and set-like. Since $\mathrm{Ult}(V;\mathcal{U})$ also satisfies (the relevant instance of) Lemma 3.2.8, this ultrapower is isomorphic to a transitive class. We denote this transitive class by $\mathrm{Ult}(\mathrm{Ult}(V;\mathcal{U}); i_{\mathcal{U}}(\mathcal{U}))$, and we denote the canonical elementary embedding of $\mathrm{Ult}(V;\mathcal{U})$ into $\mathrm{Ult}(\mathrm{Ult}(V;\mathcal{U}); i_{\mathcal{U}}(\mathcal{U}))$ by $i^{\mathrm{Ult}(V;\mathcal{U})}_{i_{\mathcal{U}}(\mathcal{U})}$.

In general, suppose that M is a transitive class satisfying ZFC, that A and $\mathcal{V} \in M$, and that $M \models$ "$\mathcal{V}$ is an ultrafilter on A." We will denote the element represented by f in the ultrapower taken inside M of M with respect to $\mathcal{V}$ by $[\![f]\!]^M_{\mathcal{V}}$. We will denote the transitive class isomorphic to this ultrapower by $\mathrm{Ult}(M;\mathcal{V})$, and we will denote the canonical elementary embedding of M

into $\mathrm{Ult}(M;\mathcal{V})$ by $i^M_{\mathcal{V}}$. Later we will generalize these notions further, allowing M to be a model of a fragment of ZFC and not requiring that $\mathcal{V}$ belong to M. Even in these more general cases, we will use the same notation, with the "M" in $[\![f]\!]^M_{\mathcal{V}}$, in $i^M_{\mathcal{V}}$, and in $\mathrm{Ult}(M;\mathcal{V})$ signifying that the ultrapower is the ultrapower of M using the functions in M.

We can think of an elementary embedding $j : M \prec N$, for M and N (with $\in$) transitive class models, as acting on subclasses Y of M which satisfy $(\forall \alpha \in \mathrm{Ord} \cap M)\, Y \cap V_\alpha \in M$; for we can let

$$j(Y) = \bigcup_{\alpha \in \mathrm{Ord} \cap M} j(Y \cap V_\alpha).$$

Thus

$$\begin{aligned} \mathrm{Ult}(V;\mathcal{U}) &= i_{\mathcal{U}}(V); \\ \mathrm{Ult}(\mathrm{Ult}(V;\mathcal{U}); i_{\mathcal{U}}(\mathcal{U})) &= i_{\mathcal{U}}(\mathrm{Ult}(V;\mathcal{U})); \\ i^{\mathrm{Ult}(V;\mathcal{U})}_{i_{\mathcal{U}}(\mathcal{U})} &= i_{\mathcal{U}}(i_{\mathcal{U}}). \end{aligned}$$

Moreover, for any transitive M and $j : V \prec M$ we can define $j(V)$ $(= M)$ and $j(j) : M \prec j(M)$. Suppose that $j : M \prec N$ with M and N transitive and with M satisfying, say, ZFC. If $j \subseteq M$ and is definable in M from elements of M, i.e. if j is a class in M, then we have $j(M) = N$ and $j(j) : N \to j(N)$.

Let M be a transitive class model of ZFC and let $j : M \prec N$ with N transitive and j a class in M. We define inductively, for $n \in \omega$, transitive classes M^j_n and embeddings $j_n : M^j_n \prec M^j_{n+1}$ as follows:

(a) $M^j_0 = M$;

(b) $j_0 = j$;

(c) $M^j_{n+1} = j_n(M^j_n)$;

(d) $j_{n+1} = j_n(j_n)$.

We can also define $j_{m,n} : M^j_m \prec M^j_n$, for $m \leq n \in \omega$ by composition:

(i) $j_{m,m}$ is the identity;

(ii) $j_{m,n+1} = j_n \circ j_{m,n}$.

Note that each $j_{m,n}$ is a class in M^j_m. Exercise 3.3.1 concerns some properties of the $j_{m,n}$.

Let M be a transitive class model of ZFC and let $\mathcal{U} \in M$ be, in M, a countably complete ultrafilter on $A \in M$. For $n \in \omega$ we let

$$\mathrm{Ult}_n(M;\mathcal{U}) = M_n^{i_{\mathcal{U}}}.$$

Lemma 3.3.1. *Let M and $\mathcal{U}$ be as in the preceding paragraph and let $i = i_{\mathcal{U}}$. For each $n \in \omega$, $\mathcal{U}_n = i_{0,n}(\mathcal{U})$ is, in $\mathrm{Ult}_n(M;\mathcal{U})$, a countably complete ultrafilter on $i_{0,n}(A)$. Moreover each $\mathrm{Ult}_{n+1}(M;\mathcal{U}) = \mathrm{Ult}(\mathrm{Ult}_n(M;\mathcal{U});\mathcal{U}_n)$ and each $i_n = i_{\mathcal{U}_n}^{\mathrm{Ult}_n(M;\mathcal{U})}$.*

The proof of the lemma is routine, and we omit it.

Lemma 3.3.2. *Let κ be the completeness of a countably complete, non-principal ultrafilter $\mathcal{U}$ on a set A. Then, for all $n \in \omega$, $V_{\kappa+1}^{\mathrm{Ult}_n(V;\mathcal{U})} = V_{\kappa+1}$. Indeed all $\mathrm{Ult}_n(V;\mathcal{U})$ are κ-closed, i.e. ${}^{\kappa}(\mathrm{Ult}_n(V;\mathcal{U})) \subseteq \mathrm{Ult}_n(V;\mathcal{U})$.*

Proof. Let $i = i_{\mathcal{U}}$. For each $m \in \omega$, applying Lemma 3.2.11 in $\mathrm{Ult}_m(V;\mathcal{U})$ gives us that

$$V_{i_{0,m}(\kappa)+1}^{\mathrm{Ult}_{m+1}(V;\mathcal{U})} = V_{i_{0,m}(\kappa)+1}^{\mathrm{Ult}_m(V;\mathcal{U})}$$

and that

$$\mathrm{Ult}_m(V;\mathcal{U}) \cap {}^{i_{0,m}(\kappa)}(\mathrm{Ult}_{m+1}(V;\mathcal{U})) \subseteq \mathrm{Ult}_{m+1}(V;\mathcal{U}).$$

Since

$$\kappa < i_{0,1}(\kappa) < i_{0,2}(\kappa) < \cdots,$$

the lemma follows by induction. □

The first assertion of Lemma 3.3.2 follows also from the special case $n = 0$ and the fact that $\mathrm{crit}(i_{1,n+1}) > \kappa$, and so, using Lemma 3.2.14:

Lemma 3.3.3. *If $j : V \prec M$ with M transitive and $\mathrm{crit}(j) = \kappa$ then, for all $n \in \omega$, $V_{\kappa+1} \subseteq j_{0,n}(V)$.*

Let M be a transitive class model of ZFC and let $j : M \prec N$ with N transitive and j a class in M. The direct limit

$$(\tilde{\mathcal{M}}_\omega^j, \langle \tilde{\jmath}_{m,\omega} \mid m \in \omega \rangle)$$

of $(\langle (M_n^j; \in) \mid n \in \omega \rangle, \langle j_{m,n} \mid m \leq n \in \omega \rangle)$ is given as follows:

For $x \in M^j_m$ and $y \in M^j_n$ and $m \leq n$, we let

$$\langle m, x\rangle \sim \langle n, y\rangle \leftrightarrow \langle n, y\rangle \sim \langle m, x\rangle \leftrightarrow j_{m,n}(x) = y.$$

Let $[\![m, x]\!]$ be the equivalence class of $\langle m, x\rangle$ with respect to the equivalence relation $\sim$. We set

$$\tilde{M}^j_\omega = \{[\![m, x]\!] \mid m \in \omega \wedge x \in M^j_m\}.$$

For $x \in M^j_m$ and $y \in M^j_n$, we define

$$[\![m, x]\!]\, \tilde{E}^j_\omega\, [\![n, y]\!] \;\leftrightarrow \begin{cases} j_{m,n}(x) \in y & \text{if } m \leq n; \\ x \in j_{n,m}(y) & \text{if } n < m. \end{cases}$$

Let $\tilde{\mathcal{M}}^j_\omega = (\tilde{M}^j_\omega; \tilde{E}^j_\omega)$. Finally, we let $\tilde{\jmath}_{m,\omega}(x) = [\![m, x]\!]$.

Remarks:

(a) We have used the natural notation "$\langle j_{m,n} \mid m \leq n \in \omega\rangle$," but this should not be construed literally, since we want the object to be a genuine class (with sets as members). A similar comment applies to, e.g., "$\langle (M^j_n; \in$ $) \mid n \in \omega\rangle$."

(b) Since each $M^j_n = j_{0,n}(M)$, we could dispense with talk of $\langle (M^j_n; \in) \mid$ $n \in \omega\rangle$ and say that $\langle \tilde{\jmath}_{m,\omega} \mid m \in \omega\rangle$ is the direct limit of $\langle j_{m,n} \mid m \leq n \in \omega\rangle$.

Lemma 3.3.4. *Let M be a transitive class model of* ZFC *and let $j : M \prec N$ with N transitive and j a class in M. Then for all natural numbers m and n with $m \leq n$, $\tilde{\jmath}_{m,\omega} = \tilde{\jmath}_{n,\omega} \circ j_{m,n}$. Moreover $\tilde{\jmath}_{m,\omega} : (M^j_m; \in) \prec (\tilde{M}^j_\omega; \tilde{E}^j_\omega)$ for all $m \in \omega$.*

Proof. The proof of the first assertion is routine, and we omit it. The second assertion follows from the elementary chain theorem of Tarski–Vaught [1957]. Nevertheless, we give the proof: We proceed by induction on the complexity of formulas φ. The only non-trivial case is that of a formula $\varphi(v_1, \ldots, v_k)$ of the form $(\exists v_0)\, \psi(v_0, \ldots, v_k)$. Consider such a φ and ψ. Let $m \in \omega$ and let $\langle x_1, \ldots, x_k\rangle \in M^j_m$. If $M^j_m \models \varphi[x_1, \ldots, x_k]$, then there is an $x_0 \in M^j_m$ such that $M^j_m \models \psi[x_0, \ldots, x_k]$. By the induction hypothesis for ψ, we get that $\tilde{\mathcal{M}}^j_\omega \models \psi[\tilde{\jmath}_{m,\omega}(x_0), \ldots, \tilde{\jmath}_{m,\omega}(x_k)]$ and so that $\tilde{\mathcal{M}}^j_\omega \models \varphi[\tilde{\jmath}_{m,\omega}(x_1), \ldots, \tilde{\jmath}_{m,\omega}(x_k)]$. Suppose then that $\tilde{\mathcal{M}}^j_\omega \models \varphi[\tilde{\jmath}_{m,\omega}(x_1), \ldots, \tilde{\jmath}_{m,\omega}(x_k)]$. Let $\tilde{x} \in \tilde{M}^j_\omega$ be such that $\tilde{\mathcal{M}}^j_\omega \models \psi[\tilde{x}, \tilde{\jmath}_{m,\omega}(x_1), \ldots, \tilde{\jmath}_{m,\omega}(x_k)]$. Let $n \in \omega$ and $y \in M^j_n$ be such that $\tilde{x} =$

$[\![n, y]\!]$. Assume, without loss of generality, that $n \geq m$. By induction and the first assertion of the lemma, we have that $M_n^j \models \psi[y, j_{m,n}(x_1), \ldots, j_{m,n}(x_k)]$. Thus $M_n^j \models \varphi[j_{m,n}(x_1), \ldots, j_{m,n}(x_k)]$. The elementarity of $j_{m,n}$ implies that $M_m^j \models \varphi[x_1, \ldots, x_k]$. □

Lemma 3.3.5. ([Gaifman, 1974]) *Let M and j be as in the statement of Lemma 3.3.4. Then $\tilde{\mathcal{M}}_\omega^j$ is wellfounded.*

Proof. Assume that the lemma is false. There is an infinite sequence $\langle \tilde{y}_i \mid i \in \omega\rangle$ such that

$$\ldots \tilde{E}_\omega^j \, \tilde{y}_2 \, \tilde{E}_\omega^j \, \tilde{y}_1 \, \tilde{E}_\omega^j \, \tilde{y}_0.$$

Hence there is such a sequence with $\tilde{y}_0$ of the form $[\![n, y]\!]$ and so of the form $\tilde{\jmath}_{n,\omega}(y)$. If $x = V_{\mathrm{rank}(y)+1}^M$, then $j_{0,n}(x) = V_{j_{0,n}(\mathrm{rank}(y)+1)}^{M_n}$ and so $y \in j_{0,n}(x)$. Hence there is an $x \in M$ such that there is a sequence $\langle \tilde{y}_i \mid i \in \omega\rangle$ with

$$\ldots \tilde{E}_\omega^j \, \tilde{y}_2 \, \tilde{E}_\omega^j \, \tilde{y}_1 \, \tilde{E}_\omega^j \, \tilde{y}_0 = \tilde{\jmath}_{0,\omega}(x).$$

Let $x \in M$ have minimal rank with this property. Choose such a sequence and let n and $u \in M_n^j$ be such that $\tilde{y}_1 = \tilde{\jmath}_{n,\omega}(u)$. By the elementarity of $j_{0,n}$, we have that $j_{0,n}(x)$ is, in M_n^j, a w of minimal rank with the following property P: There is an infinite sequence $\langle \tilde{z}_i \mid i \in \omega\rangle$ such that

$$\ldots \tilde{E}_\omega^{j_n} \, \tilde{z}_2 \, \tilde{E}_\omega^{j_n} \, \tilde{z}_1 \, \tilde{E}_\omega^{j_n} \, \tilde{z}_0 = (\tilde{\jmath}_n)_{0,\omega}(w).$$

But $M_k^{j_n} = M_{n+k}^j$ and $(j_n)_{k,m} = j_{n+k,n+m}$ for all $k \leq m \in \omega$. Thus $P(w)$ is equivalent with the existence of a sequence $\langle \tilde{z}_i \mid i \in \omega\rangle$ such that

$$\ldots \tilde{E}_\omega^j \, \tilde{z}_2 \, \tilde{E}_\omega^j \, \tilde{z}_1 \, \tilde{E}_\omega^j \, \tilde{z}_0 = \tilde{\jmath}_{n,\omega}(w).$$

But this is a contradiction; for u also has this property, and $u \in j_{0,n}(x)$ so $\mathrm{rank}\,(u) < \mathrm{rank}(j_{0,n}(x))$. □

Lemma 3.3.6. *Let M and j be as in the statement of Lemma 3.3.4. Then $(\tilde{M}_\omega^j; \tilde{E}_\omega^j)$ is set-like.*

Proof. Let $\tilde{y} \in \tilde{M}_\omega^j$. Let $n \in \omega$ and $x \in M_n^j$ be such that $\tilde{y} = [\![n, x]\!]$, so that $\tilde{y} = \tilde{\jmath}_{n,\omega}(x)$. Suppose $\tilde{z} \, \tilde{E}_\omega^j \, \tilde{y}$. Let $m \in \omega$ and $u \in M_m^j$ be such $\tilde{z} = j_{m,\omega}(u)$. If $k \geq \max\{m, n\}$, then $\tilde{\jmath}_{k,\omega}(j_{m,k}(u)) = \tilde{z}$ and $\tilde{\jmath}_{k,\omega}(j_{n,k}(x)) = \tilde{y}$ and so the elementarity of $\tilde{\jmath}_{k,\omega}$ yields that $j_{m,k}(u) \in j_{n,k}(x)$. We have then shown that for every $\tilde{z}\,\tilde{E}^j_\omega\,\tilde{y}$ there is a $k \geq n$ and a $v \in j_{n,k}(x)$ such that $\tilde{z} = \tilde{\jmath}_{k,\omega}(v)$. By the Axiom of Replacement, $\{\tilde{z} \mid \tilde{z}\,\tilde{E}^j_\omega\,\tilde{y}\}$ is a set. $\square$

If M and j are as in the statement of Lemma 3.3.4, then Lemmas 3.3.5, 3.3.6, and 3.2.8 imply that there is a transitive class M^j_ω such that $(M^j_\omega; \in)$ is isomorphic to $(\tilde{M}^j_\omega; \tilde{E}^j_\omega)$. Thus we can extend the M^j_n and $j_{m,n}$ into the transfinite. Our next goal is to carry this out.

First we introduce some general terminology for direct limits. A *directed relation* on a set D is a transitive, reflexive relation R in D such that

$$(\forall x \in D)(\forall y \in D)(\exists z \in D)(x\,R\,z \wedge y\,R\,z).$$

A *directed system of homomorphisms* is something of the form

$$(\langle \mathcal{M}_d \mid d \in D\rangle; \langle j_{d,d'} \mid d \in D \wedge d' \in D \wedge d\,R\,d'\rangle),$$

where R is a directed relation on D, each $\mathcal{M}_d$ is a class model, each $j_{d,d'}$: $\mathcal{M}_d \to \mathcal{M}_{d'}$ is a homomorphism, and

$$(\forall d_1 \in D)(\forall d_2 \in D)(\forall d_3 \in D)(d_1\,R\,d_2\,R\,d_3 \to j_{d_1,d_3} = j_{d_2,d_3} \circ j_{d_1,d_2}).$$

Such a directed system of homomorphisms is a *directed system of elementary embeddings* if each $j_{d,d'} : \mathcal{M}_d \prec \mathcal{M}_{d'}$.

The *direct limit* of a directed system $(\langle \mathcal{M}_d \mid d \in D\rangle; \langle j_{d,d'} \mid d \in D \wedge d' \in D \wedge d\,R\,d'\rangle)$ of homomorphisms is $(\tilde{\mathcal{M}}; \langle \tilde{\jmath}_d \mid d \in D\rangle)$, where $\tilde{\mathcal{M}} = (\tilde{M}; \tilde{E})$ and the $\tilde{\jmath}_d$ are defined as follows. For $d \in D$ let $\mathcal{M}_d = (M_d, E_d)$. For $x \in M_d$ and $y \in M_{d'}$, define

$$\langle d, x\rangle \sim \langle d', y\rangle \;\leftrightarrow\; (\exists d'')(d\,R\,d'' \wedge d'\,R\,d'' \wedge j_{d,d''}(x) = j_{d',d''}(y)).$$

Note that the defining condition on the right is equivalent with

$$(\forall d'')((d\,R\,d'' \wedge d'\,R\,d'') \to j_{d,d''}(x) = j_{d',d''}(y)).$$

Let $[\![d, x]\!]$ be the equivalence class of $\langle d, x\rangle$ with respect to the equivalence relation $\sim$. We set

$$\tilde{M} = \{[\![d, x]\!] \mid d \in D \wedge x \in M_d\}.$$

For $x \in M_d$ and $y \in M_{d'}$, we define

$$[\![d, x]\!]\,\tilde{E}\,[\![d', y]\!] \;\leftrightarrow\; (\exists d'')(d\,R\,d'' \wedge d'\,R\,d'' \wedge j_{d,d''}(x)\,E_{d''}\,j_{d',d''}(y)).$$

Finally, we let $\tilde{\jmath}_d(x) = [\![d, x]\!]$.

Lemma 3.3.7. *Let* $(\langle \mathcal{M}_d \mid d \in D\rangle; \langle j_{d,d'} \mid d \in D \wedge d' \in D \wedge d\,R\,d'\rangle)$ *be a directed system of elementary embeddings. Let* $(\tilde{\mathcal{M}}; \langle \tilde{j}_d \mid d \in D\rangle)$ *be the direct limit of this directed system. Then* $\tilde{j}_d : \mathcal{M}_d \prec \tilde{\mathcal{M}}$ *for all* $d \in D$. *For all* d *and* d' *in* D, *with* $d\,R\,d'$,

$$\tilde{j}_d = \tilde{j}_{d'} \circ j_{d,d'}.$$

Moreover $\tilde{\mathcal{M}}$ *is set-like if all the* $\mathcal{M}_d$ *are set-like.*

We omit the proof of Lemma 3.3.7. The proof of the first two assertions is similar to the proof of Lemma 3.3.4. The proof of the last assertion is like that of Lemma 3.3.6.

Let M be a transitive class model of ZFC and let $j : M \prec N$ with N transitive and j a class in N. We define inductively (1) ξ_j, which will be either an ordinal number or Ord, (2) for $\alpha < \xi_j$, transitive class models M^j_α, and (3) for $\alpha \leq \beta < \xi_j$, embeddings $j_{\alpha,\beta} : M^j_\alpha \prec M^j_\beta$. (What we define inductively is, of course, not ξ_j but rather membership in ξ_j, i.e. an ordinal's being $< \xi_j$.) Our inductive definition will guarantee that the $j_{\alpha,\beta}$ commute: for $\alpha \leq \beta \leq \gamma < \xi_j$, $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$.

(i) $0 < \xi_j$ and $M^j_0 = M$.

(ii) If $\alpha < \xi_j$ then $j_{\alpha,\alpha} = \mathrm{id}$.

(iii) If $\alpha < \xi_j$ then $\alpha + 1 < \xi_j$.

(iv) If $\alpha < \xi_j$ then $j_{\alpha,\alpha+1} = j_{0,\alpha}(j)$ and $M^j_{\alpha+1} = j_{\alpha,\alpha+1}(M^j_\alpha)$. (This latter stipulation makes sense, as $j_{\alpha,\alpha+1}$ is a class in M^j_α.)

(v) If $\gamma < \alpha < \xi_j$ then $j_{\gamma,\alpha+1} = j_{\alpha,\alpha+1} \circ j_{\gamma,\alpha}$.

(vi) If $\alpha \leq \xi_j$ and α is a limit ordinal, we define $((\tilde{M}^j_\alpha; \tilde{E}^j_\alpha), \langle \tilde{j}_{\beta,\alpha} \mid \beta < \alpha\rangle)$ to be the direct limit of $(\langle (M^j_\beta; \in) \mid \beta < \alpha\rangle, \langle j_{\beta,\gamma} \mid \beta \leq \gamma < \alpha\rangle)$. If $(\tilde{M}^j_\alpha; \tilde{E}^j_\alpha)$ is not wellfounded, then $\alpha = \xi_j$. If $(\tilde{M}^j_\alpha; \tilde{E}^j_\alpha)$ is wellfounded, then, since Lemma 3.3.7 implies that it is set-like as well, we let $\pi^j_\alpha : (\tilde{M}^j_\alpha; \tilde{E}^j_\alpha) \cong (M^j_\alpha; \in)$ be given by Lemma 3.2.8. We set $j_{\gamma,\alpha} = \pi^j_\alpha \circ \tilde{j}_{\gamma,\alpha}$.

Extending our notation from the case of finite α, let us set

$$j_\alpha = j_{0,\alpha}(j)$$

for $\alpha < \xi_j$.

Lemma 3.3.8. ([Gaifman, 1974]) *Let* M *be a transitive class model of* ZFC *and let* $j : M \prec N$ *with* N *transitive and* j *a class in* M. *Then* $\xi_j \geq \mathrm{Ord} \cap M$.

Proof. The proof is similar to that of Lemma 3.3.5. Assume the lemma is false for j and M. In M define x to have minimal rank such that there is an infinite sequence $\langle \tilde{y}_i \mid i \in \omega \rangle$ such that

$$\ldots \tilde{E}^j_{\xi_j}\, \tilde{y}_2\, \tilde{E}^j_{\xi_j}\, \tilde{y}_1\, \tilde{E}^j_{\xi_j}\, \tilde{y}_0 = \tilde{\jmath}_{0,\xi_j}(x).$$

Choose such a sequence and let $\alpha < \xi_j$ and $u \in M^j_\alpha$ be such that $\tilde{y}_1 = \tilde{\jmath}_{\alpha,\xi_j}(u)$. By the elementarity of $j_{0,\alpha}$, we have that $j_{0,\alpha}(x)$ is, in M^j_α, a w of minimal rank such that there is an infinite sequence $\langle \tilde{z}_i \mid i \in \omega \rangle$ such that

$$\ldots \tilde{E}^{j_\alpha}_{\xi_{j_\alpha}}\, \tilde{z}_2\, \tilde{E}^{j_\alpha}_{\xi_{j_\alpha}}\, \tilde{z}_1\, \tilde{E}^{j_\alpha}_{\xi_{j_\alpha}}\, \tilde{z}_0 = (\tilde{\jmath}_\alpha)_{0,\xi_{j_\alpha}}(w).$$

But, since $(M_\beta)^{j_\alpha} = M^j_{\alpha+\beta}$ and $(j_\alpha)_{\beta,\gamma} = j_{\alpha+\beta,\alpha+\gamma}$ for all $\beta \leq \gamma < \xi_{j_\alpha}$, we get a contradiction as in the proof of Lemma 3.3.5: u as well as $j_{0,\alpha}(x)$ has this property, and $u \in j_{0,\alpha}(x)$ so $\mathrm{rank}\,(u) < \mathrm{rank}(j_{0,\alpha}(x))$. $\square$

Let M be a transitive class model of ZFC and let $\mathcal{U} \in M$ be in M a countably complete ultrafilter on $A \in M$. For each $\alpha < \xi_{i^M_\mathcal{U}}$, so in particular for each $\alpha \leq \mathrm{Ord} \cap M$, we let

$$\mathrm{Ult}_\alpha(M;\mathcal{U}) = M^{i^M_\mathcal{U}}_\alpha.$$

Lemma 3.3.9. *Let M and $\mathcal{U}$ be as in the preceding paragraph and let $i = i^M_\mathcal{U}$. For each $\alpha < \xi_i$, $\mathcal{U}_\alpha = i_{0,\alpha}(\mathcal{U})$ is, in $\mathrm{Ult}_\alpha(M;\mathcal{U})$, a countably complete ultrafilter on $i_{0,\alpha}(A)$. Moreover each $\mathrm{Ult}_{\alpha+1}(M;\mathcal{U}) = \mathrm{Ult}(\mathrm{Ult}_\alpha(M;\mathcal{U});\mathcal{U}_\alpha)$ and each $i_\alpha = i^{\mathrm{Ult}_\alpha(M;\mathcal{U})}_{\mathcal{U}_\alpha}$.*

Lemma 3.3.10. *Let M be a transitive class model of* ZFC. *Let κ be a cardinal of M and let $\mathcal{U} \in M$ be such that $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ." Let $i = i^M_\mathcal{U}$. Let β be a limit ordinal such that $\beta < \xi_i$. Then the set $\{i_{0,\gamma}(\kappa) \mid \gamma < \beta\}$ is unbounded in $i_{0,\beta}(\kappa)$.*

Proof. Let $\eta < i_{0,\beta}(\kappa)$. Because η belongs to $\mathrm{Ult}_\beta(M;\mathcal{U})$, there must be ordinals $\gamma < \beta$ and ν such that $\eta = i_{\gamma,\beta}(\nu)$. If $\nu \geq i_{0,\gamma}(\kappa)$, then $\eta \geq i_{\gamma,\beta}(i_{0,\gamma}(\kappa)) = i_{0,\beta}(\kappa)$. Thus $\nu < i_{0,\gamma}(\kappa)$. But $\mathrm{crit}\,(i_{\gamma,\beta}) = i_{0,\gamma}(\kappa)$, and so $\eta = \nu$ and thus $\eta < i_{0,\gamma}(\kappa)$. $\square$

Lemma 3.3.11. ([Kunen, 1968]) *Let M be a transitive class model of* ZFC. *Let κ be a cardinal of M and let $\mathcal{U} \in M$ be such that $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ." Let $i = i^M_\mathcal{U}$. Let α be a limit ordinal with $\alpha < \xi_i$. Let $X \subseteq i_{0,\alpha}(\kappa)$ with $X \in \mathrm{Ult}_\alpha(M;\mathcal{U})$. Then*

$$X \in i_{0,\alpha}(\mathcal{U}) \;\leftrightarrow\; (\exists \beta < \alpha)(\forall \gamma)(\beta \leq \gamma < \alpha \rightarrow i_{0,\gamma}(\kappa) \in X).$$

Proof. We begin by, in effect, doing Exercise 3.2.3 and half of Exercise 3.2.2. Let $\pi = \pi^M_{\mathcal{U}}$ and, for $f : \kappa \to M$ with $f \in M$, let $[\![f]\!] = [\![f]\!]^M_{\mathcal{U}}$.

First we show that $\pi([\![\mathrm{id}]\!]) = \kappa$. Since $\pi([\![c_\eta]\!]) < \pi([\![\mathrm{id}]\!])$ for every $\eta < \kappa$ by the uniformity of $\mathcal{U}$ in M, we know that $\kappa \leq \pi([\![\mathrm{id}]\!])$. If $\pi([\![f]\!]) < \pi([\![\mathrm{id}]\!])$, then the normality of $\mathcal{U}$ in M implies that $[\![f]\!] = [\![c_\eta]\!]$ for some $\eta < \kappa$. Thus $\pi([\![\mathrm{id}]\!]) \leq \kappa$.

Next we show that

$$\mathcal{U} = \{Y \subseteq \kappa \mid Y \in M \wedge \kappa \in i(Y)\}.$$

This is because, for $Y \subseteq \kappa$ with $Y \in M$,

$$Y \in \mathcal{U} \leftrightarrow \{\eta \mid \mathrm{id}(\eta) \in Y\} \in \mathcal{U} \leftrightarrow \pi([\![\mathrm{id}]\!]) \in i(Y) \leftrightarrow \kappa \in i(Y).$$

Since α is a limit ordinal, there exist $\beta < \alpha$ and $Y \in \mathrm{Ult}_\beta(M;\mathcal{U})$ such that $X = i_{\beta,\alpha}(Y)$. By the elementarity of $i_{0,\beta}$, we have that

$$Y \in i_{0,\beta}(\mathcal{U}) \leftrightarrow i_{0,\beta}(\kappa) \in i_\beta(Y).$$

Let γ be such that $\beta \leq \gamma < \alpha$. Then $X \in i_{0,\alpha}(\mathcal{U}) \leftrightarrow Y \in i_{0,\beta}(\mathcal{U}) \leftrightarrow i_{0,\beta}(\kappa) \in i_\beta(Y) \leftrightarrow i_{0,\gamma}(\kappa) \in i_{\beta,\gamma+1}(Y) \leftrightarrow i_{0,\gamma}(\kappa) \in i_{\beta,\alpha}(Y)$, where the last equivalence holds because $\mathrm{crit}(i_{\gamma+1,\alpha}) > i_{0,\gamma}(\kappa)$. □

Lemma 3.3.12. ([Kunen, 1968]) *Let M be a transitive class model of* ZFC. *Let κ be a cardinal of M and let $\mathcal{U} \in M$ be such that $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ." Let $i = i^M_{\mathcal{U}}$. Let α be a limit ordinal such that $\alpha < \xi_i$ and* $\mathrm{cf}(\alpha) > \omega$.

Then $i_{0,\alpha}(\mathcal{U})$ is the restriction to $\mathrm{Ult}_\alpha(M;\mathcal{U})$ *of the closed, unbounded filter on $i_{0,\alpha}(\kappa)$; i.e., if $X \in \mathrm{Ult}_\alpha(M;\mathcal{U})$ and $X \subseteq i_{0,\alpha}(\kappa)$, then $X \in i_{0,\alpha}(\mathcal{U})$ if and only if X has a subset that is closed and unbounded in $i_{0,\alpha}(\kappa)$.*

Proof. By the preceding lemma, it suffices to prove that $\{i_{0,\gamma}(\kappa) \mid \gamma < \alpha\}$ is closed and unbounded in $i_{0,\alpha}(\kappa)$. But this follows easily from Lemma 3.3.10. □

Lemma 3.3.13. *Let κ be a cardinal number and let $\mathcal{U}$ be a uniform normal ultrafilter on κ. Let $i = i_{\mathcal{U}}$.*

(a) For all ordinals η and α with $\alpha > 0$, $|i_{0,\alpha}(\eta)| \leq \max\{|\eta|^\kappa, |\alpha|\}$.

(b) For every ordinal $\alpha \geq 2^\kappa$, $|i_{0,\alpha}(\kappa)| = |\alpha|$.

(c) For every cardinal $\delta > 2^\kappa$, $i_{0,\delta}(\kappa) = \delta$.

(d) For every cardinal $\delta > 2^\kappa$ such that $\mathrm{cf}(\delta) > \kappa$ *and for every ordinal $\alpha < \delta$, $i_{0,\alpha}(\delta) = \delta$.*

Proof. We prove (a) by induction on α, simultaneously for all η.

If $\eta > 0$ and $[\![f]\!]_{\mathcal{U}} < [\![c_\eta]\!]_{\mathcal{U}}$, then there is a $g : \kappa \to \eta$ such that $g \sim_{\mathcal{U}} f$. This fact implies (a) for the case $\alpha = 1$.

Assume that (a) holds for α. For any η, we have that

$$\begin{aligned} |i_{0,\alpha+1}(\eta)| &\leq |{}^{i_{0,\alpha}(\kappa)}i_{0,\alpha}(\eta)|^{\mathrm{Ult}_\alpha(V;\mathcal{U})} \\ &= i_{0,\alpha}(|\eta|^\kappa) \\ &\leq \max\{(|\eta|^\kappa)^\kappa, |\alpha|\} \\ &= \max\{|\eta|^\kappa, |\alpha|\}. \end{aligned}$$

Here the first inequality is by the case $\alpha = 1$ and the elementarity of $i_{0,\alpha}$.

Finally assume that (a) holds for all $\alpha < \lambda$, where λ is a limit ordinal. For any η,

$$\begin{aligned} |i_{0,\lambda}(\eta)| &= \sup_{\alpha<\lambda}|i_{0,\alpha}(\eta)| \\ &\leq \sup_{\alpha<\lambda}\max\{|\eta|^\kappa, |\alpha|\} \\ &= \max\{|\eta|^\kappa, |\lambda|\}. \end{aligned}$$

(b) follows easily from (a).

For (c), let δ be a cardinal larger than 2^κ. Part (b) of the lemma implies that $i_{0,\gamma}(\kappa) < \delta$ for all $\gamma < \delta$. Lemma 3.3.10 then gives that $i_{0,\delta}(\kappa) = \delta$.

For (d), let $\delta > 2^\kappa$ be a cardinal with $\mathrm{cf}(\delta) > \kappa$. Let $\alpha < \delta$ be such that (d) fails for δ and α but holds for δ and all $\beta < \alpha$. Obviously $\alpha > 0$.

Suppose that $\alpha = 1$. Let f be such that $\delta \leq \pi_{\mathcal{U}}([\![f]\!]_{\mathcal{U}}) < i(\delta)$. We may assume that $f : \kappa \to \delta$. Since $\mathrm{cf}(\delta) > \kappa$, there is an $\eta < \delta$ such that $f : \kappa \to \eta$. But then $\pi_{\mathcal{U}}([\![f]\!]_{\mathcal{U}}) < i(\eta)$. But this contradicts part (a), which implies that $i(\eta) < \delta$.

The fact that $\alpha \neq 1$ implies that, for any β, $i_\beta(i_{0,\beta}(\delta)) = i_{0,\beta}(\delta)$. This means that α cannot be a successor ordinal $\beta + 1$.

Thus α must be a limit ordinal. Let ν be such that $\delta \leq \nu < i_{0,\alpha}(\delta)$. Let $\gamma < \alpha$ and ρ be such that $\nu = i_{\gamma,\alpha}(\rho)$. Then $\rho < i_{0,\gamma}(\delta) = \delta$. Hence $\rho + 1 < \delta$. But $\nu < i_{\gamma,\alpha}(\rho+1) \leq i_{0,\alpha}(\rho+1)$, and this contradicts part (a), which implies that $i_{0,\alpha}(\eta) < \delta$ for every $\eta < \delta$. $\square$

Lemma 3.3.14. *Let κ and κ' be a cardinal numbers with $\kappa < \kappa'$. Let $\mathcal{U}$ and $\mathcal{U}'$ be uniform normal ultrafilters on κ and κ' respectively. Let $i = i_{\mathcal{U}}$. For any $\alpha < \kappa'$, $i_{0,\alpha}(\mathcal{U}') = \mathcal{U}' \cap \mathrm{Ult}_\alpha(V;\mathcal{U})$.*

Proof. By part (d) of Lemma 3.3.13, $i_{0,\alpha}(\kappa') = \kappa'$ for every $\alpha < \kappa'$. It also follows easily from part (d) of Lemma 3.3.13 that, for all $\alpha < \kappa'$, the set W_α of all $\eta < \kappa'$ such that η is a fixed point of $i_{0,\alpha}$ belongs to $\mathcal{U}'$.

We prove the lemma by induction on $\alpha \geq 1$.

First consider the case $\alpha = 1$. Let $X = \pi_{\mathcal{U}}([\![f]\!]_{\mathcal{U}})$ belong to $i(\mathcal{U}')$. Then $Z \in \mathcal{U}$, where

$$Z = \{\gamma < \kappa \mid f(\gamma) \in \mathcal{U}'\}.$$

Let $Y = \bigcap_{\gamma \in Z} f(\gamma)$. The κ'-completeness of $\mathcal{U}'$ implies that $Y \in \mathcal{U}'$. Since $\{\gamma < \kappa \mid Y \subseteq f(\gamma)\}$ belongs to $\mathcal{U}$, it follows by the Łoś Theorem that $i(Y) \subseteq X$. Now $Y \cap W_1$ belongs to $\mathcal{U}'$ and is a subset of $i(Y)$, and so $i(Y) \in \mathcal{U}'$. Therefore $X \in \mathcal{U}'$.

Assume that the lemma holds for α. By the elementarity of $i_{0,\alpha}$ and the case $\alpha = 1$ of the lemma, we have that $i_{0,\alpha+1}(\mathcal{U}) = i_{0,\alpha}(\mathcal{U}) \cap \mathrm{Ult}_{\alpha+1}(V;\mathcal{U})$. This fact and our induction hypothesis imply that the lemma holds for $\alpha+1$.

Let $\lambda < \kappa'$ be a limit ordinal and assume that the lemma holds for all $\alpha < \lambda$. Let $X \in i_{0,\lambda}(\mathcal{U}')$. For some $\alpha < \lambda$ and some $Y \in \mathrm{Ult}_\alpha(V;\mathcal{U})$, $X = i_{\alpha,\lambda}(Y)$. By the elementarity of $i_{\alpha,\lambda}$ and our induction hypothesis, $Y \in \mathcal{U}'$. Hence $Y \cap W_\lambda \in \mathcal{U}'$. Since $X \subseteq Y \cap W_\lambda$, $X \in \mathcal{U}'$. □

Exercise 3.3.1. Let $j : M \prec N$ with N transitive and j a class in M.

(a) Prove that $j \circ j = j_1 \circ j$.

(b) Let $x \in N$. Prove that $j(x) = j_1(x)$ if and only if $x \in \mathrm{range}(j)$.

(c) Prove that $j_{n+1}(\alpha) \leq j_n(\alpha)$ for all $n \in \omega$ and all ordinals $\alpha \in M$.

Hint. For (c), let α be an ordinal of M and let β be the least ordinal γ such that $j(\gamma) > \alpha$. By the elementarity of j, $j(\beta)$ is the least ordinal γ such that $j_1(\gamma) > j(\alpha)$.

Exercise 3.3.2. Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal κ. Prove that $\mathrm{Ult}_\omega(V;\mathcal{U})$ is not countably closed: Prove that there is an $f : \omega \to \mathrm{Ult}_\omega(V;\mathcal{U})$ that does not belong to $\mathrm{Ult}_\omega(V;\mathcal{U})$.

Hint. Prove that $\langle (i_{\mathcal{U}})_{0,n}(\kappa) \mid n \in \omega\rangle \notin \mathrm{Ult}_\omega(V;\mathcal{U})$.

Exercise 3.3.3. Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal κ. Show that $i_{\mathcal{U}^{[n]}} = (i_{\mathcal{U}})_{0,n}$ for all $n \in \omega$.

Exercise 3.3.4. Prove that part (a) of Lemma 3.3.13 remains true for $\eta \geq \kappa$ if "$\leq$" is replaced by "$=$" in its conclusion.

3.4 Sharps

In order to prove $\mathbf{\Pi}_1^1$ determinacy from a hypothesis that is actually equivalent with $\mathbf{\Pi}_1^1$ determinacy, we need to introduce the notion of what are called *sharps*. The existence of these objects follows from measurable cardinals, but it is actually weaker. The simplest example of a sharp, $0^{\#}$, is a set of natural numbers that codes up the entire universe L of constructible sets. To introduce sharps, we must then first introduce L. We carry out this latter task in a rather sketchy fashion, letting the reader consult other works, such as [Kunen, 1980], for a complete treatement.

Gödel's constructible universe L and hierarchy of constructible sets are defined as follows:

(1) $L_0 = \emptyset$.

(2) $L_{\alpha+1}$ is the collection of all subsets of L_α that are first order definable over L_α from elements of L_α. In other words, a set x belongs to $L_{\alpha+1}$ if and only if there is a formula $\varphi(v_0, \ldots, v_n)$ of the language of set theory and there are elements $y_1, \ldots, y_n$ of L_α such that

$$x = \{y_0 \in L_\alpha \mid (L_\alpha; \in) \models \varphi[y_0, \ldots, y_n]\}.$$

(3) If α is a limit ordinal, then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.

(4) $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$.

We now give the basic facts about L. Some of the proofs we outline and some of them we omit altogether. See [Kunen, 1980] for details.

Theorem 3.4.1. (Gödel [1939]) *L is a transitive class model of* ZFC.

Proof. We briefly sketch the proof. If $x \in L_\alpha$ and $x \subseteq L_\alpha$, the formula $v_0 \in v_1$ witnesses that $x \in L_{\alpha+1}$. If L_α is transitive, it follows that $L_\alpha \subseteq L_{\alpha+1}$ and that $L_{\alpha+1}$ is transitive. By induction one easily shows that $L_\alpha \subseteq L_\beta$ whenever $\alpha < \beta$ and that each L_α is transitive. Thus L is transitive. The Axiom of Foundation holds in any transitive class. One readily constructs formulas to show that, for limit ordinals α, L_α is closed under pairing and union. Since the formulas expressing $v_0 = \{v_1, v_2\}$ and $v_0 = \bigcup v_1$ are absolute for L—and indeed for any transitive class—it follows that the Axioms of Pairing and Union hold in L. (A formula $\varphi(v_1, \ldots, v_n)$ is *absolute* for a class

N if whenever $a_1, \ldots, a_n$ are elements of N, then $N \models \varphi[a_1, \ldots, a_n]$ if and only if $V \models \varphi[a_1, \ldots, a_n]$.) By induction, using the formula "v_0 is an ordinal number," one can show that each ordinal α belongs to $L_{\alpha+1}$. Hence ω belongs to $L_{\omega+1}$ and, by absoluteness, witnesses that L is a model of the Axiom of Infinity. If $x \in L$, then $\mathcal{P}(x) \cap L$ is, by Replacement in V, a subset of some L_α and so a member of some $L_{\alpha+1}$. Thus, for each $x \in L$, the set $\mathcal{P}(x) \cap L$ witnesses that the Power Set Axiom holds in L for x. If $u \in L$ and

$$(\forall x \in u)(\exists! y \in L)\, L \models \varphi(x, y),$$

then Replacement in V gives an α such that

$$(\forall x \in u)(\exists! y \in L_\alpha)\, L \models \varphi(x, y).$$

Thus Replacement holds in L if Comprehension does. By Replacement in V, there are for each formula $\varphi(v_1, \ldots, v_n)$ arbitrarily large ordinals α such that

$$(\forall a_1 \in L_\alpha) \cdots (\forall a_n \in L_\alpha)(L_\alpha \models \varphi[a_1, \ldots, a_n] \leftrightarrow L \models \varphi[a_1, \ldots, a_n]).$$

(Such an instance of the schema called *Reflection* for L is proved by a kind of Löwenheim–Skolem argument applied to the finitely many subformulas of φ.) If $u \in L$ and we want to verify that $(\exists v)(\forall x)(x \in v \leftrightarrow (x \in u \wedge L \models \varphi[x, u, y_1, \ldots, y_n]))$, then we apply Reflection to φ to get an α with $u \in L_\alpha$ and we deduce that the desired v belongs to $L_{\alpha+1}$. The Axiom of Choice holds in L because there is a wellordering $<_L$ of L definable in L: First order by $\mathrm{rank}_L(x) = \mu\alpha(x \in L_{\alpha+1})$; for $\mathrm{rank}_L(x) = \mathrm{rank}_L(y)$, order inductively by the formulas and parameters from $L_{\mathrm{rank}_L(x)}$ witnessing x and y in $L_{\mathrm{rank}_L(x)+1}$. (We assume in the sequel that some such specific definition of $<_L$ has been fixed.) □

The *Axiom of Constructibility* asserts that $V = L$.

Theorem 3.4.2. ([Gödel, 1939]) *The Axiom of Constructibility holds in L.*

Theorem 3.4.2 is proved by showing that the formula $v \in L$ is absolute for L.

Lemma 3.4.3. ([Gödel, 1939]) *For each infinite ordinal α, the cardinal number of L_α is $|\alpha|$.*

Proof. Since each $\alpha \in L_{\alpha+1}$, we have that $|L_\alpha| \geq |\alpha|$ for all infinite α.

We prove by transfinite induction that $|L_\alpha| \leq |\alpha|$ for all infinite α. It is easy to see that $L_{n+1} = \mathcal{P}(L_n)$ for $n \in \omega$. This implies that each L_n is finite and so that $|L_\omega| = \aleph_0$. Assume that α is infinite and that $|L_\alpha| \leq |\alpha|$. Each member of $L_{\alpha+1}$ is determined by a formula and finitely many elements of L_α. Thus $|L_{\alpha+1}| \leq \max\{\aleph_0, |L_\alpha|\} = |L_\alpha| \leq |\alpha|$. Now assume that α is a limit ordinal and that $|L_\beta| \leq |\beta|$ for each infinite $\beta < \alpha$. Then $|L_\alpha| = |\bigcup_{\beta<\alpha} L_\beta| \leq \sum_{\beta<\alpha} |\beta| \leq \sum_{\beta<\alpha} |\alpha| = |\alpha| \cdot |\alpha| = |\alpha|$. $\square$

Theorem 3.4.4. ([Gödel, 1939]) *The Generalized Continuum Hypothesis holds in L.*

The proof, which we omit, proceeds by showing that

$$(\forall x \in L_\alpha)\, \mathcal{P}(x) \cap L \subseteq L_{\alpha^+},$$

where as usual α^+ is the least cardinal greater than α. The theorem then follows by Lemma 3.4.3.

Lemma 3.4.5. *(a)* ZFC$^-$ + $V = L$ *holds in* L_γ *for every uncountable regular cardinal* γ, *where* ZFC$^-$ *is* ZFC *without the Power Set Axiom.*

(b) If N is a transitive class model of ZFC$^-$ + $V = L$, *then either $N = L$ or $N = L_\alpha$ for some limit ordinal α.*

Part (a) of Lemma 3.4.5 is proved much as are Theorem 3.4.1 and Theorem 3.4.2. Part (b) is proved by showing that $V = L$ is absolute for transitive models of ZFC$^-$.

For each limit α, the formula defining the wellordering $<_L$ of L described in the proof of Theorem 3.4.1 is absolute for L_α. The proof that the formula in question defines a wellordering of L goes through in ZFC$^-$.

The following striking result of Dana Scott was the inspiration for all subsequent work about the impact of large cardinals on L.

Theorem 3.4.6. ([Scott, 1961]) *If a measurable cardinal exists, then $V \neq L$.*

Proof. Let κ be the least measurable cardinal. By Theorem 3.2.12, let $j : V \prec M$ with M transitive and $\text{crit}(j) = \kappa$. By part (b) of Lemma 3.4.5, $j(L) = L$. If $V = L$ then $M = j(L) = L = V$ and so κ is the least measurable cardinal in M. But $j(\kappa) > \kappa$, and the least measurable cardinal in M is $j(\kappa)$.

□

We now deduce some much stronger consequences of the existence of a measurable cardinal for the relation of L to V. These consequences that will give us the best possible hypotheses for the determinacy result of Chapter 4 and for some of the results of Chapter 5.

A class U is a class of *indiscernibles* for a transitive class M if

(a) $U \subseteq \mathrm{Ord} \cap M$;

(b) if $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are elements of U and $\varphi(v_1, \ldots, v_n)$ is a formula of the language of set theory, then

$$M \models \varphi[\alpha_1, \ldots, \alpha_n] \;\leftrightarrow\; M \models \varphi[\beta_1, \ldots, \beta_n].$$

Recall that a subset X of a limit ordinal λ is *closed* if X is closed in the order topology. Equivalently, X is closed if whenever $\alpha < \lambda$ and X is unbounded in α then $\alpha \in X$.

Fix some recursive bijection $\varphi \mapsto n_\varphi$ from the set Φ of formulas of the language of set theory whose free variables are among $v_1, v_2, \ldots$ to the set ω. If there is a closed unbounded subset C of ω_1 such that C is a set of indiscernibles for L_{ω_1}, then $0^{\#}$ is

$$\{n_{\varphi(v_1,\ldots,v_{i_\varphi})} \mid L_{\omega_1} \models \varphi[\alpha_1, \ldots, \alpha_{i_\varphi}]\}.$$

Here i_φ is the greatest i such that v_i is free in φ if φ is not a sentence and 0 otherwise, and $\alpha_1 < \cdots < \alpha_n$ are members of C. Since the intersection of two closed unbounded sets is closed and unbounded, there is no dependence on the choice of C. If such a C does not exist, then there is no $0^{\#}$.

Remark. "$0^{\#}$" is pronounced as "0 sharp." It would perhaps then be better if it were written "$0^{\sharp}$," and it sometimes is. The notation with # has its origin in the fundamental [Solovay, 1967]. $0^{\#}$ was introduced—and most of the results of this section were proved—in Jack Silver's dissertation, [Silver, 1966], published in abridged form as [Silver, 1971].

Lemma 3.4.7. ([Rowbottom, 1964]) *If κ is a measurable cardinal and $\mathcal{U}$ is a uniform normal ultrafilter on κ, then there is subset X of κ such that $X \in \mathcal{U}$ and such that X is a set of indiscernibles for L_κ.*

Proof. Let κ be a measurable cardinal and let $\mathcal{U}$ be a uniform normal ultrafilter on κ. For $n \in \omega$ and $q \in [\kappa]^n$, let $(q)_1 < \cdots < (q)_n$ be the elements of q. For each formula $\varphi \in \Phi$, let n be minimal such that φ is $\varphi(v_1, \ldots, v_n)$, i.e. such that the free variables of φ are among $v_1, \ldots, v_n$. Let

$$Y_\varphi = \{q \in [\kappa]^n \mid L_\kappa \models \varphi[(q)_1, \ldots, (q)_n]\}.$$

Let $Z_\varphi = Y_\varphi$ if $Y_\varphi \in \mathcal{U}^{[n]}$ and let $Z_\varphi = [\kappa]^n \setminus Y_\varphi$ otherwise. (The Rowbottom ultrafilter $\mathcal{U}^{[n]}$ is defined on page 136.) By the definition of $\mathcal{U}^{[n]}$, let $X_\varphi \subseteq \kappa$ be such that $X_\varphi \in \mathcal{U}$ and $[X_\varphi]^n \subseteq Z_\varphi$. Let $X = \bigcap_\varphi X_\varphi$. Clearly $X \in \mathcal{U}$ and X is a set of indiscernibles for L_κ. $\square$

If $\varphi(v_0, v_1, \ldots, v_n)$ is a formula of the language of set theory, then let us define $f_\varphi : {}^nL \to L$ by

$$f_\varphi(x_1, \ldots, x_n) = \begin{cases} \mu x_0\, L \models \varphi[x_0, x_1, \ldots, x_n] & \text{if } (\exists x_0)\, L \models \varphi[x_0, x_1, \ldots, x_n]; \\ \emptyset & \text{otherwise.} \end{cases}$$

Here the μ-operator is being applied to the ordering $<_L$. We also define $f_\varphi^\alpha : {}^nL_\alpha \to L_\alpha$ by replacing "L" by "L_α" in the definition of f_φ.

For α a limit ordinal and $X \subseteq L_\alpha$, let $\mathcal{H}(L_\alpha, X)$ be the closure of X under all the functions f_φ^α. Note that

$$\mathcal{H}(L_\alpha, X) = \bigcup_{\varphi(v_0, \ldots v_n)} \{f_\varphi^\alpha(x_1, \ldots, x_n) \mid \langle x_1, \ldots, x_n \rangle \in {}^nX\}.$$

This is because compositions of the f_φ^α are also among the f_φ^α. Note also that $\mathcal{H}(L_\alpha, X) \prec L_\alpha$, i.e. that $\mathrm{id} : \mathcal{H}(L_\alpha, X) \prec L_\alpha$. More generally, $\mathcal{H}(L_\alpha, X) \prec \mathcal{H}(L_\alpha, Y)$ when $X \subseteq Y$. Any class model $(M; E)$ of $\mathrm{ZFC}^- + V = L$ has its own internally defined version of the f_φ. Let us call the closure of $X \subseteq M$ under these functions $\mathcal{H}((M; E), X)$. (Note that when M is a proper class, e.g. when $M = L$, then we are going beyond the language of ZFC in making this definition.)

If $(M; E)$ is a class model and $X \subseteq M$, let us say that X *generates* $(M; E)$ if for every $a \in M$ there are $n \in \omega$, $f : {}^nM \to M$ and $x_1, \ldots, x_n$ belonging to X such that f is definable (without parameters) in $(M; E)$ and $a = f(x_1, \ldots, x_n)$. This is equivalent to saying the closure of X under the definable functions is all of M. If $(M; E)$ is a class model of $\mathrm{ZFC}^- + V = L$ and $X \subseteq M$, then it is easy to see that X generates $(M; E)$ if and only if $\mathcal{H}((M; E), X) = M$. If α is a limit ordinal and $X \subseteq L_\alpha$, then X generates L_α (i.e. X generates $(L_\alpha; \in)$) just in case $\mathcal{H}(L_\alpha, X) = L_\alpha$.

Theorem 3.4.8. ([Silver, 1966]) *The following are equivalent:*

(i) $0^\#$ *exists.*

(ii) There is an uncountable regular cardinal γ *such that there is an unbounded subset of* γ *that is a set of indiscernibles for* L_γ*.*

(iii) There is a closed unbounded proper class C *such that* C *is a class of indiscernibles for* L *that generates* L *and such that, for every uncountable cardinal* η, $\mathcal{H}(L, C \cap \eta) = L_\eta$*.*

Proof. Clearly (i) implies (ii) and (iii) implies (i). We need then only show that (ii) implies (iii). Let γ be an uncountable regular cardinal and let $X \subseteq \gamma$ be unbounded in γ and a set of indiscernibles for L_γ. Since γ is regular, X has order type γ. Let $\alpha \mapsto x_\alpha$ be the order preserving bijection between γ and X.

We may assume that X has been chosen with the minimal possible value of x_ω, i.e. that, for any unbounded subset X' of γ such that X' is a set of indiscernibles for L_γ, the ωth element of X' is at least as large as x_ω.

We next show that we can replace X with a Y that has, in addition to the properties of X, the additional property of generating L_γ.

By Lemma 3.2.4, let

$$\pi : \mathcal{H}(L_\gamma, X) \cong N,$$

with N transitive. By part (a) of Lemma 3.4.5, $N \models \mathrm{ZFC}^- + V = L$. By part (b) of the same lemma, $N = L_\alpha$ for some limit ordinal α. Since $\pi(\beta) \leq \beta$ for each ordinal $\beta \in \mathcal{H}(L_\gamma, X)$, we must have $\alpha \leq \gamma$. But $|X| = \gamma$, so we get that $\alpha = \gamma$. Let

$$Y = \{\pi(x) \mid x \in X\}.$$

Clearly Y is unbounded in γ. Since $\pi : \mathcal{H}(L_\gamma, X) \cong L_\gamma$ and $\mathcal{H}(L_\gamma, X) \prec L_\gamma$, we get that Y is a set of indiscernibles for L_γ. Let $\alpha \mapsto y_\alpha$ be the order preserving bijection between γ and Y. We have then that

(a) Y is an unbounded subset of γ;

(b) Y is a set of indiscernibles for L_γ;

(c) Y has minimal ωth element among sets with properties (a) and (b);

(d) Y generates L_γ.

.

We next prove some useful facts about Y and the f^γ_φ.

(1) $\text{rank}_L(f^\gamma_\varphi(y_1, \ldots, y_n)) < y_{n+1}$.

(2) If $\text{rank}_L(f^\gamma_\varphi(y_1, \ldots, y_j, y_{j+1}, \ldots, y_{j+m})) < y_{j+1}$, then

$$f^\gamma_\varphi(y_1, \ldots, y_j, y_{j+1}, \ldots, y_{j+m}) = f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_1}, \ldots, y_{\alpha_m})$$

for any $\alpha_1, \ldots, \alpha_m$ such that $j < \alpha_1 < \cdots < \alpha_m < \gamma$.

If (1) fails, then indiscernibility gives that $\text{rank}_L(f^\gamma_\varphi(y_1, \ldots, y_n)) \geq y_\alpha$ for every $\alpha < \gamma$. Hence $f^\gamma_\varphi(y_1, \ldots, y_n) \notin L_\gamma$.

Suppose that the hypothesis of (2) holds and that the conclusion fails. Thus we have $j < \alpha_1 < \cdots < \alpha_m < \gamma$ and

$$y_{j+1} > f^\gamma_\varphi(y_1, \ldots, y_j, y_{j+1}, \ldots, y_{j+m}) \neq f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_1}, \ldots, y_{\alpha_m}).$$

Let $\alpha_m < \beta_1 < \cdots < \beta_m$. If

$$f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_1}, \ldots, y_{\alpha_m}) = f^\gamma_\varphi(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}),$$

then indiscernibility gives the contradiction that

$$f^\gamma_\varphi(y_1, \ldots, y_j, y_{j+1}, \ldots, y_{j+m}) = f^\gamma_\varphi(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}).$$

We have then shown that for any $\alpha_1 < \cdots < \alpha_m$ and $\beta_1 < \cdots < \beta_m$ such that $j < \alpha_1$, $\alpha_m < \beta_1$, and $\beta_m < \gamma$,

$$f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_1}, \ldots, y_{\alpha_m}) \neq f^\gamma_\varphi(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}).$$

By indiscernibility, we have that one of the following holds for all such $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$:

$$\begin{array}{lcl} f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_1}, \ldots, y_{\alpha_m}) & <_L & f^\gamma_\varphi(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}) \\ f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_1}, \ldots, y_{\alpha_m}) & >_L & f^\gamma_\varphi(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}) \end{array}$$

Let $\alpha_{\rho,k} = j + m\rho + k$ for $\rho < \gamma$ and $0 \leq k < m$. If it is the second inequality that holds, then $\langle f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_{n,0}}, \ldots, y_{\alpha_{n,m-1}}) \mid n \in \omega \rangle$ is an infinite descending sequence with respect to the wellordering $<_L$. Thus it is the first inequality that holds. By indiscernibility and the fact that $z <_L z'$ implies $\text{rank}_L(z) \leq \text{rank}_L(z')$, we get that one of the following holds for all any $\alpha_1 < \cdots < \alpha_m < \beta_1 < \cdots < \beta_m$ such that $j < \alpha_1$ and $\beta_m < \gamma$:

$$\begin{array}{l} \text{rank}_L(f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_1}, \ldots, y_{\alpha_m})) = \text{rank}_L(f^\gamma_\varphi(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m})) \\ \text{rank}_L(f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_1}, \ldots, y_{\alpha_m})) < \text{rank}_L(f^\gamma_\varphi(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m})) \end{array}$$

If it is the equation that holds, then there is an $\eta < \gamma$ such that

$$\{f^{\gamma}_{\varphi}(y_1, \ldots, y_j, y_{\alpha_{\rho,0}}, \ldots, y_{\alpha_{\rho,m-1}}) \mid \rho < \gamma\} \subseteq L_\eta.$$

This is impossible, since $|L_\eta| < \gamma$. Hence it is the inequality that holds. But then

$$\{\mathrm{rank}_L(f^{\gamma}_{\varphi}(y_1, \ldots, y_j, y_{\alpha_{\rho,0}}, \ldots, y_{\alpha_{\rho,m-1}})) \mid \rho < \gamma\}$$

is readily seen to be an unbounded subset of γ that is a set of indiscernibles for L_γ. Moreover the ωth element of this set is

$$f^{\gamma}_{\varphi}(y_1, \ldots, y_j, y_\omega, \ldots, y_{\omega+m-1}) <_L y_\omega.$$

This contradicts property (c) of Y.

We use Y to generate a proper class model $(M; E)$ as follows: Suppose that $\varphi(v_0, \ldots, v_n)$ and $\psi(v_0, \ldots, v_m)$ are formulas and $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_m$ are ordinal numbers. Let

$$q : \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\} \to |\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\}|$$

be the order preserving bijection. Set

$$\begin{array}{l}\langle \varphi, \alpha_1, \ldots, \alpha_n\rangle \sim \langle \psi, \beta_1, \ldots, \beta_m\rangle \;\leftrightarrow \\ f^{\gamma}_{\varphi}(y_{q(\alpha_1)}, \ldots, y_{q(\alpha_n)}) = f^{\gamma}_{\psi}(y_{q(\beta_1)}, \ldots, y_{q(\beta_m)}).\end{array}$$

Let $[\![\varphi, \alpha_1, \ldots, \alpha_n]\!]$ be the equivalence class of $\langle \varphi, \alpha_1, \ldots, \alpha_n\rangle$ with respect to the equivalence relation $\sim$, fixed up *à la* Scott to be a set. Let M be the class of all the $[\![\varphi, \alpha_1, \ldots, \alpha_n]\!]$. Define, for q as above,

$$\begin{array}{l}[\![\varphi, \alpha_1, \ldots, \alpha_n]\!] \, E \, [\![\psi, \beta_1, \ldots, \beta_m]\!] \leftrightarrow \\ f^{\gamma}_{\varphi}(y_{q(\alpha_1)}, \ldots, y_{q(\alpha_n)}) \in f^{\gamma}_{\psi}(y_{q(\beta_1)}, \ldots, y_{q(\beta_m)}).\end{array}$$

Suppose for a contradiction that $(M; E)$ is not wellfounded. Then there exists a sequence $\langle a_i \mid i \in \omega\rangle$ such that each $a_{i+1} \, E \, a_i$. Let $a_i = [\![\varphi_i, \delta_{i,1}, \ldots \delta_{i,n_i}]\!]$. Let g be the order preserving bijection between a countable ordinal and $\{\delta_{i,j} \mid i \in \omega \wedge j \leq n_i\}$. For each i, let $e_i = f^{\gamma}_{\varphi_i}(y_{g^{-1}(\delta_{i,1})}, \ldots, y_{g^{-1}(\delta_{i,n_i})})$. We have the contradiction that each $e_{i+1} \in e_i$.

For a set A of ordinals, let $\mathrm{ot}(A)$ be the order type of A, let $\alpha \mapsto a_\alpha$ be the order isomorphism between $\mathrm{ot}(A)$ and A, and let

$$M_A = \{[\![\varphi, a_{\alpha_1}, \ldots, a_{\alpha_n}]\!] \mid \varphi \text{ a formula } \wedge \alpha_1 < \cdots < \alpha_n < \mathrm{ot}(A)\}.$$

Suppose that $\mathrm{ot}(A) \leq \gamma$. It is clear from the definition of $(M; E)$ that the correspondence given by

$$f_\varphi^\gamma(y_{\alpha_1}, \ldots, y_{\alpha_n}) \mapsto [\![\varphi, a_{\alpha_1}, \ldots, a_{\alpha_n}]\!]$$

gives an isomorphism

$$h_A : (\mathcal{H}(L_\gamma, \{y_\alpha \mid \alpha < \mathrm{ot}(A)\}); \in) \cong (M_A; E).$$

For any ordinal δ, let $z_\delta = [\![v_0 = v_1, \delta]\!]$. Note that $h_A(y_\alpha) = z_{a_\alpha}$ for each $\alpha < \gamma$.

If $A \subseteq B \subseteq \mathrm{Ord}$ and $\mathrm{ot}(B) \leq \gamma$, then the fact that

$$(\mathcal{H}(L_\gamma, \{y_\alpha \mid \alpha < \mathrm{ot}(A)\}); \in) \prec (\mathcal{H}(L_\gamma, \{y_\alpha \mid \alpha < \mathrm{ot}(B)\}); \in)$$

implies that

$$(M_A; E) \prec (M_B; E).$$

Thus, in particular,

$$\langle (M_A; E) \mid A \subseteq \mathrm{Ord} \wedge A \text{ finite } \rangle,$$

together with the inclusions, forms a directed system of elementary embeddings. It follows from Lemma 3.3.7 that, for each finite A,

$$(M_A; E) \prec (\bigcup_{A \text{ finite}} M_A; E) = (M; E).$$

Hence, for every finite A,

$$h_A : (\mathcal{H}(L_\gamma, \{y_\alpha \mid \alpha < \mathrm{ot}(A)\}); \in) \prec (M; E).$$

We use properties (1) and (2) of Y to show that $(M; E)$ is set-like. Let $b = [\![\varphi, \delta_1, \ldots, \delta_n]\!] \in M$. Let $A = \{\delta_1, \ldots, \delta_n, \delta_n + 1\}$. By property (1) of Y we have that $(M; E) \models \mathrm{rank}_L(h_A(f_\varphi^\gamma(y_{\delta_1}, \ldots, y_{\delta_n}))) < h_A(y_{\delta_n+1})$, i.e. that $(M; E) \models \mathrm{rank}_L(b) < z_{\delta_n+1}$. Suppose that $a \, E \, b$. Then $(M; E) \models \mathrm{rank}_L(a) < z_{\delta_n+1}$. Let $a = [\![\psi, \rho_1, \ldots, \rho_m]\!]$. We may assume that there is a j with $1 \leq j \leq m$ such that $\rho_{j+1} = \delta_n + 1$. Using h_B with $B = \{\rho_1, \ldots, \rho_m, \xi_{j+1}, \ldots, \xi_m\}$ for arbitrary ordinals $\xi_{j+1}, \ldots, \xi_m$ and using property (2) of Y, we get that $a = [\![\psi, \rho_1, \ldots, \rho_j, \xi_{j+1}, \ldots, \xi_m]\!]$ for any $\xi_{j+1} < \cdots < \xi_m$ with $\rho_j < \xi_{j+1}$. Thus we have shown that each a such that $a \, E \, b$ is determined by numbers j and

m, a formula ψ, and a j-tuple of ordinals smaller than $\delta_n + 1$. The collection of such a thus forms a set.

By Lemma 3.2.8, $(M; E)$ is isomorphic to $(N; \in)$ for some transitive class N. Since the z_δ are distinct elements of M, we have that M is a proper class and so that N also is a proper class. By part (b) of Lemma 3.4.5, $N = L$. Let $\pi : (M; E) \cong (L; \in)$. For ordinals α let $c_\alpha = \pi(z_\alpha)$. Let $C = \{c_\alpha \mid \alpha \in \mathrm{Ord}\}$.

For each finite set $A = \{a_1, \ldots, a_n\}$ of ordinals with $a_1 < \cdots < a_n$, let $h^*_A = \pi \circ h_A$. We have that

$$h^*_A : \mathcal{H}(L_\gamma, \{y_1, \ldots, y_n\}) \prec L.$$

Moreover $h^*_A(y_i) = c_{a_i}$ for each i. Using such functions h^*_A, it is easy to see that C is a class of indiscernibles for L, that $[\![\varphi, \alpha_1, \ldots, \alpha_n]\!] = f_\varphi(c_{\alpha_1}, \ldots, c_{\alpha_n})$—and so that C generates L—and that C has properties (1) and (2), i.e. that

(1) $\mathrm{rank}_L(f^\gamma_\varphi(c_1, \ldots, c_n)) < c_{n+1}$.

(2) If $\mathrm{rank}_L(f^\gamma_\varphi(c_1, \ldots, c_j, c_{j+1}, \ldots, c_{j+m})) < c_{j+1}$, then

$$f^\gamma_\varphi(c_1, \ldots, c_j, c_{j+1}, \ldots, c_{j+m}) = f^\gamma_\varphi(c_1, \ldots, c_j, c_{\alpha_1}, \ldots, c_{\alpha_m})$$

for any $\alpha_1, \ldots, \alpha_m$ such that $j < \alpha_1 < \cdots < \alpha_m$.

We know that $|\mathcal{H}(L, \{c_\beta \mid \beta < \eta\})| = |\eta|$ for each limit ordinal η. By properties (1) and (2), we get that $\mathcal{H}(L, \{c_\beta \mid \beta < \eta\}) = \mathcal{H}(L, C) \cap L_{\sup_{\beta<\eta} c_\beta} = L_{\sup_{\beta<\eta} c_\beta}$ for all limit ordinals η. If η is an uncountable cardinal, these facts imply that

$$L_\eta = \mathcal{H}(L, \{c_\beta \mid \beta < \eta\}) = \mathcal{H}(L, C \cap \eta).$$

It remains only to show that C is closed. Suppose that α is a limit ordinal and that $\xi < c_\alpha$ (so that $\xi \in L_{c_\alpha}$). Let

$$\xi = f_\varphi(c_{\beta_1}, \ldots, c_{\beta_j}, c_\alpha, c_{\delta_1}, \ldots, c_{\delta_{j+m}}),$$

where $\beta_1 < \cdots < \beta_j < \alpha < \delta_1 < \ldots < \delta_{j+m}$. By property (2) of C, $\xi \in L_{c_{\beta_j+1}}$, i.e. $\xi < c_{\beta_j+1}$. This shows that the c_β, $\beta < \alpha$, are cofinal in c_α. □

Corollary 3.4.9. *If a measurable cardinal exists, then $0^\#$ exists.*

Proof. By Lemma 3.2.15 and Lemma 3.4.7, clause (ii) of the theorem holds for any measurable cardinal γ. □

Corollary 3.4.10. *Assume that* $0^\#$ *exists. If* κ *and* λ *are uncountable cardinals with* $\kappa < \lambda$, *then* $L_\kappa \prec L_\lambda \prec L$.

Proof. This follows directly from clause (iii) of the theorem. □

It is easy to see that the class C of clause (iii) of Theorem 3.4.8 is unique. Indeed, the whole structure of $\mathcal{H}(L, C)$ is determined directly by $0^\#$, which gives us the relations among the values $f_\varphi^{\omega_1}(c_{\alpha_1}, \ldots, c_{\alpha_n})$ and so, since $L_{\omega_1} \prec L$, among the values $f_\varphi(c_{\alpha_1}, \ldots, c_{\alpha_n})$. We will refer to C as the *Silver class of indiscernibles* for L and to the members of C as the *Silver indiscernibles* for L. Note that the uncountable cardinals are among the indiscernibles for L.

The concepts and theorems of this section can easily be relativized.

Let a be any set. Let the language $\mathcal{L}$ be the result of adding to the language of set theory a one-place predicate symbol P. We can expand any transitive class model $(M; \in)$ to a class model $(M; \in, a)$ for $\mathcal{L}$ by interpreting P by the property of belonging to a. We define:

(1) $L_0[a] = \emptyset$.

(2) $L_{\alpha+1}[a]$ is the collection of all subsets x of $L_\alpha[a]$ such that x is first order definable over $(L_\alpha; \in, a)$ from elements of $L_\alpha[a]$.

(3) $L[a] = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha[a]$.

In general, a need not belong to $L[a]$. But it is always the case that $a \cap L[a] \in L[a]$. Moreover $L[a] = L[a \cap L[a]]$. If $a \subseteq V_\omega$ or a is a set of ordinals, then $a \in L[a]$. From large cardinal hypotheses it follows that ${}^\omega\omega$ does not belong to $L[{}^\omega\omega]$; this will be shown in Chapter 9.

We omit the proofs of the following results. These proofs are essentially the same as those of the corresponding unrelativized results.

Theorem 3.4.11. *For every* a, $L[a]$ *is a transitive class model of* ZFC.

Theorem 3.4.12. *For every* a, $L[a] \models V = L[a \cap L[a]]$; *thus* $L[a] \models V = L[a]$ *if* $a \in L[a]$. *Furthermore, if* $b \cap L[a] = a$, *then* $L[a] = L[b]$.

Lemma 3.4.13. *For each* a *and each infinite ordinal* α, *the cardinal number of* $L_\alpha[a]$ *is* $|\alpha|$.

Theorem 3.4.14. *If ν is an ordinal and $a \subseteq \nu$, then for all cardinals κ of $L[a]$ such that $\nu \leq \kappa$, $L[a] \models 2^\kappa = \kappa^+$. For each $a \subseteq V_\omega$, the Generalized Continuum Hypothesis holds in $L[a]$.*

Lemma 3.4.15. *Let a be any set.*

(a) $\mathrm{ZFC}^- + V = L[a \cap L[a]]$ *holds in $L_\gamma[a]$ for every uncountable regular cardinal γ such that $a \cap L[a] \in L_\gamma[a]$.*

(b) If $a \in N$ and N is a transitive class model of $\mathrm{ZFC}^- + V = L[a]$, *then either $N = L[a]$ or $N = L_\alpha[a]$ for some limit ordinal α.*

Theorem 3.4.16. *If κ is a measurable cardinal and $a \subseteq V_\kappa$, then $V \neq L[a]$.*

For any set a, let $\mathrm{tclos}(a)$ be the smallest transitive set of which a is a member. In the special case when a is a set of ordinals, $\mathrm{tclos}(a) = \{a\} \cup \beta$, where β is the least ordinal of which a is a subset.

If M is a transitive class and if $a \in M$, then a class U is a class of *indiscernibles* for M, a if

(a) $U \subseteq \mathrm{Ord} \cap M$;

(b) if $b_1, \ldots, b_m$ are elments of $\mathrm{tclos}(a)$, if $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are elements of U, and if $\varphi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{m+n})$ is a formula of the language of set theory, then

$$M \models \varphi[b_1, \ldots, b_m, \alpha_1, \ldots, \alpha_n] \leftrightarrow M \models \varphi[b_1, \ldots, b_m, \beta_1, \ldots, \beta_n].$$

Let $\varphi \mapsto n_\varphi$ be as in the definition of $0^\#$. Let a be a set such that $a \in L[a]$. Let ν be the least ordinal such that $a \in L_{\nu^+}[a]$. If there is a closed, unbounded subset C of ν^+ such that C is a set of indiscernibles for $L_{\nu^+}[a], a$, then $a^\#$ is the set of all

$$\langle n_{\varphi(v_1,\ldots,v_{m+n})}, \langle b_1, \ldots, b_m\rangle\rangle$$

such that

$$(\forall i)(1 \leq i \leq m \to b_i \in \mathrm{tclos}(a)) \wedge L_{\nu^+}[a] \models \varphi[b_1, \ldots, b_m, \alpha_1, \ldots, \alpha_n]\},$$

where $\alpha_1 < \cdots < \alpha_n$ are members of C. If there is no such C, then there is no $a^\#$.

Remarks:

(a) If $a \subseteq V_\omega$, then all members of $\mathrm{tclos}(a)$ other than a itself are definable in $L_{\omega_1}[a]$. Thus $a^\#$ is determined by $\{n \in \omega \mid \langle n, \langle a\rangle\rangle \in a^\#\}$, and so $a^\#$ is often defined to be this subset of ω.

(b) We have not yet defined $a^\#$ or even what it means for $a^\#$ to exist when $a \notin L[a]$. We will do so later in this section.

Lemma 3.4.17. *Let κ be a measurable cardinal, let $\mathcal{U}$ be a uniform normal ultrafilter on κ, and let $a \in V_\kappa$ with $a \in L[a]$. Then $a \in L_\kappa[a]$ and there is subset X of κ such that $X \in \mathcal{U}$ and such that X is a set of indiscernibles for $L_\kappa[a], a$.*

For any set a, $L[a]$ has a wellordering internally definable from $a \cap L[a]$. Using the definition of this ordering we can define $\mathcal{H}(L[a], X)$ and $\mathcal{H}(L_\alpha[a], X)$ for sets X such that $a \cap L[a] \in X$, and we will have, e.g., $\mathcal{H}(L_\alpha[a], X) \prec L_\alpha[a]$.

Theorem 3.4.18. *Let ν be an infinite cardinal and let a be such that $a \in V_{\nu^+}$ and $a \in L[a]$. The following are equivalent:*

(i) $a^\#$ exists.

(ii) There is an uncountable regular cardinal $\gamma > \nu$ such that $a \in L_\gamma[a]$ and there is an unbounded subset of γ that is a set of indiscernibles for $L_\gamma[a], a$.

(iii) $a \in L_{\nu^+}[a]$. Moreover there is a closed, unbounded proper class C^a such that C^a is a class of indiscernibles for $L[a], a$, such that $C^a \cup$ $\mathrm{tclos}(a)$ generates $L[a]$ and such that, for every uncountable cardinal $\eta > \nu$, $\mathcal{H}(L[a]; (C^a \cap \eta) \cup \mathrm{tclos}(a)) = L_\eta[a]$.

Corollary 3.4.19. *If κ is a measurable cardinal, then $a^\#$ exists for every $a \in V_\kappa$ such that $a \in L[a]$.*

Corollary 3.4.20. *Let ν be a cardinal, let $a \in V_{\nu^+}$, and assume that $a^\#$ exists. If κ and λ are uncountable cardinals with $\nu < \kappa < \lambda$, then $L_\kappa[a] \prec L_\lambda[a] \prec L[a]$.*

The C^a of clause (iii) of Theorem 3.4.18 is unique. We call it the *Silver class of indiscernibles* for $L[a], a$ and we call its members the *Silver indiscernibles* for $L[a], a$. It is the same as the Silver class of indiscernibles defined earlier in the special case $a \subseteq V_\omega$.

When $a \notin L[a]$, we can still make sense of $a^{\#}$. This is done as follows.

Let a be any set. Let the language $\mathcal{L}_a$ be the result of adding to the language of set theory a one-place predicate symbol P_b for each $b \in \mathrm{tclos}(a)$. We can expand any transitive class model $(M; \in)$ to a class model $(M; \in, b, \ldots)$ for $\mathcal{L}_a$ by interpreting each P_b by the property of belonging to b. We define:

(1) $L_0(a) = \emptyset$.

(2) $L_{\alpha+1}(a)$ is the collection of all subsets x of $L_\alpha(a)$ such that x is first order definable over $(L_\alpha(a); \in, b, \ldots)$ from elements of $L_\alpha(a)$.

(3) $L(a) = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha(a)$.

Note that $a \in L(a)$ and moreover that $a \in L_\nu(a)$ if $a \in V_\nu$.

$L(a)$ need not satisfy the Axiom of Choice, but we still have the following fact.

Theorem 3.4.21. *For every a, $L(a)$ is a transitive class model of* ZF.

Theorem 3.4.22. *For every a, $L(a) \models V = L(a)$.*

Lemma 3.4.23. *Let a be any set.*

(a) $\mathrm{ZF}^- + V = L(a)$ *holds in $L_\gamma(a)$ for every uncountable regular cardinal γ such that $a \in V_\gamma$ and $\gamma > |\mathrm{tclos}(a)|$.*

(b) If $a \in N$ and N is a transitive class model of $\mathrm{ZFC}^- + V = L(a)$, *then either $N = L(a)$ or $N = L_\alpha(a)$ for some limit ordinal α.*

Let $\varphi \mapsto n_\varphi$ be as in the definition of $0^{\#}$. Let a be any set. Let ν be the least cardinal such that $a \in V_{\nu^+}$ and $|\mathrm{tclos}(a)| \leq \nu$. If there is a closed, unbounded subset C of ν^+ such that C is a set of indiscernibles for $L_{\nu^+}(a), a$, then $a^{\#}$ is the set of all

$$\langle n_{\varphi(v_1,\ldots,v_{m+n})}, \langle b_1, \ldots, b_m \rangle \rangle$$

such that

$$(\forall i)(1 \leq i \leq m \rightarrow b_i \in \mathrm{tclos}(a)) \;\wedge\; L_{\nu^+}(a) \models \varphi[b_1, \ldots, b_m, \alpha_1, \ldots, \alpha_n]\},$$

where $\alpha_1 < \cdots < \alpha_n$ are members of C. If there is no such C, then there is no $a^{\#}$.

Lemma 3.4.24. *Let κ be a measurable cardinal, let $\mathcal{U}$ be a uniform normal ultrafilter on κ, and let $a \in V_\kappa$. Then $a \in L_\kappa(a)$ and there is a subset X of κ such that $X \in \mathcal{U}$ and such that X is a set of indiscernibles for $L_\kappa(a), a$.*

Since $L(a)$ need not satisfy Choice, it certainly need not have an internally definable wellordering. Nevertheless, every element of any $L_\alpha(a)$ is definable in $L(a)$ from ordinals smaller than α and elements of $\mathrm{tclos}(a)$ by a formula absolute for $L_\alpha(a)$. Thus we can define $\mathcal{H}(L(a), X)$ and $\mathcal{H}(L_\alpha(a), X)$ for any X such that $\mathrm{tclos}(a) \subseteq X$, and we have, e.g., that $\mathcal{H}(L_\alpha(a), X) \prec L_\alpha(a)$.

Theorem 3.4.25. *Let ν be an infinite cardinal and let $a \in V_{\nu^+}$ be such that $|\mathrm{tclos}(a)| \leq \nu$. The following are equivalent:*

- *(i) $a^\#$ exists.*
- *(ii) There is an uncountable regular cardinal $\gamma > \nu$ such that there is an unbounded subset of γ that is a set of indiscernibles for $L_\gamma(a), a$.*
- *(iii) There is a closed, unbounded proper class C^a such that C^a is a class of indiscernibles for $L(a), a$, such that $C^a \cup \mathrm{tclos}(a)$ generates $L(a)$ and such that, for every uncountable cardinal $\eta > \nu$, $\mathcal{H}(L(a); (C^a \cap \eta) \cup \mathrm{tclos}(a)) = L_\eta(a)$.*

Corollary 3.4.26. *If κ is a measurable cardinal, then $a^\#$ exists for every $a \in V_\kappa$.*

Corollary 3.4.27. *Let ν be a cardinal and let $a \in V_{\nu^+}$ be such that $|\mathrm{tclos}(a)| \leq \nu$. Assume that $a^\#$ exists. If κ and λ are uncountable cardinals with $\nu < \kappa < \lambda$, then $L_\kappa(a) \prec L_\lambda(a) \prec L(a)$.*

The C^a of clause (iii) of Theorem 3.4.25 is unique. We call it the *Silver class of indiscernibles* for $L[a], a$ and we call its members the *Silver indiscernibles* for $L[a], a$.

Exercise 3.4.1. Assume $0^\#$ exists. Prove that $(\omega_1)^L < \omega_1$. Indeed, prove that ω_1 is inaccessible in L.

Hint. Use the fact that all the uncountable cardinals are Silver indiscernibles for L.

3.5 $L[\mathcal{U}]$

Lemma 3.5.1. *Let $\mathcal{U}$ be a uniform normal ultrafilter on a measurable cardinal κ. Then $L[\mathcal{U}] \models$ ZFC + "$\mathcal{U} \cap L[\mathcal{U}]$ is a uniform normal ultrafilter on κ."*

Proof. That $L[\mathcal{U}] \models$ ZFC follows from Lemma 3.4.11. It is easy to see that $L[\mathcal{U}] \models$ "$\mathcal{U} \cap L[\mathcal{U}]$ is a uniform ultrafilter on κ," and any counterexample to normality in the model $L[\mathcal{U}]$ would be a counterexample to normality in V. $\square$

To state an easy generalization of Lemma 3.5.1, let us introduce some notation. For a function $\langle a_j \mid j \in J \rangle$, let us write

$$\prec a_j \mid j \in J \succ$$

for

$$\{\langle j, b \rangle \mid j \in J \wedge b \in a_j\}.$$

Let us also write, for sets $a_1, a_2, \ldots, a_n$,

$$L[a_1, a_2, \ldots, a_n]$$

for

$$L[\prec a_i \mid 1 \leq i \leq n \succ].$$

To see the point of this notation, note that $L[\langle a_1, a_2 \rangle] = L$ and that if $\mathcal{U}_\beta$, $\beta < \alpha$, are uniform normal ultrafilters then $L[\langle \mathcal{U}_\beta \mid \beta < \alpha \rangle] = L$. (See Exercise 3.5.1.)

Lemma 3.5.2. *Let $\langle \kappa_\beta \mid \beta < \alpha \rangle$ and $\langle \mathcal{U}_\beta \mid \beta < \alpha \rangle$ be such that each $\mathcal{U}_\beta$ is a uniform normal ultrafilter on the measurable cardinal κ_β. Let a be any set. Then $L[a, \prec \mathcal{U}_\beta \mid \beta < \alpha \succ] \models$ ZFC + "for all $\beta < \alpha$, $\mathcal{U}_\beta \cap L[a, \prec \mathcal{U}_\beta \mid \beta < \alpha \succ]$ is a uniform normal ultrafilter on κ_β."*

The next two theorems, which will be used in Chapter 5, are from [Kunen, 1968].

Theorem 3.5.3. *Let κ and $\mathcal{U}$ be such that $L[\mathcal{U}] \models$ "$\mathcal{U} \cap L[\mathcal{U}]$ is a uniform normal ultrafilter on κ." In $L[\mathcal{U}]$, $\mathcal{U} \cap L[\mathcal{U}]$ is the unique uniform normal ultrafilter on κ.*

Proof. Let us assume for notational simplicity that $\mathcal{U} = \mathcal{U} \cap L[\mathcal{U}]$. Assume that the conclusion of the Theorem fails. Let $X \subseteq \kappa$ be least in the canonical wellordering of $L[\mathcal{U}]$ such that $X \in \mathcal{U} \leftrightarrow X \notin \mathcal{V}$ for some uniform normal ultrafilter $\mathcal{V}$ on κ in $L[\mathcal{U}]$. Let $\mathcal{V}$ witness this fact. By Exercise 3.2.3, whose proof is given during the proof of Lemma 3.3.11, this means that

$$\kappa \in i_{\mathcal{U}}(X) \leftrightarrow \kappa \notin i_{\mathcal{V}}(X),$$

where we write $i_{\mathcal{U}}$ and $i_{\mathcal{V}}$ for $I_{\mathcal{U}}^{L[\mathcal{U}]}$ and $i_{\mathcal{V}}^{L[\mathcal{U}]}$ respectively. Now let

$$\begin{aligned} M &= \mathrm{Ult}_{(2^\kappa)^+}(\mathrm{Ult}(L[\mathcal{U}];\mathcal{U}); i_{\mathcal{U}}(\mathcal{U})); \\ N &= \mathrm{Ult}_{(2^\kappa)^+}(\mathrm{Ult}(L[\mathcal{U}];\mathcal{V}); i_{\mathcal{V}}(\mathcal{U})). \end{aligned}$$

Since the canonical elementary embeddings $j : \mathrm{Ult}(L[\mathcal{U}];\mathcal{U}) \prec M$ and $k : \mathrm{Ult}(L[\mathcal{U}];\mathcal{V}) \prec N$ do not move κ, it follows that

$$\kappa \in j(i_{\mathcal{U}}(X)) \leftrightarrow \kappa \notin k(i_{\mathcal{V}}(X)).$$

Part (c) of Lemma 3.3.13, applied in $L[\mathcal{U}]$ and in $\mathrm{Ult}(L[\mathcal{U}];\mathcal{V})$, yields that $j(i_{\mathcal{U}}(\kappa)) = (2^\kappa)^+ = k(i_{\mathcal{V}}(\kappa))$. Lemma 3.3.12 thus implies that $j(i_{\mathcal{U}}(\mathcal{U})) = \mathcal{F} \cap M$ and $k(i_{\mathcal{V}}(\mathcal{U})) = \mathcal{F} \cap N$, where $\mathcal{F}$ is the closed, unbounded filter on $(2^\kappa)^+$. The elementarity of $j \circ i_{\mathcal{U}}$ gives that $M = L[j(i_{\mathcal{U}}(\mathcal{U}))]$ and therefore that $M = L[\mathcal{F} \cap M] = L[\mathcal{F}]$, by Theorem 3.4.12. Similarly $N = L[\mathcal{F}]$. From this it follows that $j(i_{\mathcal{U}}(\mathcal{U})) = k(i_{\mathcal{V}}(\mathcal{U}))$ and that $M = N$. But X is definable in $L[\mathcal{U}]$ from $\mathcal{U}$; thus $j(i_{\mathcal{U}}(X)) = k(i_{\mathcal{V}}(X))$, a contradiction. $\square$

Theorem 3.5.4. *Let $\langle \kappa_\gamma \mid \gamma < \alpha \rangle$ be a strictly increasing sequence of ordinal numbers such that $\alpha < \kappa_0$. Let $a \in V_{\kappa_0}$. Let $\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle$ be such that $L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle] \models$ "$\mathcal{U}_\gamma \cap L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ is a uniform normal ultrafilter on κ_γ," for each $\gamma < \alpha$.*

In $L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$, $\mathcal{U}_\gamma \cap L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ is the unique uniform normal ultrafilter on κ_γ for each $\gamma < \kappa$.

Proof. We may assume that $a = a \cap L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ and that $\mathcal{U}_\gamma = \mathcal{U}_\gamma \cap L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ for every $\gamma < \alpha$. As in the proof of Theorem 3.5.3, argue by contradiction. Let $\delta < \alpha$ and suppose that $X \subseteq \kappa_\delta$ is least in the canonical wellordering of $L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ such that $X \in \mathcal{U}_\delta \leftrightarrow X \notin \mathcal{V}$ for some $\mathcal{V}$ that in $L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ is a uniform normal ultrafilter on κ_δ.

Let $\mathcal{V}$ witness this fact. Taking iterated ultrapowers with respect to $\mathcal{U}_{\delta+1}$, if necessary, we may assume that $\kappa_{\delta+1} > (2^{\kappa_\delta})^+$. Let

$$\begin{aligned} M &= \mathrm{Ult}_{(2^{\kappa_\delta})^+}(\mathrm{Ult}(L[a, \langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle]; \mathcal{U}_\delta); i_{\mathcal{U}_\delta}(\mathcal{U}_\delta)); \\ N &= \mathrm{Ult}_{(2^{\kappa_\delta})^+}(\mathrm{Ult}(L[a, \langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle]; \mathcal{V}); i_{\mathcal{V}}(\mathcal{U}_\delta)). \end{aligned}$$

Let $j : \mathrm{Ult}(L[a, \langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle]; \mathcal{U}_\alpha) \prec M$ and $k : \mathrm{Ult}(L[a, \langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle]; \mathcal{V}) \prec N$ be the canonical elementary embeddings. If we can show that $j(i_{\mathcal{U}_\delta}(\mathcal{U}_\beta)) = k(i_{\mathcal{V}}(\mathcal{U}_\beta))$ for all $\beta < \alpha$ such that $\beta \neq \delta$, then we can obtain a contradiction as in the proof of Theorem 3.5.3. (Note that, e.g., $k(i_{\mathcal{V}}(\langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle)) = \langle\!\langle (k(i_{\mathcal{V}}))(\mathcal{U}_\gamma) \mid \gamma < \alpha \rangle\!\rangle$, since $\alpha < \kappa_0$.) For $\beta < \delta$, what we need to show follows from the fact that both $\mathrm{crit}(j \circ i_{\mathcal{U}_\delta})$ and $\mathrm{crit}(k \circ i_{\mathcal{V}})$ are greater than κ_β. For $\beta > \delta$ it follows from Lemma 3.3.14 applied in $L[a, \langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle]$ and in $\mathrm{Ult}(L[a, \langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle]; \mathcal{V})$. $\square$

Remark. In [Kunen, 1968], stronger uniqueness theorems are proved. For example, it is shown that if $L[\mathcal{U}] \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ" and $L[\mathcal{V}] \models$ "$\mathcal{V}$ is a uniform normal ultrafilter on κ" then $\mathcal{U} = \mathcal{V}$.

Suppose that there is a $\mathcal{U} \in L[\mathcal{U}]$ such that $L[\mathcal{U}]$ satisfies "$\mathcal{U}$ is a uniform normal ultrafilter on κ" for some ordinal κ. If $\mathcal{U}^\#$ exists, then the set of all $n \in \omega$ such that $\langle n, \langle \mathcal{U} \rangle \rangle \in \mathcal{U}^\#$ is called $0^\dagger$. (See page 174 for the definition of $\mathcal{U}^\#$. The name "$0^\dagger$" is due to R. Solovay.) This definition of $0^\dagger$ might seem to depend upon the choice of $\mathcal{U}$, but it does not; see Exercise 3.5.2. If there are two measurable cardinals, then $0^\dagger$ exists. The existence of $0^\dagger$ is equivalent to the determinacy of a certain class of games. (See Exercise 5.3.5.) One can also define the notion of $a^\dagger$ and one can define analogues of $0^\dagger$ for models $L[\langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle]$ as in Theorem 3.5.4. These generalizations also appear in determinacy results in Chapter 5.

Exercise 3.5.1. (a) Show that, for any finite set a, $L[a] = L$. (It follows that, with any of the standard definitions of n-tuples, $L[\langle a_1, \ldots, a_n \rangle] = L$ for any sets $a_1, \ldots, a_n$.)

(b) Let α be an ordinal number. For $\beta < \alpha$, let $\mathcal{U}_\beta$ be a uniform normal ultrafilter on a measurable cardinal κ_β. Show that $L[\langle \mathcal{U}_\beta \mid \beta < \alpha \rangle] = L$. (*Hint.* No $\mathcal{U}_\beta$ belongs to L.)

Exercise 3.5.2. Prove that $0^\dagger$ is well-defined, i.e., that $0^\dagger$ as defined above does not depend on the choice of $\mathcal{U}$. (*Hint.* Use Lemma 3.3.12.)

Chapter 4

$\mathbf{\Pi}_1^1$ Games

The classes $\mathbf{\Pi}_1^1$ and $\mathbf{\Sigma}_1^1$ were defined on page 84. (We will repeat these definitions in §4.1 below.) By Theorem 2.2.7, if $A \subseteq \lceil T \rceil$ belongs both to $\mathbf{\Pi}_1^1$ and to $\mathbf{\Sigma}_1^1$ (that is, if $A \in \mathbf{\Delta}_1^1$), then $G(A;T)$ is determined. We would like to extend this determinacy result to sets that belong to only one of $\mathbf{\Pi}_1^1$ and $\mathbf{\Sigma}_1^1$. The use of a dummy first move, as in the proof of Theorem 1.2.4, shows that the determinacy of all $\mathbf{\Pi}_1^1$ games is equivalent with the determinacy of all $\mathbf{\Sigma}_1^1$ games. Hence we may restrict our attention to proving the former. Unfortunately, $\mathbf{\Pi}_1^1$ determinacy, even in countable trees, is not provable in ZFC. (See Exercise 4.1.1.) If we are to prove $\mathbf{\Pi}_1^1$ determinacy, we must then assume principles that go beyond the ZFC axioms.

In §4.1 we prove the determinacy of all $\mathbf{\Pi}_1^1$ games in an arbitrary game tree T from the hypothesis that there is a measurable cardinal larger than $|T|$. In the remainder of the chapter, we present three variants of this proof. In §4.2 we show that the proof of §4.1 may be organized in terms of the machinery of *semicoverings*, a machinery similar to that of Chapter 2. In Chapter 5 we will use semicoverings to get determinacy proofs for wider classes of games. The result of §4.2 will play in Chapter 5 a role analogous to the role that Lemma 2.1.7 played in Chapter 2. In §4.3 we reorganize the proof of §4.2 in terms of the machinery of *homogeneous trees*. The determinacy proofs of Chapters 8 and 9 will use this machinery. (Those of Chapter 5, however, will make no use of it.) In §4.4 we show that–as Robert Solovay observed in the case of countable $|T|$–our $\mathbf{\Pi}_1^1$ determinacy proof in §4.1 goes through with the existence of a measurable cardinal larger than $|T|$ replaced by the existence of $a^\#$ for every subset a of $|T|$. By a a theorem of Leo Harrington, for countable T, this is equivalent with the determinacy of all $\mathbf{\Pi}^1_1$ games in T. (See Exercises 4.4.1.) The only later section that depends upon §4.4 is §5.3.

4.1 $\mathbf{\Pi}^1_1$ Determinacy

Recall that a subset A of $\lceil T \rceil$ belongs to $\mathbf{\Pi}^1_1$ if and only if there is a closed $C \subseteq \lceil T \rceil \times {}^\omega\omega$ ($= \lceil T \rceil \times \lceil {}^{<\omega}\omega \rceil$) such that

$$(\forall x \in \lceil T \rceil)(x \in A \leftrightarrow (\forall y \in {}^\omega\omega)\, \langle x, y \rangle \notin C).$$

Recall also that $A \subseteq \lceil T \rceil$ belongs to $\mathbf{\Sigma}^1_1$ just in case $\lceil T \rceil \setminus A \in \mathbf{\Pi}^1_1$.

The following lemma was proved in the course of proving Theorem 2.2.3. The two sentences of the lemma state the propositions (a) and (b) occurring in that proof. For the definition of *open-separated union*, see page 80.

Remark. The reader who has skipped §2.2 should now read the proof of Lemma 2.2.3. Such a reader may ignore the the parts of that lemma and this one that are concerned with open-separated unions.

Lemma 4.1.1. *Every clopen set belongs to $\mathbf{\Pi}^1_1$ and to $\mathbf{\Sigma}^1_1$. Both $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$ are closed under countable unions and open-separated unions.*

Remark. It will be convenient to deal with game trees T such that there are no terminal positions in T. To see that this is justified, let T be a game tree. Consider the tree

$$T' = T \cup \{p^\frown\langle 0, \ldots, 0\rangle \mid p \text{ is terminal in } T\}.$$

The obvious $f : T' \to T$ induces a homeomorphism from $\lceil T' \rceil$ to $\lceil T \rceil$ such that, for each $A \subseteq \lceil T \rceil$, $G(A; T)$ is determined if and only if $G(f^{-1}(A); T')$ is determined.

We next prove two standard representations of $\mathbf{\Pi}^1_1$ sets. The second of these will be useful for determinacy proofs. To avoid unnecessary details, we do not state the most general versions of these lemmas. For later applications, however, we state the lemmas in slightly more general form than we will need in this section.

If T is a game tree, let us denote by $[T]$ the set of all infinite plays in T. If there are no terminal positions in T, then $[T] = \lceil T \rceil$. If $\mathbf{T}$ is a game tree with taboos and every play normal in $\mathbf{T}$ is infinite, then $[T] = \lceil \mathbf{T} \rceil$.

Lemmas 4.1.2 and 4.1.4 give ways to characterize the members of a $\mathbf{\Pi}^1_1$ set that belong to $[T]$. With a little more complex characterizations, we could remove the qualification "that belong to $[T]$."

Lemma 4.1.2. *Let T be a game tree and let $A \subseteq \lceil T \rceil$. Then $A \in \mathbf{\Pi}^1_1$ if and only if there is a relation $R \subseteq T \times {}^{<\omega}\omega$, such that*

(a) $R(\emptyset, \emptyset)$;

(b) $(\forall p \in T)(\forall s \in {}^{<\omega}\omega)(R(p,s) \rightarrow \ell\mathrm{h}(p) = \ell\mathrm{h}(s))$;

(c) $(\forall p \in T)(\forall s \in {}^{\ell\mathrm{h}(p)}\omega)(\forall n < \ell\mathrm{h}(p))(R(p,s) \rightarrow R(p \restriction n, s \restriction n))$.

(d) $(\forall x \in [T])(x \in A \leftrightarrow (\forall y \in {}^{\omega}\omega)(\exists n \in \omega)\, \neg R(x \restriction n, y \restriction n))$.

(Here we have written $R(p,s)$ for $\langle p, s\rangle \in R$.)

Proof. Suppose that C witnesses that $A \in \mathbf{\Pi}^1_1$. Let $R(p,s)$ hold if and only if $\ell\mathrm{h}(p) = \ell\mathrm{h}(s)$ and

$$p = s = \emptyset \;\vee\; (\exists x \in [T])(\exists y \in {}^{\omega}\omega)(p \subseteq x \,\wedge\, s \subseteq y \,\wedge\, \langle x, y\rangle \in C).$$

It is trivial that R has properties (a), (b), and (c). Let $x \in [T]$. If $x \notin A$, then there is a $y \in {}^{\omega}\omega$ such that $\langle x, y\rangle \in C$; thus $R(x \restriction n, y \restriction n)$ holds for every $n \in \omega$. If $x \in A$, then the fact that C is closed guarantees that for all $y \in {}^{\omega}\omega$ there is an n such that $(\lceil T_{x \restriction n} \rceil \times \lceil ({}^{<\omega}\omega)_{y \restriction n} \rceil) \cap C = \emptyset$. Thus R has property (d).

Suppose now that there is an R with properties (a), (b), (c), and (d). Let

$$C = \{\langle x, y\rangle \mid x \in \lceil T \rceil \setminus [T] \,\vee\, (\forall n \in \omega)\, R(x \restriction n, y \restriction n)\}.$$

It is easy to see that C witnesses that $A \cap [T] \in \mathbf{\Pi}^1_1$. Since $A \setminus [T]$ is open, it follows by Lemma 4.1.1 that $A \in \mathbf{\Pi}^1_1$. $\square$

To prove what for us will be the most useful characterization of $\mathbf{\Pi}^1_1$ sets, we need the definition and the lemma that follow.

The *Brouwer–Kleene ordering* $<^{\mathrm{BK}}$ of ${}^{<\omega}\omega$ is defined by

$$s <^{\mathrm{BK}} t \leftrightarrow (s \supsetneq t \,\vee\, (\exists n < \min\{\ell\mathrm{h}(s), \ell\mathrm{h}(t)\})(s \restriction n = t \restriction n \,\wedge\, s(n) < t(n))).$$

The Brouwer–Kleene ordering is a linear ordering of ${}^{<\omega}\omega$. It agrees with the lexicographic ordering $<_{\mathrm{lex}}$ except that when $s \supsetneq t$ then $t <_{\mathrm{lex}} s$ but $s <^{\mathrm{BK}} t$.

Lemma 4.1.3. *Let S be a subtree of ${}^{<\omega}\omega$. Then S is wellfounded (i.e., $[S] = \emptyset$) if and only if the restriction to S of $<^{\mathrm{BK}}$ is a wellordering.*

Proof. Assume first that S is not wellfounded. Let $y \in [S]$. Then $\langle y \restriction n \mid n \in \omega\rangle$ is an infinite descending sequence with respect to $<^{\mathrm{BK}}$, and so $<^{\mathrm{BK}}$ is not a wellordering.

Now assume that $<^{\mathrm{BK}} \restriction S$ is not a wellordering. Since it is a linear ordering, it must not be wellfounded. Let $\langle t_i \mid i \in \omega\rangle$ be an infinite descending sequence with respect to $<^{\mathrm{BK}}$ with each $t_i \in S$.

We prove by induction on $m \in \omega$:

(i) for all but finitely many $i \in \omega$, $\ell\mathrm{h}(t_i) \geq m$;

(ii) $\lim_i (t_i \restriction m)$ exists.

(i) and (ii) trivially hold for $m = 0$. Suppose that (i) and (ii) hold for m. Let i_m be such that

$$(\forall i \geq i_m)(\ell\mathrm{h}(t_i) \geq m \ \wedge\ t_i \restriction m = t_{i_m} \restriction m).$$

At most one of the t_i, $i \geq i_m$, can be $t_{i_m} \restriction m$ and this one, if it exists, must be t_{i_m}. Therefore $\ell\mathrm{h}(t_i) \geq m+1$ for every $i > i_m$. By the definition of $<^{\mathrm{BK}}$,

$$t_{i_m+1}(m) \geq t_{i_m+2}(m) \geq \cdots .$$

Thus $\lim_i t_i(m)$ exists.

Now let $y \in {}^{\omega}\omega$ be given by

$$y(m) \ = \ \lim_i t_i(m).$$

Since each $y \restriction n$ is extended by some t_i, it follows that each $y \restriction n$ belongs to S. Thus $y \in [S]$, and hence S is not wellfounded. $\square$

The following characterization of $\mathbf{\Pi}^1_1$ sets is a well-known variant of those of [Kleene, 1955] and [Lusin and Sierpiński, 1923].

Lemma 4.1.4. *Let T be a game tree and let $A \subseteq \lceil T\rceil$. Then $A \in \mathbf{\Pi}^1_1$ if and only if there is a function $p \mapsto <_p$ with domain T such that*

(1) for all $p \in T$, $<_p$ is a linear ordering of $\ell\mathrm{h}(p)$ with greatest element 0 (if $\ell\mathrm{h}(p) > 0$);

(2) for all $p \subseteq q \in T$, $<_p$ is the restriction of $<_q$ to $\ell\mathrm{h}(p)$;

(3) $(\forall x \in [T])(x \in A \leftrightarrow\ <_x$ is a wellordering), where $<_x$ is the relation $\bigcup_{n\in\omega} <_{x\restriction n}$.

Proof. Suppose first that $A \in \mathbf{\Pi}^1_1$.

Let $n \mapsto s_n$ be a bijection from ω to ${}^{<\omega}\omega$ such that whenever $s_m \subseteq s_n$ then $m \leq n$, i.e. such that no sequence is enumerated before a sequence it properly extends. Let $R \subseteq T \times {}^{<\omega}\omega$ be given by Lemma 4.1.2.

For $p \in T$ and m and n smaller than $\ell\mathrm{h}(p)$, we let $m <_p n$ just in case one of the following holds:

(i) $\neg R(p \restriction \ell\mathrm{h}(s_m), s_m) \ \wedge\ R(p \restriction \ell\mathrm{h}(s_n), s_n)$;

(ii) $m < n \ \wedge\ \neg R(p \restriction \ell\mathrm{h}(s_m), s_m) \ \wedge\ \neg R(p \restriction \ell\mathrm{h}(s_n), s_n)$;

(iii) $s_m <^{\mathrm{BK}} s_n \ \wedge\ R(p \restriction \ell\mathrm{h}(s_m), s_m) \ \wedge\ R(p \restriction \ell\mathrm{h}(s_n), s_n)$.

In other words, we place all the $n < \ell\mathrm{h}(p)$ such that $\neg R(p \restriction \ell\mathrm{h}(s_n), s_n)$ at the beginning of the ordering $<_p$ in their natural order, and we then order the remaining numbers $n < \ell\mathrm{h}(p)$ according to the Brouwer–Kleene ordering of the corresponding sequences s_n.

The fact that no s precedes in the enumeration any of its proper initial segments guarantees that $\ell\mathrm{h}(s_n) \leq n$ and so that $<_p$ is well-defined. It is clear that $<_p$ is a linear ordering of $\ell\mathrm{h}(p)$ for every $p \in T$. Since $s_0 = \emptyset$, property (a) of R and the fact that $\emptyset$ is maximal with respect to $<^{\mathrm{BK}}$ imply that 0 is maximal with respect to every $<_p$. Thus (1) holds.

Condition (2) is obvious from the definition.

To prove (3), fix $x \in [T]$ and let

$$S = \{t \in {}^{<\omega}\omega \mid R(x \restriction n, t)\}.$$

Properties (b) and (c) of R imply that S is a subtree of ${}^{<\omega}\omega$. Property (d) implies that

$$x \in A \leftrightarrow [S] = \emptyset.$$

By Lemma 4.1.3,

$$[S] = \emptyset \ \leftrightarrow\ <^{\mathrm{BK}} \restriction S \text{ is a wellordering.}$$

Since $<_x$ is isomorphic to the natural ordering of $\{n \mid s_n \notin S\}$ followed by $<^{\mathrm{BK}} \restriction S$,

$$<^{\mathrm{BK}} \restriction S \text{ is a wellordering } \leftrightarrow\ <_x \text{ is a wellordering.}$$

For the other half of the lemma, suppose there is a function $p \mapsto <_p$ satisfying (1), (2), and (3). Define $C \subseteq \lceil T \rceil \times {}^\omega\omega$ by

$$\langle x, y\rangle \in C \;\leftrightarrow\; (x \notin [T] \vee (\forall n \in \omega)(y(n+1) <_x y(n))).$$

Clearly we have that

$$(\forall x \in \lceil T \rceil)((x \in [T] \wedge <_x \text{ is a wellordering}) \;\leftrightarrow\; (\forall y \in {}^\omega\omega)\, \langle x, y\rangle \notin C).$$

Thus C witnesses that $A \cap [T] \in \mathbf{\Pi}^1_1$. Lemma 4.1.1 implies that $A \in \mathbf{\Pi}^1_1$.

For later use, we also give a proof using Lemma 4.1.2 of this "if" half of the lemma: Let $<_p$ have properties (1), (2), and (3). Define $R \subseteq T \times {}^{<\omega}\omega$ by letting $R(p, s)$ hold if and only if $\ell\mathrm{h}(p) = \ell\mathrm{h}(s)$ and, for all n such that $n + 1 < \ell\mathrm{h}(p)$,

$$(s(n) < \ell\mathrm{h}(p) \wedge s(n+1) < \ell\mathrm{h}(p)) \;\rightarrow\; s(n+1) <_p s(n).$$

It is easy to check that R satisfies (a)–(d) of Lemma 4.1.2. That lemma thus gives that $x \in A$. $\square$

We will sometimes want to point out sharper lightface versions of our theorems. We make the following definitions explicitly only for subsets of ${}^\omega\omega$, but the definitions extend in an obvious way to subsets of finite products of ω and ${}^\omega\omega$. A subset A of ${}^\omega\omega$ belongs to Π^1_1 if there is a relation $R \subseteq {}^{<\omega}\omega \times {}^{<\omega}\omega$ with properties (a)–(d) of Lemma 4.1.2 such that R is recursive. A set $A \subseteq {}^\omega\omega$ belongs to Σ^1_1 if ${}^\omega\omega \setminus A \in \Pi^1_1$. For $x \in {}^\omega\omega$, the classes $\Pi^1_1(x)$ and $\Sigma^1_1(x)$ are similarly defined, with "recursive in x" replacing "recursive." It is easy to see that

$$\begin{aligned} \mathbf{\Pi}^1_1 &= \bigcup_{x \in {}^\omega\omega} \Pi^1_1(x); \\ \mathbf{\Sigma}^1_1 &= \bigcup_{x \in {}^\omega\omega} \Sigma^1_1(x). \end{aligned}$$

Here is the lightface version of Lemma 4.1.4:

Lemma 4.1.5. *Let $A \subseteq {}^{<\omega}\omega$. Then $A \in \Pi^1_1$ if and only if there is a recursive function $p \mapsto <_p$ with domain ${}^{<\omega}\omega$ such that*

(1) for all $p \in {}^{<\omega}\omega$, $<_p$ is a linear ordering of $\ell\mathrm{h}(p)$ with greatest element 0 (if $\ell\mathrm{h}(p) > 0$);

(2) for all $p \subseteq q \in {}^{<\omega}\omega$, $<_p$ *is the restriction of* $<_q$ *to* $\ell\mathrm{h}(p)$;

(3) $(\forall x \in {}^{\omega}\omega)(x \in A \leftrightarrow\ <_x$ *is a wellordering), where* $<_x$ *is the relation* $\bigcup_{n\in\omega} <_{x\restriction n}$.

The proof of the "only if" part of Lemma 4.1.5 is just like that of Lemma 4.1.4, except that the function $n \mapsto s_n$ must be chosen to be recursive. The proof of the "if" part of Lemma 4.1.5 is just like the second proof of the "if" part of Lemma 4.1.4.

Lemma 4.1.4 and the Rowbottom ultrafilters introduced in §3.1 provide us with the tools for proving $\mathbf{\Pi}^1_1$ determinacy from the existence of large enough measurable cardinals.

Theorem 4.1.6. ([Martin, 1970]) *Let* T *be a game tree. Assume there is a measurable cardinal larger than* $|T|$. *Then all* $\mathbf{\Pi}^1_1$ *games in* T *are determined.*

Proof. As we argued on page 182, we may assume that there are no terminal positions in T and so that $\lceil T\rceil = [T]$.

Let $A \subseteq \lceil T\rceil$ with $A \in \mathbf{\Pi}^1_1$. Let $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.4.

Let κ be a measurable cardinal with $|T| < \kappa$ and, by Lemma 3.1.7, let $\mathcal{U}$ be a uniform normal ultrafilter on κ.

We describe a game tree T^* by describing the legal plays in T^*:

$$\begin{array}{lccccccc} \mathrm{I} & \langle a_0,\xi_0\rangle & & \langle a_2,\xi_1\rangle & & \langle a_4,\xi_2\rangle & & \dots \\ \mathrm{II} & & a_1 & & a_3 & & \dots & \end{array}$$

Each $\langle a_i \mid i < n\rangle$ must be a legal position in T. Each ξ_i must be an ordinal number smaller than κ. Let $\pi : T^* \to T$ be given by

$$\pi(\langle\langle a_0,\xi_0\rangle, a_1,\dots,a_{2n-1}\,[,\langle a_{2n},\xi_n\rangle]\rangle) = \langle a_0,a_1,\dots,a_{2n-1}\,[,a_{2n}]\rangle.$$

π induces a continuous function, which we also call π, from $\lceil T^*\rceil$ to $\lceil T\rceil$.

Consider the set $A^* \subseteq \lceil T^*\rceil$ given by

$$\begin{aligned} &\langle\langle a_0,\xi_0\rangle, a_1, \langle a_2,\xi_1\rangle, a_3,\dots\rangle \in A^* \leftrightarrow \\ &\quad (\forall m\in\omega)(\forall n\in\omega)(m <_{\langle a_i | i\in\omega\rangle} n \leftrightarrow \xi_m < \xi_n). \end{aligned}$$

I wins a play x^* of $G(A^*;T^*)$ if his ordinal moves ξ_i give an embedding of $(\omega;<_{\pi(x^*)})$ into $(\kappa;<)$. In particular this means that $\pi(A^*) \subseteq A$. Thus I wins a play x^* of $G(A^*; T^*)$ if he not only succeeds in making $\pi(x^*) \in A$ but also *verifies* that $\pi(x^*) \in A$ by embedding of $(\omega; <_{\pi(x^*)})$ into $(\kappa; <)$.

The set A^* is closed; so, by Lemma 1.2.4, $G(A^*; T^*)$ is determined.

Suppose first that $G(A^*; T^*)$ is a win for I. Let σ^* be a winning strategy for I for $G(A^*; T^*)$. Let σ be a strategy for I in T such that $\sigma(\pi(p^*))$ is the first component of $\sigma^*(p^*)$ for every p^* consistent with σ^*. (This condition fixes σ on all positions consistent with σ.) A play consistent with σ is thus the image under π of a play consistent with σ^*. It follows that every play consistent with σ belongs to A and so that σ is a winning strategy for $G(A; T)$.

Suppose now that $G(A^*; T^*)$ is a win for II. Let τ^* be a winning strategy for II for $G(A^*; T^*)$. We will define a strategy τ for II for $G(A; T)$.

Let $n \in \omega$, let $p = \langle a_0, a_1, \ldots, a_{2n}\rangle$ be a position in T, and let $v \in [\kappa]^{n+1}$. There is a unique

$$q^*(p, v) = \langle\langle a_0, \xi_0\rangle, a_1, \ldots, \langle a_{2n}, \xi_n\rangle\rangle$$

such that $\pi(q^*(p, v)) = p$ and $i \mapsto \xi_i$ embeds $(n + 1; <_p)$ into $(v; <)$. Let

$$\tau(p) = a \;\leftrightarrow\; \{v \in [\kappa]^{n+1} \mid \tau^*(q^*(p, v)) = a\} \in \mathcal{U}^{[n+1]},$$

where $\mathcal{U}^{[n+1]}$ is the Rowbottom ultrafilter defined from $\mathcal{U}$ as on page 136. Since $|T| < \kappa$ and $\mathcal{U}^{[n+1]}$ is κ-complete, $\tau(p)$ is defined. Let

$$Z_p = \{v \in [\kappa]^{n+1} \mid \tau(p) = \tau^*(q^*(p, v))\}.$$

Z_p belongs to $\mathcal{U}^{[n+1]}$.

Remark. Equivalently, we may define $\tau(p)$ by

$$\tau(p) = \int \tau^*(q^*(p, v))d\mu^{[n+1]},$$

where $\mu^{[n+1]}$ is the measure on $[\kappa]^{n+1}$ given by

$$\mu^{[n+1]}(X) = \begin{cases} 1 & \text{if } X \in \mathcal{U}^{[n+1]}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus one might call the technique by which τ is obtained from τ^* *integration.*

Since each Z_p belongs to $\mathcal{U}^{[n+1]}$, we may, by the definition of $\mathcal{U}^{[n+1]}$, let $X_p \subseteq \kappa$ be such that $X_p \in \mathcal{U}$ and $[X_p]^{n+1} \subseteq Z_p$. Let

$$X = \bigcap\{X_p \mid p \in T \wedge \ell\text{h}(p) \text{ odd}\}.$$

Since $\mathcal{U}$ is κ-complete and $|T| < \kappa$, we have that $X \in \mathcal{U}$. Moreover, for every n and every $p \in T$ of length $2n+1$,

$$(\forall v \in [X]^{n+1})\, \tau(p) = \tau^*(q^*(p, v)).$$

To show that τ is a winning strategy for II for $G(A; T)$, let $x \in \lceil T \rceil$ be consistent with τ. Assume for a contradiction that $x \in A$. Since $|X| = \kappa > \aleph_0$, let $i \mapsto \xi_i$ embed the wellordering $(\omega; <_x)$ into $(X; <)$. Let x^* be the play in T^* with these values of the ξ_i and with $\pi(x^*) = x$. Clearly x^* is a win for I in $G(A^*; T^*)$. But, for each $p^* \subseteq x^*$ of odd length,

$$x^*(\ell\mathrm{h}(p^*)) = \tau(\pi(p^*)) = \tau^*(q^*(\pi(p^*), \{\xi_i \mid 2i < \ell\mathrm{h}(p^*)\})) = \tau^*(p^*).$$

Thus x^* is a play consistent with the winning strategy τ^*, and this contradicts the fact that x^* is a win for I. $\square$

Exercise 4.1.1. Assume that all $\mathbf{\Pi}^1_1$ games in countable trees are determined and prove that ω_1 is inaccessible in L. Deduce that $\mathbf{\Pi}^1_1$ determinacy is not provable in ZFC.

Hint. First show that every uncountable $\mathbf{\Pi}^1_1$ subset of ${}^\omega 2$ $(= \lceil {}^{<\omega}2 \rceil)$ has a perfect subset (a nonempty subset without isolated points). To do this, let $A \subseteq {}^\omega 2$ with $A \in \mathbf{\Pi}^1_1$ and let R witness that $A \in \mathbf{\Pi}^1_1$. Consider a game tree T plays in which are as follows:

$$\begin{array}{lccccccc} \mathrm{I} & s_0 & & s_1 & & s_2 & & \ldots \\ \mathrm{II} & & e_0 & & e_1 & & \ldots & \end{array}$$

Each e_i must belong to 2 $(= \{0, 1\})$. Each s_i must belong to ${}^{<\omega}2$, and each s_{i+1} must satisfy

$$s_{i+1} \supseteq s_i{}^\frown\langle e_i \rangle.$$

Let $G(B; T)$ be the game that I wins if and only if $\bigcup_{i \in \omega} s_i \in A$. Prove that $G(B; T)$ is a win for I if and only if A has a perfect subset. Prove that $G(B; T)$ is a win for II if and only if A is countable. (This argument, the same as alluded to in Exercises 1.1.2 and 1.1.4, is from [Davis, 1964]. See pages 295–297 of [Moschovakis, 1980].)

Now prove the result of Gödel [1938] that if $V = L$ then there is an uncountable subset of ${}^\omega 2$ belonging to Π^1_1 and without a perfect subset. Note that the proof works under the weaker hypothesis that $(\omega_1)^L = \omega_1$. Relativize the proof to get an uncountable set in $\Pi^1_1(x)$ without a perfect subset if $(\omega_1)^{L[x]} = \omega_1$ for any $x \in {}^\omega\omega$. (See page 283 of [Moschovakis, 1980].)

The results of the preceding two paragraphs and the assumption of the exercise imply that $(\omega_1)^{L[x]}$ is countable for every $x \in {}^\omega\omega$, and this implies that that ω_1 is inaccessible in L.

The first part of the exercise implies that, under the assumption of the exercise, $L_{\omega_1} \models$ ZFC. The last part of the exercise follows by the Second Incompleteness Theorem of Gödel. Of course, the last part follows simply from the fact that $\mathbf{\Pi}^1_1$ determinacy is false in L.

Exercise 4.1.2. Prove that the determinacy of all Π^1_1 games in ${}^{<\omega}\omega$ is not provable in ZFC.

Hint. Show that Π^1_1 determinacy is false in L, by coding as games in ${}^{<\omega}\omega$ the games $G(B;T)$ of the hint to Exercise 4.1.1.

4.2 Semicoverings

The proof of Theorem 4.1.6 is reminiscent of those of Chapter 2. The function π occurring in the former plays a role like that of the functions π that are components of coverings. The construction of the strategies σ and τ from the strategies σ^* and τ^* respectively gives an operation like the ϕ component of a covering. Nevertheless, these operations do not in general give rise to a covering. The difficulty concerns the Ψ component of a covering. In the proof of Theorem 4.1.6, we were given a play x consistent with τ and we constructed an x^* consistent with τ^* such that $\pi(x^*) = x$, but we were able do this only because we made the assumption that $x \in A$. This assumption provided us with the ordinals ξ_i used to define x^*.

It is a theorem of Itay Neeman that, under a large cardinal hypothesis, every $\mathbf{\Pi}^1_1$ set can be unraveled by a covering. We will discuss this theorem on page 309.

The proof of Theorem 4.1.6 does give rise to what we will call *semicoverings.* Semicoverings are enough like coverings that (1) any set A unraveled by what we will call an A-semicovering is determined and (2) certain operations on semicoverings yield semicoverings. In this section we will define semicoverings and use them to reprove $\mathbf{\Pi}^1_1$ determinacy from measurable cardinals. In Chapter 5 we will use semicoverings to get further determinacy results.

As we did with coverings, we will define semicoverings in terms of game trees with taboos. If $\mathbf{T}$ is a game tree with taboos, then a *semicovering of* $\mathbf{T}$ is a quadruple $\langle\tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ such that

(a) $\tilde{\mathbf{T}}$ is a game tree with taboos;

(b) $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$;

(c) $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$;

(d) $\Psi : \text{domain}\,(\Psi) \to \lceil\tilde{T}\rceil$, the domain of Ψ is a subset of

$$\{\langle\tilde{\sigma}, x\rangle \mid \tilde{\sigma} \in \mathcal{S}(\tilde{T}) \wedge x \in \lceil T\rceil \wedge x \text{ is consistent with } \phi(\tilde{\sigma})\},$$

and, for all $\langle\tilde{\sigma}, x\rangle \in \text{domain}\,(\Psi)$,

(i) $\Psi(\tilde{\sigma}, x)$ is consistent with $\tilde{\sigma}$;

(ii) $\pi(\Psi(\tilde{\sigma}, x)) \subseteq x$;

(iii) either (1) $\pi(\Psi(\tilde{\sigma}, x)) = x$ and $\Psi(\tilde{\sigma}, x)$ and x are both normal or both taboo for the same player or (2) $\Psi(\tilde{\sigma}, x)$ is taboo for the player for whom $\tilde{\sigma}$ is a strategy.

The only difference between a covering of $\mathbf{T}$ and a semicovering of $\mathbf{T}$ is that clause (d) above, unlike the clause (d) on page 66, does not demand that $\Psi(\tilde{\sigma}, x)$ be defined for every pair $\langle\tilde{\sigma}, x\rangle$ such that $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ and x is a play in T consistent with $\phi(\tilde{\sigma})$. On the one hand, this means that every covering of $\mathbf{T}$ is a semicovering of $\mathbf{T}$. On the other hand, it means that the notion of a semicovering is a very weak one. For example, domain (Ψ) may be empty. Thus a semicovering will not be of much use to us unless the domain of Ψ satisfies some further conditions.

We say that a semicovering $\langle\tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ of $\mathbf{T}$ *unravels* a subset A of $\lceil\mathbf{T}\rceil$ if $\boldsymbol{\pi}^{-1}(A)$ is a clopen subset of $\lceil\tilde{\mathbf{T}}\rceil$. Here $\boldsymbol{\pi} : \lceil\tilde{\mathbf{T}}\rceil \to \lceil\mathbf{T}\rceil$ is defined as on page 65.

The existence of a mere semicovering of $\mathbf{T}$ that unravels A does not imply the determinacy of $G(A; \mathbf{T})$. (See Exercise 4.2.1.) For $A \subseteq \lceil\mathbf{T}\rceil$, let us then define an *A-semicovering* of $\mathbf{T}$ to be a semicovering $\langle\tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ of $\mathbf{T}$ such that

(e) If $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ and x witnesses that $\phi(\tilde{\sigma})$ is not a winning strategy for $G(A; \mathbf{T})$ (i.e., if x is a play consistent with $\phi(\tilde{\sigma})$ that is a loss in $G(A; \mathbf{T})$ for the player for whom $\tilde{\sigma}$ is a strategy), then $\langle\tilde{\sigma}, x\rangle \in \text{domain}\,(\Psi)$.

An A-semicovering is exactly what is needed to make the proof of Lemma 2.1.3 go through:

Lemma 4.2.1. *If $A \subseteq \lceil \mathbf{T} \rceil$ and there is an A-semicovering of $\mathbf{T}$ that unravels A, then $G(A; \mathbf{T})$ is determined.*

Proof. The proof of Lemma 2.1.3 works here too, since we may assume that the play x occurring in that proof is a win for the bad player. □

Just as we needed the notion of a k-covering, we will need in Chapter 5 the notion of a *k-semicovering.* Let $\mathbf{T}$ be a game tree with taboos and let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a semicovering of $\mathbf{T}$. For $k \in \omega$, $\mathcal{C}$ is a *k-semicovering of* $\mathbf{T}$ if

(i) ${}_k\tilde{\mathbf{T}} = {}_k\mathbf{T}$;

(ii) $\pi \restriction {}_k\tilde{T}$ is the identity;

(iii) $\phi \restriction \mathcal{S}({}_k\tilde{T})$ is the identity.

The proof of Theorem 4.1.6 adapts fairly easily to give an A-semicovering of $\mathbf{T}$ that unravels A (under the hypothesis that there is a measurable cardinal larger than $|T|$). But for our unraveling results in Chapter 5 for sets more complicated than $\mathbf{\Pi}^1_1$ sets, we will need a stronger result for $\mathbf{\Pi}^1_1$. Of course we will need A k-semicoverings for arbitrary $k \in \omega$ and we will need a bound on the size of the $\tilde{T}$ of the semicovering, but we will need even more. To state this stronger result, we require another definition:

If $\mathbf{T}$ is a game tree with taboos and if A and B are subsets of $\lceil \mathbf{T} \rceil$, then an (A, B)-*semicovering* of $\mathbf{T}$ is an A-semicovering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ such that

(f) for every $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ and for every $x \in B$ such that x is consistent with $\phi(\tilde{\sigma})$, the pair $\langle \tilde{\sigma}, x \rangle$ belongs to the domain of Ψ;

(g) every normal play in $\tilde{\mathbf{T}}$ belongs to $\boldsymbol{\pi}^{-1}(B)$.

Lemma 4.2.2. *Let $\mathbf{T}$ be a game tree with taboos. Let $B \subseteq \lceil \mathbf{T} \rceil$ with $B \in \mathbf{\Pi}^1_1$. Let $k \in \omega$. Suppose that κ is a measurable cardinal larger than $|T|$.*

(i) There is a (B, B)-k-semicovering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ such that $|\tilde{T}| \leq \kappa$.

(ii) There is a $(\lceil \mathbf{T} \rceil \setminus B, B)$-$k$-semicovering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ such that $|\tilde{T}| \leq \kappa$.

Proof. We give the proof of (i). The proof of (ii) is similar, with the roles of the players reversed. Also, to keep the proof closer to that of Theorem 4.1.6, we do the case $k = 0$. (We will indicate briefly how to handle the case $k > 0$.)

Let $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.4. Let $\mathcal{U}$ be a uniform normal ultrafilter on κ.

Define $\tilde{\mathbf{T}}$ as follows. Plays in $\tilde{T}$ are of the form

$$\begin{array}{lccccccc} \mathrm{I} & \langle a_0, \xi_0\rangle & & \langle a_2, \xi_1\rangle & & \langle a_4, \xi_2\rangle & & \ldots \\ \mathrm{II} & & a_1 & & a_3 & & \ldots & \end{array}$$

Each $\langle a_i \mid i < n\rangle$ must be a legal position in T. Each ξ_i must be an ordinal number smaller than κ. So far the definition is like that of T^* in the proof of Theorem 4.1.6. But we impose a further restriction. We demand that

$$m <_{\langle a_0, \ldots, a_{2n}\rangle} m' \leftrightarrow \xi_m < \xi_{m'}$$

for all m and m' no greater than n. Thus all legal positions $\tilde{p}$ in $\tilde{T}$ are such that $i \mapsto \xi_i$ embeds $(n; <_{\pi(\tilde{p} \restriction n)})$ into $(\kappa; <)$, where

$$\pi(\langle\langle a_0, \xi_0\rangle, a_1, \langle a_2, \xi_1\rangle, a_3, \ldots\rangle = \langle a_0, a_1, a_2, a_3, \ldots\rangle$$

and where n is the greatest number such that $2n \leq \ell\mathrm{h}(p) + 1$. If a terminal position $\tilde{p}$ in $\tilde{T}$ is such that $\pi(\tilde{p})$ is taboo in $\mathbf{T}$, then $\tilde{p}$ is taboo for the same player in $\tilde{\mathbf{T}}$. If $\tilde{p}$ is terminal in $\tilde{T}$ and $\pi(\tilde{p})$ is terminal and normal in $\mathbf{T}$, then $\tilde{p}$ is taboo for I in $\tilde{\mathbf{T}}$ if $\pi(\tilde{p}) \notin B$ and $\tilde{p}$ is normal in $\tilde{\mathbf{T}}$ if $\pi(\tilde{p}) \in B$. If $\tilde{p}$ is terminal in $\tilde{T}$ but $\pi(\tilde{p})$ is not terminal in $\mathbf{T}$ then $\tilde{p}$ is taboo for I in $\tilde{\mathbf{T}}$. (Such a $\tilde{p}$ must have some even length $2i$. It occurs when I cannot play ξ_i so as to obey the order restriction.)

Clearly $|\tilde{T}| \leq \kappa$.

If $\tilde{x}$ is an infinite play in $\tilde{T}$, then $i \mapsto \xi_i$ embeds $(\omega; <_{\pi(\tilde{x})})$ into $(\kappa; <)$, so $\boldsymbol{\pi}(\tilde{x}) \in B$. If $\tilde{x}$ is a finite normal play in $\tilde{\mathbf{T}}$, then also $\boldsymbol{\pi}(x) \in B$. Thus $\boldsymbol{\pi}^{-1}(B) = \lceil \tilde{\mathbf{T}} \rceil$.

To define ϕ and Ψ, suppose first that $\tilde{\sigma} \in \mathcal{S}_{\mathrm{I}}(\tilde{T})$. For positions $\tilde{p}$ consistent with $\tilde{\sigma}$, let $(\phi(\tilde{\sigma}))(\pi(\tilde{p}))$ be the first component of $\tilde{\sigma}(\tilde{p})$. For other positions $p \in T$, define $\phi(\tilde{\sigma})$ arbitrarily, subject to the constraints of clause (iii) in the definition of $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$. (In the proof for $k > 0$, the constraints of clause (iii) in the definition of a k-covering must be met also.) We define $\Psi(\tilde{\sigma}, x)$ for every play x consistent with $\phi(\tilde{\sigma})$. Fix such an x. There is a unique play $\tilde{x}$ such that $\tilde{x}$ is consistent with $\tilde{\sigma}$ and $\pi(\tilde{x}) \subseteq x$. Let $\Psi(\tilde{\sigma}, x) = \tilde{x}$.

Now suppose that $\tilde{\tau} \in \mathcal{S}_{\mathrm{II}}(\tilde{T})$. Let $n \in \omega$, let $p = \langle a_0, a_1, \ldots, a_{2n}\rangle$ be a position in T, and let $v \in [\kappa]^{n+1}$. As in the proof of Theorem 4.1.6 there is a unique

$$\tilde{q}(p, v) = \langle\langle a_0, \xi_0\rangle, a_1, \ldots, \langle a_{2n}, \xi_n\rangle\rangle$$

such that $\pi(\tilde{q}(p, v)) = p$ and $i \mapsto \xi_i$ is order-preserving from $(n + 1; <_p)$ to $(v; <)$. Let

$$(\phi(\tilde{\tau}))(p) = a \;\leftrightarrow\; \{v \in [\kappa]^{n+1} \mid \tilde{\tau}(\tilde{q}(p, v)) = a\} \in \mathcal{U}^{[n+1]}.$$

(Here, as in the proof of Theorem 4.1.6, we are using the fact that $|T| < \kappa$.) Let

$$Z_p = \{v \in [\kappa]^{n+1} \mid (\phi(\tilde{\tau}))(p) = \tilde{\tau}(\tilde{q}(p, v))\}.$$

Z_p belongs to $\mathcal{U}^{[n+1]}$. Let $X_p \subseteq \kappa$ be such that $X_p \in \mathcal{U}$ and $[X_p]^{n+1} \subseteq Z_p$. Let

$$X = \bigcap\{X_p \mid p \in T \wedge \ell\mathrm{h}(p) \text{ odd}\}.$$

We have that $X \in \mathcal{U}$ and that, for every n and every $p \in T$ of length $2n+1$,

$$(\forall v \in [X]^{n+1})\, \tilde{\tau}(\tilde{q}(p, v)) = (\phi(\tilde{\tau}))(p).$$

We define $\Psi(\tilde{\tau}, x)$ for every play x consistent with $\phi(\tilde{\tau})$ such that either $x \in B$ or else x is finite. Fix such an x. If $x \in B$ and x is infinite, let $i \mapsto \xi_i$ embed $(\omega; <_x)$ into $(X; <)$. If x is finite, let $i \mapsto \xi_i$ embed $(\ell\mathrm{h}(x); <_x)$ into $(X; <)$. Let $\Psi(\tilde{\tau}, x)$ be the play $\tilde{x}$ with these values of the ξ_i and with $\pi(\tilde{x}) = x$.

We leave to the reader the easy verification that our functions π, ϕ, and Ψ have the required properties.

For $k > 0$, the main change is that plays in $\tilde{T}$ are of the form

$$\begin{array}{lcccccccc}
\mathrm{I} & a_0 & & \ldots & & \langle a_{2j}, \xi_0\rangle & & \langle a_{2j+2}, \xi_2\rangle & & \ldots \\
\mathrm{II} & & a_1 & & \ldots & & a_{2j+1} & & \ldots &
\end{array}$$

where j is large enough that $2j \geq k$. Other changes are the obvious ones. For example, if $p = \langle a_0, \ldots, a_{2j+2n}\rangle$, then $\tilde{q}(p, v)$ is defined for $v \in [\kappa]^{n+1}$; if $\ell\mathrm{h}(p) \leq 2j$ then $\tilde{q}(p, v) = p$. $\square$

Lemmas 4.2.1 and 4.2.2 give a different proof of Theorem 4.1.6. But the importance of these lemmas is that they provide ingredients for proving in Chapter 5 the determinacy of wider classes of games. For the sharpest results in Chapter 5, we need the following refinment of Lemma 4.2.2.

Lemma 4.2.3. *Let* $\mathbf{T}$ *be a game tree with taboos. Let* $B \subseteq \lceil \mathbf{T} \rceil$ *with* $B \in \mathbf{\Pi}^1_1$. *Let* $k \in \omega$ *and* $m \in \omega$. *Suppose that* κ *is a measurable cardinal such that* $|T| \leq \kappa$ *and such that*

$$(\forall p \in T)(\ell\mathrm{h}(p) > m \to |T_p| < \kappa).$$

(i) There is a (B, B)-k-*semicovering* $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ *of* $\mathbf{T}$ *such that* $|\tilde{T}| \leq \kappa$ *and such that*

$$(\forall \tilde{p} \in \tilde{T})(\ell\mathrm{h}(\tilde{p}) > \max\{k, m\} + 1 \to |\tilde{T}_{\tilde{p}}| < \kappa).$$

(ii) There is a $(\lceil \mathbf{T} \rceil \setminus B, B)$-$k$-*semicovering* $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ *of* $\mathbf{T}$ *such that* $|\tilde{T}| \leq \kappa$ *and such that*

$$(\forall \tilde{p} \in \tilde{T})(\ell\mathrm{h}(\tilde{p}) > \max\{k, m\} + 1 \to |\tilde{T}_{\tilde{p}}| < \kappa).$$

Proof. The proof is like that of Lemma 4.2.2. We indicate only the changes. Define $\tilde{\mathbf{T}}$ as in the proof of Lemma 4.2.2, but with $\max\{k, m\}$ as the k of that proof, i.e. with moves $\langle a_{2(j+i)}, \xi_i \rangle$ for the least j with $2j \geq \max\{k, m\}$.

The fact that $|T| < \kappa$ was used twice in the proof of Lemma 4.2.2. The first time was to guarantee, for each position p in T of odd length, that $\tilde{\tau}(\tilde{q}(p, v))$ took fewer that κ values. But in the present situation $\tilde{q}(p, v) = p$ unless $\ell\mathrm{h}(p) > 2j \geq m$. If $\ell\mathrm{h}(p) > 2j$ then, since the values $\tilde{\tau}(\tilde{q}(p, v))$ belong to the set T_p whose size is less than κ, we get the desired conclusion. The other use of the fact that $|T| < \kappa$ was to guarantee that the set $X = \bigcap\{X_p \mid p \in T \wedge \ell\mathrm{h}(p) \text{ is odd}\}$ belonged to $\mathcal{U}$. Instead of considering X, we can define sets

$$X^p = \bigcap\{X_{p'} \mid p' \in T_p \,\wedge\, \ell\mathrm{h}(p') \text{ is odd}\}$$

for $p \in T$. If $\ell\mathrm{h}(p) > m$, then $X^p \in \mathcal{U}$. In defining $\Psi(\tilde{\tau}, x)$, we can replace X by X^p for $p \subseteq x$ and $\ell\mathrm{h}(p) = m + 1$.

Let us verify that any $|\tilde{T}_{\tilde{p}}| < \kappa$ for every $\tilde{p} \in \tilde{T}$ such that $\ell\mathrm{h}(\tilde{p}) > 2j$. Since, $2j \leq \max\{k, m\} + 1$, this will complete the proof. Recall that 0 is maximal in $<_p$ for every $p \in T$. Thus every legal position in $\tilde{T}$ is such that $\xi_i < \xi_0$ for each $i > 0$. Therefore, if $\tilde{p} \in \tilde{T}$ and $\ell\mathrm{h}(\tilde{p}) > 2j$, then $|\tilde{T}_{\tilde{p}}|$ is no greater than the maximum of (a) $|T_{\pi(\tilde{p})}|$, (b) the cardinal of the ξ_0 given by $\tilde{p}$, and (c) $\aleph_0$. But all these cardinal numbers are smaller than κ. $\square$

Remark. The replacement described above of X by X^p would work for the original proof of Lemma 4.2.2 also, with any value of n. Indeed the use in question of the hypothesis that $|T| < \kappa$ was unnecessary altogether: Instead of considering X or the X^p, we could have defined, for each play $x \in \lceil T \rceil$, the set $X^x = \bigcap\{X_p \mid p \subseteq x \wedge \ell\text{h}(p) \text{ is odd}\}$. The set X^x is a countable intersection of sets in $\mathcal{U}$, and so it belongs to $\mathcal{U}$. In defining $\Psi(\tilde{\tau}, x)$, we could then have used X^x in place of X or an X^p. We did not do this because of a later application of the method (Lemma 5.2.12), where we will not be able to use the X^x.

Exercise 4.2.1. Let $\mathbf{T}$ be any game tree with taboos. Show that there is a semicovering of $\mathbf{T}$ that unravels every subset of $\lceil \mathbf{T} \rceil$.

Hint. Let $\tilde{T} = \{\emptyset\}$.

Exercise 4.2.2. Prove that the semicovering of $\mathbf{T}$ constructed in the proof of Lemma 4.2.2 need not extend to a covering of $\mathbf{T}$. Indeed prove that there need not be any covering of $\mathbf{T}$ extending the $\tilde{\mathbf{T}}$ and π constructed in that proof. *Hint.* Let $\mathbf{T} = \langle {}^{<\omega}2, \emptyset, \emptyset \rangle$, let $A = \{x \mid (\exists n \in \omega)\, x(2n) = 1\}$, and let $k = 0$. Let $\tilde{\tau}$ be the strategy for II in $\tilde{T}$ such that

$$\tilde{\tau}(\tilde{p}) = 1 \leftrightarrow \tilde{p}^\frown\langle 1 \rangle \text{ is terminal in } \tilde{T}.$$

4.3 Homogeneous Trees

In this section we present still another way to organize the proof of Theorem 4.1.6. No use of the ideas and results of this section will be made until Chapter 8, but they will be the basis of all our determinacy proofs from that point on.

Let X and Y be arbitrary sets. If $B \subseteq X \times Y$, then

$$\text{p}B = \{x \in X \mid (\exists y \in Y)\, \langle x, y \rangle \in B\}.$$

Thus pB is the projection of B onto the first coordinate.

If E is a topological space and κ is a cardinal number, a subset A of E is *κ-Souslin* if there is a closed $C \subseteq \lceil T \rceil \times {}^\omega\kappa$ such that $A = \text{p}B$.

We will mainly be interested in κ-Souslin subsets of $\lceil T \rceil$, where T has no terminal positions. As we have remarked on page 182, this involves no loss of generality as far as determinacy results are concerned.

Lemma 4.3.1. *Let T be a game tree and let κ be an infinite cardinal number.*

(a) If $\lambda < \kappa$ and $A \subseteq \lceil T \rceil$ is λ-Souslin, then A is κ-Souslin.

(b) Every $\mathbf{\Sigma}^1_1$ subset of $\lceil T \rceil$ is κ-Souslin.

(c) The class of κ-Souslin subsets of $\lceil T \rceil$ is closed under unions of size $\leq \kappa$.

(d) The class of κ-Souslin subsets of $\lceil T \rceil$ is closed under countable intersections.

(e) Both the class of κ-Souslin subsets of $\lceil T \rceil$ and the class of co-κ-Souslin subsets of $\lceil T \rceil$ (the class of complements of κ-Souslin subsets of $\lceil T \rceil$) are closed under open-separated unions.

Proof. (a) is obvious, since if $\lambda < \kappa$ then ${}^\omega\lambda$ is a closed subset of ${}^\omega\kappa$.

It is immediate from the definitions that

$$A \text{ is } \aleph_0\text{-Souslin} \;\leftrightarrow\; A \in \mathbf{\Sigma}^1_1.$$

From this (b) follows with the help of (a).

The proofs of (c) and (d) are a trivial modification of the proof of the proposition (b)(i) occurring in the proof of Theorem 2.2.3, and the proof of (e) is a trivial modification that of (b)(ii) of the the proof of Theorem 2.2.3. We leave them to the reader. (The reader who has skipped §2.2 may skip (e).) □

To present and study an alternative characterization of κ-Souslin sets, we need to make a few definitions:

If T is a game tree, then the *field* of T, field$\,(T)$, is

$$\{p(i) \mid p \in T \wedge i < \ell\text{h}(p)\}.$$

If X is a set, then a *tree on* X is a game tree T such that field$\,(T) \subseteq X$.

If p and q are finite sequences of the same length , then we let

$$\langle\!\langle p, q \rangle\!\rangle = \langle \langle p(n), q(n) \rangle \mid n < \ell\text{h}(p) \rangle.$$

If T is a game tree and p is a finite sequence, then let

$$T[p] = \{q \mid \langle\!\langle p, q \rangle\!\rangle \in T\}.$$

If T is a game tree and if x is an infinite sequence, then let

$$T(x) = \bigcup_{n \in \omega} T[x \restriction n].$$

Note that $T(x)$ is always a game tree if it is nonempty.

Lemma 4.3.2. *Let T be a game tree, let $A \subseteq [T]$, and let κ be a cardinal number. Then A is κ-Souslin if and only if there is a tree U on* field $(T) \times \kappa$ *such that*

$$A = \{x \in [T] \mid [U(x)] \neq \emptyset\}.$$

Proof. If C witnesses that A is κ-Souslin, then let

$$U = \{\langle\!\langle p, s \rangle\!\rangle \mid (\exists x \supseteq p)(\exists g \supseteq s)\, \langle x, g\rangle \in C\}.$$

If U is as in the statement of the lemma, then let

$$C = \{\langle x, g\rangle \mid x \in [T] \wedge (\forall n \in \omega)\, \langle\!\langle x \restriction n, g \restriction n \rangle\!\rangle \in U\}.$$

□

Remark. Recall that a game tree is *wellfounded* if $\supsetneq\restriction T$ is wellfounded on T, i.e. if every nonempty subset X of T has an element p with no proper extensions in X. The Axiom of Choice gives that T is wellfounded if and only if there are no infinite plays in T. Thus the last line of the statement of Lemma 4.3.2 can be reformulated as

$$A = \{x \in [T] \mid U(x) \text{ is not wellfounded}\}.$$

It will be useful to have a third way to express this relation between A and U. For any tree T, any set Y, and any tree U on field $(T) \times Y$, let us say that the T-*projection* of U is $\{x \in [T] \mid U(x) \neq \emptyset\}$. Thus the last line of the statement of Lemma 4.3.2 says that A is the T-projection of U.

We have already mentioned the fact that $\mathbf{\Sigma}^1_1$ is identical with the class of all $\aleph_0$-Souslin sets. We now show, as is essentially proved in Shoenfield [1961], that all $\mathbf{\Pi}^1_1$ sets are $\aleph_1$-Souslin.

Lemma 4.3.3. *If T is a game tree and if $A \subseteq \lceil T\rceil$ with $A \in \mathbf{\Pi}^1_1$, then A is $\aleph_1$-Souslin, and so A is κ-Souslin for every uncountable κ.*

Proof. Let T be a game tree and let $A \subseteq \lceil T\rceil$ with $A \in \mathbf{\Pi}^1_1$. Though the last clause of the statement of the lemma follows from the preceding one, we will directly verify the last clause. Let κ be an uncountable cardinal number.

The set $A \backslash [T]$ is open and so is κ-Souslin by Lemma 4.3.1. By Lemma 4.1.1, $A \cap [T] \in \mathbf{\Pi}^1_1$. By Lemma 4.3.1 again, it suffices to prove that $A \cap [T]$ is κ-Souslin. Without loss of generality, let us then assume that $A \subseteq [T]$.

Let the functions $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.4.

We could get a tree U witnessing that A is κ-Souslin directly from the tree T^* occurring in the proof of Lemma 4.1.6 by replacing the moves a_i of player II in T^* by $\langle a_i, 0\rangle$. Instead we prefer to use a slightly different tree. Let

$$U = \{\langle\!\langle p, s\rangle\!\rangle \mid p \in T \wedge s \text{ embeds } (\ell\mathrm{h}(p); <_p) \text{ into } (\kappa; <)\}.$$

Let $x \in [T]$. If $g \in [T(x)]$, then g embeds $(\omega; <_x)$ into $(\kappa; <)$, and so $x \in A$. If $x \in A$, then there is a g embedding the wellordering $(\omega; <_x)$ into $(\kappa; <)$, and any such g belongs to $[T(x)]$. □

Remark. Not every $\aleph_1$-Souslin set belongs to $\mathbf{\Pi}^1_1$. See Exercises 4.3.2 and 4.3.3.

A set is *Souslin* if it is κ-Souslin for some κ. Sometimes the word "Souslin" is used in a more restricted sense, synonymous with "$\aleph_0$-Souslin." We will mostly talk of Souslin sets in the context of the concepts that we now define.

Suppose $\mathcal{U}$ and $\mathcal{V}$ are countably complete ultrafilters on sets A and B respectively. Suppose that $\chi : B \to A$ is such that

$$(\forall X \in \mathcal{U})\, \{b \in B \mid \chi(b) \in X\} \in \mathcal{V}.$$

Then we say that $\mathcal{V}$ *projects* to $\mathcal{U}$ by χ. In this situation, we can define

$$i_{\mathcal{U},\mathcal{V},\chi} : \prod_{\mathcal{U}}(V; \in) \prec \prod_{\mathcal{V}}(V; \in)$$

by

$$i_{\mathcal{U},\mathcal{V},\chi}([\![f]\!]_{\mathcal{U}}) = [\![\chi^*(f)]\!]_{\mathcal{V}},$$

where

$$(\chi^*(f))(b) = f(\chi(b)).$$

We omit the routine verification that $i_{\mathcal{U},\mathcal{V},\chi}$ is well-defined and is an elementary embedding.

Let T be a game tree, let Y be a nonempty set, and let U be a tree on field$(T) \times Y$. We say that U is *homogeneous* for T if there is a system

$$\langle \mathcal{U}_p \mid p \in T\rangle$$

satisfying the following conditions:

(1) Each $\mathcal{U}_p$ is a countably complete ultrafilter on $U[p]$.

(2) The $\mathcal{U}_p$ are *compatible*: For all $p \subseteq q \in T$, $\mathcal{U}_q$ projects to $\mathcal{U}_p$ by $\chi_{q,p}$, where $\chi_{q,p} : U[q] \to U[p]$ is given by $\chi_{q,p}(s) = s \restriction \ell\mathrm{h}(p)$.

(3) Let $x \in [T]$ and let $\langle Z_n \mid n \in \omega \rangle$ be such that each Z_n belongs to $\mathcal{U}_{x \restriction n}$. Then
$$[U(x)] \neq \emptyset \ \rightarrow \ (\exists f : \omega \to Y)(\forall n \in \omega)\, f \restriction n \in Z_n.$$

Remarks:

(a) Implicit in clause (1) is the requirement that $U[p]$ be nonempty for each $p \in T$. There is an variant notion of homogeneity that does not make this requirement. (See Exercise 4.3.5.)

(b) Each $U[p]$ is a subset of ${}^{\ell\mathrm{h}(p)}Y$, and so each $\mathcal{U}_p$ induces—and is essentially the same as—an ultrafilter on ${}^{\ell\mathrm{h}(p)}Y$.

(c) The "$\rightarrow$" in the last line of condition (3) can be replaced by a "$\leftrightarrow$," since any f satisfying the right hand side must belong to $[U(x)]$.

There is an equivalent of condition (3) that will be of use to us later: Suppose that (1) and (2) are satisfied. For $p \in T$, let $\pi_p = \pi_{\mathcal{U}_p} : \prod_{\mathcal{U}_p}(V; \in) \cong (\mathrm{Ult}(V; \mathcal{U}_p); \in)$. For $p \subseteq q \in T$, let
$$i_{p,q} = \pi_q \circ i_{\mathcal{U}_p, \mathcal{U}_q, \chi_{q,p}} \circ {\pi_p}^{-1}.$$

For each $x \in [T]$, let
$$(\mathcal{M}_x; \langle i^x_{x \restriction n} \mid n \in \omega \rangle)$$
be the direct limit of the directed system of elementary embeddings
$$(\langle \mathrm{Ult}(V; \mathcal{U}_{x\restriction n}) \mid n \in \omega \rangle; \langle i_{x \restriction m, x \restriction n} \mid m \leq n \in \omega \rangle).$$

(3$'$) $(\forall x \in [T])([U(x)] \neq \emptyset \ \rightarrow \ \mathcal{M}_x$ is wellfounded).

Lemma 4.3.4. *If (1) and (2) hold of T and $\langle \mathcal{U}_p \mid p \in T \rangle$ and if $x \in [T]$, then x witnesses the falsity of (3) if and only if x witnesses the falsity of (3$'$). Thus a tree U on* field$(T) \times Y$ *is homogeneous for T if and only if there is a system $\langle \mathcal{U}_p \mid p \in T \rangle$ satisfying (1), (2), and (3$'$).*

Proof. Let U be a tree on field$(T) \times Y$ and let $\langle \mathcal{U}_p \mid p \in T \rangle$ satisfy (1) and (2).

Suppose first that x and $\langle Z_n \mid n \in \omega\rangle$ witness the failure of (3). Let

$$S = \{s \in U(x) \mid (\forall n \leq \ell\mathrm{h}(s))\, s \restriction n \in Z_n\}.$$

S is a game subtree of $U(x)$ with no infinite plays. Thus S is wellfounded. By the elementarity of $i^x_\emptyset$,

$$\mathcal{M}_x \models i^x_\emptyset(S) \text{ is wellfounded.}$$

But let

$$s_n = i^x_{x\restriction n}(\pi_{x\restriction n}([\![\mathrm{id_{U[x\restriction n]}}]\!]_{\mathcal{U}_{x\restriction n}})).$$

It is easy to see that each $s_n \in i^x_\emptyset(S)$ and that, for $m \leq n \in \omega$, $s_m \subsetneq s_n$. Thus $\bigcup_{n\in\omega} s_n$ belongs to $[i^x_\emptyset(S)]$, and so $i^x_\emptyset(S)$ is not really wellfounded. Thus $\|i^x_\emptyset(S)\|$ as computed in $\mathcal{M}_x$ is an "ordinal" of $\mathcal{M}_x$ that is not wellordered by the membership relation $i^x_\emptyset(\in)$ of $\mathcal{M}_x$. (See page 25 for the definition of $\|S\|$.) This implies that $\mathcal{M}_x$ is not wellfounded.

Now suppose that x witnesses that (3′) fails. Let $\langle z_n \mid n \in \omega\rangle$ be an infinite descending sequence with respect to $i^x_\emptyset(\in)$. For each $n \in \omega$, let m_n and $a_n \in \mathrm{Ult}(V;\mathcal{U}_{x\restriction m_n})$ be such that $z_n = i^x_{x\restriction m_n}(a_n)$. Without loss of generality, we may assume that

$$(\forall n' \in \omega)(\forall n \in \omega)(n' < n \to m_{n'} < m_n).$$

Let $g_n \in {}^{U[x\restriction m_n]}V$ be such that

$$\pi_{x\restriction m_n}([\![g_n]\!]_{\mathcal{U}_{x\restriction m_n}}) = a_n.$$

For each $n \in \omega$, let

$$Z_{m_{n+1}} = \{s \in U[x \restriction m_{n+1}] \mid g_{n+1}(s) \in g_n(s \restriction m_n)\}.$$

For each $m \in \omega$ such that m is not of the form m_{n+1}, let $Z_m = U[x \restriction m]$. For each $m \in \omega$, we have that $Z_m \in \mathcal{U}_{x\restriction m}$. But if $f : \omega \to Y$ is such that $(\forall m \in \omega)\, f \restriction m \in Z_m$, then $\langle f(m_n) \mid n \in \omega\rangle$ is an infinite descending sequence with respect to $\in$. Thus no such f exists, and we have a counterexample to (3). $\square$

Remark. By Lemma 4.3.4 and our earlier remark about (3), the "$\to$" in condition (3′) can be replaced by "$\leftrightarrow$."

For T a game tree, Y a set, and κ a cardinal number, a tree U on field $(T)\times Y$ is *κ-homogeneous* for T if there is a system $\langle \mathcal{U}_p \mid p \in T\rangle$ witnessing that U is homogeneous for T and having the further property that each $\mathcal{U}_p$ is κ-complete.

Let T be a game tree and let $A \subseteq [T]$. A is *homogeneously Souslin* if there is a tree U on field $(T)\times Y$ for some Y such that U is homogeneous for T and A is the T-projection of U. For cardinal numbers κ, A is *κ-homogeneously Souslin* if it is the T-projection of a κ-homogeneous tree.

Remark. Note that the last of the definitions just given says nothing about the size of the κ-homogeneous tree. A set can be κ-homogeneously Souslin without being κ-Souslin.

Theorem 4.3.5. *For any game tree T, all $|T|^+$-homogeneously Souslin games in T are determined.*

Proof. Without loss of generality, we may restrict ourselves to game trees T without terminal positions. Let T be such a tree and let $A \subseteq [T]$ be $|T|^+$-homogeneously Souslin. Let U and Y be such that U is a tree on field $(T)\times Y$, U is $|T|^+$-homogeneous for T, and $A = \{x \in [T] \mid [U(x)] \neq \emptyset\}$. Let $\langle \mathcal{U}_p \mid p \in T\rangle$ witness that U is $|T|^+$ homogeneous for T. Let T^* be the game tree plays in which are as follows:

$$\begin{array}{lccccccc} I & \langle a_0, b_0\rangle & & \langle a_2, b_1\rangle & & \langle a_4, b_2\rangle & & \ldots \\ II & & a_1 & & a_3 & & \ldots & \end{array}$$

Each $\langle a_i \mid i < n\rangle$ must belong to T and each b_i must belong to Y.

Define $\pi : T^* \to T$ and the induced $\pi : [T^*] \to [T]$ as in the proof of Theorem 4.1.6.

Let A^* be the set of plays in T^* such that each $\langle\langle a_i, b_i\rangle \mid i < n\rangle$ belongs to U. The game $G(A^*;T^*)$ is closed and so is determined.

Suppose first that σ^* is a winning strategy I for $G(A^*;T^*)$. Let σ be a strategy for I in T gotten as in the proof of Theorem 4.1.6. If x is a play consistent with σ, then there is an x^* consistent with σ^* such that $\pi(x^*) = x$. Since $x^* \in A^*$, x^* gives an element $\langle\langle a_i, b_i\rangle \mid i \in \omega\rangle$ of $[U]$ with $x = \langle a_i \mid i \in \omega\rangle$. Thus $\langle b_i \mid i \in \omega\rangle \in [U(x)]$, and so $x \in A$.

Suppose now that τ^* is a winning strategy for II for $G(A^*;T^*)$. For each $p = \langle a_i \mid i \leq 2n\rangle \in T$ and each $s \in {}^{n+1}Y$, let

$$q^*(p, s) = \langle\langle a_0, s(0)\rangle, a_1, \ldots, \langle a_{2n}, s(n)\rangle\rangle.$$

Each $q^*(p,s)$ is a position in T^* and is such that $\pi(q^*(p,s)) = p$. Define a strategy τ for II in T setting

$$\tau(p) = a \leftrightarrow \{s \in U[p \restriction n+1] \mid \tau^*(q^*((p,s)) = a\} \in \mathcal{U}_{p\restriction n+1},$$

for $p \in T$ with $\ell\mathrm{h}(p) = 2n+1$. Since $\mathcal{U}_{p\restriction n+1}$ is $|T|^+$-complete, τ is well-defined. To see that τ is a winning strategy for II for $G(A;T)$, let x be a play consistent with τ. Assume for a contradiction that $x \in A$, i.e. that $[U(x)] \neq \emptyset$. For each $n \in \omega$ let

$$Z_{n+1} = \{s \in U[x \restriction n+1] \mid \tau^*(q^*(x \restriction 2n+1, s)) = x(2n+1),$$

and let $Z_0 = \{\emptyset\}$. For each $n \in \omega$, $Z_n \in \mathcal{U}_{x\restriction n}$. Hence clause (3) in the definition of homogeneous trees gives us an $f : \omega \to Y$ such that $f \restriction n \in Z_n$ for every $n \in \omega$. By the definition of τ, this means that

$$x^* = \langle\langle x(0), f(0)\rangle, x(1), \langle x(2), f(1), x(3), \ldots\rangle$$

is a play in T^* consistent with τ^*. Since $x^* \in A^*$, we have our contradiction. $\square$

Theorem 4.3.6. *If T is a game tree and κ is a measurable cardinal greater than $|T|$, then every $\mathbf{\Pi}^1_1$ subset of $[T]$ is κ-homogeneously Souslin and is witnessed to be κ-homogeneously Souslin by a tree on* field $(T) \times \kappa$.

Proof. Let T be a game tree, let $\kappa > |T|$ be a measurable cardinal, let $\mathcal{V}$ be a uniform normal ultrafilter on κ, and let $A \subseteq [T]$ with $A \in \mathbf{\Pi}^1_1$. Let $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.4. Let U be defined as in the proof of Lemma 4.3.3:

$$U = \{\langle\!\langle p, s\rangle\!\rangle \mid p \in T \wedge s \text{ embeds } (\ell\mathrm{h}(p); <_p) \text{ into } (\kappa; <).$$

Let $p \in T$. For each $v \in [\kappa]^{\ell\mathrm{h}(p)}$, there is a unique bijection $s^p_v : \ell\mathrm{h}(p) \to v$ such that $\langle\!\langle p, s^p_v\rangle\!\rangle \in U$. Define an ultrafilter $\mathcal{U}_p$ on $U[p]$ by

$$X \in \mathcal{U}_p \leftrightarrow \{v \in [\kappa]^{\ell\mathrm{h}(p)} \mid s^p_v \in X\} \in \mathcal{V}^{[\ell\mathrm{h}(p)]}.$$

We know from the proof of Lemma 4.3.3 that A is the T-projection of U. The system $\langle \mathcal{U}_p \mid p \in T\rangle$ obviously has property (1) in the definition of homogeneity, and it is easy to check that it has property (2). For (3), let

$x \in [T]$ and let $\langle Z_n \mid n \in \omega \rangle$ be such that each $Z_n \in \mathcal{U}_{x \restriction n}$. Fix for the moment $n \in \omega$. Let

$$\bar{Z}_n = \{v \in [\kappa]^n \mid s_v^{x \restriction n} \in Z_n\}.$$

By the definition of $\mathcal{U}_{x \restriction n}$, we have that $\bar{Z}_n \in \mathcal{V}^{[n]}$. By the definition of $\mathcal{V}^{[n]}$, let $X_n \in \mathcal{V}$ be such that $[X_n]^n \subseteq \bar{Z}_n$. Now let $X = \bigcap_{n \in \omega} X_n$. Thus $X \in \mathcal{V}$ and, for all n, $[X] \subseteq \bar{Z}_n$. Assume that $[U(x)] \neq \emptyset$. Then $x \in A$, and so $<_x$ is a wellordering of ω. Let f embed $(\omega; <_x)$ into $(X; <)$. To see that f is as required by (3), let Let $n \in \omega$. We have that $f \restriction n \in U[x \restriction n]$ and range$(f \restriction n) \in \bar{Z}_n$. But then $f \restriction n \in Z_n$. □

Remark. The name "homogeneous tree" may seem not to be descriptive of the concept: It is not clear that homogeneous trees need be homogeneous in any standard sense. Historically, the paradigm example of a homogeneous tree was essentially the tree U of the proof just given. The tree U is homogeneous in the straightforward sense that

$$(\forall X \subseteq \kappa)(|X| = \kappa \rightarrow U \restriction X \cong U),$$

where $U \restriction X = U \cap \{\langle\!\langle p, s \rangle\!\rangle \mid \text{range}(s) \subseteq X\}$. A related property is that membership in $U[p]$ of $s \in {}^{\ell\text{h}(p)}\kappa$ depends only on the order type of the sequence s. It is these homogeneity properties that made possible the verification of (3). A.S. Kechris and Martin began applying "homogeneous" to trees like U and trees whose fields consist of transfinite sequences and which have similar homogeneity properties. Such trees arose in work on the Axiom of Determinacy by Kenneth Kunen and later by Martin. Finally Kechris and Martin independently abstracted from the particular class of examples and began using "homogeneous tree" in the current sense (Kechris in [Kechris, 1981], and Martin in lectures). The idea is that a tree must have some kind of homogeneity if (3) in the definition of homogeneous trees is to be satisfied. The definition leaves the nature of this homogeneity unspecified.

Exercise 4.3.1. Let $A \subseteq \lceil T \rceil$. Show that

$$(\exists n \in \omega)\, A \text{ is } n\text{-Souslin} \;\leftrightarrow\; A \text{ is 1-Souslin} \;\leftrightarrow\; A \text{ is closed.}$$

Exercise 4.3.2. A subset X of $\lceil T \rceil$ belongs to $\mathbf{\Sigma}^1_2$ if there is a subset B of $\lceil T \rceil \times {}^\omega\omega$ such that $B \in \mathbf{\Pi}^1_1$ and $A = \mathrm{p}B$. Prove that every $A \in \mathbf{\Sigma}^1_2$ is $\aleph_1$-Souslin.

Exercise 4.3.3. It is relatively consistent with the ZFC axioms that in countable trees $\mathbf{\Sigma}^1_2$ is identical with the class of $\aleph_1$-Souslin sets. The hypothesis $\mathrm{MA}_{\aleph_1} + (\omega_1)^L = \omega_1$, for example, implies that this is the case. (See [Martin and Solovay, 1970].)

(a) Show, on the other hand, that the continuum hypothesis implies that every subset of $\lceil T \rceil$ is $\aleph_1$-Souslin if T is countable.

(b) Deduce that it is relatively consistent with the ZFC axioms that not every $\aleph_1$-Souslin subset of ${}^\omega\omega$ belongs to $\mathbf{\Sigma}^1_2$.

(c) Show that the determinacy of all $\mathbf{\Pi}^1_1$ games in countable trees implies that not every $\aleph_1$-Souslin subset of ${}^\omega\omega$ belongs to $\mathbf{\Sigma}^1_2$

Hint. First prove that, for any game tree T, every subset A of $\lceil T \rceil$ is $|A|$-Souslin. This gives (a). For (b), use (a) and the fact that the class of all $\mathbf{\Sigma}^1_2$ subsets of ${}^\omega\omega$ has size $2^{\aleph_0}$. For (c) prove that the determinacy of all $\mathbf{\Pi}^1_1$ games in ${}^{<\omega}\omega$ implies that every uncountable $\mathbf{\Sigma}^1_2$ subset of ${}^\omega 2$ has a perfect subset. To do this, let $B \in \mathbf{\Pi}^1_1$ witness that $A \subseteq {}^\omega 2$ belongs to $\mathbf{\Sigma}^1_2$. Use the game of Exercise 4.1.1 modified so that I's moves have extra components belonging to ω. I wins the modified game if and only if the extra components form a y such that $\langle \bigcup_{i\in\omega} s_i, y\rangle \in B$. (This trick, called *unfolding*, is due to Robert Solovay, though this is not his application of it.) To complete (c) construct, by a diagonalization, an uncountable subset of ${}^\omega 2$ without a perfect subset, then from this get a set of size $\aleph_1$ with no perfect subset.

Exercise 4.3.4. Show that it is consistent with the ZFC axioms that the only homogeneously Souslin subset of $[T]$ is $[T]$ itself. (But see Exercise 4.3.5.)

Exercise 4.3.5. Redefine the concept of a tree's being homogeneous for T as follows: Replace $\langle \mathcal{U}_p \mid p \in T\rangle$ by $\langle U_p \mid p \in T'\rangle$, where T' is allowed to be an arbitrary subtree of T. Require that if $x \in [T]$ and $[U(x)] \neq \emptyset$ then $x \restriction n \in T'$ for every $n \in \omega$. Replace "T" by "T'" in clause (2) of the original definition.

Prove in ZFC that every closed set of $[T]$ is homogeneously Souslin in this modified sense. Prove that if a measurable cardinal exists then the same sets are homogeneously Souslin under the original and the modified definitions.

The modified definition is perhaps more natural, and it has other virtues. We do not adopt it as our official definition simply because it would make our notation more cumbersome.

4.4 Sharps and $\mathbf{\Pi}^1_1$ Determinacy

We have already seen in Exercise 4.1.1 that the determinacy of all Π^1_1 games is not provable from the ZFC axioms alone. On the other hand, Theorem 4.1.6 shows that it is provable if we adjoin the hypothesis that there is a measurable cardinal. That hypothesis is, however, stronger than necessary. When I told Robert Solovay the proof from a measurable cardinal, he observed that the proof needed only the existence of $0^{\#}$.[1]

Similarly the proof—which we will present in this section—of $\mathbf{\Pi}^1_1$ determinacy in a tree T goes through in ZFC plus the hypothesis that every subset of $|T|$ has a sharp. A theorem of Harrington shows that this hypothesis is optimal for countable T in that it follows from the determinacy of all $\mathbf{\Pi}^1_1$ games in ${}^{<\omega}\omega$. (See Exercise 4.4.1.) For $T = {}^{<\omega}\omega$, this equivalence is lightface. (See Theorem 4.4.3 and Exercise 4.4.1.)

We first prove a well-known lemma about the existence of definable strategies for open and closed games.

If T is a game tree, then let us say that a subset D of T *generates* an open subset A of $\lceil T \rceil$ if

$$A = \{x \in \lceil T \rceil \mid (\exists d \in D)\, d \subseteq x\}.$$

Lemma 4.4.1. *Let T be a game tree and let $A \subseteq \lceil T \rceil$. Let $D \subseteq T$ be such that D generates A or D generates $\neg A$. Let M be any transitive class model of* ZFC *such that $T \in M$ and $D \in M$. Let $\prec$ be any wellordering of* field (T) *that belongs to M. Then there is a strategy $\sigma \in M$ that is definable in M from T, D, and $\prec$ and is a winning strategy for $G(A;T)$. Moreover, $M \models$ "σ is a winning strategy for $G(\{x \in \lceil T \rceil \mid (\exists d \in D)\, d \subseteq x\}; T)$."*

Proof. We may assume that D generates A.

We now give what is essentially the construction of Exercise 1.2.4. For each ordinal number α, we define P_α, a set of positions of even length in T. The definition proceeds by transfinite induction on α. Let $p \in P_0$ if and only if $(\exists d \in D)\, d \subseteq p$. For $\alpha > 0$, let $p \in P_\alpha$ if and only if $p \in P_0$ or there is a Move q at p such that either (i) $q \in D \cap \lceil T \rceil$ or (ii) $q \notin \lceil T \rceil$ and, for every Move r at q, $r \in \bigcup_{\beta<\alpha} P_\beta$.

[1] I should have given credit for this to Solovay in [Martin, 1970].

If we turn this inductive definition into an explicit definition in the standard way, then it is absolute for M: For $\alpha \in \mathrm{Ord} \cap M$, the P_α defined in M is the same as that defined in V.

It is clear that

$$\beta < \alpha \to P_\beta \subseteq P_\alpha.$$

Since each $P_\alpha \subseteq T$, Comprehension in M gives that $\bigcup_{\alpha \in \mathrm{Ord} \cap M} P_\alpha \in M$. By Σ_1 Replacement in M, there must then be an ordinal $\gamma \in M$ such that $P_\gamma = P_{\gamma+1}$. From this it follows that $(\forall \alpha \geq \gamma)\, P_\alpha = P_\gamma$. Let $P_\infty = P_\gamma$.

Suppose first that $\emptyset \in P_\infty$. Define a strategy σ for I as follows: If $p \in P_\infty \setminus P_0$, Let a be the $\prec$-least element of the field of T such that either (i) $p^\frown\langle a \rangle \in D \cap \lceil T \rceil$ or (ii), for every Move r at $p^\frown\langle a \rangle$,

$$r \in P_\infty \;\wedge\; \mu\beta\,(r \in P_\beta) < \mu\beta\,(p \in P_\beta).$$

If p does not belong to $P_\infty \setminus P_0$, then let $\sigma(p)$ be the $\prec$-least element of field$\,(T)$. Evidently σ satisfies the definability condition. It is easy to show by induction that every position consistent with σ belongs to P_∞. To see that σ is a winning strategy for $G(A;T)$, let x be a play consistent with σ. For $n \in \omega$ and $2n \leq \ell\mathrm{h}(x)$, let

$$\beta_n = \mu\beta\,(x \restriction 2n \in P_\beta).$$

For each such n, it follows from the definitions that one of the following holds:

(a) $\beta_n = 0$ and so $(\exists m \leq 2n)\, x \restriction m \in D$;

(b) $\ell\mathrm{h}(x) = 2n+1$ and $x \in D$;

(c) $\ell\mathrm{h}(x) \geq 2n+2$ and $\beta_{n+1} < \beta_n$.

Since (c) cannot hold for every $n \in \omega$, there is an n for which (a) or (b) holds. Thus $x \in A$. Notice that the argument shows that $(\exists d \subseteq x)\, d \in D$. Hence in M the strategy σ is winning in the game $G(\{x \in \lceil T \rceil \mid (\exists d \in D)\, d \subseteq x\}; T)$.

Now suppose that $\emptyset \notin P_\infty$. If $p \in T \setminus P_\infty$ and if $\ell\mathrm{h}(p)$ is even, then for every Move q at p either $q \in \lceil T \rceil \setminus A$ or else there is an $a \in$ field$\,(T)$ such that $q^\frown\langle a \rangle \in T \setminus P_\infty$. Define a strategy σ for II by letting $\sigma(q)$ be the $\prec$-least a such that $q^\frown\langle a \rangle \in T \setminus P_\infty$ if such an a exists and 0 otherwise. It is easy to check that σ has the required properties. $\square$

Remark. The proof does not really require that M is a model of full ZFC.

Theorem 4.4.2. *Let λ be an infinite cardinal number. Assume that*

$$(\forall a \subseteq \lambda)\, a^{\#} \text{ exists.}$$

Then, for every game tree T such that $|T| \leq \lambda$, all $\mathbf{\Pi}^1_1$ games in T are determined.

Proof. Let T be a game tree with $|T| \leq \lambda$. Without loss of generality, assume that field$(T) \subseteq \lambda$ and that T has no terminal positions. Let $A \subseteq \lceil T \rceil = [T]$ with $A \in \mathbf{\Pi}^1_1$. Let $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.4.

Let T^* be defined exactly as in the proof of Theorem 4.1.6, but with $\kappa = \lambda^+$. Let $A^* \subseteq \lceil T^* \rceil$ be defined as in the proof of Theorem 4.1.6. As in that proof, A^* is closed and so $G(A; T^*)$ is determined.

The proof that if I has a winning strategy for $G(A^*; T^*)$ then I also has a winning strategy for $G(A; T)$ is exactly like the corresponding part of the proof of Theorem 4.1.6.

Suppose that $G(A^*; T^*)$ is a win for II.

Let $g : {}^{<\omega}\lambda \times \omega \times \omega \to \lambda$ be one-one and such that $g \in L$. Let

$$a = \{g(\langle p, m, n \rangle) \mid p \in T \wedge m <_p n\}.$$

Since

$$T = \{p \in {}^{<\omega}\lambda \mid g(p, 1, 0) \in a\},$$

we have that $T \in L[a]$ and that T is definable from a in $L[a]$. Since T^* is definable from T and λ^+ in any transitive class model of ZFC to which both T and λ^+ belong, it follows that T^* is definable from a and λ^+ in $L[a]$. Let D^* be the set of all $p^* \in T^*$ such that, for some n with $2n < \ell\text{h}(p^*)$, the function $i \mapsto \xi_i$ given by p^* does not embed $(n+1; <_{\pi(p^*)\restriction n+1})$ into $(\lambda^+; <)$. We also have that D^* is definable from a and λ^+ in $L[a]$. Let $\prec^*$ be the restriction to field(T^*) of the wellordering of Ord $\cup$ (Ord $\times$ Ord) which is the natural ordering of Ord followed by the lexicographic ordering of Ord $\times$ Ord. The relation $\prec^*$ is definable from a and λ^+ in $L[a]$.

Let τ^* be the σ given by Lemma 4.4.1 with T^* for T, A^* for A, $L[a]$ for M, D^* for D, and $\prec^*$ for $\prec$. Since $G(A^*; T^*)$ is a win for II, τ^* is a strategy for II. Thus τ^* is a winning strategy for II for $G(A^*; T^*)$, and τ^* is definable from a and λ^+ in $L[a]$.

Define the positions $q^*(p, v)$ as in the proof of Theorem 4.1.6. It is easy to see that the function q^* is definable from a and λ^+ in $L[a]$.

By the hypothesis of the theorem, $a^\#$ exists. Let C^a be the Silver class of indiscernibles for $L[a], a$. Let $\alpha \mapsto c^a_\alpha$ be the order-preserving bijection between Ord and C^a. It follows from (iii) of Lemma 3.4.18 that $c^a_{\lambda^+} = \lambda^+$.

Define a strategy τ for II in T as follows. For $n \in \omega$ and $p \in T$ with $\ell\text{h}(p) = 2n + 1$, let

$$\tau(p) = \tau^*(q^*(p, \{c^a_0, \dots, c^a_n\})).$$

By indiscernibility and the fact that range$(\tau^*) \subseteq \lambda$, we have that

$$(\forall v \in [C^a \cap \lambda^+]^{n+1})\, \tau(p) = \tau^*(q^*(p, v)).$$

To show that τ is a winning strategy for $G(A; T)$, let x be a play consistent with τ. Assume for a contradiction that $x \in A$. Then $<_x$ is a wellordering of ω. Let $i \mapsto \xi_i$ embed $(\omega; <_x)$ into $(C^a \cap \lambda^+; <)$. Let x^* be the play in T^* with these values of the ξ_i and with $\pi(x^*) = x$. As in the proof of Theorem 4.1.6, one can show that x^* is consistent with τ^*, contradicting the assumption that $x \in A$. □

Here is the lightface version of Theorem 4.4.2.

Theorem 4.4.3. *If* $0^\#$ *exists then all* Π^1_1 *games in* ${}^{<\omega}\omega$ *are determined.*

Proof. Let $A \subseteq {}^\omega\omega$ with $A \in \Pi^1_1$. Let $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.5. Proceed as in the proof of Theorem 4.4.2, with $T = {}^{<\omega}\omega$ and $\lambda = \omega$. The a we get is definable in L. Thus $L[a] = L$ and the Silver indiscernibles for L are the Silver indiscernibles for $L[a], a$. Using the existence of $0^\#$, we can then proceed as in the proof of Lemma 4.4.2. □

Exercise 4.4.1. Show that the determinacy of all Π^1_1 games in ${}^{<\omega}\omega$ implies that $0^\#$ exists. From this, the main result of [Harrington, 1978], and from Theorem 4.4.3, it follows that Π^1_1 determinacy is equivalent with the existence of $0^\#$.

Hint. First show that the existence of $0^\#$ follows from the existence of an $a \in {}^\omega\omega$ such that every a-admissible ordinal is a cardinal in L. (This result is due to Jack Silver, but the proof we now sketch is due to J.B. Paris.)

Let $a \in {}^\omega\omega$ be such that every a-admissible is a cardinal in L. Work in $L[a]$. Let $X \prec L_{\omega_3}[a]$ with $|X| = \aleph_1$ and ${}^\omega X \subseteq X$. Let $\pi : X \cong L_\alpha[a]$. Note that α is a-admissible.

Let $j = \pi^{-1} : L_\alpha[a] \prec L_{\omega_3}[a]$. Let $\gamma = \text{crit}(j)$. Let

$$\mathcal{U} = \{Y \in L_\alpha[a] \mid Y \subseteq \gamma \wedge \gamma \in j(Y)\}.$$

Show that $\mathcal{U}$ is a uniform normal L-ultrafilter on γ: i.e., that $\mathcal{U}$ is a filter on γ, that every subset of γ in L belongs to $\mathcal{U}$ or else its complement does, that for all $f : \gamma \to \mathcal{P}(\gamma)$ with $f \in L$ the set $\{\beta < \gamma \mid f(\beta) \in \mathcal{U}\}$ belongs to L, and that $\mathcal{U}$ is uniform and normal in the obvious senses.

Prove that Rowbottom's result, Lemma 3.1.8, holds for $\mathcal{U}$ in the following sense. If $n \in \omega$ and $Z \in L$ is a subset of $[\gamma]^n$, then there is a $Y \in \mathcal{U} \cap L$ such that either $[Y]^n \subseteq Z$ or $[Y]^n \cap Z = \emptyset$. Use this fact and the countable closure of X to get a set of indiscernibles for L of size γ.

Remark. The notion of an L-ultrafilter is from [Kunen, 1968]. There Kunen proves that the existence of $0^\#$ follows from the existence of an elementary embedding $j : L \prec L$. Kunen's proof begins by using j to define an L-ultrafilter $\mathcal{U}$. But it then proceeds by forming iterated iterated ultrapowers $\text{Ult}_\alpha(L;\mathcal{U})$, showing that they are all wellfounded, and showing that $\{i_{\mathcal{U}0,\beta}(\gamma) \mid \beta \in \text{Ord}\}$ is a closed unbounded class of indiscernibles for L. Kunen's method also works here, and it could replace the argument suggested in the preceding paragraph.

Now consider the following game G in ${}^{<\omega}\omega$. For each play of G let I's part of the play code a relation R in ω and let II's part code a relation E in ω. If R is not a wellordering of ω, then I loses. If R is a wellordering of ω, let β be its order type. Then II wins if and only if $(\omega; E)$ is a model of Extensionality and there is a

$$g : L_\beta \to \omega$$

that embeds $(L_\beta; \in)$ into $(\omega; E)$ as an initial segment, i.e. such that

(a) $(\forall u \in L_\beta)(\forall v \in L_\beta)(u \in v \leftrightarrow g(u)\, E\, g(v))$;

(b) $(\forall u \in L_\beta)(\forall m\, E\, g(u))(\exists v \in L_\beta)\, m = g(v)$.

Note that g is unique if it exists.

Show that G is Π^1_1. Assume that σ is a winning strategy for I for G. Show that there is a countable ordinal γ such that the β given by any play consistent with σ is smaller than γ. Use this fact to get a contradiction.

Assume Π^1_1 determinacy, getting that G is a win for II. Let τ be a winning strategy for II for G. Let $a \in {}^\omega\omega$ code τ. To show that $0^\#$ exists, it is enough to prove that every a-admissible ordinal is a cardinal in L. By absoluteness under the collapse of cardinals, it is enough to prove that every countable a-admissible ordinal is a cardinal in L.

Suppose that $\gamma < \beta < \omega_1$, that b is a subset of L_γ belonging to L_β, and that z is a play consistent with τ whose associated R is a wellordering of ω of order type β. Show that $b \in L_{\gamma+\omega}[z]$. To do this, first let g witness that z is a win for II and prove that $g \restriction L_\gamma \in L_{\gamma+\omega}$.

For each ordinal α, let $(\mathbf{Q}(\alpha); \leq_\alpha)$ be the following partial ordering: The members of $\mathbf{Q}(\alpha)$ are those pairs $\langle t, h\rangle$ such that

(i) t is a finite tree on ω;

(ii) $h : t \to \omega\alpha \cup \{\infty\}$;

(iii) $h(\emptyset) = \infty$;

(iv) $(\forall r \in t)(\forall s \in t)((r \subsetneq s \wedge h(r) \neq \infty) \to h(s) < h(r))$.

Let

$$\langle t, h\rangle \leq_\alpha \langle t', h'\rangle \;\leftrightarrow\; (t' \subseteq t \wedge h \restriction t' = h').$$

Show that if $\mathbf{G}$ is sufficiently $\mathbf{Q}(\alpha)$-generic and T and H are respectively the union of all the first components of elements $\mathbf{G}$ and the union of all the second components of elements of $\mathbf{G}$, then (1) T is a tree on ω, (2) H is a surjection from T onto $\omega\alpha \cup \{\infty\}$, and (3) $(\forall s \in T)\, H(s) = \|s\|^T$. (Here $\|s\|^T$ is $\|T_s\|$ if T_s is wellfounded and is ∞ otherwise.)

If $p = \langle t, h\rangle \in \mathbf{Q}(\alpha)$ and $\xi < \alpha$, define $p(\xi) \in \mathbf{Q}(\xi)$ by $p(\xi) = \langle t, h'\rangle$, where

$$h'(s) = \begin{cases} h(s) & \text{if } h(s) < \omega\xi; \\ \infty & \text{if } h(s) \geq \omega\xi \end{cases}$$

and where we consider $\infty > \beta$ for every ordinal β.

Let $p \in \mathbf{Q}(\alpha)$, $p' \in \mathbf{Q}(\alpha')$, and $\xi \leq \min\{\alpha, \alpha'\}$. Suppose that $p(\xi+1) = p'(\xi+1)$. Prove that

$$(*) \qquad (\forall q \leq_\alpha p)(\exists q' \leq_{\alpha'} p')\, q(\xi) = q'(\xi).$$

Define a class $\mathcal{S}$, the class of *ranked sentences*, and an ordinal *rank* of each element of $\mathcal{S}$ as follows:

(a) If $s \in {}^{<\omega}\omega$, then $\boldsymbol{s} \in \boldsymbol{T}$ is a ranked sentence of rank 1.

(b) If $S \subseteq \mathcal{S}$ then $\bigwedge S \in \mathcal{S}$ and

$$\mathrm{rank}(\bigwedge S) = \sup\{\mathrm{rank}(\varphi) + 1 \mid \varphi \in S\}.$$

(c) If $\varphi \in \mathcal{S}$, then $\neg\varphi \in \mathcal{S}$ and $\mathrm{rank}(\neg\varphi) = \mathrm{rank}(\varphi) + 1$.

For any tree T on ω, each member of $\mathcal{S}$ has an obvious interpretation.

Define a *forcing* relation $\Vdash_\alpha$ between elements p of $\mathbf{Q}(\alpha)$ and sentences $\varphi \in \mathcal{S}$ inductively as follows:

(a) $p \Vdash_\alpha \boldsymbol{s} \in \boldsymbol{T}$ if and only if $p = \langle t, h\rangle$ and

$$s \in t \ \vee \ (\exists r \subseteq s)(\ell\mathrm{h}(r) + 1 = \ell\mathrm{h}(s) \ \wedge \ h(r) \neq 0).$$

(b) $p \Vdash_\alpha \bigwedge S$ if and only if $(\forall \varphi \in S)\, p \Vdash_\alpha \varphi$.

(c) $p \Vdash_\alpha \neg\varphi$ if and only if $(\forall q \leq_\alpha p)\, q \nVdash_\alpha \varphi$.

Prove that if $\xi \leq \alpha$, $\xi \leq \alpha'$, $p \in \mathbf{Q}_\alpha$, $p' \in \mathbf{Q}_{\alpha'}$, and φ is a sentence of rank ξ, then

$$(\dagger) \qquad p(\xi) = p'(\xi) \ \rightarrow \ (p \Vdash_\alpha \varphi \leftrightarrow p' \Vdash_{\alpha'} \varphi).$$

Proceed by induction on ξ, using $(*)$. (This result is from [Steel, 1976], where the partial orderings $\mathbf{Q}(\alpha)$ are introduced.)

Let $\alpha < \omega_1$ be a-admissible. Assume for a contradiction that α is not a cardinal in L. Then there are ordinals $\gamma < \alpha$ and $\beta < \omega_1$ and there is a set $b \in L_\beta$ such that $b \subseteq \gamma$ and b codes a wellordering of γ of order type α. Let $\mathbf{G}$ be $\mathbf{Q}(\beta+1)$-generic over $L_{\omega\beta+\omega}[a]$. Let $\langle T, H\rangle$ be given by $\mathbf{G}$. There is an $s \in T$ such that $\|s\|^T = \beta$. Hence there is an $x \in {}^\omega\omega$ such that x is recursive in T and x codes a wellordering of ω of order type β. Let z be the play of G consistent with τ in which I plays x. Then $b \in L_{\gamma+\omega}[z]$ and so $b \in L_{\gamma+\omega}[a, T]$. Prove that, for some $n \in \omega$, there is in $L_{\gamma+\omega}[a]$ a function that associates with each $\delta < \gamma$ a ranked sentence which we call $\boldsymbol{\delta} \in \boldsymbol{b}$ such that

(i) $\mathrm{rank}(\boldsymbol{\delta} \in \boldsymbol{b}) < \omega(\gamma + n)$;

(ii) $\boldsymbol{\delta} \in \boldsymbol{b}$ is true for T if and only if $\delta \in b$.

To get the sentence $\boldsymbol{\delta} \in \boldsymbol{b}$, let $n > 0$ be such that $b \in L_{\gamma+n}[a, T]$. There is a formula $\psi(v)$ with parameters from $L_{\gamma+n-1}[a]$ that defines b over $L_{\gamma+n-1}[a, T]$. Show that for $\delta < \gamma$ there is a ranked formula of rank $< \omega(\gamma + n)$ that is true for any T' if and only if $L_{\gamma+n-1}[a, T'] \models \psi[\delta]$. Consider the ranked sentence φ:

$$\bigwedge(\{\boldsymbol{\delta} \in \boldsymbol{b} \mid \delta \in \gamma \cap b\} \cup \{\neg\,\boldsymbol{\delta} \in \boldsymbol{b} \mid \delta \in \gamma \setminus b\}).$$

The sentence φ belongs to $L_{\beta+\omega}[a]$ and has some rank $\xi < \omega(\gamma+\omega)$. Since φ is true for T, there is some $p \in \mathbf{G}$ such that $p \Vdash_{\beta+1} \varphi$. By (†) it follows that $p(\xi) \Vdash_\xi \varphi$. Hence $p(\xi) \Vdash_\xi \boldsymbol{\delta} \in \boldsymbol{b}$ if $\delta \in b$ and $p(\xi) \Vdash_\xi \neg\,\boldsymbol{\delta} \in \boldsymbol{b}$ if $\delta \notin b$ Since $\{\boldsymbol{\delta} \in \boldsymbol{b} \mid \delta \in \gamma\}$ belongs to $L_{\gamma+\omega}[a]$, it follows that $b \in L_{\gamma+\omega}[a]$. But b codes a wellordering of γ of order type α, and so this contradicts the a-admissiblity of α.

Exercise 4.4.2. *If every $\mathbf{\Pi}^1_1$ game in ${}^\omega\omega$ is determined, then $a^\#$ exists for every $a \in {}^\omega\omega$.*

The proof is like that of Exercise 4.4.1. The conclusion of the theorem—and so its hypothesis—implies that every countable subset of ω_1 has a sharp. I don't know whether the converse of Theorem 4.4.2 is true.

Chapter 5

α–$\mathbf{\Pi}^1_1$ Games

Let T be a game tree and let α be a countable ordinal. Recall that a subset A of $\lceil T \rceil$ belongs to α–$\mathbf{\Pi}^1_1$, the αth level of the difference hierarchy on $\mathbf{\Pi}^1_1$, if there exists $\langle A_\beta \mid \beta < \alpha \rangle$ such that each $A_\beta \subseteq \lceil T \rceil$ and such that

(1) each $A_\beta \in \mathbf{\Pi}^1_1$;

(2) $(\forall x \in \lceil T \rceil)(x \in A \leftrightarrow \mu\beta(x \notin A_\beta \vee \beta = \alpha)$ is odd).

Recall also that $\mathbf{Diff}(\mathbf{\Pi}^1_1) = \bigcup_{\alpha<\omega_1} \alpha$–$\mathbf{\Pi}^1_1$. These definitions make sense for arbitrary topological spaces in place of $\lceil T \rceil$.

In this chapter we aim to deduce α–$\mathbf{\Pi}^1_1$ determinacy from the weakest possible large cardinal hypotheses. Ultimately we will succeed in this. We will get implications whose converses are also theorems of ZFC.

Because such optimal results require the technical concepts of sharps and iterated ultrapowers, we will begin with stronger hypotheses than we need. In §5.1 we deduce $(\omega\alpha)$–$\mathbf{\Pi}^1_1$ determinacy for games in T from the existence of (an increasing sequence of) α measurable cardinals larger that $|T|$. This proof makes use of the concepts and results of §4.2. It needs no material on measurable cardinals beyond what is found in §3.1.

In §5.2 we use iterated ultrapowers and $\mathbf{\Delta}^1_1$ determinacy (Theorem 2.2.8) to prove the determinacy of all games $G(A;T)$ such that both A and $\neg A$ are $(\omega^2\alpha + 1)$–$\mathbf{\Pi}^1_1$ from the existence of α measurable cardinals larger than $|T|$. A little use is made in §5.2 of the concept of relative constructibility that we introduced in §3.4.

In §5.3 we prove the determinacy of α–$\mathbf{\Pi}^1_1$ games in T for all $\alpha < \omega^2$ from the existence of $a^\#$ for all subsets a of $|T|$. We then combine this proof with the methods of §5.2 to get that

$$(\forall\beta < \omega^2(\alpha+1)) \text{ all } \beta\text{–}\mathbf{\Pi}^1_1 \text{ games in } T \text{ are determined}$$

from the existence, for each $b \subseteq |T|$, of indiscernibles for an inner model M such that $b \in M$ and such that $M \models$ "there are α measurable cardinals larger than $|T|$." The theorems of §5.3, including the lightface versions of those just mentioned, are optimal: the converses also hold. (See Exercises 5.3.4 and 5.3.5.) All the results in the text of §5.1–§5.3 are due to the author. All except Theorems 5.2.31 and 5.2.32 were proved during the 1970's. Several of the exercises of §5.3 are concerned with work of Derrick DuBose giving determinacy equivalents of large cardinal hypotheses stronger than the existence of $0^\#$ and weaker than the existence of a measurable cardinal.

In §5.4 we derive from large cardinal hypotheses the determinacy of games in $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ (the class of countable unions of Boolean combinations of $\mathbf{\Pi}^1_1$ sets) and a little more. In Exercise 5.4.2, this is extended to games in $\mathbf{Diff}(\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1))$, where $\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1)$ is the dual class of $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$. The large cardinals involved are those in the hierarchy of generated from measurable cardinals by the operation of taking measurable limits. These results, due to John Simms, are mostly known to be optimal. For converses, due mainly to Simms and John Steel, see Exercises 5.4.5, 5.4.3, and 5.4.4.

5.1 α–$\mathbf{\Pi}^1_1$ Determinacy

In this section we will use Lemma 4.2.2 to construct semicoverings unraveling sets belonging to $\mathbf{Diff}(\mathbf{\Pi}^1_1)$. To do this we first give some operations on semicoverings analogous to the operations on coverings introduced in §2.1.

Suppose that $\mathcal{C}_1 = \langle \mathbf{T}_1, \pi_1, \phi_1, \Psi_1 \rangle$ is a semicovering of $\mathbf{T}_0$ and that $\mathcal{C}_2 = \langle \mathbf{T}_2, \pi_2, \phi_2, \Psi_2, \rangle$ is a semicovering of $\mathbf{T}_1$. We define the *composition* $\mathcal{C}_1 \circ \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ to be

$$\langle \mathbf{T}_2, \pi_1 \circ \pi_2, \phi_1 \circ \phi_2, \Psi \rangle,$$

where $\Psi(\sigma, x) = \Psi_2(\sigma, \Psi_1(\phi_2(\sigma), x))$. (This definition implicitly determines domain(Ψ).) Here is an analogue of Lemma 2.1.5:

Lemma 5.1.1. *Let $\mathbf{T}_0$ be a game tree with taboos. Let A, B_0, and B_1 be subsets of $\lceil \mathbf{T}_0 \rceil$. Let $\mathcal{C}_1 = \langle \mathbf{T}_1, \pi_1, \phi_1, \Psi_1 \rangle$ be an (A, B_0)–semicovering of $\mathbf{T}_0$ and let $\mathcal{C}_2 = \langle \mathbf{T}_2, \pi_2, \phi_2, \Psi_2, \rangle$ be a $(\boldsymbol{\pi}_1^{-1}(A), \boldsymbol{\pi}_1^{-1}(B_1))$–semicovering of $\mathbf{T}_1$.*

Then $\mathcal{C}_1 \circ \mathcal{C}_2$ is an $(A, B_0 \cap B_1)$ semicovering of $\mathbf{T_0}$. If k_1 and k_2 are natural numbers and $\mathcal{C}_1$ and $\mathcal{C}_2$ are k_1- and k_2-semicoverings respectively, then $\mathcal{C}_1 \circ \mathcal{C}_2$ is a $\min\{k_1, k_2\}$-semicovering.

Proof. The proof is routine, so we omit it, except for the verification that $\mathcal{C}_1 \circ \mathcal{C}_2$ is an A semicovering. For that verification, let $\sigma \in \mathcal{S}(T_2)$ and let $x \in \lceil T_0 \rceil$ witness that $\phi_1(\phi_2(\sigma))$ is not a winning strategy for $G(A; \mathbf{T}_0)$. Then $\Psi_1(\phi_2(\sigma), x)$ is defined, and clause (d) in the definition of a semicovering implies that $\Psi_1(\phi_2(\sigma), x)$ witnesses that $\phi_2(\sigma)$ is not a winning strategy for $G(\boldsymbol{\pi}_1^{-1}(A); \mathbf{T}_1)$. Thus $\Psi_2(\sigma, \Psi_1(\phi_2(\sigma), x))$ is defined, as required. $\square$

The next lemma is the analogue of Lemma 2.1.6.

Lemma 5.1.2. *Let $k \in \omega$. Let $\mathbf{T}_i$, $i \in \omega$, be game trees with taboos. Let A and B_i, $i \in \omega$, be subsets of $\lceil \mathbf{T}_0 \rceil$. Let $\langle k_{j,i}, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \mid i \leq j \in \omega \rangle$ be such that*

(1) *if $i \leq j \in \omega$ then $\mathcal{C}_{j,i} = \langle \mathbf{T}_j, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \rangle$ is a*

$$(\boldsymbol{\pi}_{0,i}{}^{-1}(A), \boldsymbol{\pi}_{0,i}{}^{-1}(\bigcap_{i \leq n < j} (B_n))$$

$k_{j,i}$-semicovering of $\mathbf{T}_i$;

(2) *if $i_1 \leq i_2 \leq i_3 \in \omega$ then $\mathcal{C}_{i_3,i_1} = \mathcal{C}_{i_2,i_1} \circ \mathcal{C}_{i_3,i_2}$;*

(3) $\inf_{i \leq j \in \omega} k_{j,i} \geq k$;

(4) $\underline{\lim}_{j \in \omega} \inf_{j' \geq j} k_{j',j} = \infty$; *i.e., for all $n \in \omega$ there is an $i \in \omega$ such that $k_{j',j} \geq n$ for all $j' \geq j \geq i$.*

Then there is a $\mathbf{T}_\infty$ with $|T_\infty| \leq \sum_{i \in \omega} |T_i|$ and there is a system

$$\langle \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \mid i \in \omega \rangle$$

such that each $\mathcal{C}_{\infty,i} = \langle \mathbf{T}_\infty, \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \rangle$ is a

$$(\boldsymbol{\pi}_{\infty,i}{}^{-1}(A), \boldsymbol{\pi}_{\infty,i}{}^{-1}(\bigcap_{i \leq n \in \omega} B_n))$$

k-semicovering of $\mathbf{T}_i$ and such that, for $i \leq j \in \omega$, $\mathcal{C}_{\infty,i} = \mathcal{C}_{j,i} \circ \mathcal{C}_{\infty,j}$.

Proof. The proof is similar to that of Lemma 2.1.6, and we omit it. $\square$

We begin for simplicity's sake with an application of Lemmas 4.2.2, 5.1.1, and 5.1.2, getting (by an appeal to Lemma 4.2.1) a proof of α–$\mathbf{\Pi}^1_1$ determinacy for games in $\mathbf{T}$ from the existence of α measurable cardinals larger than $|T|$. Later in this section, we will use Lemma 4.2.3 to improve the α of the conclusion to $\omega\alpha$. In §5.2 we will use more sophisticated methods to get still another factor of ω.

We first document a simple but useful fact about (A, B) semicoverings and α–$\mathbf{\Pi}^1_1$ sets.

Lemma 5.1.3. *Let $\mathbf{T}$ be a game tree with taboos. Let α be a countable ordinal and let $\langle A_\beta \mid \beta < \alpha\rangle$ witness that $A \in \alpha$–$\mathbf{\Pi}^1_1$. Let $\gamma < \alpha$ and let A' be the set witnessed to belong to γ–$\mathbf{\Pi}^1_1$ by $\langle A_\beta \mid \beta < \gamma\rangle$.*

Then every $(A', \bigcap_{\beta<\gamma} A_\beta)$ semicovering of $\mathbf{T}$ is an $(A, \bigcap_{\beta<\gamma} A_\beta)$ semicovering of $\mathbf{T}$.

Proof. Let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be an $(A', \bigcap_{\beta<\gamma} A_\beta)$ semicovering of $\mathbf{T}$. Let $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ and let x be a play in T consistent with $\phi(\tilde{\sigma})$ such that x is a loss in $G(A; \mathbf{T})$ for the player for whom $\tilde{\sigma}$ is a strategy. We must show that $\langle \tilde{\sigma}, x\rangle \in \text{domain}\,(\Psi)$. If $x \notin \bigcap_{\beta<\gamma} A_\beta$, then x is a loss for the same player in $G(A'; \mathbf{T})$, and so the fact that $\mathcal{C}$ is an A' semicovering gives that $\langle \tilde{\sigma}, x\rangle \in \text{domain}\,(\Psi)$. If $x \in \bigcap_{\beta<\gamma} A_\beta$, then the fact that $\mathcal{C}$ is an $(A', \bigcap_{\beta<\gamma} A_\beta)$ semicovering gives directly that $\langle \tilde{\sigma}, x\rangle \in \text{domain}\,(\Psi)$. $\square$

Theorem 5.1.4. *Let α be a countable ordinal ≥ 1. Let $\mathbf{T}$ be a game tree with taboos and let $\langle \kappa_\beta \mid \beta < \alpha\rangle$ be a strictly increasing sequence of measurable cardinals with $\kappa_0 > |T|$. Let $\langle A_\beta \mid \beta < \alpha\rangle$ witness that $A \subseteq \lceil \mathbf{T} \rceil$ belongs to α–$\mathbf{\Pi}^1_1$. Let $k \in \omega$. Then there is an $(A, \bigcap_{\beta<\alpha} A_\beta)$ k-semicovering $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ of $\mathbf{T}$ such that $|\tilde{T}| \leq \sup_{\beta<\alpha} \kappa_\beta$.*

Proof. We proceed by induction on α. Assume then that the lemma holds for every β with $0 < \beta < \alpha$, for all trees, all sequences of measurable cardinals, all sequences of $\mathbf{\Pi}^1_1$ sets, and all natural numbers.

Suppose first that $\alpha = \gamma + 1$. We consider only the case that γ is even. The other case is similar, with the roles of the two players reversed and with the appeal to Lemma 4.2.2 (i) replaced by an appeal to Lemma 4.2.2 (ii).

If $\gamma > 0$, we apply our induction hypothesis to $\mathbf{T}$, to $\langle \kappa_\beta \mid \beta < \gamma \rangle$, to $\langle A_\beta \mid \beta < \gamma \rangle$, to the associated γ–$\mathbf{\Pi}^1_1$ set A', and to k, getting

$$\mathcal{C}' = \langle \mathbf{T}', \pi', \phi', \Psi' \rangle.$$

If $\gamma = 0$, let $\mathcal{C}' = \langle \mathbf{T}', \pi', \phi', \Psi' \rangle$ be the trivial covering of $\mathbf{T}$. If $\gamma > 0$ then $\mathcal{C}'$ is an $(A', \bigcap_{\beta<\gamma} A_\beta)$ k-semicovering of $\mathbf{T}$ such that $|T'| \leq \sup_{\beta<\gamma} \kappa_\beta < \kappa_\gamma$. Whether or not $\gamma > 0$, we have that $|T'| < \kappa_\gamma$. By Lemma 5.1.3, we get that $\mathcal{C}'$ is an $(A, \bigcap_{\beta<\gamma} A_\beta)$ semicovering of $\mathbf{T}$.

We next apply Lemma 4.2.2 (i) with $\mathbf{T}'$ as the $\mathbf{T}$ of that lemma, with $\boldsymbol{\pi}'^{-1}(A_\gamma)$ as the B, with κ_γ as the κ, and with k as the k. Let $\mathcal{C}^* = \langle \mathbf{T}^*, \pi^*, \phi^*, \Psi^* \rangle$ be the $(\boldsymbol{\pi}'^{-1}(A_\gamma), \boldsymbol{\pi}'^{-1}(A_\gamma))$ k-semicovering of $\mathbf{T}'$ given by that lemma. Since every normal play in $\mathbf{T}'$ belongs to $\boldsymbol{\pi}^{-1}(\bigcap_{\beta<\gamma} A_\beta)$, we get that $\boldsymbol{\pi}'^{-1}(A) = \boldsymbol{\pi}'^{-1}(A_\gamma)$. Thus $\mathcal{C}^*$ is also a $(\boldsymbol{\pi}'^{-1}(A), \boldsymbol{\pi}'^{-1}(A_\gamma))$ k-semicovering of $\mathbf{T}'$. By Lemma 5.1.1, we have that $\mathcal{C} = \mathcal{C}' \circ \mathcal{C}^*$ is an $(A, \bigcap_{\beta<\alpha} A_\beta)$ k-semicovering of $\mathbf{T}$. Since $|T^*| \leq \kappa_\gamma$, $\mathcal{C}$ satisfies the conditions of the lemma.

Now suppose that α is a limit ordinal. Let $\langle \xi_i \mid i \in \omega \rangle$ be an increasing sequence of nonzero ordinals such that $\sup_{i\in\omega}\xi_i = \alpha$.

We are going to do a construction analogous to the one that occurs in the proof of Lemma 2.1.8. Let $\mathbf{T}_0 = \mathbf{T}$. By induction on $j' \in \omega$, we define $\mathbf{T}_{j'}$ and $\mathcal{C}_{j',j} = \langle \mathbf{T}_{j'}, \pi_{j',j}, \phi_{j',j}, \Psi^{j,j'} \rangle$ for $j \leq j'$ such that $\mathcal{C}_{j',i} = \mathcal{C}_{j,i} \circ \mathcal{C}_{j',j}$ for all $i \leq j \leq j'$, such that each $\mathcal{C}_{j',j}$ is a $({\boldsymbol{\pi}_{j,0}}^{-1}(A), {\boldsymbol{\pi}_{j,0}}^{-1}(\bigcap_{\xi_j \leq \beta < \xi_{j'}} A_\beta))$ $(k+j)$-semicovering of $\mathbf{T}_j$, such that $|T_{j'}| \leq \sup_{\beta<\xi_{j'}}\kappa_\beta$ for $j' > 0$, and such that $\mathcal{C}_{j',j'}$ is the trivial covering.

Suppose that we have defined $\mathbf{T}_{j'}$ and the $\mathcal{C}_{j',j}$ for all $j' \leq n$. For $\gamma < \alpha$ let A'_γ be the subset of $\lceil \mathbf{T} \rceil$ witnessed to be γ–$\mathbf{\Pi}^1_1$ by $\langle A_\beta \mid \beta < \gamma \rangle$. By our induction hypothesis, let $\mathcal{C}_n = \langle \mathbf{T}^*, \pi^*, \phi^*, \Psi^* \rangle$ be a $({\boldsymbol{\pi}_{n,0}}^{-1}(A'_{\xi_{n+1}}), {\boldsymbol{\pi}_{n,0}}^{-1}(\bigcap_{\xi_n \leq \beta < \xi_{n+1}} A_\beta))$ $(k+n)$-semicovering of $\mathbf{T}_n$ such that $|T^*| \leq \sup_{\beta<\xi_{n+1}}\kappa_\beta$. By Lemma 5.1.3, we have that $\mathcal{C}_n$ is a $({\boldsymbol{\pi}_{n,0}}^{-1}(A), {\boldsymbol{\pi}_{n,0}}^{-1}(\bigcap_{\xi_n \leq \beta < \xi_{n+1}} A_\beta))$ $(k+n)$-semicovering of $\mathbf{T}_n$. For $j \leq n$, let $\mathcal{C}_{n+1,j} = \mathcal{C}_{n,j} \circ \mathcal{C}_n$; let $\mathcal{C}_{n+1,n+1}$ be the trivial covering. The required properties of the $\mathcal{C}_{n+1,j}$ follow from Lemma 5.1.1.

If we let $k_{j,i} = k + i$, then the hypotheses of Lemma 5.1.2 hold. Let $\mathbf{T}_\infty$ and, for $i \in \omega$, $\mathcal{C}_{\infty,i} = \langle \mathbf{T}_\infty, \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \rangle$ be as given by Lemma 5.1.2. If $\mathcal{C} = \mathcal{C}_{\infty,0}$, then $\mathcal{C}$ is satisfies the conditions of the present lemma. □

Corollary 5.1.5. *Let $\mathbf{T}$ be a game tree with taboos. Let α be a nonzero countable ordinal. If the class of measurable cardinals larger than $|T|$ has* *order type at least* α*, then every* α–$\mathbf{\Pi}^1_1$ *subset* A *of* $\lceil\mathbf{T}\rceil$ *is unraveled by an* A*-semicovering of* $\mathbf{T}$*.*

Proof. Assume that $\langle\kappa_\beta \mid \beta<\alpha\rangle$ is a strictly increasing sequence of measurable cardinals with $\kappa_0 > |T|$. Let $A\subseteq\lceil\mathbf{T}\rceil$ and let $\langle A_\beta \mid \beta<\alpha\rangle$ witness that $A\in\alpha$–$\mathbf{\Pi}^1_1$. Let $\mathcal{C}=\langle\tilde{\mathbf{T}},\pi,\phi,\Psi\rangle$ be as given by Theorem 5.1.4. We need only show that $\mathcal{C}$ unravels A. But $\boldsymbol{\pi}(\lceil\tilde{\mathbf{T}}\rceil)\subseteq\bigcap_{\beta<\alpha}A_\beta$, and this set either is contained in A or is disjoint from A, depending on whether α is odd or even. Thus $\boldsymbol{\pi}^{-1}(A)$ either is $\lceil\tilde{\mathbf{T}}\rceil$ or else is empty. $\square$

Corollary 5.1.6. *Let* $\mathbf{T}$ *be a game tree with taboos. If* $0<\alpha<\omega_1$ *and if the class of measurable cardinals larger than* $|T|$ *has order type* $\geq\alpha$*, then all* α–$\mathbf{\Pi}^1_1$ *games in* $\mathbf{T}$ *are determined.*

Proof. The corollary is an immediate consequence of Corollary 5.1.5 and Lemma 4.2.1. $\square$

Our original proof of Corollary 5.1.6 was somewhat different from the one given here. For each α, we directly constructed the $\tilde{T}$ of what we now call the $(A,\bigcap_{\beta<\alpha}A_\beta)$ semicovering of Theorem 5.1.4, and we used the determinacy of what we now call $G(\boldsymbol{\pi}^{-1}(A);\tilde{\mathbf{T}})$ to prove the determinacy of $G(A;T)$. After we rearranged in terms of coverings our original proof (Martin [1975]) of Borel determinacy, we noticed that our proof of α–$\mathbf{\Pi}^1_1$ determinacy could be similarly rearranged. The new way of presenting the proof has a number of advantages, but it has the disadvantage that the structure of games in $\tilde{\mathbf{T}}$ is not directly exhibited. In the next paragraph, we try to make up for this omission by describing the $\tilde{\mathbf{T}}$ given by the proof of Theorem 5.1.4.

In $\tilde{T}$ the players—in addition to making their moves a_i in T—are choosing ordinals $\xi^\beta_i<\kappa_\beta$ for $\beta<\alpha$. I chooses the ξ^β_i for even β; II chooses the ξ^β_i for odd β. The purpose of the ξ^β_i is to verify that $x=\langle a_i \mid i\in\omega\rangle$ belongs to A by making $i\mapsto\xi^\beta_i$ embed the ordering $(\omega;<^\beta_x)$ into $(\kappa_\beta;<)$. If one of the players is unable to do this, then the play is taboo for the first player who reaches a position where it is impossible to to choose a required ξ^β_i so as to extend the embedding for β. If no one loses in this way and no taboo position $\langle a_i \mid i<n\rangle$ in $\mathbf{T}$ is reached, then I wins the game $G(\boldsymbol{\pi}^{-1}(A);\tilde{\mathbf{T}})$ if and only if α is odd.

We close this section by using Lemma 4.2.3 to improve the conclusion of Theorem 5.1.4 by a factor of ω. What Lemma 4.2.3 allows us to do is to reuse infinitely often a single measurable cardinal to unravel $\mathbf{\Pi}^1_1$ sets.

Theorem 5.1.7. *Let α be a countable ordinal ≥ 1. Let $\mathbf{T}$ be a game tree with taboos and let $\langle \kappa_\beta \mid \beta < \alpha \rangle$ be a strictly increasing sequence of measurable cardinals with $\kappa_0 > |T|$. Let $1 \leq \delta \leq \omega\alpha$. Let $\langle A_\beta \mid \beta < \delta \rangle$ witness that $A \subseteq \lceil \mathbf{T} \rceil$ belongs to δ–$\mathbf{\Pi}^1_1$. Let $k \in \omega$. Then there is an $(A, \bigcap_{\beta<\delta} A_\beta)$ k-semicovering $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ such that*

(a) if $\delta = \omega\gamma$, then $|\tilde{T}| \leq \sup_{\beta<\gamma} \kappa_\beta$;

(b) if $n \in \omega$ and $\delta = \omega\gamma + n$, then $|\tilde{T}| \leq \kappa_\gamma$ and

$$(\exists \tilde{m} \in \omega)(\forall \tilde{p} \in \tilde{T})(\ell\mathrm{h}(\tilde{p}) > \tilde{m} \;\rightarrow\; |\tilde{T}_{\tilde{p}}| < \kappa_\gamma).$$

Proof. The proof is like that of Theorem 5.1.4. An induction on δ replaces the induction on α in the earlier proof, and the appeal to Lemma 4.2.2 in the successor case of the earlier proof is replaced by an appeal to Lemma 4.2.3. We omit the details. $\square$

The $\tilde{\mathbf{T}}$ given by the proof of Theorem 5.1.7 differs from the corresponding tree (described above) given by the proof of Theorem 5.1.4 in that the ordinals $\xi_i^{\omega\gamma+n}$ are all chosen from the same measurable cardinal κ_γ. The trick that makes this possible is embedded in the proof of Lemma 4.2.3, but we can say how it works as follows: For $n < n' \in \omega$, the ordinal $\xi_0^{\omega\gamma+n}$ is chosen before any of the ordinals $\xi_i^{\omega\gamma+n'}$. This means that, for plays consistent with some strategy for the player choosing the $\xi_i^{\omega\gamma+n}$, all moves $\xi_i^{\omega\gamma+n}$ that depend on opponent's moves $\xi_j^{\omega\gamma+n'}$ are bounded by the ordinal $\xi_0^{\omega\gamma+n} < \kappa_\gamma$. We call this trick *the ordering trick.* Doing Exercise 5.1.2 is a good way to see how the ordering trick works.

Corollary 5.1.8. *Let $\mathbf{T}$ be a game tree with taboos. Let α be a nonzero countable ordinal. If the class of measurable cardinals larger than $|T|$ has order type at least α, then every $\omega\alpha$–$\mathbf{\Pi}^1_1$ subset of $\lceil \mathbf{T} \rceil$ is unraveled by an A semicovering of $\mathbf{T}$.*

Proof. The proof is just like that of Corollary 5.1.5, with the use of Theorem 5.1.4 replaced by a use of Theorem 5.1.7. $\square$

Corollary 5.1.9. *Let $\mathbf{T}$ be a game tree with taboos. If $0 < \alpha < \omega_1$ and if the class of measurable cardinals larger than $|T|$ has order type $\geq \alpha$, then all $\omega\alpha$–$\mathbf{\Pi}^1_1$ games in $\mathbf{T}$ are determined.*

Proof. The corollary is an immediate consequence of Corollary 5.1.8 and Lemma 4.2.1. $\square$

Exercise 5.1.1. Give a proof of Corollary 5.1.6 in the style suggested by the remarks following that corollary. (For A an α–$\mathbf{\Pi}^1_1$ subset of $\lceil \mathbf{T} \rceil$, directly define $\tilde{\mathbf{T}}$ and π and directly prove that the determinacy of $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ implies that of $G(A; T)$.)

Exercise 5.1.2. Do for Corollary 5.1.9 what you did in Exercise 5.1.1 for Corollary 5.1.6.

5.2 A Factor of ω

Limits are known on how much determinacy can follow from measurable cardinals. The existence of a measurable cardinal does not (if consistent) imply the determinacy of all $(\omega^2 + 1)$–$\mathbf{\Pi}^1_1$ games in countable trees. (See Exercise 5.3.5). In general $(\omega^2\alpha + 1)$–$\mathbf{\Pi}^1_1$ determinacy in trees of size λ does not follow from the existence of α measurable cardinals larger than λ. (See Exercise 5.3.6.) The goal of this section is to get the strongest determinacy consequences of measurable cardinals that are not ruled out by these negative theorems. We will first show that the existence of a measurable cardinal larger than $|T|$ does imply that all ω^2–$\mathbf{\Pi}^1_1$ games in T are determined. Then we use the results of §2.2 to improve the conclusion to the determinacy of all $G(A; T)$ such that both A and $\neg A$ belong to $(\omega^2 + 1)$–$\mathbf{\Pi}^1_1$. Finally we deduce, for countable ordinals α, the determinacy of all $G(A; T)$ such that both A and $\neg A$ belong to $(\omega^2\alpha + 1)$–$\mathbf{\Pi}^1_1$ from the hypothesis that there are α measurable cardinals larger than $|T|$.

A good deal of preparation must be done before we can prove any of these theorems. To see why this is so, let us consider informally the problem of deducing ω^2–$\mathbf{\Pi}^1_1$ determinacy for games in $\mathbf{T}$ from the existence of a measurable cardinal larger than $|T|$. Let $\mathcal{U}$ be a uniform normal ultrafilter on a measurable cardinal $\kappa > |T|$.

The ordering trick allows us to prove ω–$\mathbf{\Pi}^1_1$ determinacy in $\mathbf{T}$, but clearly that is all the work it will do. No ordinal larger than ω can be mapped in an order preserving manner into ω. Another way of making this same point is as follows: The number m in the hypothesis of Lemma 4.2.3 is smaller than the number $\max\{k, m\} + 1$ of the conclusion of that lemma. Hence the lemma licenses only ω reuses of a single measurable cardinal.

To get ω^2–$\mathbf{\Pi}^1_1$ determinacy, we need somehow to come up with infinitely many copies of our measurable cardinal κ. The plan is to do this with the aid of iterated ultrapowers. Let $j = i_{\mathcal{U}}$. The $j_{0,n}(\kappa)$, $n \in \omega$ (as defined on page 153), are the desired infinitely many copies of κ.

Using the sequence $\langle j_{0,n}(\kappa) \mid n \in \omega\rangle$ to replace the sequence $\langle \kappa_n \mid \in \omega\rangle$ from the $\alpha = \omega$ case of Theorem 5.1.7 does not, however, allow us to prove the literal conclusion of that theorem. For $n > 0$ the $j_{0,n}(\kappa)$ are not really measurable cardinals. The filters $j_{0,n}(\mathcal{U})$ are ultrafilters only in the corresponding model M^j_n. Thus they do not make it possible to prove the relevant cases of Lemma 4.2.3.

It turns out, nevertheless, that the $j_{0,n}(\mathcal{U})$ are sufficiently like ultrafilters to yield analogues of Theorem 5.1.7 and Corollary 5.1.8 that are strong enough to imply the desired determinacy result. The reasons why this is so turn up when we try to imitate the proof of the $\alpha = \omega$ case of Theorem 5.1.7 using the $j_{0,n}(\kappa)$ in the roles of the κ_n. The first point is that we do not need $\phi(\tilde{\sigma})$ to be defined everywhere in $\mathcal{S}(\tilde{T})$. We wish only to prove that a game of the form $G(A; \mathbf{T})$ is determined, and an inspection of the proof of Lemma 4.2.1 (i.e., of the proof of Lemma 2.1.3) reveals that this requires only that $\phi(\tilde{\sigma})$ be defined for some winning strategy $\tilde{\sigma}$ for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$. The second point is that the definitions of $\tilde{\mathbf{T}}$ and π depended only on $\mathbf{T}$, $\langle p \mapsto <^\beta_p \mid \beta < \omega^2\rangle$, $\langle A_\beta \cap T \mid \beta < \omega^2\rangle$, and $\langle \kappa_n \mid n \in \omega\rangle$; i.e., these definitions did not involve the $\mathcal{U}_n$. Thus $\tilde{\mathbf{T}}$ and π belong to any transitive class model of ZFC that contains $\mathbf{T}$, $\langle p \mapsto <^\beta_p \mid \beta < \omega^2\rangle$, and $\langle \kappa_n \mid n \in \omega\rangle$. In particular, they belong to all the M^j_n. The third point is that, because of the second point, there is a winning strategy for the open or closed game $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ that belongs to all the M^j_n. The final point is that to carry out the construction of $\phi(\tilde{\sigma})$ and to define the corresponding $\Psi(\tilde{\sigma}, x)$ for such a $\tilde{\sigma}$, it is enough that in each model M^j_n the filter $j_{0,n}(\mathcal{U})$ is an ultrafilter.

In our actual constructions, we will not use the $j_{0,n}(\kappa)$ and the $j_{0,n}(\mathcal{U})$. Instead we will use the $j_{0,\omega_1(n+1)}(\kappa)$ and the $j_{0,\omega_1(n+1)}(\mathcal{U})$. This change in the sketch just given is in preparation for the next section. For the results of this section, the change is not necessary. (See Exercise 5.2.4.)

There will be very little in the way of really new ideas in this section, which will consist primarily of adapting the constructions of §4.2 and §5.1 to fit into the plan just sketched. Nevertheless, the section will be very long, and so it will be divided into subsections to help orient the reader.

5.2.1 Semicoverings with Respect to Models

We first define the appropriate weak version of a semicovering:

If $\tilde{\mathbf{T}}$ and $\mathbf{T}$ are game trees with taboos and M is a class, we write $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S},M}{\Rightarrow} \mathbf{T}$ to mean that

(i) $\phi : \text{domain}\,(\phi) \to \mathcal{S}(T),\ \text{domain}\,(\phi) \subseteq \mathcal{S}(\tilde{T})$, and

$$\{\tilde{\sigma} \in \mathcal{S}(\tilde{T}) \mid (\forall n \in \omega)\, \tilde{\sigma} \restriction {}_n\tilde{T} \in M\} \subseteq \text{domain}\,(\phi);$$

(ii) each $\phi(\tilde{\sigma})$ is a strategy for the same player as is $\tilde{\sigma}$;

(iii) for each $n \in \omega$, the restriction of $\phi(\tilde{\sigma})$ to postions of length $< n$ depends only on the restriction of $\tilde{\sigma}$ to positions of length $< n$; that is, if $\tilde{\sigma}$ and $\tilde{\sigma}'$ both belong to domain (ϕ) and agree on positions of length $< n$, then $\phi(\tilde{\sigma})$ and $\phi(\tilde{\sigma}')$ agree on positions of length $< n$.

This definition differs from that of $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$ only in clause (i). The corresponding clause in the definition of $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$ requires that domain (ϕ) be all of $\mathcal{S}(\tilde{T})$. For almost all our applications, we could weaken the last clause of (i) to require only that $\mathcal{S}(T) \cap M \subseteq \text{domain}\,(\phi)$. (See Exercise 5.2.4.)

If M is a class and $\mathcal{T}$ is a game tree with taboos, then a *semicovering of* $\mathbf{T}$ *with respect to* M is a quadruple $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ that satisfies the definition of a semicovering of $\mathbf{T}$ on page 191, with the following two changes: Clause (c) is replaced by the condition that $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S},M}{\Rightarrow} \mathbf{T}$. Clause (d) is modified by replacing "$\tilde{\sigma} \in \mathcal{S}(\tilde{T})$" by "$\tilde{\sigma} \in \text{domain}\,(\phi)$" in the condition on the domain of Ψ.

Related concepts have the obvious definitions: Let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a semicovering of $\mathbf{T}$ with respect to M. If $A \subseteq \lceil \mathbf{T} \rceil$, then $\mathcal{C}$ *unravels* A if $\boldsymbol{\pi}^{-1}(A)$ is clopen, where $\boldsymbol{\pi}$ is defined as usual. If A is a subset of $\lceil \mathbf{T} \rceil$, then $\mathcal{C}$ is an A-*semicovering of* $\mathbf{T}$ *with respect to* M if $\langle \tilde{\sigma}, x \rangle \in \text{domain}\,(\Psi)$ whenever x witnesses that $\phi(\tilde{\sigma})$ is not a winning strategy for $G(A; \mathbf{T})$. If A and B are subsets of $\lceil \mathbf{T} \rceil$, then $\mathcal{C}$ is an (A, B) *semicovering of* $\mathbf{T}$ *with respect to* M if it is an A semicovering of $\mathcal{T}$ with respect to M and

(f) for every $\tilde{\sigma} \in \text{domain}\,(\phi)$ and for every $x \in B$ such that x is consistent with $\phi(\tilde{\sigma})$, the pair $\langle \tilde{\sigma}, x \rangle$ belongs to the domain of Ψ;

(g) every normal play in $\tilde{\mathbf{T}}$ belongs to $\boldsymbol{\pi}^{-1}(B)$.

These are just clauses (f) and (g) in the definition of an (A, B) semicovering, except that clause (f) has been modified in the obvious way. $\mathcal{C}$ is also a *k-semicovering of $\mathbf{T}$ with respect to M* if

(i) ${}_k\tilde{\mathbf{T}} = {}_k\mathbf{T}$;

(ii) $\pi \restriction {}_k\tilde{T}$ is the identity;

(iii) $\phi \restriction \{\tilde{\rho} \in \mathcal{S}({}_k\tilde{T}) \mid (\exists\tilde{\sigma} \in \text{domain}\,(\phi))\, \tilde{\rho} \subseteq \tilde{\sigma}\}$ is the identity.

These are just the clauses (i), (ii), and (iii) defining a k-semicovering of $\mathbf{T}$, except that (iii) has been modified in the natural way.

In all the definitions above, the only real change from the earlier concepts concerns domain (ϕ). There is no change in the requirements on Ψ that is not the direct result of the weakened demands on domain (ϕ). For example, if $\mathcal{C}$ is an A-semicovering with respect to M and $\tilde{\sigma} \in$ domain (ϕ), then $\Psi(\tilde{\sigma}, x)$ is defined for every $x \in V$ that witnesses that $\phi(\tilde{\sigma})$ is not a winning strategy for $G(A; \mathbf{T})$, not just for every $x \in M$ with this property. Furthermore, the components of $\mathcal{C}$ do not have to belong to M, though in our applications $\tilde{\mathbf{T}}$ and π—and sometimes the restrictions of ϕ to the $\mathcal{S}({}_n\tilde{T})$—will belong to M.

5.2.2 Unraveling, Determinacy, and Codes

Here is the basic fact that makes semicoverings with respect to models useful for proving determinacy.

Lemma 5.2.1. *Let M be a transitive class model of* ZFC. *Let $\mathbf{T}$ be a game tree with taboos. Let $A \subseteq \lceil\mathbf{T}\rceil$. Let $\langle\tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be an A-semicovering of $\mathbf{T}$ with respect to M such that $\tilde{\mathbf{T}} \in M$. Assume that there is a winning strategy $\tilde{\sigma}$ for $G(\boldsymbol{\pi}^{-1}(A); \mathbf{T})$ such that $\tilde{\sigma} \restriction {}_n\tilde{T}$ belongs to M for each $n \in \omega$. Then $G(A; \mathbf{T})$ is determined.*

Proof. Clause (i) of the definition of $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S},M}{\Rightarrow} \mathbf{T}$ implies that $\tilde{\sigma} \in$ domain (ϕ). As in the proof of Lemma 4.2.1, i.e., as in that of of Lemma 2.1.3, $\phi(\tilde{\sigma})$ is a winning strategy for $G(A; \mathbf{T})$. $\square$

Remarks:

(a) We could have weakened the hypothesis that $\tilde{\mathbf{T}} \in M$, but our applications give us no reasons to do so.

(b) The hypothesis of Lemma 5.2.1 concerning $\tilde\sigma$ is true if $\tilde\sigma$ is a winning strategy for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ that actually belongs to M. This will be the case in most, but not quite all, of our applications.

We want to use Lemma 5.2.1 to get an appropriate version of Lemma 4.2.1. In proving this and subsequent results, we will make use of the absoluteness of wellfoundedness. Let us first officially document this simple but important fact. Recall that a relation R is *wellfounded* if every nonempty set has an element x that is minimal with respect to R (i.e. to which nothing bears R).

Lemma 5.2.2. *Wellfoundedness is absolute for transisitive class models of* ZFC*; that is, if M is a transitive class model of* ZFC *and $R \in M$ is a relation, then R is wellfounded if and only if $M \models$ "R is wellfounded."*

Proof. Suppose first that $M \models$ "R is not wellfounded." Let then $Y \in M$ be such that $M \models$ "Y is a nonempty set with no R-minimal element." By easy absoluteness facts, Y is a nonempty set with no R-minimal element. Thus R is not wellfounded.

Now suppose that $M \models$ "R is wellfounded." Then we may define in M by transfinite recursion on R $\| \ \|^R : \text{field}\,(R) \to \text{Ord} \cap M$ by

$$\|x\|^R = \sup\{\|y\|^R + 1 \mid y\, R\, x\}.$$

(See Theorem 5.6 of Kunen [1980].) To see that R is wellfounded in V, let $Y \neq \emptyset$. Let $x \in Y$ be such that $\|x\|^R$ is minimal. Clearly x is an R-minimal element of Y. □

Now we prove our analogue of Lemma 4.2.1.

Lemma 5.2.3. *Let M be a transitive class model of* ZFC*. Let $\mathbf{T}$ be a game tree with taboos. Let $A \subseteq \lceil \mathbf{T} \rceil$. Let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be an A-semicovering of $\mathbf{T}$ with respect to M that unravels A and is such that $\tilde{\mathbf{T}} \in M$. Let $\tilde D \in M$ be a subset of $\tilde T$ that generates an open subset B of $\lceil \tilde T \rceil$ with $B \cap \lceil \tilde{\mathbf{T}} \rceil = \boldsymbol{\pi}^{-1}(A)$. Then $G(A; \mathbf{T})$ is determined.*

Proof. By Lemma 5.2.1, it is enough to show that there is a winning strategy $\tilde\sigma$ for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ such that $\tilde\sigma$ belongs to M.

To get such a $\tilde\sigma$, we argue as follows. In the model M, the set $\tilde D$ generates $B \cap M$ and $\boldsymbol{\pi}^{-1}(A) \cap M = B \cap M \cap \lceil \tilde{\mathbf{T}} \rceil$. Thus in M the open game

$G(\boldsymbol{\pi}^{-1}(A) \cap M; \tilde{\mathbf{T}})$ is determined. In M let $\tilde{\sigma}$ be a winning strategy for $G(\boldsymbol{\pi}^{-1}(A) \cap M; \tilde{\mathbf{T}})$. We will argue that $\tilde{\sigma}$ is also in V a winning strategy for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$.

Assume first that $\tilde{\sigma}$ is a strategy for I. If $\tilde{x}$ is an infinite play consistent with $\tilde{\sigma}$ such that $\tilde{x} \in M$, then $\tilde{x}$ extends some $\tilde{d} \in \tilde{D}$. Thus the tree

$$\tilde{S} = \{\tilde{p} \in \tilde{T} \mid \tilde{p} \text{ is consistent with } \tilde{\sigma} \wedge \neg(\exists \tilde{d} \in \tilde{D})\, \tilde{d} \subseteq \tilde{p}\}$$

is wellfounded in the model M. By the absoluteness of wellfoundedness, $\tilde{S}$ is wellfounded in V as well. Hence every infinite play consistent with $\tilde{\sigma}$ extends some $\tilde{d} \in \tilde{D}$ and so belongs to $\boldsymbol{\pi}^{-1}(A)$. But every finite play consistent with $\tilde{\sigma}$ belongs to M and so belongs to $A \cap M$ or else is taboo for II in $\tilde{\mathbf{T}}$. Thus $\tilde{\sigma}$ is a winning strategy for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$.

Now assume that $\tilde{\sigma}$ is a strategy for II. Suppose $\tilde{x}$ is an infinite play consistent with $\tilde{\sigma}$ such that $\tilde{x} \in \boldsymbol{\pi}^{-1}(A)$. Then there is a $\tilde{d} \in \tilde{D}$ such that $\tilde{d} \subseteq \tilde{x}$. Fix such a $\tilde{d}$. In the model M the tree

$$\tilde{S}^{\tilde{d}} = \{\tilde{p} \in \tilde{T}_{\tilde{d}} \mid \tilde{p} \text{ is consistent with } \tilde{\sigma}\}$$

is wellfounded. By absoluteness, we get the contradiction that $\tilde{S}^{\tilde{p}}$ is wellfounded in V also. As in the first case, it follows that $\tilde{\sigma}$ is a winning strategy for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$. □

Remarks:

(a) It looks at first as if the existence of $\tilde{\sigma}$ follows directly from Lemma 4.4.1. But the earlier lemma is not about game trees with taboos, and $G(\pi^{-1}(A); \tilde{\mathbf{T}})$ is not necessarily open as a game in $\tilde{T}$. There are various ways to deal with this fact, but the simple ones yield proofs of Lemma 5.2.3 that appeal both to Lemma 4.4.1 and to the absoluteness of wellfoundedness. Therefore we gave a direct proof using the absoluteness of wellfoundedness.

(b) As with Lemma 4.4.1, we do not need that M is a model of full ZFC.

We did not actually use the clopenness of $\boldsymbol{\pi}^{-1}(A)$. Only the openness of $\boldsymbol{\pi}^{-1}(A)$ was used. But it was crucial that the open set $\boldsymbol{\pi}^{-1}(A)$ was generated by a set $\tilde{D} \in M$. We next want to get a more general result, with openness replaced by quasi-Borelness or—equivalently, by Theorem 2.2.3—by membership in $\boldsymbol{\Delta}^1_1$. For this we need an appropriate notion of generating a quasi-Borel set or of generating a $\boldsymbol{\Delta}^1_1$ set. With an eye to other applications, we will deal with $\boldsymbol{\Delta}^1_1$ sets.

Let us say that $\mathbf{c}$ is a $\boldsymbol{\Pi}^1_1$ *code* if $\mathbf{c}$ is a triple $\langle \mathbf{T}, E, f\rangle$, where

(a) $\mathbf{T}$ is a game tree with taboos;

(b) $E \subseteq T \cap \lceil \mathbf{T} \rceil$;

(c) f is a function $p \mapsto <_p$ with domain T such that (1) for all $p \in T$, $<_p$ is a linear ordering of $\ell\mathrm{h}(p)$ with greatest element 0 (if and (2), for all $p \subseteq q \in T$, $<_p$ is the restriction of $<_q$ to $\ell\mathrm{h}(p)$.

If $\mathbf{c} = \langle \mathbf{T}, E, f \rangle$ is a $\mathbf{\Pi}^1_1$ code, then $\mathbf{c}$ is a $\mathbf{\Pi}^1_1$ *code for* A (equivalently, A is *the* $\mathbf{\Pi}^1_1$ *set coded by* $\mathbf{c}$) if

$$A = E \cup \{x \in [T] \mid <_x \text{ is a wellordering}\}.$$

By a $\mathbf{\Delta}^1_1$ *code* we mean a pair $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ of $\mathbf{\Pi}^1_1$ codes with the same first components $\mathbf{T}$, such that the $\mathbf{\Pi}^1_1$ sets coded by $\mathbf{c}_1$ and $\mathbf{c}_2$ are complementary subsets of $\lceil \mathbf{T} \rceil$. If $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ is a $\mathbf{\Delta}^1_1$ code, then $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ is a $\mathbf{\Delta}^1_1$ *code for* A (equivalently, A is *the* $\mathbf{\Delta}^1_1$ *set coded by* $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$) if A is the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}_1$ (and so $\neg A$ is the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}_2$).

Lemma 5.2.4. *Let M be a transitive class model of* ZFC. *Let $\mathbf{c} = \langle \mathbf{T}, E, f \rangle$ belong to M. Then (1) $\mathbf{c}$ is a $\mathbf{\Pi}^1_1$ code if and only if $M \models$ "$\mathbf{c}$ is a $\mathbf{\Pi}^1_1$ code," and (2) if $\mathbf{c}$ is a $\mathbf{\Pi}^1_1$ code then, for all $x \in \lceil \mathbf{T} \rceil \cap M$, x belongs to the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$ if and only if $M \models$ "x belongs to the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$."*

Remark. The lemma says precisely that the formulas expressing being a $\mathbf{\Pi}^1_1$ code and belonging to the $\mathbf{\Pi}^1_1$ set coded by a $\mathbf{\Pi}^1_1$ code are absolute for transitive class models of ZFC.

Proof. (1) is easy to verify. For (2), assume that $\mathbf{c}$ is a $\mathbf{\Pi}^1_1$ code. We have, for all $x \in \lceil \mathbf{T} \rceil$, that x belongs to the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$ if and only if $x \in E$ or else $x \in [T]$ and $<_x$ is a wellordering, where $<_x$ is given by f. Membership in E and being an infinite element of $\lceil T \rceil$ are easily seen to be absolute. Thus it is enough to prove the absoluteness of "$<_x$ is a wellordering of ω." But this is another example of the absoluteness of wellfoundedness, since being a linear ordering of ω is easily seen to be absolute. □

Lemma 5.2.5. *let M be a transitive class model of* ZFC *with $\omega_1 \in M$. Let $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle \in M$. Then (1) $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ is a $\mathbf{\Delta}^1_1$ code if and only if $M \models$ "$\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ is a $\mathbf{\Delta}^1_1$ code," and (2) if $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ is a $\mathbf{\Delta}^1_1$ code then, for all $x \in \lceil \mathbf{T} \rceil \cap M$, x belongs to the $\mathbf{\Delta}^1_1$ set coded by $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ if and only if $M \models$ "x belongs to the $\mathbf{\Delta}^1_1$ set coded by $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$."*

Proof. By part (1) of Lemma 5.2.4, we may assume without loss of generality that $\mathbf{c}_1$ and $\mathbf{c}_2$ are $\mathbf{\Pi}^1_1$ codes, both in V and in M. Let A be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}_1$ and let B be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}_2$. By part (2) of Lemma 5.2.4 the $\mathbf{\Pi}^1_1$ sets coded in the model M by $\mathbf{c}_1$ and $\mathbf{c}_2$ are $A \cap M$ and $B \cap M$. It is clear that if A and B are complementary in V then $A \cap M$ and $B \cap M$ are complementary in M. Assume then that $A \cap M$ and $B \cap M$ are complementary in M. Let S be the set of all $\langle p, \langle q_1, q_2 \rangle\rangle$ such that $p \in T$ and, for $i \in \{1, 2\}$, q_i is an embedding of $(\ell\mathrm{h}(p); <^i_p)$ into $(\omega_1; <)$, where $<^i_p$ is given by the third component of $\mathbf{c}_i$. Since $\omega_1 \in M$ it follows that $S \in M$. Since $A \cap B \cap [T] \cap M = \emptyset$, where $\mathbf{T}$ is the first component of the $\mathbf{c}_i$, we have that S is in M a wellfounded game tree. By absoluteness, it follows that S is wellfounded in V as well and so that $A \cap B \cap [T] = \emptyset$. Hence $A \cap B = \emptyset$. Now let S' be the set of all $\langle p, \langle s_1, s_2 \rangle\rangle$ such that $p \in T$ and, for $i \in \{1, 2\}$, $s_i : \ell\mathrm{h}(p) \to \omega$ and

$$(\forall n < \ell\mathrm{h}(p))(\forall n' < n)((s_i(n) < \ell\mathrm{h}(p) \ \wedge \ s_i(n') < \ell\mathrm{h}(p)) \ \to \ s_i(n) <^i_p s_i(n')).$$

Then $S' \in M$. Moreover any member of $[S']$ would give a member x of $[T]$ and a witness that neither $<^1_x$ nor $<^2_x$ was a wellordering. Since $(A \cap M) \cup (B \cap M) = \lceil \mathbf{T} \rceil \cap M$, it follow that S' is in M a wellfounded game tree. From this and absoluteness we get that $A \cup B = \lceil \mathbf{T} \rceil$. This completes the proof of (1).

(2) follows from part (2) of Lemma 5.2.4. □

Remark. Lemma 5.2.5 remains true if the hypothesis that $\omega_1 \in M$ is replaced by the weaker hypothesis that M is uncountable. The same is true of Lemma 5.2.6 below. (See Exercises 5.2.1 and 5.2.2.)

Lemma 5.2.6. *Let M be a transitive class model of* ZFC *with $\omega_1 \in M$. Let $\mathbf{c} = \langle \mathbf{T}, E, f \rangle$ belong to M and be a $\mathbf{\Pi}^1_1$ code. Let A be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$. Let $\sigma \in \mathcal{S}(T) \cap M$. Then σ is a winning strategy for $G(A; \mathbf{T})$ if and only if $M \models$ "σ is a winning strategy for $G(A \cap M; \mathbf{T})$."*

Proof. The "only if" part of the lemma follows easily from Lemma 5.2.4. For the "if" part, assume that $M \models$ "σ is a winning strategy for $G(A \cap M; \mathbf{T})$."

Suppose first that σ is a strategy for I. Let f be $p \mapsto <_p$. Let $\bar{S}'$ be the tree of all $\langle p, s \rangle$ such that $p \in T$, $s : \ell\mathrm{h}(p) \to \omega$, and

$$(\forall n < \ell\mathrm{h}(p))(\forall n' < n)((s(n) < \ell\mathrm{h}(p) \ \wedge \ s(n') < \ell\mathrm{h}(p)) \ \to \ s(n) <_p s(n')).$$

Then $\bar{S}' \in M$ and $\bar{S}'$ is in M a wellfounded game tree. By absoluteness, $\bar{S}'$ is wellfounded in V also. Thus there is no infinite play that witnesses that σ is not a winning strategy. As in the proof of Lemma 5.2.3, it follows that σ is a winning strategy.

Now suppose that σ is a strategy for II. Let $\bar{S}$ be the tree of all $\langle p, q\rangle$ such that p is a position consistent with σ and such that q embeds $(\ell\text{h}(p); <_p)$ into $(\omega_1; <)$. In the model M, $\bar{S}$ is a wellfounded game tree. Thus $\bar{S}$ is in V a wellfounded game tree. Once again it follows that σ is a winning strategy. $\square$

Lemma 5.2.7. *Let M be a transitive class model of* ZFC*. Let $\mathbf{T}$ be a game tree with taboos. Let $A \subseteq \lceil \mathbf{T} \rceil$. Let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be an A-semicovering of $\mathbf{T}$ with respect to M such that $\tilde{\mathbf{T}} \in M$. Let $\langle \mathbf{c}_1, \mathbf{c}_2\rangle \in M$ be a $\mathbf{\Delta}^1_1$ code for $\boldsymbol{\pi}^{-1}(A)$. Then $G(A; \mathbf{T})$ is determined.*

Proof. By Lemma 5.2.1, it is enough to show that there is a winning strategy $\tilde{\sigma}$ for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ that belongs to M.

To get such a $\tilde{\sigma}$, we first apply Theorem 2.2.8 in the model M to establish the determinacy in M of the game $G(\boldsymbol{\pi}^{-1}(A) \cap M; \tilde{\mathbf{T}})$. Theorem 2.2.8 is applicable, since by Lemma 5.2.5 we have that $\boldsymbol{\pi}^{-1}(A) \cap M$ is in M a $\mathbf{\Delta}^1_1$ set, namely the $\mathbf{\Delta}^1_1$ set coded by the $\mathbf{\Delta}^1_1$ code $\langle \mathbf{c}_1, \mathbf{c}_2\rangle \in M$. Let then $\tilde{\sigma}$ be in M a winning stategy for $G(\boldsymbol{\pi}^{-1}(A) \cap M; \tilde{\mathbf{T}})$.

By Lemma 5.2.6, $\tilde{\sigma}$ is also in V a winning strategy for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$. $\square$

5.2.3 Operations for Unraveling $\mathbf{\Pi}^1_1$ Sets

Our next goal is to get a result that will play the role Lemma 4.2.3 played in §5.1. One of the things we want to pay attention to is the operations that gave us the components of the semicoverings of Lemma 4.2.2. We also want to pay attention to the absoluteness of these operations. The basic picture will be the following:

(a) The operations $\mathcal{F}_{\text{t}}$ and $\mathcal{F}_{\text{pi}}$ that gave the components $\tilde{\mathbf{T}}$ and π are quite absolute.

(b) The operation that gave ϕ can be relativized to any model M of ZFC. The operation defined by this relativized definition generates an operation that we call $\mathcal{F}^M_{\text{phi}}$. The values of $\mathcal{F}^M_{\text{phi}}$ are functions $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S},M}{\Rightarrow} \mathbf{T}$.

(c) The operation that gave Ψ can be relativized to any model M of ZFC. Moreover a restriction of the operation given by this relativized definition can be extended to an operation that we call $\mathcal{F}^M_{\mathrm{psi}}$. The values of $\mathcal{F}^M_{\mathrm{psi}}$, together with those of the other operations, constitute (B, B) semicoverings with respect to M.

We first consider the operations $\mathcal{F}_{\mathrm{t}}$ and $\mathcal{F}_{\mathrm{pi}}$ given by the proof of Lemma 4.2.3. The common domain of these two operations is the set of all $\langle \mathbf{c}, \kappa, m, k, i\rangle$ such that

(i) $\mathbf{c}$ is a $\mathbf{\Pi}^1_1$ code;

(ii) κ is an ordinal number;

(iii) $m \in \omega$;

(iv) $k \in \omega$;

(v) $i \in \{1, 2\}$.

For $i = 1$ or 2, $\mathcal{F}_{\mathrm{t}}(\mathbf{c}, \kappa, m, k, i)$ and $\mathcal{F}_{\mathrm{pi}}(\mathbf{c}, \kappa, m, k, i)$ are respectively the tree $\tilde{\mathbf{T}}$ and the function π given by the proof of part (i) of Lemma 4.2.3, with B as the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$ and with the obvious values of the other parameters.

The construction and proof of Lemma 4.2.3 yields the following two lemmas.

Lemma 5.2.8. *The operations $\mathcal{F}_{\mathrm{t}}$ and $\mathcal{F}_{\mathrm{pi}}$ are absolute for transitive class models of* ZFC*. That is, if M is a transitive class model of* ZFC*, then* domain $(\mathcal{F}_{\mathrm{t}})$ *as defined in M is just* domain $(\mathcal{F}_{\mathrm{t}}) \cap M$*, and the two operations as defined in M are just the restrictions of the operations to* domain $(\mathcal{F}_{\mathrm{t}}) \cap M$.

Lemma 5.2.9. *Let M be a transitive class model of* ZFC*. Let*

$$\langle \mathbf{c}, \kappa, m, k, i\rangle \in \mathrm{domain}\,(\mathcal{F}_{\mathrm{t}}) \cap M$$

with $\mathbf{c} = \langle \mathbf{T}, E, f\rangle$. Let $\tilde{\mathbf{T}} = \mathcal{F}_{\mathrm{t}}(\mathbf{c}, \kappa, m, k, i)$ and let $\pi = \mathcal{F}_{\mathrm{pi}}(\mathbf{c}, \kappa, m, k, i)$. Then

(a) $\tilde{\mathbf{T}}$ is a game tree with taboos;

(b) $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$;

(c) both $\tilde{\mathbf{T}}$ and π belong to M;

(d) if $M \models |T| \leq |\kappa|$ then $M \models |\tilde{T}| \leq \kappa$

(e) *if* $(\forall p \in T)(\ell\mathrm{h}(p) > m \to M \models |T_p| < |\kappa|)$*, then* $(\forall \tilde{p} \in \tilde{T})(\ell\mathrm{h}(\tilde{p}) > \max\{k, m\} + 1 \to M \models |\tilde{T}_{\tilde{p}}| < |\kappa|)$*;*

(f) ${}_k\tilde{\mathbf{T}} = {}_k\mathbf{T}$*;*

(g) $\pi \restriction {}_k\tilde{T}$ *is the identity;*

(h) *Every normal play in* $\tilde{\mathbf{T}}$ *belongs to the* $\mathbf{\Pi}^1_1$ *set coded by* $\mathbf{c}$.

Proof. Clauses (a), (b), (f), (g), and (h) follow from the proof of Lemma 4.2.3. Clauses (d) and (e) follow from the same proof as applied in M, and to apply it in M is legitimate by Lemma 5.2.8, which also implies clause (c). □

The proof of Lemma 4.2.3 also gives an operation $\mathcal{F}_{\mathrm{phi}}$. The domain of $\mathcal{F}_{\mathrm{phi}}$ is the set of all $\langle \mathbf{c}, \kappa, m, k, i, \mathcal{U}\rangle$ such that

(i) $\langle \mathbf{c}, \kappa, m, k, i\rangle \in \mathrm{domain}\,(\mathcal{F}_{\mathrm{t}})$;

(ii) $(\forall p \in T)(\ell\mathrm{h}(p) > m \to |T_p| < \kappa)$, where $\mathbf{T}$ is the first component of $\mathbf{c}$;

(iii) $\mathcal{U}$ is a uniform normal ultrafilter on κ.

The value $\mathcal{F}_{\mathrm{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U})$ is the ϕ that comes from the proof of Lemma 4.2.2. The following lemma then comes from the obvious application of the proof of Lemma 4.2.3.

Lemma 5.2.10. *Let* $\langle \mathbf{c}, \kappa, m, k, i, \mathcal{U}\rangle \in \mathrm{domain}\,(\mathcal{F}_{\mathrm{phi}})$ *with* $\mathbf{c} = \langle \mathbf{T}, E, f\rangle$*. Let* $\tilde{\mathbf{T}} = \mathcal{F}_{\mathrm{t}}(\mathbf{c}, \kappa, m, k, i)$ *and let* $\phi = \mathcal{F}_{\mathrm{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U})$*. Then*

(a) $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$*;*

(b) $\phi \restriction \mathcal{S}({}_k\tilde{T})$ *is the identity.*

This operation is absolute for transitive models of ZFC, but that fact does not interest us here, for we will be applying our operations in models where there is a measurable cardinal that may not be measurable in V. For M a transitive class model of ZFC, let us then define an operation $\mathcal{F}^M_{\mathrm{phi}}$ whose domain is the set of all $\langle \mathbf{c}, \kappa, m, k, i, \mathcal{U}\rangle \in M$ such that

(i) $\langle \mathbf{c}, \kappa, m, k, i\rangle \in \mathrm{domain}\,(\mathcal{F}_{\mathrm{t}})$;

(ii) $M \models (\forall p \in T)(\ell\mathrm{h}(p) > m \to |T_p| < \kappa)$, where $\mathbf{T}$ is the first component of $\mathbf{c}$;

(iii) $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ."

To define $\mathcal{F}^M_{\mathrm{phi}}$, first let $(\mathcal{F}_{\mathrm{phi}})^M$ be the operation given by the definition of $\mathcal{F}_{\mathrm{phi}}$ as applied in M. Now let $\langle \mathbf{c}, \kappa, m, k, i, \mathcal{U}\rangle$ belong to M and satisfy (i)–(iii). We let

$$\mathrm{domain}\,(\mathcal{F}^M_{\mathrm{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U})) \;=\; \{\tilde{\sigma} \in \mathcal{S}(\tilde{T}) \mid (\forall k \in \omega)\, \tilde{\sigma} \restriction {}_k\tilde{T} \in M\}.$$

For $\tilde{\sigma} \in \mathrm{domain}\,(\mathcal{F}^M_{\mathrm{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U}))$ and for $k \in \omega$, we set

$$(\mathcal{F}^M_{\mathrm{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U}))(\tilde{\sigma}) \;=\; ((\mathcal{F}_{\mathrm{phi}})^M(\mathbf{c}, \kappa, m, k, i, \mathcal{U}))(\tilde{\sigma}'),$$

for some $\tilde{\sigma}' \in M$ such that $\tilde{\sigma} \restriction {}_k\tilde{T} = \tilde{\sigma}' \restriction {}_k\tilde{T}$. By clause (a) of Lemma 5.2.10, applied in M, and by clause (iii) of the definition of $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$, there is no dependence on the choice of $\tilde{\sigma}'$.

The close relationship between the operations $(\mathcal{F}_{\mathrm{phi}})^M$ and $\mathcal{F}^M_{\mathrm{phi}}$ provides at least some justification for our using almost the same notation for the two.

In all transitive class models N of ZFC such that $M \subseteq N$ and M is a class of N (is definable in N from members of N), one can define the operation $(\mathcal{F}_{\mathrm{phi}})^M$. If one applies the definition of $\mathcal{F}^M_{\mathrm{phi}}$ in such an N, it may not give the true $\mathcal{F}^M_{\mathrm{phi}}$, but only because there may be strategies $\tilde{\sigma}$ that do not belong to N but all of whose restrictions to positions of length k do belong to N.

Clause (c) of the following lemma does not correspond to any clause of Lemma 5.2.10. This clause follows from the fact that $\phi \restriction \mathcal{S}({}_n\tilde{T}) = \phi \restriction (\mathcal{S}({}_n\tilde{T}) \cap M) = ((\mathcal{F}_{\mathrm{phi}})^M(\mathbf{c}, \kappa, m, k, i, \mathcal{U})) \restriction (\mathcal{S}({}_n\tilde{T}) \cap M)$.

Lemma 5.2.11. *Let M be a transitive class model of* ZFC*. Let*

$$\langle \mathbf{c}, \kappa, m, k, i, \mathcal{U}\rangle \in \mathrm{domain}\,(\mathcal{F}^M_{\mathrm{phi}})$$

be such that $\mathbf{c} = \langle \mathbf{T}, E, f\rangle$. Let $\tilde{\mathbf{T}} = \mathcal{F}_{\mathrm{t}}(\mathbf{c}, \kappa, m, k, i)$ and let $\phi = \mathcal{F}^M_{\mathrm{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U})$. Then

(a) $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S},M}{\Rightarrow} \mathbf{T}$;

(b) $\phi \restriction (\mathcal{S}({}_k\tilde{T}) \cap M)$ *is the identity.*

(c) $(\forall n \in \omega)\, \phi \restriction (\mathcal{S}({}_n\tilde{T}) \cap M) \in M$.

The proof of Lemma 4.2.2 also gives us an operation $\mathcal{F}_{\mathrm{psi}}$ whose domain is the same as that of $\mathcal{F}_{\mathrm{phi}}$. $\mathcal{F}_{\mathrm{psi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U})$ is the Ψ given by the proof of Lemma 4.2.2. We could catalogue the properties of this operation and its relations to those already defined. Instead we proceed directly to the operations in which we are really interested. For each transitive class model M of ZFC, we will define an operation $\mathcal{F}^M_{\text{psi}}$. The domain of $\mathcal{F}^M_{\text{psi}}$ is the set of all $\langle \mathbf{c}, \kappa, m, k, i, \mathcal{U}\rangle \in \text{domain}\,(\mathcal{F}^M_{\text{phi}})$ such that every intersection (in V) of countably many elements of $\mathcal{U}$ is nonempty. Since the complements of singletons belong to $\mathcal{U}$, this condition implies that every intersection of countably many elements of $\mathcal{U}$ is uncountable. If $\langle \mathbf{c}, \kappa, m, k, i, \mathcal{U}\rangle$ belongs to $\text{domain}\,(\mathcal{F}^M_{\text{psi}})$, then the domain of $\mathcal{F}^M_{\text{psi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U})$ is the set of all pairs $\langle \tilde{\sigma}, x\rangle$ such that $\tilde{\sigma} \in \mathcal{S}(\tilde{\mathbf{T}})$ and $(\forall k \in \omega)\, \tilde{\sigma} \restriction {}_k\tilde{T} \in M$, such that x is a play (not necessarily in M) that is consistent with $(\mathcal{F}^M_{\text{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U}))(\tilde{\sigma})$, and such that at least one of the following holds

(i) $i = 1$ and $\tilde{\sigma}$ is a strategy for I;

(ii) $i = 2$ and $\tilde{\sigma}$ is a strategy for II;

(iii) x is finite;

(iv) x belongs to the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$.

Let us see how the proof of Lemma 4.2.3 yields such a function. Let $\langle \mathbf{c}, \kappa, m, k, i, \mathcal{U}\rangle \in \text{domain}\,(\mathcal{F}^M_{\text{psi}})$. Let $p \mapsto <_p$ be the third component of $\mathbf{c}$. Let $\tilde{T} = \mathcal{F}_{\text{t}}(\mathbf{c}, \kappa, m, k, i)$, let $\pi = \mathcal{F}_{\text{pi}}(\mathbf{c}, \kappa, m, k, i)$, and let $\phi = \mathcal{F}^M_{\text{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U})$. We want to define

$$\Psi \;=\; \mathcal{F}^M_{\text{psi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U}).$$

First let $\tilde{\sigma} \in \mathcal{S}_I(\tilde{T})$ be such that each $\tilde{\sigma} \cap {}_k\tilde{T} \in M$. Then it is easy to see that, for every play x consistent with $\phi(\tilde{\sigma})$, there is a unique play $\tilde{x}$ consistent with $\tilde{\sigma}$ such that $\pi(\tilde{x}) \subseteq x$. Let $\Psi(\tilde{\sigma}, x) = \tilde{x}$.

Next let $\tilde{\tau} \in \mathcal{S}_{II}(\tilde{T})$ be such that each $\tilde{\tau} \cap {}_k\tilde{T} \in M$. Let B be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$. For $p \in T$ with $\ell\text{h}(p)$ odd, let the set X_p be defined as in the proof of Lemma 4.2.2. Each X_p belongs to $\mathcal{U}$. For $p \in T$, define, as in the proof of Lemma 4.2.3,

$$X^p \;=\; \bigcap\{X_{p'} \mid p' \in T_p \,\wedge\, \ell\text{h}(p') \text{ is odd}\}.$$

Since $\tilde{\tau}$ need not belong to M, we cannot conclude that $X^p \in \mathcal{U}$ if $\ell\text{h}(p) > m$. Nevertheless we have for each odd $n > m$ that

$$\bigcap\{X_{p'} \mid p' \in T_p \,\wedge\, \ell\text{h}(p') = n\} \;\in\; \mathcal{U}.$$

Thus for $\ell\text{h}(p) > m$ the set X^p is an intersection of countably many elements of $\mathcal{U}$. By the definition of $\text{domain}\,(\mathcal{F}^M_{\text{psi}})$, it follows that X^p is uncountable if

$\ell\mathrm{h}(p) > m$. Finite plays all belong to M, so we can define $\Psi(\tilde{\tau}, x)$ for all finite x by carrying out the proof of Lemma 4.2.2 in M. Thus we need consider only infinite plays $x \in V$ such that x is consistent with $\phi(\tilde{\tau})$ and $x \in B$. Fix such an x. Then $(\omega; <_x)$ is a wellordering. Let $p \subseteq x$ with $\ell\mathrm{h}(p) = m + 1$. Since X^p is uncountable, its order type is $\geq \omega_1$. Thus there is a function $i \mapsto \xi_i$ embedding $(\omega; <_x)$ into $(X^p; <)$. We may then let $\Psi(\tilde{\tau}, x)$ be a play $\tilde{x}$ with $\pi(\tilde{x}) = x$ and with the ξ_i given by such an embedding.

The following lemma generalizes Lemma 4.2.3, and provides the basic ingredient for using measurable cardinals in iterated ultrapowers to replace genuine measurable cardinals.

Lemma 5.2.12. *Let M be a transitive class model of* ZFC*. Let $\mathbf{c} = \langle \mathbf{T}, E, f\rangle$ be a $\mathbf{\Pi}^1_1$ code belonging to M and let $B \subseteq \lceil \mathbf{T} \rceil$ be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$. Let κ be an ordinal that is a measurable cardinal in the model M. Let $\mathcal{U} \in M$ be such that $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ" and such that every intersection of countably many elements of $\mathcal{U}$ is nonempty. Let m and k belong to ω. Let $\tilde{m} = \max\{k, m\} + 1$. Suppose that $N \models |T| \leq \kappa$, and that*

$$(\forall p \in T)(\ell\mathrm{h}(p) > m \rightarrow (M \models |T_p| < \kappa)).$$

For $i \in \{1, 2\}$, let

$$\begin{aligned} \tilde{\mathbf{T}}_i &= \mathcal{F}_{\mathrm{t}}(\mathbf{c}, \kappa, m, k, i); \\ \pi_i &= \mathcal{F}_{\mathrm{pi}}(\mathbf{c}, \kappa, m, k, i,); \\ \phi_i &= \mathcal{F}^M_{\mathrm{phi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U}); \\ \Psi_i &= \mathcal{F}^M_{\mathrm{psi}}(\mathbf{c}, \kappa, m, k, i, \mathcal{U}). \end{aligned}$$

Then

(i) $\langle \tilde{\mathbf{T}}_1, \pi_1, \phi_1, \Psi_1\rangle$ is a (B, B) k-semicovering of $\mathbf{T}$ with respect to M such that

- *(a) both $\tilde{\mathbf{T}}_1$ and π_1 belong to M;*
- *(b) $M \models |\tilde{T}_1| \leq \kappa$;*
- *(c) $(\forall \tilde{p} \in \tilde{T}_1)(\ell\mathrm{h}(\tilde{p}) > \tilde{m} \rightarrow (M \models |(\tilde{T}_1)_p| < \kappa))$.*
- *(d) $(\forall n \in \omega)\, \phi_1 \restriction (\mathcal{S}({}_n\tilde{T}_1) \cap M) \in M$.*

(ii) $\langle \tilde{\mathbf{T}}_2, \pi_2, \phi_2, \Psi_2\rangle$ is a $(\lceil \mathbf{T} \rceil \setminus B, B)$ k-semicovering of $\mathbf{T}$ with respect to M such that

(a) *both* $\tilde{\mathbf{T}}_2$ *and* π_2 *belong to* M*;*
(b) $M \models |\tilde{T}_2| \leq \kappa$*;*
(c) $(\forall \tilde{p} \in \tilde{T}_2)(\ell\mathrm{h}(\tilde{p}) > \tilde{m} \to (M \models |(\tilde{T}_2)_p| < \kappa))$*;*
(d) $(\forall n \in \omega)\, \phi_2 \restriction (\mathcal{S}({}_n\tilde{T}_2) \cap M) \in M$*.*

Proof. The proof can readily be gotten from the proof of Lemma 4.2.2, together with the arguments given in the course of defining $\mathcal{F}^M_{\mathrm{psi}}$. □

5.2.4 Composition and Limit Operations

Our next task is to assemble the two basic kinds of operations needed for iterating the operations $\mathcal{F}_{\mathrm{t}}$, etc.: composition and the formation of limits.

Suppose that $\mathcal{C}_1 = \langle \mathbf{T}_1, \pi_1, \phi_1, \Psi_1 \rangle$ is a semicovering of $\mathbf{T}_0$ with respect to M_0 and that $\mathcal{C}_2 = \langle \mathbf{T}_2, \pi_2, \phi_2, \Psi_2, \rangle$ is a semicovering of $\mathbf{T}_1$ with respect to M_1. We define the *composition* $\mathcal{C}_1 \circ \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ to be

$$\langle \mathbf{T}_2, \pi_1 \circ \pi_2, \phi_1 \circ \phi_2, \Psi \rangle,$$

where $\Psi(\sigma, x) = \Psi_2(\sigma, \Psi_1(\phi_2(\sigma), x))$. (This definition implicitly determines domain (ϕ) and domain (Ψ).)

Lemma 5.2.13. *Let* $\mathbf{T}_0$ *be a game tree with taboos. Let* A*,* B_0*, and* B_1 *be subsets of* $\lceil \mathbf{T}_0 \rceil$*. Let* M_1 *be a transitive class model of* ZFC*. Let* $M_2 \subseteq M_1$*. Let* $\mathcal{C}_1 = \langle \mathbf{T}_1, \pi_1, \phi_1, \Psi_1 \rangle$ *be an* (A, B_0) *semicovering of* $\mathbf{T}_0$ *with respect to* M_1 *such that* $\mathbf{T}_1 \in M_1$*. Let* $\mathcal{C}_2 = \langle \mathbf{T}_2, \pi_2, \phi_2, \Psi_2, \rangle$ *be a* $(\boldsymbol{\pi}_1^{-1}(A), \boldsymbol{\pi}_1^{-1}(B_1))$ *semicovering of* $\mathbf{T}_1$ *with respect to* M_2 *such that* $\mathbf{T_2}$ *and* π_2 *belong to* M_1 *and such that* $\phi(\sigma) \restriction_n T_1$ *belongs to* M_1 *for every* $\sigma \in \mathcal{S}(T_2) \cap M_2$ *and every* $n \in \omega$*.*

Then $\mathcal{C}_1 \circ \mathcal{C}_2$ *is an* $(A, B_0 \cap B_1)$ *semicovering of* $\mathbf{T}$ *with respect to* M_2*. If* k_1 *and* k_2 *are natural numbers and if, for* $i \in \{1, 2\}$*,* $\mathcal{C}_i$ *is a* k_i*-semicovering with respect to* M_i*, then* $\mathcal{C}_1 \circ \mathcal{C}_2$ *is a* $\min\{k_1, k_2\}$*-semicovering with respect to* M_2*.*

Proof. Most of the proof is like the proofs of Lemmas 2.1.5 and 5.1.1. In addition we must show, for every strategy $\sigma \in \mathcal{S}(T_2)$ such that every $\sigma \restriction {}_n T_2 \in M_2$, that σ belongs to the domain of $\phi_1 \circ \phi_2$. This is true because the $\phi_2(\sigma) \restriction {}_n T_1$ belong to M_1 and $\mathcal{C}_1$ is a semicovering with respect to M_1. We leave the rest of the proof to the reader. □

The proof of Lemma 5.1.2 (or, more precisely, the construction given in the proof of Lemma 2.1.6) gives us four limit operations.

First of all, it gives us operations $\mathcal{I}_{\rm t}$ and $\mathcal{I}_{\rm pi}$. The domain of $\mathcal{I}_{\rm t}$ is the set of all

$$\langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i} \mid i \leq j \in \omega\rangle\rangle$$

such that

(1) if $i \leq j \in \omega$ then $\pi_{j,i} : \mathbf{T}_j \Rightarrow \mathbf{T}_i$, and $\pi_{i,i}$ is the identity;

(2) if $i_1 \leq i_2 \leq i_3 \in \omega$ then $\pi_{i_3,i_1} = \pi_{i_2,i_1} \circ \pi_{i_3,i_2}$;

(3) There exists $\langle k_{j,i} \mid i \leq j \in \omega\rangle$ such that each $k_{j,i} \in \omega$ and

  (a) $\underline{\lim}_{j\in\omega} \inf_{j'\geq j} k_{j',j} = \infty$;

  (b) if $i \leq j \in \omega$ then ${}_{k_{j,i}}\mathbf{T}_j = {}_{k_{j,i}}\mathbf{T}_i$;

  (c) if $i \leq j \in \omega$ then $\pi_{j,i} \restriction {}_{k_{j,i}}T_j$ is the identity.

The domain of $\mathcal{I}_{\rm pi}$ is domain$(\mathcal{I}_{\rm t}) \times \omega$. If $\mathbf{d} \in$ domain$\,\mathcal{I}_{\rm t}$, then $\mathcal{I}_{\rm t}(\mathbf{d})$ is the $\mathbf{T}_\infty$ given by the proof of 5.1.2, and for $j \in \omega$ the value $\mathcal{I}_{\rm pi}(\mathbf{d}, j)$ is the $\pi_{\infty,j}$ given by that proof. $\langle \mathcal{I}_{\rm t}(\mathbf{d}), \langle \mathcal{I}_{\rm pi}(\mathbf{d}, j) \mid j \in \omega\rangle\rangle$ is really just the inverse limit of the system $\mathbf{d}$. The existence of the inverse limit is guaranteed by clause (3).

We omit the proofs of the following two lemmas.

Lemma 5.2.14. *The operations $\mathcal{I}_{\rm t}$ and $\mathcal{I}_{\rm pi}$ are absolute for transitive class models of* ZFC.

Lemma 5.2.15. *Let M be a transitive class model of* ZFC*. Let $\mathbf{d} = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i} \mid i \leq j \in \omega\rangle\rangle \in$ domain$(\mathcal{I}_{\rm t}) \cap M$. Let $\mathbf{T}_\infty = \mathcal{I}_{\rm t}(\mathbf{d})$ and for $i \in \omega$ let $\pi_{\infty,i} = \mathcal{I}_{\rm pi}(\mathbf{d}, i)$.*

(1) $\mathbf{T}_\infty$ is a game tree with taboos;

(2) if $i \in \omega$, then $\pi_{\infty,i} : \mathbf{T}_\infty \Rightarrow \mathbf{T}_i$;

(3) $(\forall i \in \omega)(\forall j \in \omega)(i \leq j \to \pi_{\infty,i} = \pi_{j,i} \circ \pi_{\infty,j})$;

(4) if $\langle k_{j,i} \mid i \leq j \in \omega\rangle$ witnesses for $\mathbf{d}$ clause (3) in the definition of domain$(\mathcal{I}_{\rm t})$ and if $\inf_{i\leq j\in\omega} k_{j,i} \geq k$, then

  (a) for all $i \in \omega$, ${}_k\mathbf{T}_\infty = {}_k\mathbf{T}_i$;

  (b) $\pi_{\infty,i} \restriction {}_k\mathbf{T}_\infty$ is the identity.

If $\mathbf{d} = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i} \mid i \leq j \in \omega\rangle\rangle$ and if $n \in \omega$, then let

$$\mathbf{d}^n = \langle\langle \mathbf{T}_{n+i} \mid i \in \omega\rangle, \langle \pi_{n+j,n+i} \mid i \leq j \in \omega\rangle\rangle.$$

Similarly, if $\mathbf{d}^* = \langle \mathbf{T}_\infty, \langle \pi_{\infty,j} \mid i \leq j \in \omega\rangle\rangle$ and if $n \in \omega$, then let

$$(\mathbf{d}^*)^n = \langle \mathbf{T}_\infty, \langle \pi_{\infty,n+j} \mid i \leq j \in \omega\rangle\rangle.$$

An inspection of the construction in the proof of Lemma 2.1.6 shows that the following lemma holds:

Lemma 5.2.16. *If* $\mathbf{d} \in \operatorname{domain}(\mathcal{I}_{\mathrm{t}})$ *and* $n \in \omega$, *then* $\mathbf{d}^n \in \operatorname{domain}(\mathcal{I}_{\mathrm{t}})$, $\mathcal{I}_{\mathrm{t}}(\mathbf{d}^n) = (\mathcal{I}_{\mathrm{t}}(\mathbf{d}))^n$, *and*

$$\langle \mathcal{I}_{\mathrm{pi}}(\mathbf{d}^n, i) \mid i \in \omega\rangle = (\langle \mathcal{I}_{\mathrm{pi}}(\mathbf{d}, i) \mid i \in \omega\rangle)^n.$$

Note that the T_∞ given by the proof of Lemma 5.1.2 is a subset of the union of the T_i. Thus we have the following absoluteness result:

Lemma 5.2.17. *Let* $\mathbf{d} = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i} \mid i \leq j \in \omega\rangle\rangle$ *belong to the domain of* $\mathcal{I}_{\mathrm{t}}$. *Let* M *be any transitive class model of* ZFC *to which* $\mathbf{T}_\infty = \mathcal{I}_{\mathrm{t}}(\mathbf{d})$ *and* $\langle \mathbf{T}_i \mid i \in \omega\rangle$ *belong. Then*

$$M \models |T_\infty| \leq \sup_{i\in\omega} |T_i|.$$

A third kind of operation that can be extracted from the construction of Lemma 2.1.6 is given as follows. Let $\langle M_i \mid i \in \omega\rangle$ be a sequence of transitive class models of ZFC such that $M_i \supseteq M_j$ for $i \leq j \in \omega$ and such that, for all $i \in \omega$, the subsequence $\langle M_{i+j} \mid j \in \omega\rangle$ is a class in M_i, in the sense that $\{\langle j, a\rangle \mid a \in M_{i+j}\}$ is a class in M_i (is a subclass of M_i and is definable in M_i from members of M_i). Then we have an operation $\mathcal{I}_{\mathrm{phi}}^{\langle M_i \mid i\in\omega\rangle}$ whose domain is the set of all $\langle \mathbf{d}', n\rangle$ with $n \in \omega$ and with

$$\mathbf{d}' = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i} \mid i \leq j \in \omega\rangle\rangle,$$

where

(1) for each $i \in \omega$, $\langle\langle \mathbf{T}_j \mid i \leq j \in \omega\rangle, \langle \pi_{j',j} \mid i \leq j \leq j' \in \omega\rangle\rangle \in M_i$;

(2) $\langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i} \mid i \leq j \in \omega\rangle\rangle \in \operatorname{domain}(\mathcal{I}_{\mathrm{t}})$;

(3) if $i \le j \in \omega$ then domain $(\phi_{j,i}) = \{\sigma \in \mathcal{S}(T_j) \mid (\forall n \in \omega)\, \sigma \restriction {}_nT_j \in M_j\}$ and $\phi : \tilde{\mathbf{T}}_j \overset{\mathcal{S},M_j}{\Rightarrow} \mathbf{T}_i$; $\phi_{i,i}$ is the identity;

(4) if $i \le j \in \omega$, if $\sigma \in \mathcal{S}(T_j) \cap M_j$, and if $n \in \omega$, then $\phi_{j,i}(\sigma) \restriction {}_nT_i \in M_i$;

(5) if $i_1 \le i_2 \le i_3 \in \omega$ then $\phi_{i_3,i_1} = \phi_{i_2,i_1} \circ \phi_{i_3,i_2}$;

(6) there exists $\langle k_{j,i} \mid i \le j \in \omega\rangle$ such that

(a) $\langle k_{j,i} \mid i \le j \in \omega\rangle$ satisfies for $\langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i} \mid i \le j \in \omega\rangle\rangle$ clause (3) in the definition of domain $(\mathcal{I}_{\mathrm{t}})$;

(b) if $i \le j \in \omega$ then $\phi_{j,i} \restriction \mathcal{S}({}_{k_{j,i}}T) \cap M_j$ is the identity.

Remark. Clause (4) actually follows from clauses (3) and (5).

For $\langle \mathbf{d}', n\rangle \in$ domain $(\mathcal{I}_{\mathrm{phi}}^{\langle M_i \mid i \in \omega\rangle})$, the value $\mathcal{I}_{\mathrm{phi}}^{\langle M_i \mid i \in \omega\rangle}(\mathbf{d}', n)$ is the $\phi_{\infty,n}$ given by the proof of Lemma 5.1.2. The $\phi_{\infty,n}$ have the common domain

$$\{\sigma \in \mathcal{S}(T_\infty) \mid (\forall n \in \omega)\, \sigma \restriction {}_nT_\infty \in \bigcap_{i \in \omega} M_i\}.$$

Lemma 5.2.18. *Let $\langle M_i \mid i \in \omega\rangle$ be such that $\mathcal{I}_{\mathrm{phi}}^{\langle M_i \mid i \in \omega\rangle}$ is defined. Let $\mathbf{d}' = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i} \mid i \le j \in \omega\rangle\rangle$ be such that, for $j \in \omega$, $\langle \mathbf{d}', j\rangle$ belongs to the domain of $\mathcal{I}_{\mathrm{phi}}^{\langle M_i \mid i \in \omega\rangle}$. Let $\mathbf{d}$ be the member of* domain $(\mathcal{I}_{\mathrm{t}})$ *associated with $\mathbf{d}'$. Let $\mathbf{T}_\infty = \mathcal{I}_{\mathrm{t}}(\mathbf{d})$. For $i \in \omega$ let $\pi_{\infty,i} = \mathcal{I}_{\mathrm{pi}}(\mathbf{d}, i)$ and let $\phi_{\infty,i} = \mathcal{I}_{\mathrm{phi}}^{\langle M_i \mid i \in \omega\rangle}(\mathbf{d}', i)$. Then*

(a) if $n \in \omega$ then $\langle \phi_{\infty,i} \restriction {}_nT_\infty \mid i \in \omega\rangle \in M_0$;

(b) if $i \in \omega$ then $\phi_{\infty,i} : \mathbf{T}_\infty \overset{\mathcal{S},\bigcap_{j\in\omega} M_j}{\Longrightarrow} \mathbf{T}_i$;

(c) if $i \le j \in \omega$ then $\phi_{\infty,i} = \phi_{j,i} \circ \phi_{\infty,j}$;

(d) if $i \in \omega$, if $\sigma \in$ domain $(\phi_{\infty,i})$, and if $n \in \omega$, then $\phi_{\infty,i}(\sigma) \restriction {}_nT_i \in M_i$; if, for each $i \in \omega$ and each $n \in \omega$, $\langle \phi_{j',j} \restriction (\mathcal{S}({}_nT_{j'}) \cap M_{j'}) \mid i \le j \le j' \in \omega\rangle \in M_i$, then $\langle \phi_{\infty,j} \restriction \mathcal{S}({}_nT_\infty) \mid i \le j \in \omega\rangle$ belongs to M_i;

(e) if $\langle k_{j,i} \mid i \le j \in \omega\rangle$ witnesses clause (6) in the definition of domain $(\mathcal{I}_{\mathrm{phi}}^{\langle M_i \mid i \in \omega\rangle})$*, and if $\inf_{i \le j \in \omega} k_{j,i} \ge k$, then, for each $i \in \omega$, $\phi_{\infty,i} \restriction (\mathcal{S}({}_kT_i) \cap M_i)$ is the identity.*

If $\mathbf{d}' = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i} \mid i \leq j \in \omega\rangle\rangle$ and if $n \in \omega$, then let

$$\mathbf{d}'^n = \langle\langle \mathbf{T}_{n+i} \mid i \in \omega\rangle, \langle \pi_{n+j,n+i}, \phi_{j+n,i+n} \mid i \leq j \in \omega\rangle\rangle.$$

Similarly, if $\mathbf{d}'^* = \langle \mathbf{T}_\infty, \langle \pi_{\infty,j}, \phi_{\infty,j} \mid i \leq j \in \omega\rangle\rangle$ and if $n \in \omega$, then let

$$(\mathbf{d}'^*)^n = \langle \mathbf{T}_\infty, \langle \pi_{\infty,n+j}, \phi_{\infty,n+j} \mid i \leq j \in \omega\rangle\rangle.$$

As with Lemma 5.2.16, we have the following:

Lemma 5.2.19. *If $n \in \omega$ and if $\mathbf{d}'$ is such that $\langle \mathbf{d}', j\rangle$ belongs to* domain $(\mathcal{I}_{\mathrm{phi}}^{\langle M_{n+i} | i\in\omega\rangle})$ *for all $j \in \omega$, then $(\forall j \in \omega)\langle \mathbf{d}'^n, j\rangle \in$* domain $(\mathcal{I}_{\mathrm{phi}}^{\langle M_i | i\in\omega\rangle})$ *and*

$$\langle \mathcal{I}_{\mathrm{phi}}^{\langle M_{n+i} | i\in\omega\rangle}(\mathbf{d}^n, i) \mid i \in \omega\rangle \;=\; (\langle \mathcal{I}_{\mathrm{phi}}^{\langle M_i | i\in\omega\rangle}(\mathbf{d}, i) \mid i \in \omega\rangle)^n.$$

If $\langle M_i \mid i \in \omega\rangle$ is such that the operation $\mathcal{I}_{\mathrm{phi}}^{\langle M_i | i\in\omega\rangle}$ is defined, then the construction in the proof of Lemma 2.1.6 yields also an operation $\mathcal{I}_{\mathrm{psi}}^{\langle M_i | i\in\omega\rangle}$. The domain of $\mathcal{I}_{\mathrm{psi}}^{\langle M_i | i\in\omega\rangle}$ is the set of all $\langle \mathbf{d}'', n\rangle$ with $n \in \omega$ and with

$$\mathbf{d}'' \;=\; \langle\langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \mid i \leq j \in \omega\rangle\rangle,$$

where

(1) $\langle\langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i}, \mid i \leq j \in \omega\rangle\rangle, n\rangle$ belongs to the domain of $\mathcal{I}_{\mathrm{phi}}^{\langle M_i | i\in\omega\rangle}$;

(2) if $i \leq j \in \omega$, then $\langle \mathbf{T}_j, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j}\rangle$ satisfies clause (d) in the definition of a semicovering of $\mathbf{T}_i$;

(3) if $i_1 \leq i_2 \leq i_3 \in \omega$ and $\langle \sigma, x\rangle \in$ domain (Ψ^{i_1,i_3}), then $\Psi^{i_1,i_3}(\sigma, x) = \Psi^{i_2,i_3}(\sigma, \Psi^{i_1,i_2}(\phi_{i_3,i_2}(\sigma), x))$.

If $\langle \mathbf{d}'', n\rangle \in$ domain $(\mathcal{I}_{\mathrm{psi}}^{\langle M_i | i\in\omega\rangle})$, then $\mathcal{I}_{\mathrm{psi}}^{\langle M_i | i\in\omega\rangle}(\mathbf{d}'', n)$ is just the $\Psi^{n,\infty}$ given by the proof of Lemma 5.1.2.

Lemma 5.2.20. *Let $k \in \omega$. Let $\mathbf{T}_i$, $i \in \omega$, be game trees with taboos. Let M_i, $i \in \omega$ be transitive class models of* ZFC *such that, for each $i \in \omega$, the sequence $\langle M_j \mid i \leq j \in \omega\rangle$ is a a class in M_i (in the sense described on page 238). Let A and B_i, $i \in \omega$, be subsets of $\lceil \mathbf{T}_0 \rceil$. Let $k \in \omega$ and $\langle k_{j,i}, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \mid i \leq j \in \omega\rangle$ be such that*

(1) if $i \leq j \in \omega$ then $\mathcal{C}_{j,i} = \langle \mathbf{T}_j, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \rangle$ is a

$$(\boldsymbol{\pi}_{0,i}{}^{-1}(A), \boldsymbol{\pi}_{0,i}{}^{-1}(\bigcap_{i \leq n < j} (B_n))$$

$k_{j,i}$-semicovering of $\mathbf{T}_i$ with respect to M_j;

(2) if $i_1 \leq i_2 \leq i_3 \in \omega$ then $\mathcal{C}_{i_3,i_1} = \mathcal{C}_{i_2,i_1} \circ \mathcal{C}_{i_3,i_2}$;

(3) $\inf_{i \leq j \in \omega} k_{j,i} \geq k$;

(4) $\underline{\lim}_{j \in \omega} \inf_{j' \geq j} k_{j',j} = \infty$;

(5) $(\forall i \in \omega)\, \langle \mathbf{T}_j, \pi_{j,j'} \mid i \leq j' \leq j \in \omega \rangle \in M_i$;

(6) if $i \leq j \in \omega$, then

(a) $\mathrm{domain}\,(\phi_{j,i}) = \{\sigma \in \mathcal{S}(T_j) \mid (\forall n \in \omega)\, \sigma \restriction_n T_j \in M_j\}$;

(b) if $\sigma \in \mathcal{S}(T_j) \cap M_j$ and $n \in \omega$, then $\phi_{j,i}(\sigma) \restriction_n T_i \in M_i$.

Let

$$\mathbf{T}_\infty = \mathcal{I}_{\mathrm{t}}(\langle \mathbf{T}_i \mid i \in \omega \rangle, \langle \pi_{j,i} \mid i \leq j \in \omega \rangle).$$

For $n \in \omega$, let

$$\begin{array}{rcl} \pi_{\infty,n} & = & \mathcal{I}_{\mathrm{pi}}(\langle \langle \mathbf{T}_i \mid i \in \omega \rangle, \langle \pi_{j,i} \mid i \leq j \in \omega \rangle \rangle, n); \\ \phi_{\infty,n} & = & \mathcal{I}_{\mathrm{phi}}^{\langle M_i \mid i \in \omega \rangle}(\langle \langle \mathbf{T}_i \mid i \in \omega \rangle, \langle \pi_{j,i}, \phi_{j,i} \mid i \leq j \in \omega \rangle \rangle, n); \\ \Psi^{n,\infty} & = & \mathcal{I}_{\mathrm{psi}}^{\langle M_i \mid i \in \omega \rangle}(\langle \langle \mathbf{T}_i \mid i \in \omega \rangle, \langle \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \mid i \leq j \in \omega \rangle \rangle, n). \end{array}$$

Then

(a) each $\mathcal{C}_{\infty,i} = \langle \mathbf{T}_\infty, \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \rangle$ is a

$$(\boldsymbol{\pi}_{\infty,i}{}^{-1}(A), \boldsymbol{\pi}_{\infty,i}{}^{-1}(\bigcap_{i \leq n \in \omega} B_n))$$

k-semicovering of $\mathbf{T}_i$ with respect to $\bigcap_{i \leq j \in \omega} M_j$;

(b) $(\forall i \in \omega)(\forall j \in \omega)(i \leq j \to \mathcal{C}_{\infty,i} = \mathcal{C}_{j,i} \circ \mathcal{C}_{\infty,j})$.

Proof. The only thing left to check is the properties of the $\Psi^{i,\infty}$. We leave this task to the reader. □

5.2.5 Operations for Unraveling ω–$\mathbf{\Pi}^1_1$ Sets

We have now assembled all the ingredients necessary for iterating the operations $\mathcal{F}_{\mathrm{t}}$, etc. Our first step in this iteration process is to define some operations $\bar{\mathcal{F}}_{\mathrm{t}}$, etc. that do for ω–$\mathbf{\Pi}^1_1$ pretty much what the operations $\mathcal{F}_{\mathrm{t}}$, etc. do for $\mathbf{\Pi}^1_1$.

First we define operations $\bar{\mathcal{F}}_{\mathrm{t}}$ and $\bar{\mathcal{F}}_{\mathrm{pi}}$. The common domain of these two operations is the set of all $\langle\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k\rangle$ such that

(i) $\langle \mathbf{c}_i \mid i \in \omega\rangle$ is a sequence of $\mathbf{\Pi}^1_1$ codes, all with the same first component;

(ii) κ is an ordinal number;

(iii) $k \in \omega$.

For $\langle\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k\rangle$ belonging to this common domain, we define $\bar{\mathcal{F}}_{\mathrm{t}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$ and $\bar{\mathcal{F}}_{\mathrm{pi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$ as follows:

First we define by induction a system

$$\mathbf{d} = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i} \mid i \leq j \in \omega\rangle\rangle.$$

Let $\mathbf{T}_0$ be the first component of the $\mathbf{c}_i$. Assume that $\mathbf{T}_i$ and $\pi_{j,i}$ have been defined for all $i \leq j \leq n \in \omega$ and that they satisfy:

(1) if $i \leq j \leq n$ then $\pi_{j,i} : \mathbf{T}_j \Rightarrow \mathbf{T}_i$, and $\pi_{i,i}$ is the identity;

(2) if $i_1 \leq i_2 \leq i_3 \leq n$ then $\pi_{i_3,i_1} = \pi_{i_2,i_1} \circ \pi_{i_3,i_2}$;

For $i \in \omega$, let $\mathbf{c}_i = \langle \mathbf{T}_0, E_i, f_i\rangle$, where f_i is $p \mapsto <^i_p$. For $i \in \omega$, let

$$\mathbf{c}^n_i = \langle \mathbf{T}_n, \boldsymbol{\pi}^{-1}_{n,0}(E_i), f^n_i\rangle,$$

where

$$f^n_i(p) = <^i_{\pi_{n,0}(p)} .$$

Thus $\mathbf{c}^n_i$ is a $\mathbf{\Pi}^1_1$ code for $\boldsymbol{\pi}^{-1}_{n,0}(p)$.

Now let

$$\begin{aligned} \mathbf{T}_{n+1} &= \mathcal{F}_{\mathrm{t}}(\mathbf{c}^n_n, \kappa, m_n, k+n, i_n); \\ \pi_{n+1,n} &= \mathcal{F}_{\mathrm{pi}}(\mathbf{c}^n_n, \kappa, m_n, k+n, i_n), \end{aligned}$$

where $m_n = k + n$, and where i_n is 1 if n is even and 2 if n is odd. For $i < n$, let $\pi_{n+1,i} = \pi_{n,i} \circ \pi_{n+1,n}$. It is clear that (1) and (2) are satisfied.

Conditions (1), (2), and (3) on membership in domain $(\mathcal{I}_{\rm t})$ are satisfied by the system $\mathbf{d}$ we have defined. The values $k_{j,i} = k + i$ witness that condition (4) is satisfied as well. Thus we may let

$$\begin{aligned}\bar{\mathcal{F}}_{\rm t}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k) &= \mathcal{I}_{\rm t}(\mathbf{d});\\ \bar{\mathcal{F}}_{\rm pi}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k) &= \mathcal{I}_{\rm pi}(\mathbf{d}, 0).\end{aligned}$$

By the definitions and by Lemmas 5.2.8 and 5.2.14, we get the following lemma:

Lemma 5.2.21. *The operations $\bar{\mathcal{F}}_{\rm t}$ and $\bar{\mathcal{F}}_{\rm pi}$ are absolute for transitive class models of* ZFC.

We also have the following analogue of Lemma 5.2.9:

Lemma 5.2.22. *Let M be a transitive class model of* ZFC. *Let*

$$\langle\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k\rangle \in \text{domain}\,(\bar{\mathcal{F}}_{\rm t}) \cap M,$$

with $\mathbf{T}$ the common first component of the $\mathbf{c}_i$. Let $\tilde{\mathbf{T}} = \bar{\mathcal{F}}_{\rm t}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$ and let $\pi = \bar{\mathcal{F}}_{\rm pi}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$. Then

(a) $\tilde{\mathbf{T}}$ is a game tree with taboos;

(b) $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$;

(c) $\tilde{\mathbf{T}}$ and π belong to M;

(d) if $M \models |T| < |\kappa|$ then $M \models |\tilde{T}| \leq |\kappa|$;

(e) ${}_k\tilde{\mathbf{T}} = {}_k\mathbf{T}$;

(f) $\pi \restriction {}_k\tilde{T}$ is the identity;

(g) Every normal play in $\tilde{\mathbf{T}}$ belongs to the intersection of the $\mathbf{\Pi}^1_1$ sets coded by the $\mathbf{c}_i$.

Proof. The lemma follows easily from the definitions and Lemmas 5.2.9, 5.2.15, 5.2.17, and 5.2.21. Perhaps a word about clause (d) is in order. The hypothesis of (d) guarantees that the hypothesis of clause (d) of Lemma 5.2.9 holds, with $m = 0$, for the $\mathbf{T}_0$ of our inductive construction, namely for $\mathbf{T}$. The construction gives inductively that the hypothesis of clause (d) holds, with $m = k + n$, for the $\mathbf{T}_n$ of the construction. The concnlusion of clause (d) then follows by Lemma 5.2.17.

□

We next define, for each transitive class model M of ZFC, an operation $\bar{\mathcal{F}}^M_{\rm phi}$ whose domain is the set of all $\langle\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U}\rangle \in M$ such that

(i) $\langle\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k\rangle \in \text{domain}\,(\mathcal{F}_{\rm t})$;

(ii) $M \models |T| < \kappa$, where T is the common first component of the $\mathbf{c}_i$;

(iii) $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ."

To define $\bar{\mathcal{F}}^M_{\rm phi}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U})$, we repeat the construction of the $\mathbf{T}_i$ and the $\pi_{j,i}$ used in defining $\bar{\mathcal{F}}_{\rm t}$ and $\bar{\mathcal{F}}_{\rm pi}$, except that we also define $\phi_{j,i}$ for $i \leq j \in \omega$, thus producing a system

$$\mathbf{d}' = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i} \mid i \leq j \in \omega\rangle\rangle.$$

We assume inductively that

(1) if $i \leq j \leq n$ then $\phi_{j,i} : \mathbf{T}_j \overset{\mathcal{S},M}{\Rightarrow} \mathbf{T}_i$;

(2) if $i_1 \leq i_2 \leq i_3 \leq n$ then $\phi_{i_3,i_1} = \phi_{i_2,i_1} \circ \phi_{i_3,i_2}$.

(3) if $m \in \omega$ and $i \leq j \leq n$ then $\phi_{j,i} \restriction \mathcal{S}({}_mT_j) \in M$;

In the induction step, we set

$$\phi_{n+1,n} = \mathcal{F}^M_{\rm phi}(\mathbf{c}^n_n, \kappa, m_n, k+n, i_n, \mathcal{U}).$$

For $i < n$ we let $\phi_{n+1,i} = \phi_{n,i} \circ \phi_{n+1,n}$. We finally define

$$\bar{\mathcal{F}}^M_{\rm phi}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U}) \;=\; \mathcal{I}^{\langle M \mid i \in \omega\rangle}_{\rm phi}(\mathbf{d}', 0).$$

Lemma 5.2.23. *Let M be a transitive class model of* ZFC*. Let*

$$\langle\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U}\rangle \in \text{domain}\,(\bar{\mathcal{F}}^M_{\rm phi})$$

be such that $\mathbf{T}$ is the first component of all the $\mathbf{c}_i$. Let $\tilde{\mathbf{T}} = \bar{\mathcal{F}}_{\rm t}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$ and let $\phi = \bar{\mathcal{F}}^M_{\rm phi}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U})$. Then

(a) $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S},M}{\Rightarrow} \mathbf{T}$;

(b) $\phi \restriction (\mathcal{S}({}_k\tilde{T}) \cap M)$ is the identity;

(c) for all $n \in \omega$, $\phi \restriction (\mathcal{S}({}_n\tilde{T}) \cap M) \in M$.

Finally we define, for each transitive class model M of ZFC, an operation $\bar{\mathcal{F}}^M_{\mathrm{psi}}$ whose domain is the set of all $\langle\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U}\rangle \in \mathrm{domain}\,(\bar{\mathcal{F}}^M_{\mathrm{phi}})$ such that every intersection of countably many elements of $\mathcal{U}$ is nonempty. The definition of $\bar{\mathcal{F}}^M_{\mathrm{psi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U})$ is the obvious one: We repeat the construction of the sytem $\mathbf{d}'$, execept that we also define $\Psi^{i,j}$ for $i \leq j \in \omega$, producing a system

$$\mathbf{d}'' = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \mid i \leq j \in \omega\rangle\rangle.$$

In the induction step, we set

$$\Psi^{n,n+1} = \mathcal{F}^M_{\mathrm{psi}}(\mathbf{c}^n_n, \kappa, k, \mathcal{U}).$$

$\Psi^{i,n+1}(\sigma, x) = \Psi^{n,n+1}(\sigma, \Psi^{i,n}(\phi_{n+1,n}(\sigma), x))$. We finally define

$$\bar{\mathcal{F}}^M_{\mathrm{psi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U}) = \mathcal{I}^{\langle M_i \mid i \in \omega\rangle}_{\mathrm{psi}}(\mathbf{d}'', 0).$$

We are almost ready to state our analogue of Lemma 5.2.12. Before doing so, we note that Lemma 5.1.3 holds also for semicoverings with respect to M:

Lemma 5.2.24. *Let $\mathbf{T}$ be a game tree with taboos and let M be any class. Let α be a countable ordinal and let $\langle A_\beta \mid \beta < \alpha\rangle$ witness that $A \in \alpha$–$\mathbf{\Pi}^1_1$. Let $\gamma < \alpha$ and let A' be the set witnessed to belong to γ–$\mathbf{\Pi}^1_1$ by $\langle A_\beta \mid \beta < \gamma\rangle$.*

Then every $(A', \bigcap_{\beta<\gamma} A_\beta)$ semicovering of $\mathbf{T}$ with respect to M is an $(A, \bigcap_{\beta<\gamma} A_\beta)$ semicovering of $\mathbf{T}$ with respect to M.

Proof. The proof of Lemma 5.1.3 goes through without change. □

Here finally is the analogue of Lemma 5.2.12. The only extra ingredient beyond what would be in a pure analogue of the earlier lemma is the appearance of the model N. In our applications, the model N will be $\mathrm{Ult}_{\omega_1}(M; \mathcal{U})$.

Lemma 5.2.25. *Let N and $M \supseteq N$ be transitive class models of* ZFC. *Let $\langle \mathbf{c}_i \mid i \in \omega\rangle \in N$ be a sequence of $\mathbf{\Pi}^1_1$ codes with the same first component $\mathbf{T}$. For $i \in \omega$ let A_i be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}_i$. Let A be the subset of $\lceil \mathbf{T} \rceil$ that $\langle A_i \mid i \in \omega\rangle$ witnesses to belong to ω–$\mathbf{\Pi}^1_1$. Let κ be an ordinal that is a measurable cardinal in the model M. Let $\mathcal{U} \in M$ be such that $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ" and such that every intersection of countably* *many elements of $\mathcal{U}$ is nonempty. Let $k \in \omega$. Suppose that $M \models |T| < \kappa$. Let*

$$\begin{aligned}
\tilde{\mathbf{T}} &= \bar{\mathcal{F}}_{\mathrm{t}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k);\\
\pi &= \bar{\mathcal{F}}_{\mathrm{pi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k);\\
\phi &= \bar{\mathcal{F}}^M_{\mathrm{phi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U});\\
\Psi &= \bar{\mathcal{F}}^M_{\mathrm{psi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U}).
\end{aligned}$$

Then

- *(i) $\langle \tilde{\mathbf{T}}, \pi_1, \phi_1, \Psi_1\rangle$ is a $(A, \bigcap_{i\in\omega} A_i)$ k-semicovering of $\mathbf{T}$ with respect to M ;*
- *(ii) both $\tilde{\mathbf{T}}$ and π belong to N, and $N \models |\tilde{T}| \leq \kappa$;*
- *(iii) for all $n \in \omega$, $\phi \restriction (\mathcal{S}({}_n\tilde{T}) \cap M)$ belongs to M.*

Proof. The lemma follows easily from our earlier lemmas. In particular, Lemma 5.2.24 giving that the $\langle \mathbf{T}_i, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j}\rangle$ occurring in the definition of Ψ are $(\boldsymbol{\pi}_{i,0}(A), \bigcap_{i\leq n<j} A_n)$ semicoverings, as required for applying Lemma 5.2.20. That $N \models |T| \leq \kappa$ can be seen as follows. Since κ is a cardinal in M and $M \models |T| < \kappa$, it must be that $N \models |T| < \kappa$. Thus Lemma 5.2.22 is applicable with N as the M of that Lemma. Clause (c) of that lemma gives that $N \models |\tilde{T}| \leq \kappa$. □

5.2.6 Operations for Unraveling ω^2–$\mathbf{\Pi}^1_1$ Sets

We are now going to define operations $\bar{\bar{\mathcal{F}}}_{\mathrm{t}}$, etc. that do for ω^2–$\mathbf{\Pi}^1_1$ something like what the operations $\bar{\mathcal{F}}_t$, etc. do for ω–$\mathbf{\Pi}^1_1$. In doing so, we will at last use iterated ultrapowers with respect to an ultrafilter $\mathcal{U}$.

We begin with the operations $\bar{\bar{\mathcal{F}}}_{\mathrm{t}}$ and $\bar{\bar{\mathcal{F}}}_{\mathrm{pi}}$. The common domain of these two operations is the set of all $\langle\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle \lambda_i \mid i \in \omega\rangle, k\rangle$ such that

- (i) $\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle$ is a sequence of $\mathbf{\Pi}^1_1$ codes, all with the same first component;
- (ii) $\langle \lambda_i \mid i \in \omega\rangle$ is an increasing sequence of ordinal numbers;
- (iii) $k \in \omega$.

For $\langle\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle\lambda_i \mid i \in \omega\rangle, k\rangle$ belonging to this common domain, we define $\bar{\bar{\bar{\mathcal{F}}}}_{\mathrm{t}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle\lambda_i \mid i \in \omega\rangle, k)$ and $\bar{\bar{\bar{\mathcal{F}}}}_{\mathrm{pi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle\lambda_i \mid i \in \omega\rangle, k)$ as follows.

First we define by induction a system

$$\mathbf{d} = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i} \mid i \leq j \in \omega\rangle\rangle.$$

Let $\mathbf{T}_0$ be the first component of the $\mathbf{c}_\beta$. Assume that $\mathbf{T}_i$ and $\pi_{j,i}$ have been defined for all $i \leq j \leq n \in \omega$ and that they satisfy:

(1) if $i \leq j \leq n$ then $\pi_{j,i} : \mathbf{T}_j \Rightarrow \mathbf{T}_i$, and $\pi_{i,i}$ is the identity;

(2) if $i_1 \leq i_2 \leq i_3 \leq n$ then $\pi_{i_3,i_1} = \pi_{i_2,i_1} \circ \pi_{i_3,i_2}$;

For $\beta < \omega^2$, let $\mathbf{c}_\beta = \langle \mathbf{T}_0, E_\beta, f_\beta\rangle$, where f_β is $p \mapsto <^\beta_p$. For $\beta < \omega^2$, let

$$\mathbf{c}^n_\beta = \langle \mathbf{T}_n, \boldsymbol{\pi}^{-1}_{n,0}(E_\beta), f^n_\beta\rangle,$$

where

$$f^n_\beta(p) = <^\beta_{\pi_{n,0}(p)} .$$

If $\mathbf{c}_\beta$ is a $\mathbf{\Pi}^1_1$ code for a set B, then $\mathbf{c}^n_\beta$ is a $\mathbf{\Pi}^1_1$ code for $\boldsymbol{\pi}^{-1}_{n,0}(B)$.

Now let

$$\begin{aligned} \mathbf{T}_{n+1} &= \bar{\mathcal{F}}_{\mathrm{t}}(\langle \mathbf{c}^n_{\omega n+i} \mid i \in \omega\rangle, \lambda_n, k+n) \\ \pi_{n+1} &= \bar{\mathcal{F}}_{\mathrm{pi}}(\langle \mathbf{c}^n_{\omega n+i} \mid i \in \omega\rangle, \lambda_n, k+n). \end{aligned}$$

For $i < n$, let $\pi_{n+1,i} = \pi_{n,i} \circ \pi_{n+1,n}$. It is clear that (1) and (2) are satisfied.

Conditions (1), (2), and (3) on membership in domain $(\mathcal{I}_{\mathrm{t}})$ are satisfied by the system $\mathbf{d}$ we have defined. The values $k_{j,i} = k + i$ witness that condition (4) is satisfied as well. Thus we may let

$$\begin{aligned} \bar{\bar{\bar{\mathcal{F}}}}_{\mathrm{t}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle\lambda_i \mid i \in \omega\rangle, k) &= \mathcal{I}_{\mathrm{t}}(\mathbf{d}); \\ \bar{\bar{\bar{\mathcal{F}}}}_{\mathrm{pi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle\lambda_i \mid i \in \omega\rangle, k) &= \mathcal{I}_{\mathrm{pi}}(\mathbf{d}, 0). \end{aligned}$$

By the definitions and by Lemmas 5.2.21 and 5.2.14, we get the following lemma:

Lemma 5.2.26. *The operations $\bar{\bar{\bar{\mathcal{F}}}}_{\mathrm{t}}$ and $\bar{\bar{\bar{\mathcal{F}}}}_{\mathrm{pi}}$ are absolute for transitive class models of* ZFC.

We also have the following analogue of Lemma 5.2.22:

Lemma 5.2.27. *Let M be a transitive class model of* ZFC*. Let*

$$\langle\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle\lambda_i \mid i \in \omega\rangle, k\rangle \in \text{domain}\,(\bar{\bar{\mathcal{F}}}_{\text{t}}) \cap M$$

with $\mathbf{T}$ the common first component of the $\mathbf{c}_i$. Let $\tilde{\mathbf{T}} = \bar{\bar{\mathcal{F}}}_{\text{t}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle\lambda_i \mid i \in \omega\rangle, k)$ and let $\pi = \bar{\bar{\mathcal{F}}}_{\text{pi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle\lambda_i \mid i \in \omega\rangle, k)$. Then

(a) $\tilde{\mathbf{T}}$ is a game tree with taboos;

(b) $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$;

(c) both $\tilde{\mathbf{T}}$ and π belong to M;

(d) if $M \models |T| < |\lambda|$ then $M \models |\tilde{T}| \leq \sup_{i\in\omega}|\lambda_i|$;

(e) ${}_k\tilde{\mathbf{T}} = {}_k\mathbf{T}$;

(f) $\pi \restriction {}_k\tilde{T}$ is the identity;

(g) Every normal play in $\tilde{\mathbf{T}}$ belongs to the intersection of the $\mathbf{\Pi}^1_1$ sets coded by the $\mathbf{c}_\beta$.

Proof. The lemma follows easily from the definitions and Lemmas 5.2.22, 5.2.15, and 5.2.26. □

We next define, for each uncountable transitive class model M of ZFC, an operation $\bar{\bar{\mathcal{F}}}^M_{\text{phi}}$. To make this definition, we introduce the following notation: If M is an uncountable transitive class model of ZFC satisfying "$\mathcal{U}$ is a uniform normal ultrafilter on the measurable cardinal κ," then, for all $n \in \omega$, let

$$\begin{aligned} M_n(\mathcal{U}) &= \text{Ult}_{\omega_1(n+1)}(M;\mathcal{U}); \\ \lambda_n(\mathcal{U}) &= i^M_{\mathcal{U}\ 0,\omega_1(n+1)}(\kappa) \\ \mathcal{V}_n(\mathcal{U}) &= i^M_{\mathcal{U}\ 0,\omega_1(n+1)}(\mathcal{U}). \end{aligned}$$

Remark. Lemma 3.3.12 implies that, for each $n \in \omega$, every intersection of countably many elements of $\mathcal{V}_n(\mathcal{U})$ is nonempty.

The domain of $\bar{\bar{\mathcal{F}}}^M_{\text{phi}}$ is the set of all $\langle\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \kappa, k, \mathcal{U}\rangle \in M$ such that

(i) $M \models$ "$|T| < \kappa$," where $\mathbf{T}$ is the common first element of the $\mathbf{c}_\beta$;

(ii) $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ";

(iii) $\langle\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle \lambda_i(\mathcal{U}) \mid i \in \omega\rangle, k\rangle \in \operatorname{domain}(\bar{\bar{\mathcal{F}}}_{\mathrm{t}})$.

To define $\bar{\bar{\mathcal{F}}}^M_{\mathrm{phi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \kappa, k)$, we repeat the construction of the $\mathbf{T}_i$ and the $\pi_{j,i}$ used in defining $\bar{\bar{\mathcal{F}}}_{\mathrm{t}}$ and $\bar{\bar{\mathcal{F}}}_{\mathrm{pi}}$, with $\lambda_i = \lambda_n(\mathcal{U})$, except that we also define $\phi_{j,i}$ for $i \leq j \in \omega$, thus producing a system

$$\mathbf{d}' = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i} \mid i \leq j \in \omega\rangle\rangle.$$

We assume inductively that

(1) if $i \leq j \leq n$ then $\phi_{j,i} : \mathbf{T}_j \stackrel{\mathcal{S}, M_j(\mathcal{U})}{\Rightarrow} \mathbf{T}_i$;

(2) if $i_1 \leq i_2 \leq i_3 \leq n$ then $\phi_{i_3,i_1} = \phi_{i_2,i_1} \circ \phi_{i_3,i_2}$.

(3) if $m \in \omega$ and $i \leq j \leq n$ then $\phi_{j,i} \restriction \mathcal{S}({}_mT_j) \in M_i(\mathcal{U})$;

In the induction step, we set

$$\phi_{n+1,n} = \bar{\mathcal{F}}^{M_n(\mathcal{U})}_{\mathrm{phi}}(\langle \mathbf{c}^n_{\omega n+i} \mid i \in \omega\rangle, \lambda_n(\mathcal{U}), k+n, \mathcal{V}_n(\mathcal{U})),$$

and we let $\phi_{n+1,i} = \phi_{n,i} \circ \phi_{n+1,n}$ for $i < n$. We finally define

$$\bar{\bar{\mathcal{F}}}^M_{\mathrm{phi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \kappa, k, \mathcal{U}) = \mathcal{I}^{\langle M_i(\mathcal{U}) \mid i \in \omega\rangle}_{\mathrm{phi}}(\mathbf{d}', 0).$$

Lemma 5.2.28. *Let M be an uncountable transitive class model of* ZFC*. Let*

$$\langle\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \kappa, k, \mathcal{U}\rangle \in \operatorname{domain}(\bar{\bar{\mathcal{F}}}^M_{\mathrm{phi}})$$

be such that $\mathbf{T}$ is the first component of all the $\mathbf{c}_\beta$. Let $\tilde{\mathbf{T}} = \bar{\bar{\mathcal{F}}}_{\mathrm{t}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle \lambda_i(\mathcal{U}) \mid i \in \omega\rangle, k)$, and let $\phi = \bar{\bar{\mathcal{F}}}^M_{\mathrm{phi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \kappa, k, \mathcal{U})$. Then

(a) $\phi : \tilde{\mathbf{T}} \stackrel{\mathcal{S}, \bigcap_{i\in\omega} M_i(\mathcal{U})}{\Rightarrow} \mathbf{T}$;

(b) $\phi \restriction (\mathcal{S}({}_k\tilde{T}) \cap \bigcap_{i\in\omega} M_i(\mathcal{U})$ *is the identity;*

(c) for all $n \in \omega$, $\phi \restriction \mathcal{S}({}_n\tilde{T}) \in M$.

Finally we define, for each uncountable transitive class model M of ZFC, an operation $\bar{\bar{\mathcal{F}}}^M_{\mathrm{psi}}$ whose domain is the same as that of $\bar{\bar{\mathcal{F}}}^M_{\mathrm{phi}}$. To define $\bar{\bar{\mathcal{F}}}^M_{\mathrm{psi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \kappa, k, \mathcal{U})$, we repeat the construction of the sytem $\mathbf{d}'$ used in defining $\bar{\bar{\mathcal{F}}}^M_{\mathrm{phi}}$, execept that we also define $\Psi^{i,j}$ for $i \leq j \in \omega$, producing a system

$$\mathbf{d}'' = \langle\langle \mathbf{T}_i \mid i \in \omega\rangle, \langle \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \mid i \leq j \in \omega\rangle\rangle.$$

In the induction step, we appeal to the remark after the definition of $\mathcal{V}_n(\mathcal{U})$ and set

$$\Psi^{n,n+1} = \bar{\mathcal{F}}^{M_n(\mathcal{U})}_{\mathrm{psi}}(\langle \mathbf{c}^n_{\omega n+i} \mid i \in \omega\rangle, \lambda_n(\mathcal{U}), k, \mathcal{V}_n(\mathcal{U})).$$

For $i < n$ we let $\Psi^{i,n+1}(\sigma, x) = \Psi^{n,n+1}(\sigma, \Psi^{i,n}(\phi_{n+1,n}(\sigma), x))$. We finally define

$$\bar{\bar{\mathcal{F}}}^M_{\mathrm{psi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U}) = \mathcal{I}^{\langle M_i(\mathcal{U}) \mid i\in\omega\rangle}_{\mathrm{psi}}(\mathbf{d}'', 0).$$

The next lemma is the analogue for ω^2–$\mathbf{\Pi}^1_1$ of Lemma 5.2.25. The only disanalogy is that the earlier lemma yielded a semicovering with respect to M, while the lemma below gives only a semicovering with respect to $\bigcap_{i\in\omega} M_i(\mathcal{U})$. In our applications, we will have $N \subseteq \bigcap_{i\in\omega} M_i(\mathcal{U})$.

Lemma 5.2.29. *Let N and $M \supseteq N$ be uncountable transitive class models of* ZFC*. Let $\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle \in N$ be a sequence of $\mathbf{\Pi}^1_1$ codes with the same first component $\mathbf{T}$. For $\beta < \omega^2$ let A_β be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}_\beta$. Let A be the subset of $\lceil \mathbf{T} \rceil$ that $\langle A_\beta \mid \beta < \omega^2\rangle$ witnesses to belong to ω^2–$\mathbf{\Pi}^1_1$. Let κ be an ordinal that is a measurable cardinal in the model M. Let $\mathcal{U} \in M$ be such that $M \models$ "$\mathcal{U}$ is a uniform normal ultrafilter on κ." Assume that $\langle \mathcal{V}_i(\mathcal{U}) \mid i \in \omega\rangle \in N$. Assume also that $M \models |T| < \kappa$. Let $k \in \omega$. Let*

$$\begin{aligned}
\tilde{\mathbf{T}} &= \bar{\bar{\mathcal{F}}}_{\mathrm{t}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle \lambda_i(\mathcal{U}) \mid i \in \omega\rangle, k);\\
\pi &= \bar{\bar{\mathcal{F}}}_{\mathrm{pi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle \lambda_i(\mathcal{U}) \mid i \in \omega\rangle, k);\\
\phi &= \bar{\bar{\mathcal{F}}}^M_{\mathrm{phi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \kappa, k, \mathcal{U});\\
\Psi &= \bar{\bar{\mathcal{F}}}^M_{\mathrm{psi}}(\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \kappa, k, \mathcal{U}).
\end{aligned}$$

Then

(i) $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ is a $(A, \bigcap_{\beta<\omega^2} A_\beta)$ k-semicovering of $\mathbf{T}$ with respect to $\bigcap_{i\in\omega} M_i(\mathcal{U})$;

(ii) both $\tilde{\mathbf{T}}$ and π belong to N, and $N \models |\tilde{T}| \leq \kappa$;

(iii) for all $n \in \omega$, $\phi \restriction (\mathcal{S}({}_n\tilde{T}) \cap \bigcap_{i\in\omega} M_i(\mathcal{U}))$ belongs to M.

Proof. The lemma follows easily from our earlier lemmas. □

5.2.7 The Main Theorems

Theorem 5.2.30. *If* $\mathbf{T}$ *is a game tree with taboos and there is a measurable cardinal greater than* $|T|$*, then all* ω^2–$\mathbf{\Pi}^1_1$ *games in* $\mathbf{T}$ *are determined.*

Proof. Let $\mathbf{T}$ be a game tree with taboos. Let $\langle A_\beta \mid \beta < \omega^2\rangle$ witness that $A \subseteq \lceil \mathbf{T} \rceil$ belongs to ω^2–$\mathbf{\Pi}^1_1$. Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal $\kappa > |T|$. We will show that $G(A; \mathbf{T})$ is determined. We may assume without loss of generality that field$\,(T) \subseteq \delta$ for some cardinal $\delta < \kappa$. Let $\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle$ be such that each $\mathbf{c}_\beta$ is a $\mathbf{\Pi}^1_1$ code for A_β. Let $M = V$. Let N be any transitive class model of ZFC such that

(i) $\{\mathbf{T}, \langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \langle \lambda_i(\mathcal{U}) \mid i \in \omega\rangle\} \subseteq N$;

(ii) $N \subseteq \bigcap_{n\in\omega} M_n(\mathcal{U})$.

For example, we could take N to be the smallest transitive proper class model of ZFC such that (i) is satisfied. (Formally this model is $L[a]$, where a is the set of all

$$\langle p, p_1, p_2, \beta, p', m, n, i, \lambda\rangle$$

such that $p \in T$, $p_1 \in \mathcal{T}_I$ or $p_1 = \delta$, $p_2 \in \mathcal{T}_{II}$ or $p_2 = \delta$, $\beta < \omega^2$, $p' \in E_\beta$ or $p' = \delta$, $\langle m, n\rangle \in f_\beta(p)$, $i \in \omega$, and $\lambda = \lambda_i(\mathcal{U})$, where $\mathbf{c}_\beta = \langle \mathbf{T}, E_\beta, f_\beta\rangle$.) Alternatively, N could be taken as the smallest transitive class model N' of ZFC such that $\{\langle \lambda_i(\mathcal{U}) \mid i \in \omega\rangle\} \cup \mathrm{Ult}_{\omega_1\omega}(V; \mathcal{U}) \subseteq N'$.

Apply Lemma 5.2.29 with $k = 0$. Let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be given by that lemma. Then $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ is $(A, \bigcap_{\beta<\omega^2} A_\beta)$ semicovering of $\mathbf{T}$ with respect to N. Moreover $\boldsymbol{\pi}^{-1}(A) = \emptyset$, and so $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ is an A-semicovering with respect to N that unravels A. By Lemma 5.2.26, $\tilde{\mathbf{T}} \in N$. If we set $\tilde{D} = \emptyset$, the hypotheses of Lemma 5.2.3 are satisfied. Thus $G(A; \mathbf{T})$ is determined. $\square$

Derrick DuBose noticed that Theorem 5.2.30 could be improved by an appeal to Borel determinacy. Suppose everything is as in the proof of the theorem, except that we have $\langle A_\beta \mid \beta \leq \omega^2\rangle$ witnessing that A is (ω^2+1)–$\mathbf{\Pi}^1_1$ instead of $\langle A_\beta \mid \beta < \omega^2\rangle$ witnessing that A is ω^2–$\mathbf{\Pi}^1_1$. Suppose further that A_{ω^2} is Borel. Repeat the construction as in the proof of the theorem, ignoring A_{ω^2}. The $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ we get is still an A-semicovering with respect to M, but it no longer unravels A. Nevertheless, we have that $\boldsymbol{\pi}^{-1}(A) = \boldsymbol{\pi}^{-1}(A_{\omega^2})$ is a Borel subset of $\lceil \tilde{\mathbf{T}} \rceil$ and so is determined. We can also argue that $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ has a winning strategy in N and deduce that it is determined.

The results of Section 2.2 were proved in order to make the application below of DuBose's idea.

Theorem 5.2.31. (Martin [1990]) *If* $\mathbf{T}$ *is a game tree with taboos and there is a measurable cardinal greater than* $|T|$*, then* $G(A;\mathbf{T})$ *is determined for every* $A \subseteq \lceil\mathbf{T}\rceil$ *such that both* A *and* $\lceil\mathbf{T}\rceil \setminus A$ *belong to* (ω^2+1)–$\mathbf{\Pi}^1_1$.

Proof. Let $\mathbf{T}$ be a game tree with taboos. Let $\langle B_\beta \mid \beta < \omega^2+1\rangle$ witness that $A \subseteq \lceil\mathbf{T}\rceil$ belongs to (ω^2+1)–$\mathbf{\Pi}^1_1$ and let $\langle C_\beta \mid \beta < \omega^2+1\rangle$ witness that $\lceil\mathbf{T}\rceil \setminus A$ belongs to (ω^2+1)–$\mathbf{\Pi}^1_1$. Let

$$A_{\beta+1} = C_\beta \cap B_{\beta+1} \quad \text{for } \beta < \omega^2;$$
$$A_{\omega n} = B_{\omega n} \quad \text{for } n \in \omega.$$

Note that

(a) $\langle A_\beta \mid \beta < \omega^2\rangle \cup \{\langle \omega^2, B_{\omega^2}\rangle\}$ witnesses that A belongs to (ω^2+1)–$\mathbf{\Pi}^1_1$;

(b) for all $x \in \lceil\mathbf{T}\rceil$, if $x \in \bigcap_{\beta<\omega^2} A_\beta$, then

$$x \in A \leftrightarrow x \in B_{\omega^2} \leftrightarrow x \notin C_{\omega^2}.$$

Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal $\kappa > |T|$. We will show that $G(A;\mathbf{T})$ is determined. As in the proof of Theorem 5.2.30, we may assume without loss of generality that field$\,(T) \subseteq \delta$ for some cardinal $\delta < \kappa$. Let $\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle$ be such that each $\mathbf{c}_\beta$ is a $\mathbf{\Pi}^1_1$ code for A_β. Let $\mathbf{c}'$ be a $\mathbf{\Pi}^1_1$ code for B_{ω^2} and let $\mathbf{c}''$ be a $\mathbf{\Pi}^1_1$ code for C_{ω^2}. Let $M = V$. Let N be any transitive class model of ZFC satisfying

(i) $\{\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle, \mathbf{c}', \mathbf{c}'', \langle \lambda_i(\mathcal{U}) \mid i \in \omega\rangle\} \subseteq N$;

(ii) $N \subseteq \bigcap_{n\in\omega} M_n(\mathcal{U})$.

(The second version of the N of the proof of Theorem 5.2.30 will also work for N here, and the first version can easily be modified to work here.)

As in the proof of Theorem 5.2.30, we apply Lemma 5.2.29 with V as the M of that lemma and with $k = 0$. Let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be given by that lemma. By Lemma 5.2.24, $\mathcal{C}$ is an $(A; \bigcap_{\beta<\omega^2} A_\beta)$ semicovering of $\mathbf{T}$ with respect to N. Let $\mathbf{c}' = \langle \mathbf{T}, E', f'\rangle$ and let $\mathbf{c}'' = \langle \mathbf{T}, E'', f''\rangle$. Let

$$\begin{aligned}\mathbf{c}^* &= \langle \tilde{\mathbf{T}}, \boldsymbol{\pi}^{-1}(E'), f^*\rangle;\\ \mathbf{c}^{**} &= \langle \tilde{\mathbf{T}}, \boldsymbol{\pi}^{-1}(E''), f^{**}\rangle,\end{aligned}$$

where $f^*(\tilde{p}) = f'(\pi(\tilde{p}))$ and $f^{**}(\tilde{p}) = f''(\pi(\tilde{p}))$. The pair $\langle \mathbf{c}^*, \mathbf{c}^{**} \rangle$ belongs to the model N. Moreover $\mathbf{c}^*$ and $\mathbf{c}^{**}$ are $\mathbf{\Pi}^1_1$ codes for $\boldsymbol{\pi}^{-1}(B_{\omega^2})$ and $\boldsymbol{\pi}^{-1}(C_{\omega^2})$ respectively. By (b) above, the pair $\langle \mathbf{c}^*, \mathbf{c}^{**} \rangle$ is a $\mathbf{\Delta}^1_1$ code for $\boldsymbol{\pi}^{-1}(A)$. Thus we may apply Lemma 5.2.7 with N as the M of that lemma and with $\langle \mathbf{c}^*, \mathbf{c}^{**} \rangle$ as its $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$, concluding that $G(A; \mathbf{T})$ is determined. □

An alternative way to prove Theorem 5.2.31 would have been via an A-semicovering that actually unravels A. See Exercise 5.2.3.

The next theorem gives an optimal determinacy consequence of the existence of α measurable cardinals.

Theorem 5.2.32. *Let α be a countable ordinal. If $\mathbf{T}$ is a game tree with taboos and the class of measurable cardinals greater than $|T|$ has order type $\geq \alpha$, then $G(A; \mathbf{T})$ is determined for every $A \subseteq \lceil \mathbf{T} \rceil$ such that both A and $\lceil \mathbf{T} \rceil \setminus A$ belong to $(\omega^2\alpha + 1)$–$\mathbf{\Pi}^1_1$.*

Proof. Let $\mathbf{T}$ be a game tree with taboos. Let $\langle B_\beta \mid \beta < \omega^2\alpha + 1 \rangle$ witness that $A \subseteq \lceil \mathbf{T} \rceil$ belongs to $(\omega^2\alpha + 1)$–$\mathbf{\Pi}^1_1$ and let $\langle C_\beta \mid \beta < \omega^2\alpha + 1 \rangle$ witness that $\lceil \mathbf{T} \rceil \setminus A$ belongs to $(\omega^2\alpha + 1)$–$\mathbf{\Pi}^1_1$. Let

$$\begin{aligned} A_{\beta+1} &= C_\beta \cap B_{\beta+1} \quad \text{for } \beta < \omega^2\alpha; \\ A_{\omega\gamma} &= B_{\omega\gamma} \quad \text{for } \gamma < \omega\alpha. \end{aligned}$$

Note that

(a) $\langle A_\beta \mid \beta < \omega^2\alpha \rangle ^\frown \langle B_{\omega^2\alpha} \rangle$ witnesses that A belongs to $(\omega^2\alpha + 1)$–$\mathbf{\Pi}^1_1$;

(b) for all $x \in \lceil \mathbf{T} \rceil$, if $x \in \bigcap_{\beta < \omega^2\alpha} A_\beta$, then

$$x \in A \leftrightarrow x \in B_{\omega^2\alpha} \leftrightarrow x \notin C_{\omega^2\alpha}.$$

Let $\langle \kappa_\gamma \mid \gamma < \alpha \rangle$ be an strictly increasing sequence of measurable cardinals with $\kappa_0 > |T|$. For each $\gamma < \alpha$, let $\mathcal{U}_\gamma$ be a uniform normal ultrafilter on κ_γ. We will show that $G(A; \mathbf{T})$ is determined. We may assume without loss of generality that $\mathrm{field}\,(T) \subseteq \lambda$ for some cardinal $\lambda < \kappa_0$. Let $\langle \mathbf{c}_\beta \mid \beta < \omega^2\alpha \rangle$ be such that each $\mathbf{c}_\beta$ is a $\mathbf{\Pi}^1_1$ code for A_β. Let $\mathbf{c}'$ be a $\mathbf{\Pi}^1_1$ code for $B_{\omega^2\alpha}$ and let $\mathbf{c}''$ be a $\mathbf{\Pi}^1_1$ code for $C_{\omega^2\alpha}$.

Let $a \subseteq \lambda$ be such that $\langle \mathbf{c}_\beta \mid \beta < \omega^2\alpha \rangle$, $\mathbf{c}'$, and $\mathbf{c}''$ all belong to $L[a]$ and such that α is countable in $L[a]$.

We define transitive class models M_γ, $\gamma \leq \alpha$, of ZFC and simultaneously define ordinals λ_i^γ, $\gamma < \alpha$ and $i \in \omega$. Let

$$\begin{aligned} M_0 &= L[a, \langle\!\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle\!\rangle]; \\ \lambda_i^\gamma &= \lambda_i(\mathcal{U}_\gamma \cap M_\gamma); \\ M_\gamma &= L[a, \langle\!\langle \lambda_i^\xi \mid \xi < \gamma, i \in \omega \rangle\!\rangle, \langle\!\langle \mathcal{U}_\xi \mid \gamma \leq \xi < \alpha \rangle\!\rangle]. \end{aligned}$$

By Lemma 3.5.2, $\mathcal{U}_\xi \cap M_\gamma$ is in M_γ a uniform normal ultrafilter on κ_ξ whenever $\gamma \leq \xi < \alpha$. Note that, for each $\gamma < \alpha$, the sequence $\langle M_\eta \mid \gamma \leq \eta \leq \alpha \rangle$ is (in the sense of page 238) a class in M_γ.

For each limit $\gamma \leq \alpha$, let $g_\gamma : \omega \to \gamma$ be a strictly increasing function whose range is cofinal in γ and such that $g_\gamma(0) = 0$. Choose the g_γ such that $\langle g_\gamma \mid \gamma \leq \alpha \rangle \in M_\alpha$.

For $\beta < \omega^2\alpha$, let $\mathbf{c}_\beta = \langle \mathbf{T}_0, E_\beta, f_\beta \rangle$, where f_β is $p \mapsto <_p^\beta$.

We define by induction on $\gamma \leq \alpha$ a system

$$\langle \langle \mathbf{T}_\gamma \mid \gamma < \alpha \rangle, \langle \pi_{\gamma,\xi}, \phi_{\gamma,\xi}, \Psi^{\xi,\gamma} \mid \xi \leq \gamma \leq \alpha \rangle \rangle.$$

Our definition will be such that

(i) if $\xi \leq \gamma \leq \alpha$ and
$$\mathcal{C}_{\gamma,\xi} = \langle \mathbf{T}_\gamma, \pi_{\gamma,\xi}, \phi_{\gamma,\xi}, \Psi^{\xi,\gamma} \rangle,$$
then $\mathcal{C}_{\gamma,\xi}$ is a $(\boldsymbol{\pi}_{\xi,0}^{-1}(A), \boldsymbol{\pi}_{\xi,0}^{-1}(\bigcap_{\omega\xi \leq \beta < \omega\gamma} A_\beta))$ semicovering of $\mathbf{T}_\xi$ with respect to M_γ such that $\mathbf{T}_\gamma$ and $\pi_{\gamma,\xi}$ belong to M_γ and such that $\phi_{\gamma,\xi} \restriction \mathcal{S}({}_nT_\gamma \cap M_\gamma) \in M_\xi$ for all $n \in \omega$;

(ii) if $\rho \leq \xi \leq \gamma \leq \alpha$ then $\mathcal{C}_{\gamma,\rho} = \mathcal{C}_{\xi,\rho} \circ \mathcal{C}_{\gamma,\xi}$.

We begin the inductive definition by setting $\mathbf{T}_0 = \mathbf{T}$.

Next we deal with the case of successor ordinals $\gamma + 1 \leq \alpha$. For $\beta < \omega^2\alpha$, let
$$\mathbf{c}_\beta^\gamma = \langle \mathbf{T}_\gamma, \pi_{\gamma,0}^{-1}(E_\beta), f_\beta^\gamma \rangle,$$
where $f_\beta^\gamma(p) = <_{\pi_{\gamma,0}(p)}^\beta$. We define $\mathcal{C}_{\gamma+1,\gamma}$ by setting

$$\begin{aligned} \mathbf{T}_{\gamma+1} &= \bar{\bar{\mathcal{F}}}_{\mathrm{t}}(\langle \mathbf{c}_\beta^\gamma \mid \omega^2\gamma \leq \beta < \omega^2(\gamma+1) \rangle, \langle \lambda_i^\gamma \mid i \in \omega \rangle, k(\gamma)); \\ \pi_{\gamma+1,\gamma} &= \bar{\bar{\mathcal{F}}}_{\mathrm{pi}}(\langle \mathbf{c}_\beta^\gamma \mid \omega^2\gamma \leq \beta < \omega^2(\gamma+1) \rangle, \langle \lambda_i^\gamma \mid i \in \omega \rangle, k(\gamma)); \\ \phi_{\gamma+1,\gamma} &= \bar{\bar{\mathcal{F}}}_{\mathrm{phi}}^{M_\gamma}(\langle \mathbf{c}_\beta^\gamma \mid \omega^2\gamma \leq \beta < \omega^2(\gamma+1) \rangle, \kappa_\gamma, k(\gamma), \mathcal{U}_\gamma \cap M_\gamma); \\ \Psi^{\gamma,\gamma+1} &= \bar{\bar{\mathcal{F}}}_{\mathrm{psi}}^{M_\gamma}(\langle \mathbf{c}_\beta^\gamma \mid \omega^2\gamma \leq \beta < \omega^2(\gamma+1) \rangle, \kappa_\gamma, k(\gamma), \mathcal{U}_\gamma \cap M_\gamma). \end{aligned}$$

To complete the successor case, we preserve commutativity by defining, for $\xi < \gamma$,

$$\begin{aligned}
\pi_{\gamma+1,\xi} &= \pi_{\gamma,\xi} \circ \pi_{\gamma+1,\gamma};\\
\phi_{\gamma+1,\xi} &= \phi_{\gamma,\xi} \circ \phi_{\gamma+1,\gamma};\\
\Psi^{\xi,\gamma+1}(\sigma, x) &= \Psi^{\gamma+1,\gamma}(\sigma, \Psi^{\xi,\gamma}(\phi_{\gamma+1,\gamma}(\sigma), x)).
\end{aligned}$$

To complete the definition, let γ be a limit ordinal $\leq \alpha$. For $i \in \omega$, let us write $\hat{\imath}$ for $g_\gamma(i)$. We set

$$\begin{aligned}
\mathbf{T}_\gamma &= \mathcal{I}_{\mathrm{t}}(\langle \mathbf{T}_{\hat{\imath}} \mid i \in \omega\rangle, \langle \pi_{\hat{\jmath},\hat{\imath}} \mid i \leq j \in \omega\rangle);\\
\pi_{\gamma,\hat{\imath}} &= \mathcal{I}_{\mathrm{pi}}(\langle \mathbf{T}_{\hat{\imath}} \mid i \in \omega\rangle, \langle \pi_{\hat{\jmath},\hat{\imath}} \mid i \leq j \in \omega\rangle, i);\\
\phi_{\gamma,\hat{\imath}} &= \mathcal{I}_{\mathrm{phi}}^{\langle M_{\hat{n}} \mid n \in \omega\rangle}(\langle \mathbf{T}_{\hat{\imath}} \mid i \in \omega\rangle, \langle \pi_{\hat{\jmath},\hat{\imath}}, \phi_{\hat{\jmath},\hat{\imath}} \mid i \leq j \in \omega\rangle, i);\\
\Psi^{\gamma,\hat{\imath}} &= \mathcal{I}_{\mathrm{psi}}^{\langle M_{\hat{n}} \mid n \in \omega\rangle}(\langle \mathbf{T}_{\hat{\imath}} \mid i \in \omega\rangle, \langle \pi_{\hat{\jmath},\hat{\imath}}, \phi_{\hat{\jmath},\hat{\imath}}, \Psi^{\hat{\imath},\hat{\jmath}} \mid i \leq j \in \omega\rangle, i).
\end{aligned}$$

For $i \in \omega$ and $\hat{\imath} < \xi < \widehat{i+1}$, we preserve commutativity by defining

$$\begin{aligned}
\pi_{\gamma,\xi} &= \pi_{\widehat{i+1},\xi} \circ \pi_{\gamma,\widehat{i+1}};\\
\phi_{\gamma,\xi} &= \phi_{\widehat{i+1},\xi} \circ \phi_{\gamma,\widehat{i+1}};\\
\Psi^{\xi,\gamma}(\sigma, x) &= \Psi^{\gamma,\widehat{i+1}}(\sigma, \Psi^{\xi,\widehat{i+1}}(\phi_{\gamma,\widehat{i+1}}(\sigma), x)).
\end{aligned}$$

We leave to the reader the verification that this inductive definition makes sense and has the stated properties. In particular, this means that that $\langle \mathbf{T}_\alpha, \pi_{\alpha,0}, \phi_{\alpha,0}, \Psi^{0,\alpha}\rangle$ is a an $(A, \bigcap_{\beta<\omega^2\alpha} A_\beta)$ semicovering of $\mathbf{T}$ with respect to M_α. The rest of the proof is like that of Theorem 5.2.31.

$\square$

Exercise 5.2.1. Prove the assertion that results if, in the statement of Lemma 5.2.5, the condition "$\omega_1 \in M$" is replaced by the weaker "M is uncountable."

Exercise 5.2.2. If U is a tree on field$(T) \times \omega$ and $\mathbf{c}$ is a $\mathbf{\Pi}^1_1$ code for a subset B of $\lceil U \rceil$, then we say that $\mathbf{c}$ is a $\mathbf{\Sigma}^1_2$ *code* and the $\mathbf{\Sigma}^1_2$ *set coded by* $\mathbf{c}$ is

$$\{x \in \lceil \mathbf{T} \rceil \mid U(x) \neq \emptyset\}.$$

($U(x)$ is defined on page 197.)

Prove the following version of the *Shoenfield Absoluteness Theorem* (Shoenfield [1961]): If $\mathbf{c}$ is a $\mathbf{\Sigma}^1_2$ code, then membership in the $\mathbf{\Sigma}^1_2$ set coded by $\mathbf{c}$ is absolute for uncountable transitive class models of ZFC. Use this result to do Exercise 5.2.1 and to reprove Lemma 5.2.6.

Exercise 5.2.3. Let $\mathbf{T}$ be a game tree with taboos. Let $A \subseteq \lceil\mathbf{T}\rceil$ be such that both A and $\lceil\mathbf{T}\rceil \setminus A$ belong to (ω^2+1)–$\mathbf{\Pi}^1_1$. Assume that there is a measurable cardinal larger than $\lceil\mathbf{T}\rceil$. Prove that there is a transitive class model N of ZFC such that there is an A-semicovering $\mathcal{C}^* = \langle \mathbf{T}^*, \pi^*, \phi^*, \Psi^*\rangle$ of $\mathbf{T}$ with respect to N that unravels A such that $\mathbf{T}^* \in N$. (From this it follows by Lemma 5.2.3 that $G(A;\mathbf{T})$ is determined.)

Hint. Proceed exactly as in the proof of Theorem 5.2.31, getting a transitive class model N of ZFC and a $(A, \bigcap_{\beta<\omega^2} A_\beta)$ semicovering $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ of $\mathbf{T}$ with respect to N such that there is a $\mathbf{\Delta}^1_1$ code in N for $\boldsymbol{\pi}^{-1}(A)$. Note that the proof gives that $\tilde{\mathbf{T}}$ and π belong to N. Now show that the proofs of Theorems 2.2.6 and 2.2.3 actually give a covering $\mathcal{C}' = \langle \mathbf{T}', \pi', \phi', \Psi'\rangle$ of of $\mathbf{T}$ that unravels $\boldsymbol{\pi}^{-1}(A)$ and is such that $\mathbf{T}'$, π', and ϕ' belong to N. Now apply Lemma 5.2.13 to show that $\mathcal{C} \circ \mathcal{C}'$ is the desired $\mathcal{C}^*$.

Exercise 5.2.4. Redo (and simplify) Section 5.2 as follows.

(1) In the definition of $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S},M}{\Rightarrow} \mathbf{T}$, replace the last conjunct of clause (i) by

$$\mathcal{S}(\tilde{T}) \cap M \ \subseteq\ \text{domain}\,(\phi).$$

(2) Strengthen the hypothesis of Lemma 5.2.1 to require that $\tilde{\sigma} \in M$.

(3) Redefine $\mathcal{F}^M_{\text{phi}}$ to be $(\mathcal{F}_{\text{phi}})^M$.

(4) Replace clause (c) of Lemmas 5.2.11, 5.2.23, and 5.2.28 by

$$\phi \in M.$$

Replace clause (i)(d) and (ii)(d) of Lemma 5.2.12 by the assertions that ϕ_1 and ϕ_2 belong to M. Make similar changes in the hypotheses of Lemma 5.2.13, the definition of $\mathcal{I}^{\langle M_i | i\in\omega\rangle}_{\text{phi}}$, clause (d) of Lemma 5.2.18, and the hypotheses of Lemma 5.2.20, the definitions of $\bar{\mathcal{F}}^M_{\text{phi}}$ and $\bar{\bar{\mathcal{F}}}_{\text{phi}}{}^M$, and the conclusions of Lemma 5.2.25 and 5.2.29.

(5) (Optional) Replace, in the definitions of the domains of $\mathcal{F}^M_{\text{psi}}$ and $\bar{\mathcal{F}}^M_{\text{psi}}$, the requirement that the intersection of countably many elements of $\mathcal{U}$ be nonempty by the requirement that $\kappa \geq \omega_1$. Add the latter requirement to the definition of domain$\,(\bar{\bar{\mathcal{F}}}_{\text{phi}}{}^M)$. Add $\kappa \geq \omega_1$ to the hypotheses of Lemmas 5.2.12, 5.2.25, and 5.2.29.

(6) In the definitions of $M_n(\mathcal{U})$, $\lambda_n(\mathcal{U})$, and $\mathcal{V}_n(\mathcal{U})$, replace "$\omega_1(n+1)$" by "$n$."

5.3 Reversible Implications

Let λ be an infinite cardinal number. Theorem 4.4.2 shows that if $a^{\#}$ exists for every $a \subseteq \lambda$ then all $\mathbf{\Pi}^1_1$ games in trees of size $\leq \lambda$ are determined. Our first goal in this section is to improve that theorem, replacing "$\mathbf{\Pi}^1_1$" by "$\bigcup_{\beta<\omega^2} \beta$–$\mathbf{\Pi}^1_1$" in its conclusion. Afterward we will generalize this result. We will prove, for all $\alpha < \omega_1$ and for all $\beta < \omega^2(\alpha+1)$, that the determinacy of all β–$\mathbf{\Pi}^1_1$ games in trees of size λ follows from the existence, for each $a \subseteq \lambda$, of indiscernibles for M, a, where M is a transitive class model of ZFC with α measurable cardinals. (Our first result will thus be the special case $\alpha = 0$.) For countable λ, the conclusion of this implication implies its hypothesis for every α, and so the implication is really half of an equivalence. This is also true of the lightface version. (See Exercises 4.4.1 and 5.3.4, 5.3.5, and 5.3.6 for the converses.)

The proof of Theorem 4.4.2 is like that of Theorem 4.1.6, except that λ^+ replaces the measurable cardinal of the proof of Theorem 4.1.6. Assuming the hypothesis of Theorem 4.4.2 and using

$$\lambda^+, \lambda^{++}, \ldots, \lambda^{\overbrace{+\cdots+}^{n}}$$

in place of n measurable cardinals, one can imitate the proof of Theorem 5.1.4 and prove the determinacy of n–$\mathbf{\Pi}^1_1$ games. With the aid of the ordering trick (see page 221), one can prove the determinacy of ωn–$\mathbf{\Pi}^1_1$ games for each $n \in \omega$. (One cannot get an even stronger conclusion by using infinitely many cardinals. To see why the attempt to do so breaks down, note, for example, that any finite set of cardinals belongs to L but, if $0^{\#}$ exists, then no infinite set of uncountable cardinals belongs to L.)

We present the details in terms of semicoverings. We begin with an analogue of Lemma 4.2.2.

Lemma 5.3.1. *Let b be a set such that $b \in L[b]$ and such that $b^{\#}$ exists. Let $\mathbf{T} \in L[b]$ be a game tree with taboos. Let $B \subseteq \lceil \mathbf{T} \rceil$ be such that $B \in \mathbf{\Pi}^1_1$ and such that some $\mathbf{\Pi}^1_1$ code $\mathbf{c}$ for B belongs to $L[b]$. Let $k \in \omega$. Suppose that κ is an uncountable cardinal (of V) such that b and $\mathbf{T}$ belong to $L_\kappa[b]$.*

(i) There is a (B, B) k-semicovering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ with respect to $L[b]$ such that $\tilde{\mathbf{T}}$ and π belong to $L[b]$, such that $\phi(\tilde{\sigma}) \restriction_n T$ belongs to $L[b]$ for every $\tilde{\sigma} \in$ domain(ϕ) and for every $n \in \omega$, and such that $L[b] \models |\tilde{T}| \leq \kappa$.

(ii) There is a $(\lceil\mathbf{T}\rceil \setminus B, B)$ k-semicovering $\langle\tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ of $\mathbf{T}$ with respect to $L[b]$ such that $\tilde{\mathbf{T}}$ and π belong to $L[b]$, such that $\phi(\tilde{\sigma}) \restriction {}_n T$ belongs to $L[b]$ for every $\tilde{\sigma} \in \mathrm{domain}\,(\phi)$ and for every $n \in \omega$, and such that $L[b] \models |\tilde{T}| \leq \kappa$.

Proof. As in the proof of Lemma 4.2.2, we consider only (i) and only the case $k = 0$. Define $\tilde{\mathbf{T}}$ and π as in the proof of Lemma 4.2.2, with the third component of $\mathbf{c}$ as $p \mapsto <_p$.

The domain of ϕ will be the set of all $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ such that $\tilde{\sigma} \restriction {}_n\tilde{T} \in L[b]$ for each $n \in \omega$.

For $\tilde{\sigma} \in \mathcal{S}_I(\tilde{T}) \cap L[b]$, define $\phi(\tilde{\sigma})$ as in the proof of Lemma 4.2.2. It is clear that $\phi(\tilde{\sigma}) \in L[b]$ for such $\tilde{\sigma}$. Now extend $\phi(\tilde{\sigma})$ in the obvious to its full domain. Define $\Psi(\tilde{\sigma}, x)$, for x a play consistent with $\phi(\tilde{\sigma})$, as in the proof of Lemma 4.2.2.

Suppose that $\tilde{\tau} \in \mathcal{S}_{II}(\tilde{T})$ belongs to domain$\,(\phi)$. For $p \in T$ with $\ell\mathrm{h}(p) = 2n+1$ and for $v \in [\kappa]^{n+1}$, define $\tilde{q}(p, v)$ as in the proof of Lemma 4.2.2. Let $\gamma_0, \ldots, \gamma_m$ be such that both $\tilde{\tau} \restriction {}_{2n+1}\tilde{T}$ and $\tilde{q}$ are definable in $L[b]$ from b and $c^b_{\gamma_0}, \ldots, c^b_{\gamma_m}$. Let $\beta_n < \kappa$ be larger than any γ_i that is smaller than κ. For $p \in T$ with $\ell\mathrm{h}(p) = 2n + 1$, let

$$(\phi(\tilde{\tau}))(p) \;=\; \tilde{\tau}(\tilde{q}(p, \{c^b_{\beta_n}, \ldots, c^b_{\beta_n+n}\})).$$

Note that the restriction of $\phi(\tilde{\tau})$ to positions of length $2n+1$ is defined in $L[b]$ from $b, c^b_{\beta_n}, \ldots, c^b_{\beta_n+n}, c^b_{\gamma_0}, \ldots, c^b_{\gamma_m}$. Thus $\phi(\tilde{\tau}) \restriction {}_nT$ belongs to $L[b]$ for each $n \in \omega$. For $n \in \omega$, let $X_n = \{c^b_\xi \mid \beta_n \leq \xi < \kappa\}$. By indiscernibility and the fact that $\tilde{\tau}$ takes fewer than κ values, for every $p \in T$ with $\ell\mathrm{h}(p) = 2n + 1$,

$$(\forall v \in [X_n]^{n+1})\, \tilde{\tau}(\tilde{q}(p, v)) \;=\; (\phi(\tilde{\tau}))(p).$$

For plays x consistent with $\phi(\tilde{\tau})$, define $\Psi(\tilde{\tau}, x)$ as in the proof of Lemma 4.2.2, with $\bigcap_{n\in\omega} X_n$ replacing the X of the earlier proof. □

Next we prove an analogue of Lemma 4.2.3.

Lemma 5.3.2. *Let b be a set such that $b \in L[b]$ and such that $b^\#$ exists. Let $\mathbf{T} \in L[b]$ be a game tree with taboos. Let $B \subseteq \lceil\mathbf{T}\rceil$ be such that $B \in \mathbf{\Pi}^1_1$ and such that some $\mathbf{\Pi}^1_1$ code $\mathbf{c}$ for B belongs to $L[b]$. Let $k \in \omega$ and $m \in \omega$. Suppose that κ is an uncountable cardinal such that $b \in L_\kappa[b]$, $T \subseteq L_\kappa[b]$, and*

$$(\forall p \in T)(\ell\mathrm{h}(p) > m \;\rightarrow\; T_p \in L_\kappa[b]).$$

(i) There is a (B, B) k-semicovering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ with respect to $L[b]$ such that $\tilde{\mathbf{T}}$ and π belong to $L[b]$, such that $\phi(\tilde{\sigma}) \restriction_n T$ belongs to $L[b]$ for every $\tilde{\sigma} \in \mathrm{domain}\,(\phi)$ and for every $n \in \omega$, such that $\tilde{T} \subseteq L_\kappa[b]$, and such that

$$(\forall \tilde{p} \in \tilde{T})(\ell\mathrm{h}(\tilde{p}) > \max\{k, m\} + 1 \rightarrow \tilde{T}_{\tilde{p}} \in L_\kappa[b]).$$

(ii) There is a $(\lceil \mathbf{T} \rceil \backslash B, B)$ k-semicovering $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ of $\mathbf{T}$ of with respect to $L[b]$ such that $\tilde{\mathbf{T}}$ and π belong to $L[b]$, $\phi(\tilde{\sigma}) \restriction_n T$ belongs to $L[b]$ for every $\tilde{\sigma} \in \mathrm{domain}\,(\phi)$ and for every $n \in \omega$, such that $\tilde{T} \subseteq L_\kappa[b]$, and such that

$$(\forall \tilde{p} \in \tilde{T})(\ell\mathrm{h}(\tilde{p}) > \max\{k, m\} + 1 \rightarrow \tilde{T}_{\tilde{p}} \in L_\kappa[b]).$$

Proof. Define j, $\tilde{\mathbf{T}}$, and π as in the proof of Lemma 4.2.3. Define ϕ and the X_n as in the proof of Lemma 5.3.2, but replace "$\ell\mathrm{h}(p) = 2n + 1$" by "$\ell\mathrm{h}(p) = 2(j + n) + 1$." For $p \in T$ with $\ell\mathrm{h}(p) = 2(j + n) + 1$, we have that

$$n > m \rightarrow (\forall v \in [X_n]^{n+1})\, \tilde{\tau}(\tilde{q}(p, v)) = (\phi(\tilde{\tau}))(p).$$

This allows us to define Ψ as in the proof of Lemma 5.3.1, i.e., as in the proof of Lemma 4.2.2. $\square$

Next we want to extract the operations implicit in the proof of Lemma 5.3.2.

The operations giving $\tilde{\mathbf{T}}$ and π are just those of §5.2.3, $\mathcal{F}_{\mathrm{t}}$ and $\mathcal{F}_{\mathrm{pi}}$ respectively.

The operation giving ϕ we call $\mathcal{F}^*_{\mathrm{phi}}$. The domain of $\mathcal{F}^*_{\mathrm{phi}}$ is the set of all $\langle b, \mathbf{c}, \kappa, m, k, i \rangle$, where

(i) $\langle \mathbf{c}, \kappa, m, k, i \rangle \in \mathrm{domain}\,(\mathcal{F}_{\mathrm{t}})$;

(ii) $b \in L[b]$ and $b^\#$ exists;

(iii) $\mathbf{c} \in L[b]$;

(iv) κ is an uncountable cardinal number such that $b \in L_\kappa[b]$, $T \subseteq L_\kappa[b]$, and
$$(\forall p \in T)(\ell\mathrm{h}(p) > m \rightarrow T_p \in L_\kappa[b]),$$
where $\mathbf{T}$ is the first component of $\mathbf{c}$.

The analogue of Lemma 5.2.11 holds:

Lemma 5.3.3. *Let*

$$\langle b, \mathbf{c}, \kappa, m, k, i\rangle \in \mathrm{domain}\,(\mathcal{F}^*_{\mathrm{phi}})$$

be such that $\mathbf{c} = \langle \mathbf{T}, E, f\rangle$. *Let* $\tilde{\mathbf{T}} = \mathcal{F}_{\mathrm{t}}(\mathbf{c}, \kappa, m, k, i)$ *and let* $\phi = \mathcal{F}^*_{\mathrm{phi}}(b, \mathbf{c}, \kappa, m, k, i)$. *Then*

(a) $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}, L[b]}{\Rightarrow} \mathbf{T}$;

(b) $\phi \restriction (\mathcal{S}({}_k\tilde{T}) \cap L[b])$ *is the identity;*

(c) *for all* $\tilde{\sigma} \in \mathrm{domain}\,(\phi)$ *and for all* $n \in \omega$, $\phi(\tilde{\sigma}) \restriction {}_n T$ *belongs to* $L[b]$.

The operation $\mathcal{F}^*_{\mathrm{psi}}$ giving the Ψ of the proof of Lemma 5.3.2 has the same domain as $\mathcal{F}^*_{\mathrm{phi}}$. Moreover, if $\langle b, \mathbf{c}, \kappa, m, k, i\rangle \in \mathrm{domain}\,(\mathcal{F}^*_{\mathrm{psi}})$, then the domain of $\mathcal{F}^*_{\mathrm{psi}}(b, \mathbf{c}, \kappa, m, k, i)$ is the set of all pairs $\langle \tilde{\sigma}, x\rangle$ such that $\tilde{\sigma} \in \mathrm{domain}\,(\phi)$, such that x is a play (not necessarily in $L[b]$) that is consistent with $(\mathcal{F}^*_{\mathrm{phi}}(b, \mathbf{c}, \kappa, m, k, i))(\tilde{\sigma})$, and such that at least one of the following holds

(i) $i = 1$ and $\tilde{\sigma}$ is a strategy for I;

(ii) $i = 2$ and $\tilde{\sigma}$ is a strategy for II;

(iii) x is finite;

(iv) x belongs to the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$.

The next lemma is analogous to Lemma 5.2.12.

Lemma 5.3.4. *Let b be a set such that $b \in L[b]$ and such that $b^{\#}$ exists. Let $\mathbf{c} = \langle \mathbf{T}, E, f\rangle$ be a $\mathbf{\Pi}^1_1$ code belonging to $L[b]$ and let $B \subseteq \lceil \mathbf{T} \rceil$ be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}$. Let m and k belong to ω. Let $\tilde{m} = \max\{k, m\} + 1$. Suppose that κ is an uncountable cardinal such that $b \in L_\kappa[b]$, $T \subseteq L_\kappa[b]$, and*

$$(\forall p \in T)(\ell\mathrm{h}(p) > m \rightarrow T_p \in L_\kappa[b]).$$

For $i \in \{1, 2\}$, let

$$\begin{aligned}
\tilde{\mathbf{T}}_i &= \mathcal{F}_{\mathrm{t}}(\mathbf{c}, \kappa, m, k, i);\\
\pi_i &= \mathcal{F}_{\mathrm{pi}}(\mathbf{c}, \kappa, m, k, i,);\\
\phi_i &= \mathcal{F}^*_{\mathrm{phi}}(b, \mathbf{c}, \kappa, m, k, i);\\
\Psi_i &= \mathcal{F}^*_{\mathrm{psi}}(b, \mathbf{c}, \kappa, m, k, i).
\end{aligned}$$

Then

(i) $\langle \tilde{\mathbf{T}}_1, \pi_1, \phi_1, \Psi_1 \rangle$ *is a* (B, B) *k-semicovering of* $\mathbf{T}$ *with respect to* $L[b]$ *such that*

(a) *both* $\tilde{\mathbf{T}}_1$ *and* π_1 *belong to* $L[b]$*;*
(b) $\tilde{T}_1 \subseteq L_\kappa[b]$*;*
(c) $(\forall \tilde{p} \in \tilde{T}_1)(\ell\mathrm{h}(\tilde{p}) > \tilde{m} \to (\tilde{T}_1)_p \in L_\kappa[b])$*;*
(d) $(\forall \tilde{\sigma} \in \mathrm{domain}(\phi_1))(\forall n \in \omega)\, \phi_2(\tilde{\sigma}) \restriction_n T \in L[b]$.

(v) $\langle \tilde{\mathbf{T}}_2, \pi_2, \phi_2, \Psi_2 \rangle$ *is a* $(\lceil \mathbf{T} \rceil \setminus B, B)$ *k-semicovering of* $\mathbf{T}$ *with respect to* $L[b]$ *such that*

(a) *both* $\tilde{\mathbf{T}}_2$ *and* π_2 *belong to* $L[b]$*;*
(b) $\tilde{T}_2 \subseteq L_\kappa[b]$*;*
(c) $(\forall \tilde{p} \in \tilde{T}_2)(\ell\mathrm{h}(\tilde{p}) > \tilde{m} \to (\tilde{T}_2)_p \in L_\kappa[b])$*;*
(d) $(\forall \tilde{\sigma} \in \mathrm{domain}(\phi_2))(\forall n \in \omega)\, \phi_2(\tilde{\sigma}) \restriction_n T \in L[b]$.

We do not need to define any new composition and limit operations, so we turn to the task of defining operations for unraveling ω–$\mathbf{\Pi}^1_1$ sets.

The old operations $\bar{\mathcal{F}}_{\mathrm{t}}$ and $\bar{\mathcal{F}}_{\mathrm{pi}}$ will still play their roles in the present context. We have the following refinement of clause (c) of Lemma 5.2.22.

Lemma 5.3.5. *Let* b *be a set and let*

$$\langle \langle \mathbf{c}_i \mid i \in \omega \rangle, \kappa, k \rangle \in \mathrm{domain}\,(\bar{\mathcal{F}}_{\mathrm{t}}) \cap L[b],$$

with $\mathbf{T}$ *the common first component of the* $\mathbf{c}_i$ *and with* κ *an uncountable cardinal number. Let* $\tilde{\mathbf{T}} = \bar{\mathcal{F}}_{\mathrm{t}}(\langle \mathbf{c}_i \mid i \in \omega \rangle, \kappa, k)$*. Then*

$$T \in L_\kappa[b] \;\to\; \tilde{T} \subseteq L_\kappa[b].$$

We next define a opertion $\bar{\mathcal{F}}^*_{\mathrm{phi}}$ whose domain is the set of all $\langle b, \langle \mathbf{c}_i \mid i \in \omega \rangle, \kappa, k \rangle$ such that

(i) $\langle \langle \mathbf{c}_i \mid i \in \omega \rangle, \kappa, k \rangle \in \mathrm{domain}\,(\bar{\mathcal{F}}_{\mathrm{t}})$;

(ii) $b \in L[b]$ and $b^{\#}$ exists;

(iii) $c \in L[b]$;

(iv) κ is an uncountable cardinal number such that $\mathbf{T} \in L_\kappa[b]$, where $\mathbf{T}$ is the common first component of the $\mathbf{c}_i$.

The definition of $\bar{\mathcal{F}}^*_{\mathrm{phi}}(b, \langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$ is exactly like that of $\bar{\mathcal{F}}^{L[b]}_{\mathrm{phi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U})$, with two exceptions. First, induction hypothesis (3) is replaced by

(3′) if $m \in \omega$ and $i \leq j \leq n$ then $\phi_{j,i}(\sigma) \restriction {}_m T_i$ belongs to $L[b]$ for all $\sigma \in \mathrm{domain}\,(\phi_{j,i})$.

Second, in the induction step of the definition of $\mathbf{d}'$, we set

$$\phi_{n+1,n} \;=\; \mathcal{F}^*_{\mathrm{phi}}(b, \mathbf{c}^n_n, \kappa, m_n, k+n, i_n).$$

We have the following analogue of Lemma 5.2.23.

Lemma 5.3.6. *Let*

$$\langle b, \langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k\rangle \in \mathrm{domain}\,(\bar{\mathcal{F}}^*_{\mathrm{phi}}),$$

with $\mathbf{T}$ *is the common first component of the* $\mathbf{c}_i$. *Let* $\tilde{\mathbf{T}} = \bar{\mathcal{F}}_{\mathrm{t}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$ *and let* $\phi = \bar{\mathcal{F}}^*_{\mathrm{phi}}(b, \langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$. *Then*

(a) $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S},L[b]}{\Rightarrow} \mathbf{T}$;

(b) $\phi \restriction (\mathcal{S}({}_k\tilde{T}) \cap L[b])$ *is the identity.*

(c) for all $\tilde{\sigma} \in \mathrm{domain}\,(\phi)$ *and for all* $n \in \omega$, $\phi(\tilde{\sigma}) \restriction {}_n T$ *belongs to* $L[b]$.

Proof. We content ourselves with sketching the proof of (c). Let $\mathbf{T}_i$, $i \in \omega$, be as in the definiton of $\bar{\mathcal{F}}_{\mathrm{t}}$ and let $\phi_{j,i}$, $i \leq j \in \omega$, be as in the definition of $\bar{\mathcal{F}}^M_{\mathrm{phi}}$, for $M = L[b]$. For $i \in \omega$, let $\phi_{\infty,i} : \mathcal{S}(\tilde{T}) \to \mathcal{S}(T_i)$ be the canonical function. Then $\phi = \phi_{i,0} \circ \phi_{\infty,i}$. For each n, there is an i such that $\phi_{\infty,i} \restriction \mathcal{S}({}_n\tilde{T})$ is the identity. Thus (c) follows from the definitions and clause (c) of Lemma 5.3.3. □

Finally we define an operation $\bar{\mathcal{F}}^*_{\mathrm{psi}}$ whose domain is the same as that of $\bar{\mathcal{F}}^*_{\mathrm{phi}}$. The definition of $\bar{\mathcal{F}}^*_{\mathrm{psi}}(b, \langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$ is exactly like that of $\bar{\mathcal{F}}^{L[b]}_{\mathrm{psi}}(\langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k, \mathcal{U})$, except that, in the definition of $\mathbf{d}''$, we make the changes noted above when we defined $\bar{\mathcal{F}}^*_{\mathrm{psi}}(b, \langle \mathbf{c}_i \mid i \in \omega\rangle, \kappa, k)$, and, in the induction step, we set

$$\Psi^{n,n+1} \;=\; \mathcal{F}^*_{\mathrm{psi}}(b, \mathbf{c}^n_n, \kappa, m_n, k+n, i_n).$$

We have the following analogue of Lemma 5.2.25.

Lemma 5.3.7. *Let b be a set such that $b \in L[b]$ and such that $b^{\#}$ exists. Let $\langle \mathbf{c}_i \mid i \in \omega \rangle \in L[b]$ be a sequence of $\mathbf{\Pi}^1_1$ codes with the same first component $\mathbf{T}$. For $i \in \omega$ let A_i be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}_i$. Let A be the subset of $\lceil \mathbf{T} \rceil$ that $\langle A_i \mid i \in \omega \rangle$ witnesses to belong to ω–$\mathbf{\Pi}^1_1$. Let κ be an uncountable cardinal such that b and T belong to $L_\kappa[b]$. Let $k \in \omega$. Let*

$$\begin{aligned}
\tilde{\mathbf{T}} &= \bar{\mathcal{F}}_{\mathrm{t}}(\langle \mathbf{c}_i \mid i \in \omega \rangle, \kappa, k);\\
\pi &= \bar{\mathcal{F}}_{\mathrm{pi}}(\langle \mathbf{c}_i \mid i \in \omega \rangle, \kappa, k);\\
\phi &= \bar{\mathcal{F}}^*_{\mathrm{phi}}(b, \langle \mathbf{c}_i \mid i \in \omega \rangle, \kappa, k);\\
\Psi &= \bar{\mathcal{F}}^*_{\mathrm{psi}}(b, \langle \mathbf{c}_i \mid i \in \omega \rangle, \kappa, k).
\end{aligned}$$

Then

(i) *$\langle \tilde{\mathbf{T}}, \pi_1, \phi_1, \Psi_1 \rangle$ is a $(A, \bigcap_{i \in \omega} A_i)$ k-semicovering of $\mathbf{T}$ with respect to M;*

(ii) *both $\tilde{\mathbf{T}}$ and π belong to $L[b]$ and $T \subseteq L_\kappa[b]$;*

(iii) *for all $\tilde{\sigma} \in \mathrm{domain}\,(\phi)$ and for all $n \in \omega$, $\phi(\tilde{\sigma}) \restriction_n T$ belongs to $L[b]$.*

We are now ready to deal with ωn–$\mathbf{\Pi}^1_1$ sets.

Lemma 5.3.8. *Let b be a set such that $b \in L[b]$ and such that $b^{\#}$ exists. Let $n \in \omega$. Let $\langle \mathbf{c}_\beta \mid \beta < \omega n \rangle \in L[b]$ be a sequence of $\mathbf{\Pi}^1_1$ codes with the same first component $\mathbf{T}$. For $\beta < \omega n$ let A_β be the $\mathbf{\Pi}^1_1$ set coded by $\mathbf{c}_\beta$. Let A be the subset of $\lceil \mathbf{T} \rceil$ that $\langle A_\beta \mid \beta < \omega n \rangle$ witnesses to belong to ω–$\mathbf{\Pi}^1_1$. Then $G(A; \mathbf{T})$ is determined. Indeed, there is a winning strategy σ for $G(A; \mathbf{T})$ such that $\sigma \restriction_i T$ belongs to $L[b]$ for each $i \in \omega$.*

Proof. Let κ be an uncountable cardinal such that b and T belong to $L_\kappa[b]$. Let $\kappa_1 = \kappa$ and, for $1 \le m < n$, let $\kappa_{m+1} = (\kappa_m)^+$.

For $\beta < \omega n$, let $\mathbf{c}_\beta = \langle \mathbf{T}_0, E_\beta, f_\beta \rangle$, where f_β is $p \mapsto <^\beta_p$.

By induction on $m \le n$ we define a system

$$\langle \mathbf{T}_m \mid m < n \rangle, \langle \pi_{m,m'}, \phi_{m,m'}, \Psi^{m',m} \mid m \le m' \le n \rangle \rangle.$$

Our definition will be such that

(i) if $m \le m' \le n$ and

$$\mathcal{C}_{m',m} = \langle T_{m'}, \pi_{m',m}, \phi_{m',m}, \Psi^{m,m'} \rangle,$$

then $\mathcal{C}_{m',m}$ is a $(\boldsymbol{\pi}^{-1}_{m,0}(A), \boldsymbol{\pi}^{-1}_{m,0}(\bigcap_{\omega m \leq \beta < \omega m'} A_\beta))$ semicovering of $\mathbf{T}_m$ with respect to $L[b]$ such that $\mathbf{T}_{m'}$ and $\pi_{m',m}$ belong to $L[b]$, such that $T_{m'} \subseteq L_{\kappa_{m'}}[b]$, and such that $\phi_{m',m}(\sigma) \restriction {}_i T_m$ belongs to $L[b]$ for every $\sigma \in \text{domain}\,(\phi_{m',m})$ and every $i \in \omega$.

(ii) if $m \leq m' \leq m'' \leq n$ then $\mathcal{C}_{m'',m} = \mathcal{C}_{m'',m'} \circ \mathcal{C}_{m',m}$.

We begin by setting $\mathbf{T}_0 = \mathbf{T}$.

Now let $0 \leq m < n$. For $\beta < \omega n$, set

$$\mathbf{c}^m_\beta = \langle \mathbf{T}_m, \pi^{-1}_{m,0}(E_\beta), f^m_\beta \rangle,$$

where $f^m_\beta(p) = <^\beta_{\pi_{m,0}(p)}$. We define $\mathcal{C}_{m+1,m}$ by setting

$$\begin{aligned}
\mathbf{T}_{m+1} &= \bar{\mathcal{F}}_{\rm t}(\langle \mathbf{c}^m_\beta \mid \omega m \leq \beta < \omega(m+1)\rangle, \kappa, 0\rangle);\\
\pi_{m+1,m} &= \bar{\mathcal{F}}_{\rm pi}(\langle \mathbf{c}^m_\beta \mid \omega m \leq \beta < \omega(m+1)\rangle, \kappa, 0\rangle);\\
\phi_{m+1,m} &= \bar{\mathcal{F}}^*_{\rm phi}(b, \langle \mathbf{c}^m_\beta \mid \omega m \leq \beta < \omega(m+1)\rangle, \kappa, 0\rangle);\\
\Psi^{m,m+1} &= \bar{\mathcal{F}}^*_{\rm psi}(b, \langle \mathbf{c}^m_\beta \mid \omega m \leq \beta < \omega(m+1)\rangle, \kappa, 0\rangle).
\end{aligned}$$

For $m' < m$ we define $\pi_{m+1,m'}$, $\phi_{m+1,m'}$, and $\Psi^{m,m+1}$ as required by (ii).

By (i), $\mathcal{C}_{n,0}$ is an $(A, \bigcap_{\beta < \omega n})$ semicovering of $\mathbf{T}$ with respect to $L[b]$. Since $\boldsymbol{\pi}^{-1}_{n,0}(A)$ is empty, $\mathcal{C}_{n,0}$ unravels A. $\square$

Lemma 5.3.8 gives us our strengthening of Theorem 4.4.2:

Theorem 5.3.9. *Let λ be an infinite cardinal number. Assume that*

$$(\forall a \subseteq \lambda)\, a^\# \text{ exists.}$$

Then, for every $n \in \omega$ and every game tree T such that $|T| \leq \lambda$, all ωn–$\mathbf{\Pi}^1_1$ games in T are determined.

Proof. Let $\mathbf{T}$ be a game tree with taboos with $|T| \leq \lambda$. Let $n \in \omega$. Let $\langle A_\beta \mid \beta < \omega n\rangle$ witness that $A \subseteq \lceil \mathbf{T} \rceil$ belongs to ωn–$\mathbf{\Pi}^1_1$. We will show that $G(A; \mathbf{T})$ is determined. We may assume without loss of generality that field$\,(T) \subseteq \lambda$. Let $\langle \mathbf{c}_\beta \mid \beta < \omega n\rangle$ be such that each $\mathbf{c}_\beta$ is a $\mathbf{\Pi}^1_1$ code for A_β. It is easy to see that there is a $b \subseteq \lambda$ such that both $\langle \mathbf{c}_\beta \mid \beta < \omega n\rangle$ belongs to $L_{\lambda^+}[b]$. Since $b^\#$ exists, Lemma 5.3.8 implies that $G(A; \mathbf{T})$ is determined. $\square$

Remarks:

(a) (Exercise 4.4.2) implies the converse of Theorem 5.3.9 for the case of countable λ. Thus the two theorems together give that the determinacy of all $\mathbf{\Pi}^1_1$ games in countable trees implies the determinacy of all all β–$\mathbf{\Pi}^1_1$ games in countable trees for all $\beta < \omega^2$. To my knowledge, no direct proof is known of this latter fact.

(b) A special case of Theorem 5.3.9 is due to Friedman [1971a]. (See remark (b) following Theorem 5.3.10 below.)

(c) The converse of Theorem 5.3.9 was first proved by the author. (See remark (c) following Theorem 5.3.10.)

Suppose that α is an ordinal smaller than the ordinal called ω_1^{CK}; i.e., suppose there is a recursive wellordering of a (recursive) subset of ω of order type α. If $A \subseteq {}^\omega\omega$, then A belongs to the lightface class α–Π^1_1 if there are sets A_β, $\beta < \alpha$, and there is a one-one $g : \alpha \to \omega$ such that

(1) $\{\langle g(\gamma), g(\beta)\rangle \mid \gamma < \beta < \alpha\}$ is recursive;

(2) $\{\langle g(\beta), x\rangle \mid \beta < \alpha \wedge x \in A_\beta\} \in \Pi^1_1$.

Condition (1) can be modified without changing the concept defined. One can replace "recursive" by "Δ^1_1," and, in the other direction, one can require that range $(g) = \omega$ for α infinite. There are fairly reasonable notions of α–Π^1_1 for larger classes of countable ordinals, notions that we will not discuss.

Here is the lightface version of Theorem 5.3.9.

Theorem 5.3.10. *If $0^\#$ exists then, for all $n \in \omega$, all ωn–Π^1_1 games in ${}^{<\omega}\omega$ are determined.*

Proof. The proof of Theorem 5.3.9 proves the present theorem, for our hypotheses allow us to take the b of that proof to be $\emptyset$. □

Remarks:

(a) Harrington's Exercise 4.4.1 implies the converse of Theorem 5.3.10. The two theorems together imply that, for all $n \in \omega$, the determinacy of all ωn–Π^1_1 games in ${}^{<\omega}\omega$ is a consequence of the determinacy of all Π^1_1 games in ${}^{<\omega}\omega$. No direct proof is known of this last fact.

(b) In Friedman [1971a] there is a proof of the determinacy of all 3–Π^1_1 games from the existence of $0^\#$. That proof easily generalizes to n–Π^1_1.

(Friedman apparently did not notice that the class he was considering was the third level of a hierarchy.)

(c) The converse of Theorem 5.3.10 was proved by the author before Harrington's result, but after Theorem 5.3.10 itself. He showed that the determinacy of all 3–Π^1_1 games in ${}^{<\omega}\omega$ implies the existence of $0^\#$.

We now turn to the many-measurable-cardinals generalization of Theorem 5.3.9. For this we will need a slight strengthening of the basic determinacy result, Lemma 5.2.1.

Lemma 5.3.11. *Let M be a transitive class model of* ZFC. *Let $\mathbf{T}$ be a game tree with taboos. Let $A \subseteq \lceil \mathbf{T} \rceil$. Let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be an A-semicovering of $\mathbf{T}$ with respect to M that unravels A and is such that $\tilde{\mathbf{T}} \in M$. Assume that there is a winning strategy $\tilde{\sigma}$ for $G(\boldsymbol{\pi}^{-1}(A); \mathbf{T})$ such that, for every $k \in \omega$, $\tilde{\sigma} \restriction {}_k T$ belongs to M. Then $G(A; \mathbf{T})$ is determined.*

Proof. Since $\tilde{\sigma} \in M$, $\tilde{\sigma} \in \operatorname{domain}(\phi)$. As in the proof of Lemma 4.2.1, i.e., as in that of of Lemma 2.1.3, $\phi(\tilde{\sigma})$ is a winning strategy for $G(A; \mathbf{T})$. $\square$

Theorem 5.3.12. *Let α be a countable ordinal. Let λ be an infinite cardinal number. Assume that for every $a \subseteq \lambda$ there is a transitive proper class model M of* ZFC *such that*

(i) the class of $\kappa > \lambda$ such that $M \models$ "κ is a measurable cardinal" has order type $\geq \alpha$;

(ii) there is a proper class C of indiscernibles for M, a.

Then, for every $\beta < \omega^2(\alpha + 1)$, all β–$\mathbf{\Pi}^1_1$ games in trees of size λ are determined.

Proof. Let $n \in \omega$. Let $\mathbf{T}$ be a game tree with taboos with $\operatorname{field}(T) \subseteq \lambda$. Let $\langle A_\beta \mid \beta < \omega^2\alpha + \omega n \rangle$ witness that $A \subseteq \lceil \mathbf{T} \rceil$ belongs to $(\omega^2\alpha + \omega n)$–$\mathbf{\Pi}^1_1$. For $\beta < \omega^2\alpha + \omega n$, let $\mathbf{c}_\beta$ be a $\mathbf{\Pi}^1_1$ code for A_β.

Let $a \subseteq \lambda$ be such that $\langle \mathbf{c}_\beta \mid \beta < \omega^2\alpha + \omega n \rangle$ belongs to $L[a]$ and such that α is countable in $L[a]$. Let M be as given by the hypotheses of the theorem.

Let $\langle \kappa_\gamma \mid \gamma < \alpha \rangle$ enumerate the first α measurable cardinals of M in order of magnitude. Let $\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle \in M$ be such that $M \models$ "$\mathcal{U}_\gamma$ is a uniform normal ultrafilter on κ_γ" for every $\gamma < \alpha$. Replacing M by $\mathrm{Ult}_{\omega_1}(M;\mathcal{U}_0)$ if necessary, we may assume that $\kappa_0 \geq \omega_1$. Let

$$N = L[a, \prec\mathcal{U}_\gamma \mid \gamma < \alpha\succ].$$

By Theorem 3.5.4, $\langle \mathcal{U}_\gamma \cap N \mid \gamma < \alpha\rangle$ is definable in M from a and $\langle \kappa_\gamma \mid \gamma < \alpha\rangle$. Thus property (ii) of M gives us the existence of $(\prec a, \prec\mathcal{U}_\gamma \cap N \mid \gamma < \alpha\succ\succ)^\#$.

Observe that the construction and proof of Theorem 5.2.32 go through unchanged if we replace the assumption that each $\mathcal{U}_\gamma$ is in V a uniform normal ultrafilter on κ_γ by the hypothesis that each $\mathcal{U}_\gamma \cap L[a, \prec\mathcal{U}_\gamma \mid \gamma < \alpha\succ]$ is in $L[a, \prec\mathcal{U}_\gamma \mid \gamma < \alpha\succ]$ a uniform normal ultrafilter on κ_γ, provided that we add the assumption that $\kappa_0 \geq \omega_1$. (Without this last assumption, we might have $\lambda_i^\gamma \geq \kappa_{\gamma'}$ for with $\gamma < \gamma' < \alpha$.) In the present context, we can thus repeat the earlier definitions and construction, getting a $\mathcal{C}_{\alpha,0} = \langle \mathbf{T}_\alpha, \pi_{\alpha,0}, \phi_{\alpha,0}, \Psi^{0,\alpha}\rangle$ that is an $(A, \bigcap_{\beta<\omega^2\alpha} A_\beta)$ semicovering of $\mathbf{T}$ with respect to $M_\alpha = L[a, \langle \lambda_i^\xi \mid \xi < \alpha \wedge i \in \omega\rangle]$.

Note that the existence of $(\prec a, \prec\mathcal{U}_\gamma \cap N \mid \gamma < \alpha\succ\succ)^\#$ implies the existence of $(\prec a, \langle \lambda_i^\xi \mid \xi < \alpha \wedge i \in \omega\rangle\succ)^\#$.

The hypotheses of Lemma 5.3.8 are satisfied with $\prec a, \langle \lambda_i^\xi \mid \xi < \alpha \wedge i \in \omega\rangle\succ$ as b, with and $\langle \mathbf{c}^\alpha_{\omega^2\alpha+\xi} \mid \xi < \omega n\rangle$ as $\langle \mathbf{c}_\beta \mid \beta < \omega n\rangle$, and with $\hat{A}$, the set witnessed to be ωn–$\mathbf{\Pi}^1_1$ by $\langle \boldsymbol{\pi}^{-1}_{\alpha,0}(A_{\omega^2\alpha+\xi}) \mid \xi < \omega n\rangle$, as A. Hence Lemma 5.3.8 gives us a winning strategy σ for the game $G(\hat{A}; \mathbf{T}_\alpha)$ with the property that, for every $i \in \omega$, $\sigma \restriction {}_iT_\alpha \in M_\alpha$. But $\hat{A} = \boldsymbol{\pi}^{-1}_{\alpha,0}(A)$, and so we get the determinacy of $G(A; \mathbf{T})$ as in the proofs of Theorems 5.2.31 and 5.2.32. $\square$

Remarks:

(a) For ordinals such that a lightface notion α–$\mathbf{\Pi}^1_1$ is definable, the lightface version of Theorem 5.3.12 holds. See Exercise 5.3.1.

(b) For each α and λ, the converse of Theorem 5.3.12 holds. (This is also true of the lightface version.) See Exercises 5.3.4, 5.3.5, and 5.3.6.

(c) Using the $\mathbf{Diff}^*(\mathbf{\Pi}^1_1)$ hierarchy, one can remove the condition that α be countable from Theorem 5.3.12 and its converse. See Exercises 5.3.2 and 5.3.7.

Theorem 5.3.12 and Exercise 5.3.6 imply that, for all countable ordinals α, for all infinite cardinals λ, and for all ordinals β and γ with $1 \leq \beta \leq \gamma < \omega^2$, the determinacy of $(\omega^2\alpha+\beta)$–$\mathbf{\Pi}^1_1$ games is equivalent to that of $(\omega^2\alpha+\gamma)$–$\mathbf{\Pi}^1_1$

games. Analogous results hold for uncountable α (Exercises 5.3.2 and 5.3.7) and for the lightface case. For classes of the form $(\omega^2\alpha)$–$\mathbf{\Pi}^1_1$, determinacy turns out not to be provably equivalent to the determinacy of any of the classes just mentioned. We already know half of this fact. Consider, for simplicity, the lightface case, for small enough α. The determinacy of $(\omega^2\alpha+1)$–Π^1_1 games implies the existence of indiscernibles for a transitive proper class model of ZFC with α measurable cardinals, and the strictly weaker assumption that there exists a transitive proper class model of ZFC with α measurable cardinals implies $\omega^2\alpha$–Π^1_1 determinacy.

It turns out that there is a rich structure of classes lying between $\bigcup_{n\in\omega}\omega n$–$\Pi^1_1$ and (ω^2+1)–Π^1_1 with inequivalent determinacy questions.

Derrick DuBose has found classes whose determinacy is equivalent with

(1) the existence of $0^{k\#}$, where $0^{1\#}=0^\#$ and $0^{(k+1)\#}=(0^{k\#})^\#$ for $1\leq k\in\omega$ (DuBose [1990] and Exercise 5.3.10);

(2) the existence of a proper class of indiscernibles for $L[\#_1]$, where $\#_1$ is the sharp function on ${}^\omega\omega$ (DuBose [1992] and Exercise 5.3.14;

(3) the existence of a proper class of indiscernibles for $L[\#_k]$, where $\#_k$ is the sharp function on type k objects (DuBose [1995] and Exercise 5.3.15).

Equivalence results for classes intermediate between those of (2) and (3) are in DuBose [1992a] and DuBose [199?].

DuBose and the author independently proved that determinacy for a certain class is equivalent with the existence of a proper class of indiscernibles for $L[\#]$, where $\#$ is the sharp function on arbitrary objects. See Exercise5.3.17.

Philip Welch has found a large cardinal hypothesis equivalent with the determinacy ω^2–$\mathbf{\Pi}^1_1$ games. Indeed, Welch [1996] gives large cardinal equivalents for $\omega^2\gamma$–Π^1_1 determinacy for, e.g., all recursive ordinals γ. Welch [19??] gives, for each such γ, an equivalent for the determinacy of games $G(A;{}^{<\omega}\omega)$ such that both A and $\neg A$ belong to $\omega^2\gamma$–Π^1_1.

Exercise 5.3.1. Let $\alpha<\omega_1^{\mathrm{CK}}$. Assume that there is a transitive proper class M such that $\{\kappa \mid M\models$ "κ is a measurable cardinal"$\}$ has order type $\geq\alpha$ and such that there is a proper class of indiscernibles for M. Prove that, for every $\beta<\omega^2(\alpha+1)$, all β–Π^1_1 games are determined.

Remark. The lightface version of Exercise 5.3.6 gives a strong converse to this result, with "$\beta=\omega^2\alpha+1$" replacing "for every $\beta<\omega^2(\alpha+1)$."

Exercise 5.3.2. Show that Theorem 5.3.12 remains true if the restriction that α be countable is removed, provided that "β–$\mathbf{\Pi}^1_1$" is replaced by "(β–$\mathbf{\Pi}^1_1)^*$" in its statement. (See page 50 for the definition of the generalized difference hierarchy.) This theorem of the author was proved around 1990.

Hint. Use the methods of §2.2 to generalize the limit operations of §5.2.4.

Exercise 5.3.3. Work in ZF and assume the Axiom of Determinacy. Prove that ω_1 is a measurable cardinal. Indeed prove that the filter generated by the closed unbounded subsets of ω_1 is an ultrafilter. (This result of Solovay was perhaps the main cause for set theorists' becoming interested in AD.)

Hint. Let $X \subseteq \omega_1$. We define a game $G = G(A; {}^{<\omega}\omega)$, describing A implicitly as follows. Dividing the even and odd natural numbers into infinitely many infinite sets, construe I's part of a play z as a sequence $\langle x_i^z \mid i \in \omega\rangle$ of elements of ${}^\omega\omega$ and construe II's part as a sequence $\langle y_i^z \mid i \in \omega\rangle$. For each $i \in \omega$, let $r_i^z = \{\langle m, n\rangle \mid x_i^z(m) < x_i^z(n)\}$ and let $s_i^z = \{\langle m, n\rangle \mid y_i^z(m) < y_i^z(n)\}$. If r_i^z is a wellordering, let η_i^z be its order type. If s_i^z is a wellordering, let ζ_i^z be its order type.

The winning conditions for G are as follows:

(1) If some r_i^z or s_i^z is not a wellordering, then I wins if and only if, for the least such i, r_i^z is a wellordering.

(2) If all r_i^z and all s_i^z are wellorderings, then let

$$\rho^z = \sup_{i\in\omega}\max\{\eta_i^z, \zeta_i^z\}.$$

I wins if and only if $\rho^z \in X$.

Assume for definiteness I has a winning strategy σ for G.

For $i \in \omega$, show that there is a closed unbounded subset C_i of ω_1 such that, for all plays z consistent with σ and all $\xi \in C$,

$$(\forall j < i)(\zeta_j^z \text{ is defined and } \zeta_j^z < \xi \;\rightarrow\; \eta_i^z \text{ is defined and } \eta_i^z < \xi).$$

To do this, note that if $\xi < \omega_1$ and

$$B_\xi = \{x_i^z \mid z \text{ is consistent with } \sigma \;\wedge\; (\forall j < i)(\zeta_j^z \text{ is defined and } \zeta_j^z < \xi)\},$$

then $B_\xi \in \mathbf{\Sigma}^1_1$ and the r_i^z associated with members of B_ξ are all wellorderings. By the boundedness principle (Moschovakis [1980], Exercise 4A6), the set of all order types of these wellorderings is bounded in ω_1.

Let $C = \bigcap_{i\in\omega} C_i$. Show that every member of C_i is ρ^z for some play z consistent with σ. Thus C is a closed, unbounded subset of X.

Remark. The proof outlined in the hint is Solovay's original proof. Variations of it are used in the following exercises. There are other, very different, proofs. One such proof, due to the author, uses *Turing determinacy*, i.e., $\mathcal{P}({}^\omega\omega)$ Turing determinacy as defined in Exercise 1.4.3. By Exercise 1.4.4, Turing determinacy gives a countably complete ultrafilter on the set of all degrees of unsolvability. Mapping each Turing degree $\mathbf{d}$ to, e.g., the least $\mathbf{d}$-admissible ordinal, one induces a countably complete ultrafilter on ω_1.

Exercise 5.3.4. Assume that all lightface (ω^2+1)–Π^1_1 games are determined and prove that there is a transitive proper class model for ZFC + "There is a measurable cardinal."

Hint. Divide up the natural numbers as in the hint to Exercise 5.3.3, but now let let I's part of a play z give $z(0)$ plus $\langle x^z_\beta \mid \beta \leq \omega^2\rangle$ and let II's part give $\langle y^z_\beta \mid \beta < \omega^2\rangle$. For $\beta \leq \omega^2$, define r^z_β, s^z_β, η^z_β, and ζ^z_β by analogy with (and under the same conditions as) the corresponding concepts in the earlier hint. For $n \in \omega$, let

$$\rho^z_n = \text{sup}_{i\in\omega}\text{max}\{\eta^z_{\omega n+i}, \zeta^z_{\omega n+i}\},$$

provided that all the $\eta^z_{\omega n+i}$ and all the $\zeta^z_{\omega n+i}$ are defined.

Let G be the game in ${}^{<\omega}\omega$ with the following winning conditions:

(1) If some r^z_β or s^z_β is not a wellordering, then I wins if and only if, for the least such β, r^z_β is a wellordering.

(2) If all r^z_β and all s^z_β are wellorderings, let $\kappa^z = \text{sup}_{n\in\omega}\rho^z_n$ and let $\mathcal{U}^z$ be the filter on κ^z generated by the tails of the sequence $\langle \rho^z_n \mid n \in \omega\rangle$, i.e., let

$$\mathcal{U}^z = \{X \subseteq \kappa^z \mid (\exists m)(\forall n \geq m)\, \rho^z_n \in X\}.$$

I wins if and only if

(i) there is $f : \kappa^z \to \kappa^z$ such that $f \in L_{\eta^z_{\omega^2}}[\mathcal{U}^z]$, such that $\{\gamma \mid f(\gamma) < \gamma\} \in \mathcal{U}^z$, and such that f is not constant on a set belonging to $\mathcal{U}^z$;

(ii) for f^z the $<_{L[\mathcal{U}^z]}$-least f witnessing (i),

(a) $(\forall n \in \omega)\, f^z(\rho^z_n) < \rho^z_n$;

(b) $f^z(\rho_1^z) > f^z(\rho_0^z) \;\leftrightarrow\; z(0) > 0.$

Prove that G is (ω^2+1)–Π_1^1.

Assume that σ is a winning strategy for I for G.

Use the boundedness principle to show that there is a closed unbounded subset C of ω_1 such that for every strictly increasing sequence $\langle \xi_n \mid n \in \omega \rangle$ of elements of C there is a play z consistent with σ such that $(\forall n \in \omega)\, \rho_n^z = \xi_n$.

Let $\langle \rho_\xi \mid \xi < \omega_1 \rangle$ enumerate C in order of magnitude. Let λ be a limit ordinal such that $\rho_\lambda = \lambda$. Let $\kappa = \rho_{\lambda+\omega}$. Let $\mathcal{U}$ be the filter on κ generated by the tails of the sequence $\langle \rho_{\lambda+n} \mid n \in \omega \rangle$.

Let f be least in the canonical wellordering of $L[\mathcal{U}]$ such that $\{\gamma \mid f(\gamma) < \gamma\} \in \mathcal{U}$ but such that f is not constant on an element of $\mathcal{U}$. (Because there is a play $\bar{z}$ consistent with σ such that each $\rho_n^{\bar{z}} = \rho_{\lambda+n}$, there must be such an f.)

First show that

$$(\dagger) \qquad (\forall \xi < \lambda + \omega)\, f(\rho_\xi) < \rho_\xi.$$

To do this, assume for a contradiction that $\xi < \lambda + \omega$ and $f(\rho_\xi) \geq \rho_\xi$. Let z be a play consistent with σ such that $\rho_0^z = \rho_\xi$ and $\rho_n^z = \rho_{\max\{\lambda,\xi\}+n}$ for all $n > 0$. It is clear that $\mathcal{U}^z = \mathcal{U}$. Therefore $f^z = f$, and so $f(\rho_0^z) \geq \rho_0^z$, contrary to the assumption that σ is a winning strategy for I for G.

Assume that σ calls for I to play $z(0) = 0$. There must be an $n \in \omega$ such that $f(\rho_{\lambda+n+1}) > f(\rho_{\lambda+n})$, for otherwise f would be constant on a tail of the $\rho_{\lambda+n}$ and so on an element of $\mathcal{U}$. Let m be the least such n and let z be a play consistent with σ such that $\rho_n^z = \rho_{\lambda+m+n}$ for all $n \in \omega$. Then $f^z(\rho_1^z) > f^z(\rho_0^z)$, contradicting the assumption that σ is a winning strategy.

Thus σ calls for I to play $z(0) > 0$. If there exist ξ_0 and ξ_1 such that $\xi_0 < \xi_1 < \lambda$ and such that $f(\rho_{\xi_1}) \leq f(\rho_{\xi_0})$, then one can contradict the assumption that σ is a winning strategy by letting z be consistent with σ and such that $\rho_0^z = \rho_{\xi_0}$, $\rho_1^z = \rho_{\xi_1}$, and $\rho_n^z = \rho_{\lambda+n}$ for all $n \geq 2$. Thus $f \restriction (C \cap \lambda)$ is order preserving. From $(\dagger)$ it follows that $f \restriction (C \cap \lambda) : C \cap \lambda \to \lambda$. But λ was chosen such that $C \cap \lambda$ has order type λ. Thus range$(f \restriction (C \cap \lambda))$ is unbounded in λ. Since $f(\rho_\lambda) < \lambda$ by $(\dagger)$, there must be a $\xi < \lambda$ such that $f(\rho_\xi) > f(\rho_\lambda)$. From this one can once again contradict the assumption that σ is a winning strategy.

By the determinacy assumption, let τ be a winning strategy for II for G.

Use the boundedness principle to show that there is a closed unbounded subset C of ω_1 such that for every $k \in \omega$, for every strictly increasing sequence $\langle \xi_n \mid n \in \omega \rangle$ of elements of C, and for every $\delta < \omega_1$, there is a play z consistent with G such that $z(0) = k$, such that $(\forall n \in \omega)\, \rho^z_n = \xi_n$, and such that $\eta^z_{\omega^2} \geq \delta$

Let $\langle \xi_n \mid n \in \omega \rangle$ be an increasing sequence of elements of C and let $\kappa = \sup_{n \in \omega} \xi_n$. Let $\mathcal{U}$ be the filter generated by the tails of the sequence $\langle \xi_n \mid n \in \omega \rangle$.

Assume that $\mathcal{U} \cap L[\mathcal{U}]$ is not in $L[\mathcal{U}]$ a uniform normal ultrafilter on κ. By an absoluteness argument, there is a countable δ such that $\mathcal{U} \cap L[\mathcal{U}]$ is not in $L_\delta[\mathcal{U}]$ a uniform normal ultrafilter on κ. Let f be least in the canonical wellordering of $L[\mathcal{U}]$ such that $\{\gamma \mid f(\gamma) < \gamma\} \in \mathcal{U}$ but such that f is not constant on an element of $\mathcal{U}$. Thus $f \in L_\delta[\mathcal{U}]$. Let m be least number such that $f(\xi_n) < \xi_n$ for all $n \geq m$. Let z be a play consistent with τ such that $\rho^z_n = \xi_{m+n}$ for all $n \in \omega$, such that $z(0) = 0$ if and only if $f(\xi_{m+1}) \leq f(\xi_m)$, and such that $\eta^z_{\omega^2} = \delta$. Then z is a win for I, contradicting the assumption that τ is a winning strategy for II for G.

Remark. The result of the exercise, the a partial converse of Theorem 4.4.2, is due to the author. The earliest result of this general kind was proved by Robert Solovay, who deduced the existence of a transitive proper class model for ZFC + "There is a measurable cardinal" from the hypothesis that all $\mathbf{\Pi}^1_3$ games in ${}^{<\omega}\omega$ are determined. After further results by Harvey Friedman (in Friedman [1971a]) and by the author, Solovay succeeded in weakening his hypothesis to the determinacy of all Δ^1_2 games and in improving his conclusion to get models with many measurable cardinals. All these results were based on the author's proof of the measurabilty of ω_1 under AD, the proof sketched in the remark after the hint to Exercise 5.3.3. (Note that the proof of the optimal result, that of the present exercise, is based on Solovay's original proof of the measurability of ω_1.)

To get a model with a single measurable cardinal, Solovay proceeded as follows. To each degree of unsolvability $\mathbf{d}$, he associated the sequence of the first ω $\mathbf{d}$-admissibles. He thus associated with $\mathbf{d}$ the filter generated by the tails of this sequence. Turing determinacy guarantees that any question about $\mathbf{d}$ and its associated sequence is constant on a cone. A result of Ronald Jensen guarantees that any infinite increasing sequence of $\mathbf{d}$-admissibles is the sequence of the first ω $\mathbf{d}'$-admissibles for some $\mathbf{d}'$-admissibles for some $\mathbf{d}' > \mathbf{d}$. These facts make possible an argument much like that given in the hint to the exercise.

Exercise 5.3.5. Assume that all lightface (ω^2+1)–Π^1_1 games are determined and prove that $0^\dagger$ exists. (See page 180 for the definition of $0^\dagger$.)

Hint. Consider a game G like that of the hint to Exercise 5.3.4, but modify the earlier game so that II's part of a play z also gives a $y^z_{\omega^2}$, and so an associated $s^z_{\omega^2}$. For convenience, arrange the division of ω so that the moves related only to $x^z_{\omega^2}$ and $y^z_{\omega^2}$ consititute, when all other moves are ignored, a genuine play of a game with I and II moving alternately. By regarding $r^z_{\omega^2}$ as R and $s^z_{\omega^2}$ as E, one may construe this play as a play of the G of the hint to Exercise 4.4.1 and so as a play of the G_b of the hint to Exercise 4.4.2, for any subset b of a countable ordinal.

Player I wins a play of G if (a) I wins according to conditions (1) and (2) as on page 270 or (b) the ρ^z_n, $n \in \omega$, are all defined and I wins the play of $G_{\{\rho^z_n | n\in\omega\}}$ given by $x^z_{\omega^2}$ and $y^z_{\omega^2}$.

The game G is (ω^2+1)–Π^1_1 and so is determined by assumption.

Assume that σ is a winning strategy for I for G. Use the boundedness principle and the argument of the hint to Exercise 5.3.4 to show that there is a closed unbounded subset C of ω_1 such that for every strictly increasing sequence $\langle \rho_n \mid n \in \omega\rangle$ of elements of C there is a sequence $\langle y_\beta \mid \beta < \omega^2\rangle$ of elements of ${}^\omega\omega$ such that, for every $y_{\omega^2} \in {}^\omega\omega$, the play z consistent with σ with each $y^z_\beta = y_\beta$ has the following properties:

(i) z does not satisfy condition (a), i.e., z is no a win for I via conditions (1) and (2);

(ii) $(\forall n \in \omega)\, \rho^z_n = \rho_n$;

(iii) $\eta^z_{\omega^2}$ is less than the next element of C after $\bigcup_{n\in\omega} \rho_n$.

Let $\langle \rho_n \mid n \in \omega\rangle$ be any strictly increasing sequence of elements of C. Fix $\langle y_\beta \mid \beta < \omega^2\rangle$ as given by $\langle \rho_n \mid n \in \omega\rangle$ and the stated property of C. Letting y_{ω^2} range over ${}^\omega\omega$, one sees that σ gives a winning strategy fo I for $G_{\{\rho_n | n\in\omega\}}$. But, as in the hint to Exercise 4.4.1, it is easy to see that I cannot have such a winning strategy.

Let τ be a winning strategy for II for G. Use the boundedness principle to show that there is a closed unbounded subset C of ω_1 such that for every strictly increasing sequence $\langle \rho_n \mid n \in \omega\rangle$ of elements of C there is a sequence $\langle x_\beta \mid \beta < \omega^2\rangle$ of elements of ${}^\omega\omega$ such that, for every $k \in \omega$ and every $x_{\omega^2} \in {}^\omega\omega$, the play z consistent with τ with $z(0) = k$ and each $y^z_\beta = y_\beta$ satisfies $(\forall n \in \omega)\, \rho^z_n = \rho_n$.

Let $\langle \rho_n \mid n \in \omega \rangle$ be any strictly increasing sequence of elements of C. By the argument of the hint to Exercise 5.3.4, $\mathcal{U} \cap L[\mathcal{U}]$ is in $L[\mathcal{U}]$ a uniform normal ultrafilter on $\sup_{n\in\omega}\rho_n$, where $\mathcal{U}$ is the filter generated by the tails of the sequence.

Fix $\langle x_\beta \mid \beta < \omega^2 \rangle$ as given by $\langle \rho_n \mid n \in \omega \rangle$ and the stated property of C. Letting y_{ω^2} vary over ${}^\omega\omega$, one sees that τ gives a winning strategy for $G_{\{\rho_n \mid n\in\omega\}}$. But this means that $\{\rho_n \mid n \in \omega\}^\#$ exists and thus that $(\mathcal{U}\cap L[\mathcal{U}])^\#$ exists. Thus $0^\dagger$ exists.

Remarks:

(a) The results of this exercise and the next were proved by the author shortly after Harrington proved the result of Exercise 4.4.1. Harrington's argument made it possible to weaken the hypotheses of earlier theorems of the author.

(b) By the case $\alpha = 1$ of Exercise 5.3.1. the existence of $0^\dagger$ implies the determinacy of all β–Π_1^1 games for every $\beta < \omega^2 2$. The result of the present exercise is a strong converse of that result. Combining the two, we get that the determinacy of all (ω^2+1)–Π_1^1 games implies the determinacy of all β–Π_1^1 games for every $\beta < \omega^2 2$.

Exercise 5.3.6. Let α be a countable ordinal, and let λ be an infinite cardinal number. Assume that all $(\omega^2\alpha + 1)$–$\mathbf{\Pi}_1^1$ games in trees of size $\leq \lambda$ are determined. Prove that for every $a \subseteq \lambda$ there is a transitive proper class model M of ZFC such that

(a) the class of $\kappa > \lambda$ such that $M \models$ "κ is a measurable cardinal" has order type $\geq \alpha$;

(b) there is a proper class C of indiscernibles for M, a.

Hint. The proof is a fairly routine modification of the proof for Exercise 5.3.5. (Exercise 5.3.5 is the lightface version of the case $\alpha = 1$.) One difference is in the properties of the sets C given by the boundedness principle: "for every strictly increasing sequence $\langle \rho_n \mid n \in \omega \rangle$ of elements of C" should be replaced by "for every strictly increasing sequence $\langle \rho_\gamma \mid \gamma < \omega\alpha \rangle$ of elements of C containing none of its limit points." Another difference is that the method of Exercise 4.4.2 has to be used when λ is uncountable.

Remarks:

(a) Essentially the same proof gives the lightface version of the exercise, for ordinals α such that the lightface version makes sense.

(b) Combining the exercise with Theorem 5.3.12, one gets the equivalence, for all α and λ, of (1) the hypotheses of Theorem 5.3.12 (i.e., the conclusion of the exercise), (2) the conclusion Theorem 5.3.12, and (3) the hypotheses of the exercise. By the equivalence of (2) and (3), the determinacy of all $(\omega^2\alpha+1)$–$\mathbf{\Pi}^1_1$ games in trees of size $\leq \lambda$ implies the determinacy of all β–$\mathbf{\Pi}^1_1$ games in such trees for every $\beta < \omega^2(\alpha+1)$. An analogous remark applies to the combination of the lightface version of the exercise with Exercise 5.3.1.

Exercise 5.3.7. Show the result of Exercise 5.3.6 remains true if the restriction that α be countable is removed, provided that "β–$\mathbf{\Pi}^1_1$" is replaced by "$(\beta$–$\mathbf{\Pi}^1_1)^*$" in its statement. (See page 50 for the definition of the generalized difference hierarchy.)

Hint. The method for modifying the proof of Exercise 5.3.6 is analogous to the method for modifying the proof of Exercise 4.4.1 to get that of Exercise 4.4.2.

Exercise 5.3.8. The remaining exercises for this section concern results giving equivalents of determinacy for classes between the classes ωn–Π^1_1 and the class ω^2–Π^1_1. Each of these results is due entirely or (in one case) partly to Derrick DuBose.

DuBose [1990] introduces, for classes Γ, two notions of what one might describe as "ωn–Π^1_1 with n given by a Γ condition." Both notions allow for the possibility that the "Γ condition" fails to yield a number n. The two notions differ precisely in how such failure is treated.

If Γ is a class of sets, then say a subset A of ${}^\omega\omega$ belongs to $(\Gamma)^*$ if there are $\langle A_\beta \mid \beta < \omega^2\rangle$ and $g : \omega^2 \to \omega$ witnessing that some set is ω^2–Π^1_1 and there is a $B \in \Gamma$ with $B \subseteq \omega \times {}^\omega\omega$ such that, for all $x \in {}^\omega\omega$,

$$x \in A \;\leftrightarrow\; (\exists n)(\langle n, x\rangle \in B \,\wedge\, (\forall n' < n)\, \langle n, x\rangle \notin B \,\wedge\, x \in \hat{A}_{\omega n}),$$

where, for $\beta \leq \omega^2$, $\hat{A}_\beta$ is the set witnessed β–$\mathbf{\Pi}^1_1$ by $\langle A_\gamma \mid \gamma < \beta\rangle$, and where we construe 0–Π^1_1 as having $\emptyset$ as its unique member.

Say that $A \subseteq {}^\omega\omega$ belongs to $(\Gamma)^*_+$ if there are $\langle A_\beta \mid \beta < \omega^2\rangle$, g, and B witnessing that some set A^* belongs to $(\Gamma)^*$ and there are $m \in \omega$ and $D \in \omega m$–Π^1_1 such that, for all $x \in^\omega \omega$,

$$x \in A \;\leftrightarrow\; (x \in A^* \,\vee\, ((\forall n)(n, x) \notin B \wedge x \in D)).$$

The boldface notions $(\Gamma)^{**}$ and $(\Gamma)^{**}_+$ are analogously defined, with "$\mathbf{\Pi}^1_1$" replacing "Π^1_1."

Remark. Note that the definitions for $(\Gamma)^*$ and $(\Gamma)^*_+$ agree when there is an n such that $\langle n, x\rangle \in B$. When there is no such n, the $(\Gamma)^*$ definition stipulates that $x \notin A$, but the $(\Gamma)^*_+$ definition stipulates that $x \in A \leftrightarrow x \in D$.

(a) Show that $(\Sigma^1_1)^* = \omega^2$–$\Pi^1_1$.
(b) Let $\Gamma \subseteq \Delta^1_1$ and let $A \in (\Gamma)^*$. Show that there are $\langle A_\beta \mid \beta < \omega^2\rangle$, g, and B witnessing that $A \in (\Gamma)^*$ such that

(i) if $\beta < \gamma < \omega^2$ then $A_\beta \supseteq A_\gamma$;

(ii) if $n < m \in \omega$ and $\langle n, x\rangle \in B$, then $x \in A$ if and only if $x \in \hat{A}_{\omega m}$, where the $\hat{A}_\beta$, $\beta < \omega^2$ are as above;

(iii) $\bigcap_{\beta<\omega^2} A_\beta = \emptyset$.

Hint. In (b), to arrange for (ii) and (iii), intersect the given Π^1_1 sets A'_β with

$$\{x \in {}^\omega\omega \mid (\exists n \in \omega)\, \langle n, x\rangle \in B \ \wedge\ (\forall m \in \omega)(\omega m \leq \beta \rightarrow \langle m, x\rangle \notin B)\}.$$

Exercise 5.3.9. Let $A \in (\Delta^1_1)^*$. Prove that both A and ${}^\omega\omega \setminus A$ belong to ω^2–Π^1_1.

Exercise 5.3.10. Let $1 \leq k \in \omega$, let Γ be a class of sets, and let X be a topological space. A subset B of $\omega \times X$ belongs to the class $(k * \Gamma)$ if there are $R_0, \ldots, R_{k-1}$, such that

(1) each R_i is subset of $\omega \times X$ belonging to Γ;

(2) for all $\langle n, x\rangle \in \omega \times X$, $\langle n, x\rangle$ belongs to B if and only if there are $i < k$ and $j \in \omega$ with $\langle j, x\rangle \in R_i$ and, for the lexicographically least such $\langle i, j\rangle$, $j = n$.

Let $0^{0\#} = \emptyset$ and, for $k \in \omega$, let $0^{(k+1)\#} = (0^{k\#})^\#$.
Let $1 \leq k \in \omega$. Prove that $0^{k\#}$ exists if and only if of all $(k * \Sigma^0_1)^*$ games are determined.

Remark. The definition and the theorem are from DuBose [1990]. For the case $k = 1$, the theorem gives a new equivalent of the existence of $0^\#$.

For each k, the boldface version of "only if" part gives a new equivalent of "$(\forall x \in {}^{\omega}\omega)\, x^{\#}$ exists."

Hint. For the "only if" part, proceed by induction on k as follows.

First let $b \in L[b]$ such that $b^{\#}$ exists, let $A \subseteq {}^{\omega}\omega$, and let $\mathbf{c}$ be a $(1 * \mathbf{\Sigma}^0_1)^{**}$ code (in the obvious sense) for A that belongs to $L[b]$. (See Exercise 5.3.8 for the definition of $(\Gamma)^{**}$.) Prove that there is a winning strategy for $G(A; {}^{<\omega}\omega)$ that belongs to $L[b^{\#}]$. To do this, use $\mathbf{c}$ and the proof of part (b) of Exercise 5.3.8 to get $\langle A_\beta \mid \beta < \omega^2\rangle$ and B that belong to $L[b]$, witness that $A \in (1 * \mathbf{\Sigma}^0_1)^{**}$, and satisfy (i)–(iii) of part (b) of Exercise 5.3.8. Let $\langle \mathbf{c}_\beta \mid \beta < \omega^2\rangle \in L[b]$ be such that each $\mathbf{c}_\beta$ is a $\mathbf{\Pi}^1_1$ code for A_β. Let R be the witness given by $\mathbf{c}$ that $B \in (1 * \mathbf{\Sigma}^0_1)$.

For any $p \in T$ and for any $n \in \omega$, if

$$(\forall x \in \lceil T_p \rceil)\, \langle n, x\rangle \in R,$$

then (ii) implies that $A \cap \lceil T_p \rceil$ is the same as the ωn–$\mathbf{\Pi}^1_1$ set given by $\langle A_\beta \mid \beta < \omega n\rangle$. Therefore the proof of Lemma 5.3.8, applied in $L[b^{\#}]$, shows that there is a winning strategy $G(A; T_p)$ that belongs to $L[b^{\#}]$.

Now let $k \geq 1$ and assume by induction that if $0^{k\#}$ exists then every $(k * \Sigma^0_1)^*$ game has a winning strategy belonging to $L[0^{k\#}]$. Assume that $0^{(k+1)\#}$ exists and let $A \in (k+1 * \Sigma^0_1)^*$. Let $\langle A_\beta \mid \beta < \omega^2\rangle$, g, and B witness that $A \in (k+1 * \Sigma^0_1)^*$. Let $R_0 \ldots, R_k$ witness that $B \in (k+1 * \Sigma^0_1)$. Using R_k and $R_0, \ldots, R_{k-1}$ respectively, define $\bar{A} \in (1 * \Sigma^0_1)^*$ and $\tilde{A} \in (k * \Sigma^0_1)^*$ in the obvious way.

Let E be the set of all $x \in {}^{\omega}\omega$ such that one of the following holds:

(a) $x \in \bar{A}$ and $(\forall m < k)(\forall n \in \omega)\, \langle n, x\rangle \notin R_m$;

(b) (a) fails, and there is a position $p \subseteq x$ such that $(\exists m < k)(\exists n \in \omega)(\forall x \in \lceil T \rceil)\, \langle n, x\rangle \in R_m$, and, for the shortest such p, player I has a winning strategy for $G(\tilde{A}; {}^{<\omega}\omega)$ that belongs to $L[0^{k\#}]$.

The set E has a $(1 * \mathbf{\Sigma}^0_1)^{**}$ code belonging to $L[0^{k\#}]$. Therefore there is a winning strategy for $G(E; {}^{<\omega}\omega)$ that belongs to $L[0^{(k+1)\#}]$. Show that such a strategy yields a winning strategy for $G(A; {}^{<\omega}\omega)$ that belongs to $L[0^{(k+1)\#}]$.

For the "if" direction of the Exercise, also proceed by induction. Let $k \in \omega$, and assume that $0^{k\#}$ exists.

Let $\varphi(v_0, \ldots, v_j)$ be a formula of the language of set theory. Consider a game G in ${}^{<\omega}\omega$ defined as follows. Construe I's part of a play x as giving

$\langle x^z_n \mid n \in \omega\rangle$ with each $x^z_n \in {}^\omega\omega$, and let x^z_0 code a relation r^z in ω. Similarly let II's part of the play give $\langle y^z_n \mid n \in \omega\rangle$ and let y^z_0 code a relation s^z in ω. For n and $i \in \omega$, if the restriction of r^z to $\{a \in \omega \mid a\, r^z\, x^z_{n+1}\}$ is a wellordering, then let $\eta^z_{\omega n+i}$ be the order type of this wellordering. Analogously define ordinals $\zeta^z_{\omega n+i}$. For $n \in \omega$, if η^z_β and ζ^z_β are defined for all $\beta < \omega(n+1)$, then let $\rho^z_n = \sup_{i\in\omega}\max\{\eta^z_{\omega n+i}, \zeta^z_{\omega n+i}\}$. The winning conditions for G are as follows.

(1) Let $\mathrm{ZFC}_{\hat{n}}$ be the conjunction of the first $\hat{n}$ axioms of ZFC, for some large enough natural number N, which we leave unspecified. Unless $(\omega; r^z)$ is a isomorphic to an ω-model $\mathcal{M}_I$ of $\mathrm{ZFC}_{\hat{n}} + V = L[0^{k\#}]$ and unless $(\omega; r^z) \models$ "$x^z_{n+1}(i)$ is an ordinal" for all n and $i \in \omega$, I loses. (See page 44 for the definition of *ω-model*.)

(2) Unless either I loses by (1) or else $(\omega; s^z)$ is a isomorphic to an ω-model $\mathcal{M}_{II}$ of $\mathrm{ZFC}_{\hat{n}} + V = L[0^{k\#}]$ and $(\omega; s^z) \models$ "$y^z_{n+1}(i)$ is an ordinal" for all n and $i \in \omega$, II loses.

(3) Suppose that no one loses because of (1) or (2). Suppose also that there is a k' such that $1 \le k' \le k$ and

$$(0^{k'\#})^{\mathcal{M}_I} \neq (0^{k'\#})^{\mathcal{M}_{II}}.$$

Let k^z be the least such k', and let the formula $\psi^z(v_0, \ldots, v_{m^z})$ be such that $n_{\psi^z(v_0,\ldots,v_{m^z})}$ is the smallest number d such that

$$d \in (0^{k^z\#})^{\mathcal{M}_I} \;\leftrightarrow\; d \notin (0^{k^z\#})^{\mathcal{M}_{II}}.$$

(Here and in the remaining exercises of for this secition, assume that $n_\varphi \ge m$ whenever v_m is free in φ.) If there is a $\beta < \omega(m^z+1)$ such that at least one of η^z_β and ζ^z_β is undefined, then I wins if and only if, for the least such β, η^z_β is defined. If there is no such β, then let

$$a = (0^{(k^z-1)\#})^{\mathcal{M}_I} = (0^{(k^z-1)\#})^{\mathcal{M}_{II}}.$$

Then I wins if and only if

$$L_{\rho^z_{m^z}}[a] \models \psi^z[a, \rho^z_0, \ldots, \rho^z_{m^z-1}].$$

(4) Suppose that no one loses because of (1)–(3). If there is a $\beta < \omega(n+1)$ such that at least one of η^z_β and ζ^z_β is undefined, then I wins if and only if, for the least such β, η^z_β is defined.

(5) Suppose that no one loses because of (1)–(4). Let $b = (0^{k\#})^{\mathcal{M}_I} = (0^{k\#})^{\mathcal{M}_{II}}$. Then I wins if and only if

$$L_{\rho_j^z}[b] \models \varphi[b, \rho_0^z, \ldots, \rho_{j-1}^z].$$

Show that G is $(k+1 * \Sigma_1^0)^*$. Indeed, show that $G \in (k * \Sigma_1^0)_+^*$.

Assume that σ is a winning strategy for I for G. Use the boundedness principle to show that there is a closed unbounded subset C of ω_1 such that every member of C is a Silver indiscernible for each $L[0^{k'\#}]$, $k' < k$, and such that the following condition holds: Let $\langle \xi_i \mid i \in \omega \rangle$ be any strictly increasing sequence of elements of C. Then there is a play z consistent with σ such that $\mathcal{M}_{II} \models \mathrm{ZFC}_{\hat{n}}$, such that $\mathcal{M}_{II} \cong L_\gamma$ for some $\gamma > \sup_{i \in \omega} \xi_i$, such that $(0^{k'\#})^{\mathcal{M}_I} = 0^{k'\#}$ for each $k' \leq k$, and such that, if m^z is defined and $m < m^z + 1$ or if m^z is undefined and $m < j+1$, then $\rho_m^z = \xi_m$. Deduce that, for $\langle \xi_m \mid m < n \rangle$ any strictly increasing n-tuple of elements of C,

$$L[0^{k\#}] \models \varphi[0^{k\#}, \xi_1, \ldots, \xi_n].$$

Prove an analogous result under the assumption that II has a winning strategy for G. Deduce that $0^{(k+1)\#}$ exists.

Exercise 5.3.11. Let $1 \leq k \in \omega$. Prove that the existence of $0^{k\#}$ is equivalent with the determinacy of all $(k-1 * \Sigma_1^0)_+^*$ games.

Remark. This result is also from DuBose [1990]. It would remain true if we redefined $(\Gamma)_+^*$ by requiring that $D \in \Pi_1^1$ rather than merely that $D \in \omega m\text{–}\Pi_1^1$ for some m. Then one would have to use a game like that of Exercise 4.4.1 in place of part of the game in the hint to the "if" half of Exercise 5.3.11 (the part involving the formuala φ).

Exercise 5.3.12. Generalize the both the definitions and the theorem of Exercise 5.3.11 from $k \in \omega$ to, say, the case $\alpha < \omega_1^{\mathrm{CK}}$.

Exercise 5.3.13. Let M be a transitive class model of ZFC and let $\mathbf{T} \in M$ be a game tree with taboos. A *strong M-covering* of $\mathbf{T}$ is a $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ such that

(1) in M, $\mathcal{C}$ has properties (a), (b), and (c) of a covering of $\mathbf{T}$, as defined on page 66;

(2) for every $\tilde{\sigma} \in \mathcal{S}(\tilde{T}) \cap M$ and for every $x \in \lceil T \rceil$ such that x is consistent with $\tilde{\sigma}$, $\Psi(\tilde{\sigma}, x)$ is defined and satisfies (c)(i), (c)(ii), and (c)(iii) on page 66.

Let $A \subseteq \lceil \mathbf{T} \rceil$ be such that A has a $\mathbf{\Delta}^1_1$ code that belongs to M. Prove that there is a strong M-covering of of $\mathbf{T}$ that unravels A (in the obvious sense).

Hint. Verify that when the constructions of Chapter 2 are applied in M they yield strong M-coverings.

Remark. This observation is literally due to the author, though DuBose had earlier in effect made use of special cases.

Exercise 5.3.14. Let $\#_1$ be the function $x \mapsto x^{\#}$ with domain a subset of $\mathcal{P}(\omega)$

(a) Prove that if

$$L[\#_1] \models (\forall x \subseteq \omega)\, x^{\#}\text{exists},$$

then all $(\Pi^0_1)^*$ games are determined.

(b) Prove that the existence of a proper class of indiscernibles for $L[\#_1]$ is equivalent with the determinacy of all $(\Pi^0_1)^*_+$ games.

Remark. These results are from DuBose [1992]. The boldface version of (a) gives a new equivalent of the hypothesis that every subset of ω has a sharp.

Hint. For (a) suppose that $\langle A_\beta \mid \beta < \omega^2 \rangle$, g, and B witness that A belongs to $(\Pi^0_1)^*$. Let $\langle \mathbf{c}_\beta \mid \beta < \omega^2 \rangle \in L$ be such that each $\mathbf{c}_\beta$ is a $\mathbf{\Pi}^1_1$ code for A_β. For $\beta < \omega^2$, let $\hat{A}_\beta$ be the set witnessed to be β–$\mathbf{\Pi}^1_1$ by $\langle A_\gamma \mid \gamma < \beta \rangle$.

Let $\mathbf{T} = ({}^{<\omega}\omega, \emptyset, \emptyset)$, a game tree with taboos.

Define $\tilde{\mathbf{T}}$, another game tree with taboos, as follows. A play $\tilde{x}$ in $\tilde{T}$ begins with a (possible empty) sequence of pairs of moves, with pair number i being

$$\begin{array}{lll} I & \langle m_{2i}, X_i \rangle & \\ II & & \langle \langle 2, r \rangle, m_{2i+1} \rangle \end{array}$$

The play may simply consist of infinitely many such pairs, or there may be a number $k = k_{\tilde{x}}$ such that the play continues by

$$\begin{array}{lllllll} I & \langle m_{2k}, X_k \rangle & & m_{2k+2} & & \ldots & \\ II & & \langle 1, m_{2k+1} \rangle & & m_{2k+3} & & \ldots \end{array}$$

When $k_{\tilde{x}}$ exists, the play may be finite or infinite.

For uniformity, let us regard $k_{\tilde{x}}$ as always existing, taking it to be ω when it is not a finite number. A play $\tilde{x}$ must satisfy the following conditions.

(1) Each m_j belongs to ω.

(2) Each X_n belongs to $L[\#_1]$.

(3) X_0 is a subtree of $({}^{<\omega}\omega)_{\langle m_0\rangle}$.

(4) For $n > 0$, X_n is a subtree of $(X_{n-1})_{\langle m_j | j \leq 2n\rangle}$.

(5) For $n < k_{\tilde{x}}$,

  (a) $r_n \in X_n$;
  (b) $\ell\mathrm{h}(r_n) \geq 2n + 2,$;
  (c) $r_n \supseteq \langle m_i \mid i \leq 2n + 1\rangle$;
  (d) $r_n \supseteq r_{n'}$ for all $n' < n$;
  (e) $\{x \in {}^{\omega}\omega \mid r_n \subseteq x \wedge \langle n, x\rangle \in B\} = \emptyset$.

(6) For $n < k_{\tilde{x}}$ and $i < \ell\mathrm{h}(r_n)$, $\langle m_j \mid j < i\rangle \subseteq r_n$.

(7) If $k_{\tilde{x}}$ is finite, $2k_{\tilde{x}} + 2 \leq i$, and $\langle m_j \mid j < i\rangle \notin X_{k_{\tilde{x}}}$, then $\ell\mathrm{h}(\tilde{x}) = i$ and $\tilde{x}$ is taboo for I in $\tilde{\mathbf{T}}$.

(8) If $k_{\tilde{x}}$ is finite and $\{x \in {}^{\omega}\omega \mid \langle m_j \mid j < i\rangle \subseteq x \wedge \langle k_{\tilde{x}}, x\rangle \in B\} = \emptyset$, then $\ell\mathrm{h}(\tilde{x}) = i$ and $\tilde{x}$ is taboo for II in $\tilde{\mathbf{T}}$.

(9) Unless $x \restriction i$ is terminal because of (7) or (8), $\tilde{x}$ is infinite.

Note that $\tilde{\mathbf{T}}$ belongs to $L[\#_1]$.

Show that there are π, ϕ, and Ψ such that $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ is a strong $L[\#_1]$-covering of $\mathbf{T}$. (This is a minor variation on the constructions of §2.1.)

Prove that $G(\pi^{-1}(A); \tilde{\mathbf{T}})$ has a winning strategy that belongs to $L[\#_1]$. For this observe that, for a position $\tilde{p} \subseteq \tilde{x}$ witnessing that $k_{\tilde{x}}$ is finite, $G(\pi^{-1}(A); \tilde{\mathbf{T}}_{\tilde{p}})$ is essentially identical with $G(\hat{A}_{\omega k_{\tilde{x}}}; \mathbf{T}'_{\pi(\tilde{p})})$ for a tree $\mathbf{T}' \in L[X_{k_{\tilde{x}}}]$. The existence of $(X_{k_{\tilde{x}}})^{\#}$ guarantees that the latter game is determined (and has a winning strategy belonging to $L[\#_1]$).

The proof that the existence of a proper class of indiscernibles for $L[\#_1]$ implies the determinacy of all $(\Pi^0_1)^*_+$ games (half of part (b) of the exercise) is similar to the proof of (a). Begin by deducing the hypothesis of (a) from the current hypothesis. Next ignore the set D and define $\tilde{\mathbf{T}}$ and $\mathcal{C}$ as before. Given D and given $\langle D_\beta \mid \beta < \omega m\rangle$ and h witnessing that $D \in \omega n$–Π^1_1, let

$\langle \mathbf{d}_\beta \mid \beta < \omega m\rangle \in L$ be such that each $\mathbf{d}_\beta$ is a $\mathbf{\Pi}^1_1$ code for D_β. Finally use the $(D, \bigcap_{\beta<\omega m} D_\beta)$ semicovering of $\tilde{\mathbf{T}}$ with respect to $L[\#_1]$ that is given by the proof of Lemma 5.3.8.

Now consider the other half of (b). Assume that all $(\Pi^0_1)^*$ games are determined.

Assume first that there is an $x \in \mathcal{P}(\omega) \cap L[\#_1]$ such that $x^\#$ does not exist. Let u be the $<_{L[\#_1]}$-least such x. Let $\varphi(v_0, \ldots, v_k)$ a formula such that there is no closed unbounded subset C of ω_1 such that

$$\varphi(u, \alpha_1, \ldots, \alpha_k) \leftrightarrow \varphi(u, \beta_1, \ldots, \beta_k)$$

for $\langle\alpha_1, \ldots, \alpha_k\rangle$ and $\langle\beta_1, \ldots, \beta_k\rangle$ any two increasing sequences of elements of C.

Consider the following game G in ${}^{<\omega}\omega$. Let the objects x^z_n, y^z_n, r^z, s^z, η_β, ζ_β, and ρ^z_n be as in the hint for Exercise 5.3.10. The winning conditions for G are as follows, where $\mathrm{ZFC}_{\hat{n}}$ is as in the hint for Exercise 5.3.10.

(1) Unless $(\omega; r^z)$ is a isomorphic to an ω-model $\mathcal{M}_I$ of $\mathrm{ZFC}_{\hat{n}} + V = L[\#_1]$ and unless $(\omega; r^z) \models$ "$x^z_{n+1}(i)$ is an ordinal" for all n and $i \in \omega$ with $n \geq 1$ and also $\models$ "there is an $x \in \mathcal{P}(\omega) \cap L_{x^z_1(0)}[\#_1]$ such that $x^\#$ does not exist," I loses.

(2) Unless either I loses by (1) or else $(\omega; s^z)$ is a isomorphic to an ω-model $\mathcal{M}_{II}$ of ZFC + $V = L[\#_1]$ and $(\omega; s^z) \models$ "$y^z_{n+1}(i)$ is an ordinal" for all n and $i \in \omega$ with $n \geq 1$ and also $\models$ and "there is an $x \in \mathcal{P}(\omega) \cap L_{y^z_1(0)}[\#_1]$ such that $x^\#$ does not exist," II loses.

(3) Suppose that no one loses because of (1) or (2). Suppose also that there is a $w \in \mathcal{P}(\omega) \cap \mathcal{M}_I \cap \mathcal{M}_{II}$ such that both $\mathcal{M}_I$ and $\mathcal{M}_{II}$ satisfy "$w^\#$ exists" and such that

$$(w^\#)^{\mathcal{M}_I} \neq (w^\#)^{\mathcal{M}_{II}}.$$

For any such w, let i^z_w be the element of ω sent to w by any isomorphism from $(\omega; r^z)$ to $\mathcal{M}_I$, and let j^z_w be the defined analogously from s^z and $\mathcal{M}_{II}$. Let the formula $\psi^z_w(v_0, \ldots, v_{m^z_w})$ be such that $n_{\psi^z_w(v_0,\ldots,v_m)}$ is the smallest number d such that

$$d \in (w^\#)^{\mathcal{M}_I} \leftrightarrow d \notin (w^\#)^{\mathcal{M}_{II}}.$$

Let w^z and ψ^z be such that $\langle i^z_w, j^z_w, n_{\psi^z_w(v_0,\ldots,v_{m^z_w})}\rangle$ has the least possible value, where the ordering is first by maximum and then by the lexicographic ordering.

If there is a $\beta < \omega(m^z+1)$ such that at least one of η^z_β and ζ^z_β is undefined, then I wins if and only if, for the least such β, η^z_β is defined. If there is no such β, then I wins if and only if

$$L_{\rho^z_{m^z}}[w^z] \models \psi^z[w^z, \rho^z_0, \ldots, \rho^z_{m^z-1}].$$

(4) Suppose that no one loses because of (1)–(3). If there is a $\beta < \omega(2k+1)$ such that at least one of η^z_β and ζ^z_β is undefined, then I wins if and only if, for the least such β, η^z_β is defined.

(5) Suppose that no one loses because of (1)–(4). Then ρ^z_{2k} belongs to the wellfounded parts of both $\mathcal{M}_I$ and $\mathcal{M}_{II}$. Let $\alpha^z < \eta^z_0$ be the order type of the initial segment of $u^z_I = u^{\mathcal{M}_I}$ with respect to the $(<_{L[\#_1]})^{\mathcal{M}_I}$ and let $\beta^z < \zeta^z_0$ be the corresponding ordinal for $u^z_{II} = u^{\mathcal{M}_{II}}$ and $\mathcal{M}_{II}$.

(a) Assume first that $\alpha^z < \beta^z$. It follows that $u^z_I \in \mathcal{M}_{II}$. Then II wins if and only if

$$L_{\rho^z_{2k}}[u_I] \models \varphi[u^z_I, \rho^z_0, \ldots, \rho^z_{k-1}] \leftrightarrow \varphi[u^z_I, \rho^z_k, \ldots, \rho^z_{2k-1}].$$

(b) Assume next that $\beta^z < \alpha^z$. Then I wins if and only if

$$L_{\rho^z_{2k}}[u_{II}] \models \varphi[u^z_{II}, \rho^z_0, \ldots, \rho^z_{k-1}] \leftrightarrow \varphi[u^z_{II}, \rho^z_k, \ldots, \rho^z_{2k-1}].$$

(c) Assume finally that $\alpha^z = \beta^z$. It follows that $u^z_I = u^z_{II}$. Then I wins if and only if

$$L_{\rho^z_k}[u^z_I] \models \varphi[u^z_I, \rho^z_0, \ldots, \rho^z_{k-1}].$$

Prove that G is $(\Pi^0_1)^*_+$.

Assume that σ is a winning strategy for I for G. Use the boundedness principle to show that there is a closed unbounded subset C of ω_1 such that the following condition holds: Let $\langle \xi_i \mid i \in \omega \rangle$ be any strictly increasing sequence of elements of C and let $\gamma > \sup_{i\in\omega}\xi_i$. There is a play z consistent with σ such that $\mathcal{M}_I \cong L_\gamma[\#_1]$ and such that $\rho^z_m = \xi_m$ for all m such that m^z exists and $m \le m^z$ or m^z does not exist and $m \le 2k$. Argue that, for any such play z, $u^z_I = u^z_{II} = u$ and

$$L[u] \models \varphi[u, \xi_1, \ldots, \xi_k].$$

This contradicts the definition of u.

Get a similar contradiction from the assumption that II has a winning strategy for G.

Now prove that a proper class of indiscernibles exist for $L[\#_1]$ by considering a game like G, but where clauses (4) and (5) are replaced by clauses analogous to the (4) and (5) of the hint to Exercise 5.3.10.

Remarks:

(a) The use of the machinery of Chapter 2 is not really needed for the exercise, and it was not used in DuBose's original proof (though it was used in the proof in DuBose [1992]). This machinery *is* needed, however, for the following exercise.

(b) If $(\Gamma)^*_+$ were redefined as suggested in the remarks to Exercise 5.3.11, then the result of the present exercise would still hold.

Exercise 5.3.15. For $1 \leq k \in \omega$, let $\#_k$ be the function $x \mapsto x^\#$ with domain a subset of $\mathcal{P}^k(\omega)$. (See page 94 for the definition of $\mathcal{P}^\alpha(X)$.)

(a) Prove that if $k \in \omega$ and

$$L[\#_k] \models (\forall x \in \mathcal{P}^k(\omega)\, x^\# \text{ exists},$$

then all $(\Pi^0_k)^*$ games are determined.

(b) Prove that, for $k \in \omega$, the existence of a proper class of indiscernibles for $L[\#_k]$ and is equivalent with the determinacy of all $(\Pi^0_k)^*_+$ games.

(c) Generalize the results of (a) and (b), which are proved in DuBose [199?], from the case $k \in \omega$ to the case $k < \omega_1^{\mathrm{CK}}$.

Hint. For (a) and one part of (b), use the constructions of Chapter 2 (of §2.3, in particular) followed by the construction of the hint to Exercise 5.3.14 For the other part of (b), imitate the proofs for the corresponding part of Exercise 5.3.14.

Exercise 5.3.16. (a) Let $A \in (\Pi^1_1)^*$. Prove that there are $\langle A_\beta \mid \beta < \omega^2\rangle$, g, and B witnessing that $A \in (\Pi^1_1)^*$ such that for every $x \in {}^\omega\omega$ there is at most one n such that $\langle n, x\rangle \in B$.

(b) Prove the result analogous to that of (a) with "$(\Pi^1_1)^*_+$" replacing "$(\Pi^1_1)^*$."

Remark. This observation is due to DuBose.

Hint. Let $\langle A'_\beta \mid \beta < \omega^2 \rangle$, g, and B' witnessing that $A \in (\Pi^1_1)^*$. Let B belong to Π^1_1 and uniformize B'. Let

$$x \in A_{\omega k+i} \leftrightarrow ((\exists n > k)\, \langle n, x \rangle \in B \wedge (x \in A'_{\omega k+i} \vee (\exists j \leq k)\, \langle j, x \rangle \in B')) .$$

Exercise 5.3.17. Let $\#$ be the class function $x \mapsto x^\#$ with domain a subclass of the class of all sets of ordinals.

(i) Prove that the existence of a proper class of indiscernibles for $L[\#]$ implies the determinacy of all $(\Pi^1_1)^*_+$ games.

(ii) Prove that the determinacy of all $(\Pi^1_1)^*$ games implies that

$$L[\#] \models (\forall x \subseteq \mathrm{Ord})\, x^\# \text{exists.}$$

(iii) Prove the converse of (i).

Remarks:

(a) The class $L[\#]$ would be unaffected if $\#$ were the general sharp function, defined on every set that has a $\#$. Similarly, the conclusion of (ii) would be unaffected if "$\subseteq$ Ord" were deleted.

(b) Part (i) is due to DuBose and the author jointly. The proofs outlined in the hints to Exercise 5.3.14 and 5.3.15 incorporate elements of the more recent proof of (i).

(c) Part (ii) is due to the author. Note that it implies the failure of the analogue of Exercise 5.3.14, part (a) and Exercise 5.3.15, part (a).

(d) At the time when DuBose proved the theorems of Exercises 5.3.14 and 5.3.15, Philip Welch and the author independently noticed that DuBose's methods would yield part (iii) of the present exercise. This led to the conjecture that part (i) should hold.

(e) It seems likely that the determinacy of all $(\Pi^1_1)^*$ games implies the existence of a proper class of indiscernibles for $L[\#]$ (and so implies the determinacy of all $(\Pi^1_1)^*_+$ games). Using techniques of Steel [1982] and Ronald Jensen's theorem on the Σ^1_3 correctness of the core model, the author has proved that this implication holds under the assumption that every subset of ω has a $\#$.

Hint. For (i), assume that there is a proper class of indiscernibles for $L[\#]$. For classes $X \subseteq L[\#]$, define $\mathcal{H}(L[\#], X)$ just as $\mathcal{H}(L, X)$ was defined on page 167, but using $L[\#]$ and $<_{L[\#]}$ in place of L and $<_L$. Show there is a closed unbounded proper class C of indiscernibles for $L[\#]$ such that C generates L and such that, for every uncountable cardinal η, $\mathcal{H}(L[\#], C\cap\eta) = L_\eta[\#]$. To do this, imitate the proof of Theorem 3.4.8.

First consider the case of $(\Pi^1_1)^*$. Let $\langle A_\beta \mid \beta < \omega^2\rangle$, g, and B be as given by Exercise 5.3.16. Let $\langle c_\beta \mid \beta < \omega^2\rangle$, $\langle \hat{A}_\beta \mid \beta < \omega^2\rangle$, and $\mathbf{T}$ be as in the hint to Exercise 5.3.14.

Let

$$B^* = \{x \in {}^\omega\omega \mid (\exists n) \mid \langle n, x\rangle \in B\}.$$

Let $\mathbf{c} \in L$ be a Π^1_1 code for B^*. Imitating the proof of part (i) of Lemma 5.3.1, show that there is a (B^*, B^*) semicovering $\mathcal{C}' = \langle \mathbf{T}', \pi', \psi', \Psi'\rangle$ of $\mathbf{T}$ with respect to $L[\#]$ such that $\mathbf{T}'$ and π' belong to $L[\#]$ and such that $|T'| = \aleph_1$.

Since every normal play in $\mathbf{T}'$ belongs to $\boldsymbol{\pi}'^{-1}(B^*)$, it follows that $\boldsymbol{\pi}'^{-1}(B)$ has a $\mathbf{\Delta}^1_1$ code belonging to $L[\#]$. By a sequence of applications of Exercise 5.3.13, let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be a strong $L[\#]$-covering of $\mathbf{T}'$ that unravels $\boldsymbol{\pi}'^{-1}(\{x \mid \langle n, x\rangle \in B\})$ for each $n \in \omega$. Because, $\tilde{\mathbf{T}}^\#$ belongs to $L[\#]$, the game $G(\boldsymbol{\pi}^{-1}(\boldsymbol{\pi}'^{-1}(A)); \tilde{\mathbf{T}})$ has a winning strategy $\tilde{\sigma}$ that belongs to $L[\#]$. (One way to prove this is to generalize to uncountable trees the $k = 1$ case of Exercise 5.3.10.) Since $\mathcal{C}$ is a strong $L[\#]$-covering, it follows that $\phi(\tilde{\sigma})$ is defined and is a winning strategy for $G(\boldsymbol{\pi}'^{-1}(A); \mathbf{T}')$. Since $\phi(\tilde{\sigma})$ belongs to $L[\#]$, this implies that $\phi'(\phi(\tilde{\sigma}))$ is a winning strategy for $G(A; \mathbf{T})$.

Now let $A \in (\Pi^1_1)^*_+$ and let $\langle A_\beta \mid \beta < \omega^2\rangle$, g, B, and D be as given by Exercise 5.3.16. Let $\langle c_\beta \mid \beta < \omega^2\rangle$, $\langle \hat{A}_\beta \mid \beta < \omega^2\rangle$, $\mathbf{T}$ and B^* be as above. Let D, $\langle D_\beta \mid \beta < \omega m\rangle$, h, and $\langle \mathbf{d}_\beta \mid \beta < \omega m\rangle \in L$ be as in the hint to Exercise 5.3.14. For $\beta < \omega m$, let

$$D^*_\beta = D_\beta \cup B^*.$$

Let $D^*_{\omega n} = B^*$. Let D^* be the set witnessed $(\omega m + 1)$–$\mathbf{\Pi}^1_1$ by $\langle D^*_\beta \mid \beta \leq \omega m\rangle$. Imitate the proof of Lemma 5.3.8 to show that there is a $(D^*, \bigcap_{\beta\leq\omega m} D^*_\beta)$ semicovering $\mathcal{C}' = \langle \mathbf{T}', \pi'\psi', \Psi'\rangle$ of $\mathbf{T}$ with respect to $L[\#]$ such that $\mathbf{T}'$ and π' belong to $L[\#]$ and such that $|T'| = \aleph_{m+1}$. Now let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be a strong $L[\#]$-covering of $\mathbf{T}'$ that unravels $\boldsymbol{\pi}'^{-1}(\{x \mid \langle n, x\rangle \in B\})$ for each $n \in \omega$. Argue more or less as above.

For (ii), assume that there is a set a of ordinals such that $a \in L[\#]$ and such that $a^\#$ does not exist. Let u be the $<_{L[\#]}$-least such a. Define $\varphi(v_0, \ldots, v_k)$ as in the hint to Exercise 5.3.14, with "$L[\#]$" replacing "$L[\#_1]$."

Define a game G in ${}^{<\omega}\omega$ as follows. Let the objects x^z_n, y^z_n, r^z, s^z, η_β, ξ_β, and ρ^z_n be as in the hint for the last half of part (b) of Exercise 5.3.14.

Winning conditions (1) and (2) are like (1) and (2) for the earlier game, except that "$\#_1$" is replaced by "$\#$" and "$\mathcal{P}(\omega)$" is replaced by "$\mathcal{P}(x_1^z(0)$."

(3) Recall from the definition of "ω-model" on page 44 that the wellfounded parts of $\mathcal{M}_I$ and $\mathcal{M}_{II}$ are identical with the transitive sets to which they are isomorphic. Suppose that no one loses because of (1) or (2). If η_0^z is undefined, then I loses. If η_0^z is defined and and does not belong to the wellfounded part of $\mathcal{M}_{II}$, then II loses. Assume that neither player loses for these reasons. Then there is a set $w \subseteq \eta_0^z$ such that $w \in \mathcal{M}_I \cap \mathcal{M}_{II}$ and such that at least one of the following holds.

(a) $w = u^{\mathcal{M}_I}$.

(b) $w = u^{\mathcal{M}_{II}}$.

(c) Both $\mathcal{M}_I$ and $\mathcal{M}_{II}$ satisfy "$w^\#$ exists," and $(w^\#)^{\mathcal{M}_I} \neq (w^\#)^{\mathcal{M}_{II}}$.

Let w^z be the $(<_{L[\#]})^{\mathcal{M}_I}$-least such w. If (c) holds, let d^z be the least number d such that

$$d \in ((w^z)^\#)^{\mathcal{M}_I} \leftrightarrow d \notin ((w^z)^\#)^{\mathcal{M}_{II}}.$$

Determine which player wins as in the last part of (3) in the hint to Exercise 5.3.14. If (a) or (b) holds, determine who wins as in (4) and (5) of the hint to Exercise 5.3.14.

Prove that $G \in (\Pi^1_1)^*$ and then proceed as in the hint to Exercise 5.3.14. The proof of (iii) is similar to that of the analogous part of Exercise 5.3.14.

5.4 $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ Games

In this section we go beyond the difference hierarchy on $\mathbf{\Pi}^1_1$ and present work of John Simms (Simms [1979]) concerning the first level of the σ-algebra generated by Boolean combinations of $\mathbf{\Pi}^1_1$ sets. In any topological space, let $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ be the collection of all countable unions of Boolean combinations of sets belonging to $\mathbf{\Pi}^1_1$. We prove Simms' result that the determinacy of all $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ games in a game tree T follows from the existence of a $\kappa > |T|$ such that κ is a measurable limit of measurable cardinals. Simms [1979], plus later work of John Steel, William Mitchell, and the author, further shows, for any infinite cardinal λ, that the determinacy of all $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ games in trees of size λ is equivalent to the existence, for each subset a of λ, of a proper class of indiscernibles for M, a, where M is a transitive proper class model ZFC + "There is a proper class of measurable cardinals" with $a \in M$. We prove one half of this equivalence and also a strong version, due to Simms, of this half. We outline in hints to the exercises the proofs of the other half of the equivalence and of related results. We also present lightface equivalences. In the exercises we indicate how to generalize the equivalence theorems to the difference hierarchy on the dual class $\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1)$, using the "measurable limit of" hierarchy.

The following lemma shows that "differences of $\mathbf{\Pi}^1_1$ sets" can replace "Boolean combinations of $\mathbf{\Pi}^1_1$" sets in the definition of $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$.

Lemma 5.4.1. *Let T be a game tree and let $A \subseteq \lceil T \rceil$. Then $A \in \mathbf{\Sigma}^1_0(\mathbf{\Pi}^1_1)$ if and only if A is a countable union of differences of $\mathbf{\Pi}^1_1$ sets.*

Proof. The "if" part of the lemma is trivial. For the "only if" part, let $A = \bigcup_{n\in\omega} B_n$ with each B_n a Boolean combination of $\mathbf{\Pi}^1_1$ sets. Using the basic properties of Boolean algebras, we can show that, for each $n \in \omega$,

$$B_n = \bigcup_{i \leq m_n} B_{n,i},$$

where each $m_n \in \omega$ and each $B_{n,i}$ is a finite intersection of $\mathbf{\Pi}^1_1$ sets and complements of $\mathbf{\Pi}^1_1$ sets. Since $\mathbf{\Pi}^1_1$ is closed under finite intersections and unions, each $B_{n,i}$ is a difference of $\mathbf{\Pi}^1_1$ sets. But then

$$A = \bigcup_{\substack{n \in \omega \\ i \leq m_n}} B_{n,i},$$

so A is a countable union of differences of $\mathbf{\Pi}^1_1$ sets. □

We can also define $\Sigma^0_1(\Pi^1_1)$, the lightface version of $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$. Say that $A \subseteq {}^\omega\omega$ belongs to $\Sigma^0_1(\Pi^1_1)$ if there are $C \subseteq \omega \times {}^\omega\omega$ and $D \subseteq \omega \times {}^\omega\omega$ such that C and D belong to Π^1_1 and such that

$$A = \{x \in {}^\omega\omega \mid (\exists n \in \omega)\,(n, x) \in C \setminus D\}.$$

We next present a characterization of $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ that will be a key ingredient in the determinacy proofs for games in this class. In order to do this, we will first prove a technical result.

For any class linear ordering $\mathcal{A}$, define wfo($\mathcal{A}$) as follows. If every ordinal is order isomorphic to an initial segment of $\mathcal{A}$, let wfo($\mathcal{A}$) = Ord. Otherwise let wfo($\mathcal{A}$) be the largest ordinal that is order isomorphic to an initial segment of $\mathcal{A}$.

Let us extend our notion of a $\mathbf{\Pi}^1_1$ code (defined on page 228) to the simpler case of game trees without taboos or terminal positions. For such a tree T, say that $\langle T, p \mapsto <_p\rangle$ is a $\mathbf{\Pi}^1_1$ *code* if $\langle \mathbf{T}, \emptyset, p \mapsto <_p\rangle$ is a $\mathbf{\Pi}^1_1$ code, where $\mathbf{T} = \langle T, \emptyset, \emptyset\rangle$. Let us also define a lightface notion. Say that $\langle {}^{<\omega}\omega, p \mapsto <_p\rangle$ is a Π^1_1 *code* if it is a $\mathbf{\Pi}^1_1$ code and $p \mapsto <_p$ is recursive.

Lemma 5.4.2. *Let T be a game tree without terminal positions. For each $n \in \omega$ let $\mathbf{c}_n = \langle T, p \mapsto <^n_p\rangle$ be a $\mathbf{\Pi}^1_1$ code. Then there is a $\mathbf{\Pi}^1_1$ code $\mathbf{c} = \langle T, p \mapsto <_p\rangle$ such that, for all $x \in [T]$,*

(1) the induced ordering $<_x$ of ω is not a wellordering;

(2) every non-zero $i \in \omega$ has an immediate successor with respect to $<_x$;

(3) for every $n \in \omega$, if $<^n_x$ is a wellordering then there is an $i \in \omega$ such that

$$(\omega; <^n_x) \cong (\{j \in \omega \mid j <_x i\}; <_x);$$

i.e., wfo$(\omega; <_x)$ *is larger than the order type of any of the $(\omega; <^n_x)$ that are wellorderings.*

Moreover, if $T = {}^{<\omega}\omega$ and if $\langle n, p\rangle \mapsto <^n_p$ is recursive, then $\mathbf{c}$ is a Π^1_1 code.

Remark. The essential clause of the lemma is (3). If we deleted clauses (1) and (2)—and so deleted the corresponding clauses from Lemmas 5.4.3 and 5.4.4, only slight changes would be needed in the proofs of our determinacy theorems.

Proof. Our proof will proceed in three steps:

(i) We define a continuous function $x \mapsto S^x$ with domain $[T]$. Each S^x will be a game subtree of ${}^{<\omega}\omega$.

(ii) We show that, for each $x \in [T]$, the restriction of the Brouwer–Kleene ordering $<^{\mathrm{BK}}$ to S^x has wfo larger than the order types of the $(\omega; <^n_x)$ that are wellorderings. (The ordering $<^{\mathrm{BK}}$ is defined on page 183.)

(iii) We use (ii) to construct $p \mapsto <_p$.

For step (i), let $\langle n, i\rangle \mapsto k^n_i$ be a bijection from $\omega \times \omega$ onto ω with the property that $k^n_i < k^n_j$ whenever $i < j$. For $q \in {}^{<\omega}\omega$, let $q \in S^x$ provided that, for all n such that $k^n_0 < \ell\mathrm{h}(q)$ and $q(k^n_0) = 0$,

$$(\forall i \geq 1)(k^n_{i+1} < \ell\mathrm{h}(q) \rightarrow q(k^n_{i+1}) <^n_x q(k^n_i).$$

The idea is that, for each n, the value $q(k^n_0)$ is a guess at whether or not $<^n_x$ is a wellordering. The value 0 is a guess that it is not a wellordering. If $q(k^n_0) = 0$, then the values of q on the arguments $k^n_1, k^n_2, \ldots$ must must verify the correctness of this guess by forming a strictly decreasing sequence with respect to $<^n_x$. If $q(k^n_0) > 0$, then the guess is that $<^n_x$ is a wellordering, and no conditions are placed on the values of q on arguments of the form k^n_i.

For step (ii), fix $x \in [T]$.

We show that $[S^x]$ is nonempty by defining $y \in [S^x]$ as follows. For each $n \in \omega$, set

$$y(k^n_0) = \begin{cases} 0 & \text{if } <^n_x \text{ is not a wellordering;} \\ 1 & \text{otherwise.} \end{cases}$$

If $n \in \omega$ and $y(k^n_0) = 0$, then let $\langle y(k^n_i) \mid 1 \leq i \in \omega\rangle$ be a strictly decreasing sequence with respect to $<^n_x$. If $n \in \omega$ and $y(k^n_0) > 0$, then let $y(k^n_i) = 0$ for all $i \geq 1$. Evidently y belongs to $[S^x]$.

Now let z be the *leftmost branch* of S^x, defined inductively by:

$$z(i) = \min\{m \in \omega \mid [S^x_{z\restriction i^\frown\langle m\rangle}] \neq \emptyset\}.$$

Thus $z \in [S^x]$ and, for all $y \in [S^x]$ with $y \neq z$, $z(i) < y(i)$ for the least i such that $z(i) \neq y(i)$. By the definition of $<^{\mathrm{BK}}$, this means that the subtree S^* of S^x defined by

$$S^* = \{q \in S^x \mid (\forall i \in \omega)\, q <^{\mathrm{BK}} z \restriction i\}$$

is wellfounded. By Lemma 4.1.3, it follows that the restriction of $<^{\mathrm{BK}}$ to S^* is wellfounded.

Let $n \in \omega$ be such that $<^n_x$ is a wellordering. Let $q^* = (z \restriction k^n_0)^\frown\langle 0\rangle$. Clearly $[(S^x)_{q^*}] = \emptyset$. Hence $z(k^n_0) > 0$, and so $q^* \in S^*$. We will prove that the restriction of $<^{\mathrm{BK}}$ to $\{q \in S^x \mid q < q^*\}$, i.e., to $\{q \in S^* \mid q < q^*\}$, has order type at least as great as that of $<^n_x$. Since whenever q and q' are elements of ${}^{<\omega}\omega$ with $q \subsetneq q'$ then $q' <^{\mathrm{BK}} q$, it suffices to prove that $\|q^*\|^{S^x}$ is at least as great as the order type of $<^n_x$.

Let $\bar{S}$ be the subtree of $(S^x)_{q*}$ defined by

$$\bar{S} = \{q \in (S^x)_{q^*} \mid (\forall m)(\forall i)((m \neq n \wedge k_i^m < \ell\mathrm{h}(q)) \to q(k_i^m) = z(k_i^m))\}.$$

It is enough to prove that $\|q^*\|^{\bar{S}}$ is at least as great as the order type of $<_x^n$. There is a natural embedding h of the tree U of all finite sequences that are strictly descending with respect to $<_x^n$ into the tree $\bar{S}_{q^*}$, and embedding that sends $\emptyset$ to q^*. Thus $\|U\| \leq \|\bar{S}_{q^*}\|$, and a routine induction shows that $\|U\|$ is at least as great as the order type of $<_x^n$.

For step (iii), let $i \mapsto s_i$ be a bijection from ω to ${}^{<\omega}\omega$ as on page 185. Define $x \mapsto <_x^*$ by letting $<_x^*$ be the natural ordering of $\omega \setminus \{i \mid s_i \notin S^x\}$ followed by the ordering of $\{i \mid s_i \in S^x\}$ that makes $i \mapsto s_i$ an order isomorphism into of $(S^x; <^{\mathrm{BK}})$. For each $x \in [T]$, $<_x^*$ is a linear ordering of ω that is not a wellordering, and its wellordered initial segment has order type greater than that of any of the $<_x^n$ that are wellorderings.

The function $x \mapsto <_x^*$ is continuous in the sense that $<_x^* \restriction \{i \in \omega \mid i < j\}$ is determined by $x \restriction j'$ for some $j' \in \omega$. But it is not continuous in a strong enough sense to induce the desired $p \mapsto <_p$.

We deal with this problem as follows. For each $x \in [T]$ and each $j \in \omega$, let $f_x(j)$ be the least $j' > f_x(j)$ such that $<_x^* \restriction \{i \in \omega \mid i < j\}$ is determined by $x \restriction j'$. Note that $f_x(0) = 0$ and that $f_x(1) = 1$. For each $x \in [T]$, define $<_x$ by letting

$$f_x(i+1) <_x f_x(j+1) \;\leftrightarrow\; i <_x^* j$$

and by placing $\{i \in \omega \mid i+1 \notin \mathrm{range}\,(f_x)\}$ at the beginning of $<_x$, in the natural order. Since 0 is maximal in $<_x^*$, it follows that 0 is maximal in $<_x$. It is easy to check that $<_x$ has the strong continuity property needed to induce our function $p \mapsto <_p$.

It is evident that clauses (1) and (3) of the lemma hold. We see as follows that clause (2) holds. If $S' \subseteq {}^{\omega}\omega$ is any game tree, then, as is easily verified, every $s \in S'$ except $\emptyset$ has an immediate successor with respect to $<^{\mathrm{BK}} \restriction S'$. For each $x \in {}^{\omega}\omega$, $(\omega; <_x)$ is a wellordering followed by an ordering isomorphic to such an $(S'; <^{\mathrm{BK}} \restriction S')$.

It is easy to check that, under the hypothesis of the last part of the lemma, our construction gives a recursive $p \mapsto <_p$ as required. $\square$

Remark. Lemma 5.4.2 is in effect well-known. The usual proof is based on a relativization of the construction of a subtree S of ${}^{<\omega}\omega$ such that $[S] \neq \emptyset$ but $[S] \cap \Delta_1^1 = \emptyset$. The more elementary proof we have given is due to John Steel.

The following normal form result is due to the author.

Lemma 5.4.3. *Let $A \subseteq {}^\omega\omega$. Then $A \in \Sigma^0_1(\Pi^1_1)$ if and only if there is a $B \subseteq \omega \times {}^\omega\omega$ with $B \in \Pi^1_1$ and there is a recursive function $p \mapsto <_p$ with domain ${}^{<\omega}\omega$ such that*

(1) $<_\emptyset = \emptyset$ and, for all $p \in T \setminus \{\emptyset\}$, $<_p$ is a linear ordering of $\ell\mathrm{h}(p)$ with greatest element 0;

(2) for elements p and p' of T, if $p \subseteq p'$ then $<_p \subseteq <_{p'}$;

(3) for all $x \in {}^\omega\omega$, $<_x$ is not a wellordering of ω, where

$$<_x = \bigcup_{n\in\omega} <_{x\restriction n};$$

(4) for all $x \in {}^\omega\omega$, every non-zero member of ω has an immediate successor with respect to $<_x$;

(5) for all $x \in {}^\omega\omega$, $x \in A$ if and only if

$$(\exists e \in \omega)(<_x \restriction \{e' \mid e' <_x e\} \text{ is a wellordering } \wedge \ (e,x) \notin B).$$

Proof. First assume that $A \in \mathbf{\Sigma}^0_1(\Pi^1_1)$. Let $A = \bigcup_{n\in\omega}(C_n \setminus D_n)$, where $C_n = \{x \mid (n,x) \in C\}$, $D_n = \{x \mid (n,x) \in D\}$, and C and D belong to Π^1_1.

It is easy to show that there is a recursive function $\langle n,p\rangle \mapsto <^n_p$ such that, for all $n \in \omega$, $\langle {}^{<\omega}\omega, p \mapsto <^n_p\rangle$ is a Π^1_1 code for C_n. By Lemma 5.4.2, let $\mathbf{c} = \langle {}^{<\omega}\omega, p \mapsto <_p\rangle$ be a Π^1_1 code satisfying clauses (1), (2), and (3) of the conclusion of that lemma. The fact that $\mathbf{c}$ is a $\mathbf{\Pi}^1_1$ code means that $p \mapsto <_p$ satisfies clauses (1) and (2) of the present lemma. Clause (1) of Lemma 5.4.2 gives clause (3) of the present lemma, and clause (2) of that lemma gives clause (4) of the present lemma.

Define $B \subseteq \omega \times {}^\omega\omega$ by letting (e,x) belong to B just in case, for every $n \in \omega$, if $(\omega; <^n_x)$ can be embedded into $(\{e' \mid e' <_x e\}; <_x)$, then $x \in D_n$. A routine calculation shows that $B \in \Pi^1_1$. To verify clause (5), let $x \in {}^\omega\omega$. Assume first that $x \in A$. Let n be such that $x \in C_n \setminus D_n$. By clause (3) of Lemma 5.4.2, the order type of $(\omega; <^n_x)$ is less than $\mathrm{wfo}(\omega; <_x)$. Therefore let e be such that $<_x \restriction \{e' \mid e' <_x e\}$ is a wellordering and $(\omega; <^n_x)$ can be embedded into $(\{e' \mid e' <_x e\}; <_x)$. Since $x \notin D_n$, it follows that $(e,x) \notin B$. Now assume that $x \notin A$ and let $e \in \omega$. Assume that $<_x \restriction \{e' \mid e' <_x e\}$ is a wellordering. For any $n \in \omega$, if $(\omega; <^n_x)$ can be embedded into $(\{e' \mid e' <_x e\}; <_x)$ then $x \in C_n$ and therefore $x \in D_n$. Thus $(e,x) \in B$.

The other half of the lemma is easy to verify. $\square$

Lemma 5.4.4. *If T is a game tree without terminal positions and $A \subseteq [T]$, then $A \in \mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ if and only if there are $\mathbf{\Pi}^1_1$ sets B_e, $e \in \omega$, and there is a function $p \mapsto <_p$ with domain T such that*

(1) $<_\emptyset = \emptyset$ and, for all $p \in T \setminus \{\emptyset\}$, $<_p$ is a linear ordering of $\ell\mathrm{h}(p)$ with greatest element 0;

(2) for elements p and p' of T, if $p \subseteq p'$ then $<_p \subseteq <_{p'}$;

(3) for all $x \in [T]$, $<_x$ is not a wellordering of ω, where

$$<_x = \bigcup_{n\in\omega} <_{x\restriction n};$$

(4) for all $x \in [T]$, every non-zero member of ω has an immediate successor with respect to $<_x$;

(5) for all $x \in [T]$, $x \in A$ if and only if

$$(\exists e \in \omega)(<_x \restriction \{e' \mid e' <_x e\} \text{ is a wellordering } \wedge\ x \notin B_e).$$

The proof of Lemma 5.4.4 is similar to that of Lemma 5.4.3, and we omit it.

Theorem 5.4.5. (Simms [1979]) *Let T be a game tree. If there is a measurable limit of measurable cardinals that is larger than $|T|$, then every $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ game in T is determined.*

Proof. We may assume without loss of generality that T has no terminal positions and so that $\lceil T \rceil = [T]$. (See page 182.)

By Lemma 5.4.4, let $\langle B_e \mid e \in \omega\rangle$ and $p \mapsto <_p$ witness that $A \subseteq [T]$ belongs to $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$. For each $e \in \omega$, let $p \mapsto <^e_p$ and $x \mapsto <^e_x$ be the functions given by Lemma 4.1.4 with B_e as the A of that lemma.

Let $\kappa > |T|$ be a measurable limit of measurable cardinals. Let $\langle \kappa_\alpha \mid \alpha < \kappa\rangle$ be the strictly increasing sequence of all of all measurable cardinals smaller than κ that are not limits of measurable cardinals. Let $\mathcal{U}$ be a uniform normal ultrafilter on κ and, for each $\alpha < \kappa$, let $\mathcal{V}_\alpha$ be a uniform normal ultrafilter on κ_α.

Let $h : \omega \times \omega \to \omega$ be a bijection such that

(i) $(\forall e \in \omega)\, h(e, 0) \geq e$;

(ii) $(\forall e \in \omega)(\forall m \in \omega)(\forall n \in \omega)(m < n \to h(e,m) < h(e,n))$.

We describe a game tree T^* by describing the plays in T^*:

$$\begin{array}{lccccccc} I & \langle a_0, \xi_0\rangle & & \langle a_2, \xi_1\rangle & & \langle a_4, \xi_2\rangle & & \ldots \\ II & & \langle a_1, \eta_0\rangle & & \langle a_3, \eta_1\rangle & & \ldots & \end{array}$$

Each $\langle a_i \mid i < n\rangle$ must be a position in T. Each ξ_i must be an ordinal number smaller than κ. For all elements e and n of ω it must be that

$$\eta_{h(e,n)} < \kappa_{\xi_e}.$$

Let $\pi : T^* \to T$ be given by

$$\begin{aligned} &\pi(\langle\langle a_0, \xi_0\rangle, \langle a_1, \eta_0\rangle, \ldots, \langle a_{2n-1}, \eta_{n-1}\rangle\, [, \langle a_{2n}, \xi_n\rangle]\rangle) \\ &\quad = \langle a_0, a_1, \ldots, a_{2n-1}\, [, a_{2n}]\rangle. \end{aligned}$$

The function π induces a continuous function, which we also call π, from $[T^*]$ to $[T]$.

We define $A^* \subseteq [T^*]$ as follows. Fix a play

$$x^* = \langle a_0, \xi_0\rangle, \langle a_1, \eta_0\rangle, \ldots$$

in T^*. Let $x = \pi(x^*) = \langle a_i \mid i \in \omega\rangle$. Since $<_x$ is not a wellordering, we have that

$$(\exists i \in \omega)(\exists j \in \omega)(i <_x j \not\leftrightarrow \xi_i < \xi_j).$$

Let k be the smallest $\max\{i, j\}$ for such i and j (in the sense of the natural ordering of ω). Then $x^* \in A^*$ if and only if there are elements e, m, and n of ω such that

$$h(e,m) < k \;\wedge\; h(e,n) < k \;\wedge\; (m <^e_x n \not\leftrightarrow \eta_{h(e,m)} < \eta_{h(e,n)}).$$

We can describe A^* more informally as follows. At some point in a play x^*, I must fail in the endeavor to make the ordinal moves ξ_i give an embedding of $(\omega; <_{\pi(x^*)})$ into $(\kappa; <)$. I wins just in case this failure is preceded by II's failure, for some e, to make the ordinal moves $\eta_{h(e,n)}$ give an embedding of $(\omega; <^e_{\pi(x^*)})$ into κ_{ξ_e}. (Note that, by the properties of h, the ordinal ξ_e is chosen before any of the $\eta_{h(e,n)}$.)

Let us say that $p^* \in T^*$ is *good* if the neither player has yet failed in the sense described in the preceding paragraph. In other words, $p^* = \langle\langle a_0, \xi_0\rangle, \langle a_1, \eta_0\rangle \ldots$ is good if

(a) $(\forall i)(\forall j)((2i < \ell\mathrm{h}(p^*) \wedge 2j < \ell\mathrm{h}(p^*)) \rightarrow (i <_{\pi(p^*)} j \leftrightarrow \xi_i < \xi_j))$;

(b) $(\forall i)(\forall m)(\forall n)((2h(i,m)+1 < \ell\mathrm{h}(p^*) \wedge 2h(i,n)+1 < \ell\mathrm{h}(p^*)) \rightarrow (m <_{\pi(p^*)} n \leftrightarrow \eta_{h(i,m)} < \eta_{h(i,n)}))$.

The set A^* is clopen, and so the game $G(A^*;T^*)$ is determined.

Suppose first that $G(A^*;T^*)$ is a win for II. Let τ^* be a winning strategy for II for $G(A^*;T^*)$.

We first define a strategy τ for II for $G(A;T)$. The idea is that the first components of II's moves given by τ^* are independent of the second components of I's moves, as long as these second components are in the right order and are members of a certain set belonging to $\mathcal{U}$.

Let $p \in T$ with $\ell\mathrm{h}(p) = 2k+1$. For $v \in [\kappa]^{k+1}$, if there is a good $q^* \in T^*$ such that $\pi(q^*) = p$, q^* is consistent with τ^*, and all the ξ_e belong to v, then let $q^*(p,v)$ be the unique such q^*. By Lemma 3.1.8, there is set $X_p \in \mathcal{U}$ such that one of the following holds.

(1) For all $v \in [X_p]^{k+1}$, $q^*(p,v)$ is undefined.

(2) There is an a such that, for all $v \in [X_p]^{k+1}$, $q^*(p,v)$ is defined and the first component of $\tau^*(q^*(p,v))$ is a.

Choose a strategy τ for II for $G(A;T)$ as follows.

If (2) holds for p, let $\tau(p)$ be the common first component of the $\tau^*(q^*(p,v))$ for $v \in [X_p]^{k+1}$. If (1) holds, let $\tau(p)$ be arbitrary.

It is easy to see by induction on $\ell\mathrm{h}(p)$ that (2) holds for every p that is consistent with τ.

We will prove that τ is a winning strategy for $G(A;T)$. The idea is that, for each e, the components $\eta_{h(e,n)}$ of II's moves given by τ^* are independent of the components $\xi_{\bar{e}}$ of I's moves for $e <_x \bar{e}$, as long as these moves are in the right order and are members of a certain set belonging to $\mathcal{U}$.

For $p \in T$, let us abuse notation by writing T^*_p for $\bigcup\{T^*_{q^*} \mid q^* \in T^* \wedge \pi(q^*) = p\}$.

Let $e \in \omega$ and $\bar{p} \in T$ with $\ell\mathrm{h}(\bar{p}) > e$. Let $T^e_{\bar{p}}$ be like $T^*_{\bar{p}}$ except that second components $\xi_{e'}$ and $\eta_{h(e',n)}$ are played only if $e' <_{\hat{p}} e$, where $\hat{p}$ is the longer of $\bar{p}$ and the sequence of first components of the position in $T^e_{\bar{p}}$. Let $\pi^e_{\bar{p}} : T^*_{\bar{p}} \rightarrow T^e_{\bar{p}}$ be the obvious function and also call $\pi^e_{\bar{p}}$ the associated function from $[T^*_{\bar{p}}]$ to $[T^e_{\bar{p}}]$. Write $\pi_{e,\bar{p}}$ for the obvious functions from $T^e_{\bar{p}}$ to $T_{\bar{p}}$ and from $[T^e_{\bar{p}}]$ to $[T_{\bar{p}}]$. Let us define *good* positions in $T^e_{\bar{p}}$ by the obvious modification of the definition of good positions in T^*; that is, replace "$(\forall i)$" and "$(\forall j)$" by

$(\forall i <_{\bar{p}\cup\pi_{e,\bar{p}}(p)} e)$" and "$(\forall j <_{\bar{p}\cup\pi_{e,\bar{p}}(p)} e)$" respectively in clauses (a) and (b) of the earlier definition.

For the next three paragraphs, fix $\bar{p} \in T$ with $\ell\mathrm{h}(\bar{p}) > e$ and fix $k \in \omega$ and $p \in T^e_{\bar{p}}$ with $\ell\mathrm{h}(p) = 2k+1$. Let m_p be the number of $e' \leq k$ such that $e' <_{\bar{p}\cup\pi_{e,\bar{p}}(p)} e$; i.e., let m_p be the number of $e' \leq k$ such that $p(2e')$ has a second component. Let $v \in [\kappa]^{m_p}$. If there is a good $q^* \in T^*_{\bar{p}}$ such that q^* is consistent with τ^*, $\pi^e_{\bar{p}}(q^*) = p$, and all $\xi_{e'}$ belong to v, then let $q^*(p,v)$ be the unique such q^*.

Note that the definition of $q^*(p,v)$ just given does not depend on e or $\bar{p}$. Note also that, for $p \in T \cap T^e_{\bar{p}}$, the definition just given of $q^*(p,v)$ agrees with the earlier one.

By Lemma 3.1.8, there is a set $X_p \in \mathcal{U}$ such that one of the following holds.

(1) For all $v \in [X_p]^{m_p}$, $q^*(p,v)$ is undefined.

(2) There is an a and, if $k = h(e',n)$ with $e' <_{\bar{p}\cup\pi_{e,\bar{p}}(p)} e$, there is an $\alpha < \kappa$, such that, for all $v \in [X_p]^{m_p}$, $q^*(p,v)$ is defined, the first component of $\tau^*(q^*(p,v))$ is a, and the second component of $\tau^*(q^*(p,v))$, if it exists, is α.

To see that Lemma 3.1.8 applies, observe that $h(e',n)$ must be smaller than the $\kappa_{\xi_{e'}}$ given by $q^*(p,v)$.

For $e \in \omega$ and $\bar{p} \in T$, choose a strategy $\tau^e_{\bar{p}}$ for II in $T^e_{\bar{p}}$ as follows. Let p be as in the preceding paragraphs. (a) If (2) holds for p, let $\tau^e_{\bar{p}}(p)$ be the common value of $\pi^e_{\bar{p}}(\tau^*(q^*(p,v)))$ for $v \in [X_p]^{m_p}$, provided that this is a legal move at p in $T^e_{\bar{p}}$. (The only way this proviso can fail is if $\ell\mathrm{h}(p) < \ell\mathrm{h}(\bar{p})$, and the the first component of $\tau^*(q^*(p,v))$ is different from $\bar{p}(\ell\mathrm{h}(p))$.) (b) Otherwise, let $\tau^e_{\bar{p}}(p)$ be arbitrary.

Suppose that $\bar{p}$ is consistent with τ. By induction on $\ell\mathrm{h}(p)$ we show that, for every good p of odd length that is consistent with $\tau^e_{\bar{p}}$,

(i) $\tau^e_{\bar{p}}(p)$ is defined by (a);

(ii) $p^\frown\langle\tau^e_{\bar{p}}(p)\rangle$ is a good position in $T^e_{\bar{p}}$.

Assume that p is good, that $\ell\mathrm{h}(p) = 2k+1$, and that (i) holds for $p \restriction 2k-1$ if $k > 0$. Let $X = X_{p\restriction 2k-1}$ if $k > 0$ and let $X = \kappa$ if $k = 0$. Let $v \in [X]^{m_p}$ and let v' be the image of $\{e' < k \mid e' <_{p\cup\pi_{e,\bar{p}}(p)} e\}$ under the isomorphism between $(\{e' \leq k \mid e' <_{p\cup\pi_{e,\bar{p}}(p)} e\}; <_{p\cup\pi_{e,\bar{p}}(p)})$ and $(v; <)$.

Assume for the moment that $k > 0$. Then (2) holds for $p \restriction 2k-1$, and so $q^*(p \restriction 2k-1, v')$ exists. Since $q^*(p \restriction 2k-1, v')$ is good and τ^* is a winning strategy, we get that $q^*(p \restriction 2k-1, v')^\frown\langle\tau^*(q^*(p \restriction 2k-1, v))\rangle$ is good. But

$$p \restriction 2k-1^\frown\langle\tau^e_{\bar{p}}(p \restriction 2k-1)\rangle = \pi^e_{\bar{p}}(q^*(p \restriction 2k-1, v')^\frown\langle\tau^*(q^*(p \restriction 2k-1, v))\rangle),$$

and hence $p \restriction 2k-1^\frown\langle\tau^e_{\bar{p}}(p \restriction 2k-1)\rangle$ is good. Since v' is an arbitrary member of $[X]^{m_{p\restriction 2k-1}}$, we have shown that (ii) holds for $p \restriction 2k-1$. The goodness of $q^*(p \restriction 2k-1, v')^\frown\langle\tau^*(q^*(p \restriction 2k-1, v))\rangle$ and the consistency of p with $\tau^e_{\bar{p}}$ imply that $q^*(p, v)$ exists.

Since $q^*(p, v)$ obviously exists if $k = 0$, we have established its existence whether or not $k = 0$, and so we know that (2) holds for p. Since $\bar{p}$ is consistent with τ, we have that $\pi^e_{\bar{p}}(q^*(p, v)^\frown\tau^*(q^*(p, v)))$ belongs to $T^e_{\bar{p}}$. This fact and the fact that v is an arbitrary member of X show that (i) holds for p.

Suppose that both $\bar{p}$ and $\bar{p}'$ are positions in T consistent with τ and that both have length greater than e. If $p \in T \cap T^e_{\bar{p}}$ and p is good and consistent with τ, then $\tau(p)$ is identical with the first component of $\tau^e_{\bar{p}}(p)$. If $p \in T^e_{\bar{p}} \cap T^{e'}_{\bar{p}'}$ and p is good and consistent with both $\tau^e_{\bar{p}}$ and $\tau^{e'}_{\bar{p}'}$, then the first components of $\tau^e_{\bar{p}}(p)$ and $\tau^{e'}_{\bar{p}'}(p)$ are the same, and the second components are the same if both exist.

Let $X \subseteq \kappa$ be the set of all $\beta < \kappa$ such that

(i) for all $p \in T$, $\beta \in X_p$;

(ii) for all $e \in \omega$, for all $\bar{p} \in T$ with $\ell\mathrm{h}(\bar{p}) > e$, and for all $p \in T^e_{\bar{p}}$, if every second component of a move in p is smaller than β, then $\beta \in X_p$.

Using the normality of $\mathcal{U}$, one readily verifies that $X \in \mathcal{U}$.

Let x be a play in T consistent with τ. Let E be the wellordered initial segment of ω with respect to $<_x$. We will define

(i) $\langle\xi_e \mid e \in E\rangle$;

(ii) $\langle\eta_{h(e,n)} \mid e \in E \wedge n \in \omega\rangle$.

We will arrange that the following conditions are satisfied.

(a) Each $\xi_e \in X$.

(b) The function $e \mapsto \xi_e$ embeds $(E; <_x)$ into $(\kappa; <)$.

(c) For each $e \in E$, the function $n \mapsto \eta_{h(e,n)}$ embeds $(\omega; <^e_x)$ into $(\kappa_{\xi_e}; <)$.

(d) For each $e \in E$, if x is augmented by the second components $\xi_{e'}$, $e' <_x e$, and $\eta_{h(e',n)}$, $e' <_x e$ and $n \in \omega$, then the resulting x_e is a play in $T^e_{x \restriction e+1}$ that is consistent with $\tau^e_{x \restriction e+1}$.

Since condition (c) implies that $X \in B_e$ for every $e \in E$, we will have proved that $x \notin A$.

We define ξ_e and the $\eta_{h(e,n)}$ by transfinite induction on e with respect to the wellordering $<_x$ of E.

If E is empty then there is nothing to define, so assume that E is nonempty. Note that (d) holds for the the $<_x$-least e, since $T^e_{x \restriction e+1} = T_{x \restriction e+1}$ and x is consistent with τ. Note also that, if e is not the immediate $<_x$-successor of any e' and if (d) holds for every $e' <_x e$, then (d) holds for e. This is because of the agreement mentioned earlier between the strategies $\tau^e_{x \restriction e+1}$ and $\tau^{e'}_{x \restriction e'+1}$ and because for each $k \in \omega$ there is a $e' <_x e$ such that $x_e \restriction k = x_{e'} \restriction k$.

Let $e \in E$. Assume that we have defined $\xi_{e'}$ and $\langle \eta_{h(e',n)} \mid n \in \omega \rangle$ for all $e' <_x e$ in such a manner that (b) is not violated, (a), (c), and (d) hold for all $e' <_x e$, and (d) holds for e if e is a $<_x$-successor. By the remarks in the the preceding paragraph, (d) holds for e whether or not e is a $<_x$ successor.

Let ξ_e be the least member of X that is larger than $\xi_{e'}$ for every $e' <_x e$. Note that (a) holds for e and that (b) is still not violated. Let $\bar{e}$ be the immediate successor of e with respect to $<_x$. Suppose $\eta_{h(e,n')}$ is defined for $n' < n$ so as not to violate (c) for e or (d) for $\bar{e}$. Let $i = h(e,n)$. Let $\eta_{h(e,n)}$ be the second component of $\tau^{\bar{e}}_{x \restriction \bar{e}+1}$ applied to the good position p' in $T^e_{x \restriction \bar{e}+1}$ given by $x_e \restriction 2i+1$, ξ_e, and the $\eta_{h(e,n')}$ for $n' < n$. Clearly (d) is still not violated for $\bar{e}$. Since $p' {}^\frown \langle \tau_e(p') \rangle$ is good, (c) is still not violated for e.

Suppose now that $G(A^*; T^*)$ is a win for I. Let σ^* be a winning strategy for I for $G(A^*; T^*)$.

To define a strategy for I for $G(A; T)$ and to prove this strategy is winning, we use the fact that I's moves in T given by σ^* are independent of II's moves $\eta_{h(e,n)}$ and I's moves ξ_e are independent of II's moves $\eta_{h(e',n)}$, for $e \leq_x e'$, provided that these moves by II are in the right order and are from certain members of the $\mathcal{V}_\alpha$.

Lemma 5.4.6. *Let $k \in \omega$ and let $p \in T$ with $\ell\mathrm{h}(p) = 2k$. Let $e_0 <_p \cdots <_p e_{k-1}$ be all natural numbers smaller than k.*

For all j such that $0 \leq j < k$, there are sets $X_{\alpha,p,j} \in \mathcal{V}_\alpha$, $\alpha < \kappa$, with the following property. Let $U_{p,j}$ be the set of all good positions q^* of even length, consistent with σ^*, such that $\pi(q^*) \subseteq p$, and such that $\eta_{h(e_{j'},n)} \in X_{\xi_{e_{j'}},p,j'}$ for every j' and n with $j \leq j' < k$ and $2h(e_{j'}, n) < \ell\mathrm{h}(q^*)$. For $q^* \in U_{p,j}$,

(a) the first component of $\sigma^*(q^*)$ depends only on $\pi^{e_j}_{p\restriction e_j+1}(q^*)$;

(b) for $\ell\mathrm{h}(q^*) = 2e_{\bar{\jmath}}$ with $\bar{\jmath} \leq j$, the second component of $\sigma^*(q^*)$ depends only on $\pi^{e_j}_{p\restriction e_j+1}(q^*)$.

Proof. We prove the lemma by induction on $k - j$.

Let $j < k$, and assume by induction that the $X_{\alpha,p,j'}$ are defined for $j' \geq j$ and that the lemma holds for $j+1$ if $j+1 < k$. Our induction hypothesis implies that the lemma holds for j for $q^* \in U_{p,j}$ such $\ell\mathrm{h}(q^*) \leq 2e_j$. This is because such positions q^* contain no move components of the form $\eta_{h(e_j,n)}$. One consequence of this is that, for $q^* \in U_{p,j}$ with $\ell\mathrm{h}(q^*) > 2e_j$, the ξ_{e_j} of q^* depends only on $\pi^{e_j}_{p\restriction e_j+1}(q^*)$ (in fact, only on $\pi^{e_j}_{p\restriction e_j+1}(q^*) \restriction 2e_j$).

Let $\bar{\jmath} < k$ with $e_{\bar{\jmath}} > e_j$. Let $W_{p,j,\bar{\jmath}}$ be the set of all good q^* such that $\ell\mathrm{h}(q^*) = 2e_{\bar{\jmath}}$, q^* is consistent with σ^*, $\pi(q^*) \subseteq p$, and $\eta_{h(e_{j'},n)} \in X_{\xi_{e_{j'}},p,j'}$ for every j' and n with $j < j' < k$ and $2h(e_{j'}, n) < \ell\mathrm{h}(q^*)$.

For $q^* \in W_{p,j,\bar{\jmath}}$,

(a) the first component of $\sigma^*(q^*)$ depends only on $\pi^{e_j}_{p\restriction e_j+1}(q^*)$ plus $\langle \eta_{h(e_j,n)} \mid 2h(e,j) < \ell\mathrm{h}(q^*)\rangle$;

(b) if $\bar{\jmath} < j$, then the second component of $\sigma^*(q^*)$ depends only on $\pi^{e_j}_{p\restriction e_j+1}(q^*)$ plus $\langle \eta_{h(e_j,n)} \mid 2h(e,j) < \ell\mathrm{h}(q^*)\rangle$.

Furthermore, if $\bar{\jmath} < j$ then, for $q^* \in W_{p,j,\bar{\jmath}}$, the second component of $\sigma(q^*)$ is smaller than the ξ_{e_j} of q^* (which is smaller than $\kappa_{\xi_{e_j}}$). Let

$$V_{p,j,,\bar{\jmath}} = \{r \in T_{e_j} \mid (\exists q^* \in W_{p,j,\bar{\jmath}})\, \pi^{e_j}_{p\restriction e_j+1}(q^*) = r\}.$$

Let $r \in V_{p,j,\bar{\jmath}}$ and let $\xi_{e_j}(r)$ be the ξ_{e_j} determined by it. There is a set $Y_r \in \mathcal{V}_{\xi_{e_j}(r)}$ such that, for $q^* \in W_{p,j,\bar{\jmath}}$ with $\pi^{e_j}_{p\restriction e_j+1}(q^*) = r$ and and with each $\eta_{h(e_j,n)} \in Y_r$,

(a) the first component of $\sigma^*(q^*)$ is constant;

(b) if $\bar{\jmath} < j$, then the second component of $\sigma^*(q^*)$ is constant.

For each $\bar{\jmath}$ and for each $\alpha < \kappa$, there are no more than $\max\{|T|, |\alpha|\}$ elements r of $V_{p,j,\bar{\jmath}}$ with $\xi_{e_j}(r) = \alpha$. For $\alpha < \kappa$, set

$$X_{\alpha,p,j} = \bigcap \{Y_r \mid (\exists \bar{\jmath} < k)(r \in V_{p,j,\bar{\jmath}} \wedge \xi_{e_j}(r) = \alpha\}.$$

The $X_{\alpha,p,j}$, along with the given $X_{\alpha,p,j'}$, $j' > j$, witness that the lemma holds for j. □

For $\alpha < \kappa$, let

$$X_\alpha = \bigcap_{p,j} X_{\alpha,p,j}.$$

Evidently $X_\alpha \in V_\alpha$ for each α.

Let U be the set of all good positions q^* of even length, consistent with σ^*, and such that $\eta_{h(e,n)} \in X_{\xi_e}$ for every e and n with $2h(e,n) < \ell\mathrm{h}(q^*)$. Note that $U \subseteq U_{p,j}$ for all p and j. Thus Lemma 5.4.6 gives that, for $q^* \in U$,

(1) the first component of $\sigma^*(q^*)$ depends only on $\pi(q^*)$;

(2) for $\ell\mathrm{h}(q^*) = 2e$, the second component of $\sigma^*(q^*)$ depends only on $\pi^e_{\pi(q^*)\restriction e+1}(q^*)$.

Choose a strategy σ for I for $G(A;T)$ as follows. If (a) there is a $q^* \in U$ such that $\pi(q^*) = p$, then let $\sigma(p)$ be the common first component of $\sigma^*(q^*)$ for all such q^*. If (b) no such q^* exists, let $\sigma(p)$ be arbitrary.

We show that (a) holds for every p consistent with σ. To do this we prove the stronger fact that, for every p consistent with σ and for all $\langle Y_\alpha \mid \alpha < \kappa\rangle$ with each Y_α a subset of X_α of order type κ_α, there is a q^* witnessing that (a) holds for p such that $\eta_{h(e,n)} \in Y_{\xi_e}$ for every e and n with $2h(e,n) < \ell\mathrm{h}(q^*)$. This is true trivially for $p = \emptyset$. Let $\ell\mathrm{h}(p) = 2k$ and assume that it is true for p. Let $p' \supseteq p$ with $\ell\mathrm{h}(p') = 2k+2$. Assume that p' is consistent with σ. For $\alpha < \kappa$, let Y_α be a subset of X_α whose order type is κ. For each α, let $Y_\alpha = \{\beta_{\alpha,\gamma} \mid \gamma < \kappa_\alpha\}$, where $\gamma < \gamma' \rightarrow \beta_{\alpha,\gamma} < \beta_{\alpha,\gamma'}$ for all α, γ, and γ'. For $\alpha < \kappa$, set

$$Y'_\alpha = \{\beta_{\alpha,2\gamma+1} \mid 2\gamma < \delta_\alpha\}.$$

Let q^* witness the truth of our induction hypothesis for p for the Y'_α. Let $k = h(e,n)$. Let $\sigma^*(q^*) = \langle a, \alpha\rangle$. By the definition of σ and the consistency of p' with it, $a = p'(2k)$. Clearly there is a $\beta \in Y_\alpha \setminus Y'_\alpha$ such that $q^{*\frown}\langle\sigma^*(q^*), \langle p'(2k+1), \beta\rangle\rangle$ is a good position in T^*.

For $e \in \omega$ and $\bar{p} \in T$ with $\ell\mathrm{h}(\bar{p}) > e$, choose a strategy $\sigma^e_{\bar{p}}$ for I for $G(A;T)$ as follows. If (a) there is a $q^* \in U$ such that $\pi^e_{\bar{p}}(q^*) = p$, then let $\sigma^e_{\bar{p}}(p)$ be the common value of $\pi^e_{\bar{\sigma}}(\sigma^*(q^*))$ for all such q^*, provided that this is a legal move at p in $T^e_{\bar{p}}$. If (b) no such q^* exists, let $\sigma^e_{\bar{p}}(p)$ be arbitrary.

Suppose that $\bar{p}$ is consistent with σ. Then for every good p of even length that is consistent with $\sigma^e_{\bar{p}}$,

(i) (a) holds;

(ii) $p^\frown\langle\sigma^e_{\bar{p}}(p)\rangle$ is a good position in $T^e_{\bar{p}}$.

The proof of (i) is similar to the proof of the analogous fact about σ. (ii) follows because, for good q^*, $q^{*\frown}\langle\sigma^*(q^*)\rangle$ must be good since σ is a winning strategy.

Suppose that both $\bar{p}$ and $\bar{p}'$ are consistent with σ and that both have length greater than e. If $p \in T \cap T^e_{\bar{p}}$ and p is good and consistent with σ, then $\sigma(p)$ is identical with the first component of $\sigma^e_{\bar{p}}(p)$. If $p \in T^e_{\bar{p}} \cap T^{e'}_{\bar{p}'}$ and p is good and consistent with both $\sigma^e_{\bar{p}}$ and $\sigma^{e'}_{\bar{p}'}$, then the first components of $\sigma^e_{\bar{p}}(p)$ and $\sigma^{e'}_{\bar{p}'}(p)$ are the same, and the second components can differ only in that one or the other might be absent.

Let x be a play in T consistent with σ. Assume, in order to derive a contradiction, that $x \notin A$. Let E be the wellordered initial segment of ω with respect to $<_x$. Since $x \notin A$, $x \in B_e$ for each $e \in E$. Thus $<^e_x$ is a wellordering of ω for each $e \in E$. We will define

(i) $\langle \xi_e \mid e \in E\rangle$;

(ii) $\langle \eta_{h(e,n)} \mid e \in E \wedge n \in \omega\rangle$.

We will arrange that the following conditions are satisfied.

(a) For each e, $\xi_e < \kappa$, and each $\eta_{h(e,n)} \in X_{\xi_e}$.

(b) The function $e \mapsto \xi_e$ embeds $(E; <_x)$ into $(\kappa; <)$.

(c) For each $e \in E$, the function $n \mapsto \eta_{h(e,n)}$ embeds $(\omega; <^e_x)$ into $(\kappa_{\xi_e}; <)$.

(d) For each $e \in E$, if x is augmented by the second components $\xi_{e'}$, $e' <_x e$, and $\eta_{h(e',n)}$, $e' <_x e$ and $n \in \omega$, then the resulting x_e is a play in $T^e_{x\restriction e+1}$ that is consistent with $\sigma^e_{x\restriction e+1}$.

We define ξ_e and the $\eta_{h(e,n)}$ by transfinite induction on e with respect to the wellordering $<_x$ of E.

If E is empty then there is nothing to define, so assume that E is nonempty. Note that (d) holds for the the $<_x$-least e, since $T^e_{x\restriction e+1} = T_{x\restriction e+1}$ and x is consistent with σ. Note also that, if e is not the immediate $<_x$-successor of any e' and if (d) holds for every $e' <_x e$, then (d) holds for e.

Let $e \in E$. Assume that we have defined $\xi_{e'}$ and $\langle \eta_{h(e',n)} \mid n \in \omega\rangle$ for all $e' <_x e$ in such a manner that (b) is not violated, (a), (c), and (d) hold for all $e' < e$, and (d) holds for e if e is a $<_x$-successor. By the remarks in the the preceding paragraph, (d) holds for e whether of not e is a $<_x$ successor.

Let $\bar{e}$ be the immediate successor of e with respect to $<_x$. Let ξ_e be the second component of $\sigma^{\bar{e}}_{x\restriction\bar{e}+1}(x_e \restriction 2e)$. Now $x_e \restriction 2e$ is good and is consistent with $\sigma^e_{x\restriction e+1}$ and so with $\sigma^{\bar{e}}_{x\restriction\bar{e}+1}$. Therefore $x_e \restriction 2e^\frown\langle\sigma^{\bar{e}}_{x\restriction\bar{e}+1}(x \restriction 2e)\rangle$ is a good position in $T^{\bar{e}}_{x\restriction\bar{e}+1}$. This implies that (b) is still not violated. Moreover, it is already clear that (d) will hold for $\bar{e}$ if (a) and (c) hold for e.. Let $n \mapsto \eta_{h(e,n)}$ embed the wellordering $(\omega; <^e_x)$ into X_{ξ_e}. Note that (a) and (c) hold for e.

Recall that $(\omega; <_x)$ is not a wellordering. Let e_0 be the least member of $\omega \setminus E$ with respect to the natural ordering of ω. Given $e_n \in \omega \setminus E$, let e_{n+1} be the least member e of $\omega \setminus E$ such that $e <_x e_n$. Note that $e_{n+1} > e_n$. Let ξ_{e_0} be the second component of $\sigma^*(x_{e_0} \restriction 2e_0)$. For $n \in \omega$, let $\xi_{e_{n+1}}$ be the second component of $\tau^{e_n}_{x\restriction e_n+1}(x_{e_n} \restriction 2e_{n+1})$.

We derive our contradiction by showing that $\xi_{e_{n+1}} < \xi_{e_n}$ for each $n \in \omega$. Let $q^* \in U$ be such that $\pi^{e_n}_{x\restriction e_n+1}(q^*) = x_{e_n} \restriction 2e_{n+1}+1$. The second component of $q^*(2e_{n+1})$ is $\xi_{e_{n+1}}$, and the second component of $q^*(2e_n)$ must be ξ_{e_n}. Since q^* is good, this implies that $\xi_{e_{n+1}} < \xi_{e_n}$. □

The hypothesis of Theorem 5.4.5 can be weakened to get the following result.

Theorem 5.4.7. *Let λ be an infinite cardinal number. Assume that for every $a \subseteq \lambda$ there is a transitive proper class model M of* ZFC *such that*

(i) the class of $\kappa > \lambda$ such that $M \models$ "κ is a measurable cardinal" is unbounded in the ordinals;

(ii) there is a proper class C of indiscernibles for M, a.

Then all $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ games in trees of size λ are determined.

Proof. We sketch the proof.

Let T be a game tree with field $(t) \subseteq \lambda$ and with no terminal nodes. Let A, $\langle B_e \mid e \in \omega\rangle$, $p \mapsto <_p$, and $\langle p \mapsto <^e_p \mid e \in \omega\rangle$ be as in the proof of Theorem 5.4.5. Let $a \subseteq \lambda$ be such that T, $p \mapsto <_p$, and $\langle p \mapsto <^e_p \mid e \in \omega\rangle$ are all definable from a in $L[a]$. Let M be given by the hypothesis of the theorem.

We may assume that $M \models$ "There is no $\kappa > \lambda$ such that κ is a measurable limit of measurable cardinals." To see this, assume that γ is the least measurable limit of measurable cardinals of M that is larger than λ. Let $\mathcal{W}$ be such that $M \models$ "$\mathcal{W}$ is a uniform normal ultrafilter on γ." Let $N = \bigcap_{\alpha \in \mathrm{Ord}} \mathrm{Ult}_\alpha(M;\mathcal{W})$. ($N$ is what one might call the V_{Ord} of $\mathrm{Ult}_{\mathrm{Ord}}(M;\mathcal{W})$.) In the class model N, there is a proper class of measurable cardinals, but there is no $\kappa > \lambda$ that is a measurable limit of measurables. Furthermore, $\{i^M_{\mathcal{W}0,\alpha}(\gamma) \mid \alpha \in \mathrm{Ord}\}$ is a proper class of indiscernibles for N, a.

Generalizing Theorem 3.5.4 (see Mitchell [1974]), we may assume that

$$M = L[a, \langle\!\langle \mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\rangle\!\rangle],$$

that $M \models$ "every measurable cardinal is larger than λ," and that, for each ordinal α, $M \models$ "$\mathcal{V}_\alpha$ the unique normal ultrafilter on the αth measurable cardinal κ_α."

By the hypothesis of the theorem, together with constructions and arguments like those of the proof of Theorem 3.4.8, we may assume that there is a closed proper class C of indiscernibles for M, a such that $C \cap \lambda^+$ has order type λ^+.

Let $\kappa = \lambda^+$. Define T^* and A^* as in the proof of the proof of Theorem 5.4.5. Observe that $T \in M$ and that T is definable in M from κ and a. Observe also and that there is set D that is definable in M from κ and a such that D generates the open set A^*. (See page 206.) By Lemma 4.4.1, there is a winning strategy ρ for $G(A^*;T^*)$ such that ρ is definable in M from κ and a and such that $M \models$ "ρ is a winning strategy for $G(E^*;T^*)$, where E^* is the open set generated by D."

Suppose first that ρ is a strategy σ^* for I. Applying in M the proof of Theorem 5.4.5, we get a strategy σ in T such that $\sigma \in M$ and such that M $\models$ "σ is a winning strategy for $G(E;T)$, where E is the set witnessed $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ by $p \mapsto <\, p$ and $\langle p \mapsto <^e_p \mid e \in \omega\rangle$." (The measurability of κ was not used in this part of the proof of Theorem 5.4.5.) By absoluteness, σ is a winning strategy for $G(A;T)$.

Now suppose that ρ is a strategy τ^* for II. Proceed as in the proof of Theorem 5.4.5, using for X_p the set of members of C larger than all second components of moves of p. The construction and proof go through essentially unchanged. They define a strategy τ for II for $G(A;T)$ and they show that τ is a winning strategy. □

Here is the lightface version of Theorem 5.4.7.

Theorem 5.4.8. (Simms [1979]) *If there is a transitive proper class model M of* ZFC + *"there are arbitrarily large measurable cardinals" and there is a proper class of indiscernibles for M, then every $\Sigma^0_1(\Pi^1_1)$ game is determined.*

It is unknown whether the converse of Theorem 5.4.8 holds. But Simms [1979] strengthened both Theorems 5.4.7 and 5.4.8 by deriving stronger conclusions from their hypotheses, and Simms proved the converses of the resulting theorems (except that the converse of Theorem 5.4.7 in case that λ is uncountable was proved later by the author.) The converse of Theorem 5.4.7 itself is known to hold. This follows by combining results in Steel [1982] and Mitchell [1992]. The Steel and Mitchell theorems almost, but not quite, give the converse of Theorem 5.4.8.

We will next give Simms' improved versions of Theorems 5.4.7 and 5.4.8. Exercise 5.4.5 concerns the converses of these results. Exercise 5.4.4 presents the converse of Theorem 5.4.7.

To state Simms' theorems, we need to introduce a hierarchy built on $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ and an analogous lightface hierarchy. To avoid having to prove normal form results analogous to Lemmas 5.4.3 and 5.4.4, we choose somewhat artificial defintions of these hierarchies, with the normal forms built into the definitions.

Let T be a game tree without terminal positions. For countable ordinals α and sets $A \subseteq [T]$, say that $A \in \mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$–$\alpha$ if there are sets $B_{\beta,e}$, $\beta < \alpha$ and $e \in \omega$, and there functions $p \mapsto <_{\beta,p}$, $\beta < \alpha$, such that each $p \mapsto <_{\beta,p}$ has domain T and such that

(1) for all β, $<_{\beta,\emptyset} = \emptyset$ and, for all $p \in T \setminus \{\emptyset\}$, $<_{\beta,p}$ is a linear ordering of $\ell\mathrm{h}(p)$ with greatest element 0;

(2) for all β and for elements p and p' of T, if $p \subseteq p'$ then $<_{\beta,p} \subseteq <_{\beta,p'}$;

(3) for all β and for all $x \in T$, every non-zero member of ω has an immediate successor with respect to $<_{\beta,x}$, where

$$<_{\beta,x} = \bigcup_{n\in\omega} <_{\beta,x\restriction n};$$

(4) for all $x \in [T]$, $x \in A$ if and only if, for the least β such that

(a) $(\exists e \in \omega)(<_{\beta,x} \restriction \{e' \mid e' <_{\beta,x} e\}$ is a wellordering $\wedge\ x \notin B_{\beta,e})$ or
(b) $<_{\beta,x}$ is not a wellordering of ω or
(c) $\beta = \alpha$,

then β is odd if (c) holds or (a) fails and β is even otherwise.

Remark. Suppose that we have sets and functions satisfying (1), (2), and (3), and let A be defined by clause (4). For each $\beta < \alpha$, let A_β be the set of all $x \in [T]$ such that condition (a) above holds for x and β. For each β, $A_\beta \in \mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$. For $x \in [T]$ and $\gamma \leq \alpha$, let $x \in C_\gamma$ just in case conditions (a) and (b) both fail for x and every $\beta < \gamma$. Each $C_\gamma \in \mathbf{\Pi}^1_1$. Furthermore, for all $x \in [T]$, x belongs to A if and only one of the following conditions holds.

(i) α is odd and $x \in C_\alpha$.

(ii) there is a $\beta < \alpha$ such that $x \in A_\beta$.

(iii) $x \notin C_\alpha$ and $(\forall \beta < \alpha)((\beta \text{ even } \wedge\ x \in C_\beta) \to x \in C_{\beta+1})$ and $(\forall \beta < \alpha)((\beta \text{ odd } \wedge\ x \in C_\beta) \to x \notin A_\beta)$.

It is not hard to see that the set of all x satisfying condition (i) or (ii) belongs to $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ and that the set of all x not satisfying (iii) belongs $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$. Thus $\neg A \in 2\text{–}(\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1))$. Switching "even" and "odd," we see also that $A \in 2\text{–}(\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1))$. Thus the entire $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$–$\alpha$ hierarchy is properly contained within the first two levels of the difference hierarchy on $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$, and so within the first two levels of the difference hierarchy on the dual $\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1)$.

For game trees T with terminal nodes, let $T' \supseteq T$ be the gotten from T by adding one play extending each terminal node of T. Say that $A \subseteq \lceil T \rceil$ belongs to $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$–$\alpha$ just in case A belongs to $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$–$\alpha$ as a subset of $[T]$.

For $\alpha < \omega_1^{\mathrm{CK}}$, the lightface class $\Sigma^1_1(\Pi^1_1)$–α is defined in the obvious way. We leave the formulation of the definition to the reader.

Here is Simms' improvement of Theorem 5.4.7.

Theorem 5.4.9. *Let λ be an infinite cardinal number. Assume that for every $a \subseteq \lambda$ there is a transitive proper class model M of* ZFC *such that*

(i) the class of $\kappa > \lambda$ such that $M \models$ "κ is a measurable cardinal" is unbounded in the ordinals;

(ii) $a \in M$ and there is a proper class C of indiscernibles for M, a.

Then for every $\alpha < \omega^2$ all $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$–$\alpha$ games in trees of size λ are determined.

Proof. The theorem is related to Theorem 5.4.7 as Theorem 5.3.9 is related to Theorem 4.4.2. We could prove the theorem as Theorem 5.3.9 was proved, using semicoverings. Instead we proceed in a more direct manner, using a single auxiliary game. We will content ourselves with defining this auxiliary game, leaving to the reader the task of modifying the proof of Theorem 5.4.7 to show that the determinacy of the auxiliary game implies that of the given $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$–$\omega n$ game.

Let T be a game tree with field $(t) \subseteq \lambda$ and with no terminal nodes. Let $n \in \omega$. Let $\langle B_{\beta,e} \mid \beta < \omega n \wedge e \in \omega\rangle$ and $\langle p \mapsto <_{\beta,p} | \; \beta < \omega n\rangle$ witness that $A \in \mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$–$\omega n$. For $\beta < \omega n$ and for $e \in \omega$, let $p \mapsto <^e_{\beta,p}$ and $x \mapsto <^e_{\beta,x}$ be the functions given by Lemma 4.1.4, with $B_{\beta,e}$ as the A of that lemma. Let $a \subseteq \lambda$ be such that T, $\langle p \mapsto <_{\beta,p} | \; \beta < \omega n\rangle$, and $\langle p \mapsto <^e_{\beta,p} | \; e \in \omega\rangle$ all are definable from a in $L[a]$.

Let $M = L[a, \prec\!\langle \mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\rangle\!\succ]$ and $\langle \kappa_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ be as in the proof of Theorem 5.4.7. Let $\lambda_1 = \lambda^+$ and, for $1 \leq m < n$, let $\lambda_{m+1} = \lambda^+_m$. We may assume that there is a closed proper class C of indiscernibles for M, a such that $C \cap \lambda_m$ has order type λ_m for $1 \leq m \leq n$.

Let $g : \omega n \times \omega \to \omega$ be a bijection such that

(i) $(\forall m < n)(\forall k \in \omega)(\forall k' \in \omega)(k < k' \to g(\omega m + k, 0) < g(\omega m + k', 0))$;

(ii) $(\forall \beta < \omega n)(\forall e \in \omega)(\forall e' \in \omega)(e < e' \to g(\beta, e) < g(\beta, e'))$;

(iii) $(\forall \beta < \omega n)(\forall e \in \omega)(\beta \text{ even } \leftrightarrow g(\beta, e) \text{ even })$.

Let $h : \omega n \times \omega \times \omega \to \omega$ be a bijection such that

(i) $(\forall \beta < \omega n)(\forall e \in \omega)\, h(\alpha, e, 0) \geq g(\beta, e)$;

(ii) $(\forall \beta < \omega n)(\forall e \in \omega)(\forall m \in \omega)(\forall m' \in \omega)(m < m' \to h(\beta, e, m) < h(\beta, e, m'))$;

(iii) $(\forall \beta < \omega n)(\forall e \in \omega)(\forall m \in \omega)(\beta \text{ even } \leftrightarrow h(\beta, e, m) \text{ even })$.

We describe a game tree T^* by describing the plays in T^*:

$$\begin{array}{llllllll} I & \langle a_0, \xi_0\rangle & & \langle a_2, \eta_1, \xi_2\rangle & & \langle a_4, \eta_3, \xi_4\rangle & & \ldots \\ II & & \langle a_1, \eta_0, \xi_1\rangle & & \langle a_3, \eta_2, \xi_3\rangle & & \ldots & \end{array}$$

Each $\langle a_i \mid i < n\rangle$ must be a position in T. For all $m < n$, for all $k \in \omega$, and for all $e \in \omega$, $\xi_{g(\omega m+k,e)}$ must be an ordinal number smaller than λ_{m+1}. For all $\beta < \omega n$ and for all e and m, it must be that

$$\eta_{h(\beta,e,m)} < \kappa_{\xi_{g(\beta,e)}}.$$

Let $\pi : T^* \to T$ and the associated $\pi : [T^*]$ to $[T]$ be defined in the obvious way. (See the proof of Theorem 5.4.5.)

We define $A^* \subseteq [T^*]$ as follows. Fix a play

$$x^* = \langle a_0, \xi_0\rangle, \langle a_1, \eta_0, \xi_1\rangle, \ldots$$

in T^* and let $x = \pi(x^*) = \langle a_i \mid i \in \omega\rangle$. If there is no $\beta < \omega n$ such that either

(1) $(\exists e)(\exists e')(e <_{\beta,x} e' \wedge \xi_{g(\beta,e)} \not< \xi_{g(\beta,e')})$ or

(2) $(\exists e)(\exists m)(\exists m')(m <^e_{\beta,x} m' \wedge \eta_{h(\beta,e,m)} \not< \eta_{g(\beta,e,m')})$,

then $x \notin A^*$. Suppose that there is a β satisfying (1) or (2). Let

$$\begin{aligned} i &= \min\{\max\{g(\beta,e), g(\beta,e')\} \mid \beta, e, e' \text{ as in (1)}\}; \\ j &= \min\{\max\{h(\beta,e,m), h(\beta,e,m')\} \mid \beta, e, m, m' \text{ as in (2)}\}. \end{aligned}$$

In each case we declare the min be ω if it is otherwise undefined. If $i \leq j$, then $x \in A^*$ just in case i is odd. If $j < i$, then $x \in A^*$ just in case j is even.

Remark. In playing game $G(A^*; T^*)$, the players are simultaneously playing ωn copies of the game of the proof of Theorem 5.4.7, with I in the role of the first player for even numbered copies and II in that role for odd numbered copies. The first player to lose any one of these games is the player who loses the play of $G(A^*; T^*)$.

Since A^* is easily seen to be open, $G(A^*; T^*)$ is determined. We leave it to the reader to prove that whoever has a winning strategy for $G(A^*; T^*)$ also has a winning strategy for $G(A; T)$. The proof is an elaboration of the proof of Theorem 5.4.7 (which was a slight modification of the proof of Theorem 5.4.5). The extra ingredient is that moves $\xi_{g(\beta,e)}$ and $\eta_{h(\beta,e,m)}$ given by a winning strategy are independent of the opponent's moves $\xi_{g(\beta',e')}$ and $\eta_{h(\beta',e',m')}$ for $\beta' > \beta$, provided that these moves are in the right order and are from indiscernibles in the one case and measure one sets in the other. $\square$

Here is Simms' lightface version of Theorem 5.4.9.

Theorem 5.4.10. *If there is a transitive proper class model M of* ZFC + *"there are arbitrarily large measurable cardinals" and there is a proper class of indiscernibles for M, then for every $\alpha < \omega^2$ all $\Sigma^0_1(\Pi^1_1)$–α games are determined.*

As we have already said, the converses of Theorems 5.4.9 and 5.4.10 both hold. See Exercise 5.4.5. The theorems and their converses generalize to higher levels of the $(\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1))$–$\alpha$ hierarchy, with each ω^2 steps up this hierarchy corresponding to one more measurable limit of measurable cardinals. See Exercise 5.4.5 for a precise statement of the boldface version of this general theorem.

Exercises 5.4.2 and 5.4.3 concern a more interesting generalization by Simms of the equivalence results, with $(\alpha+1)$–$(\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1))$ replacing $\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1)$ and the αth level of the "measurable limit of" hierarchy replacing the 0-th level. Exercise 5.4.4 discusses a sharpening of this theorem.

No equivalence theorems for determinacy and large cardinals have been proved for classes strictly between Simms' and the class $\mathbf{\Delta}^1_2$. For countable ordinals α, define $\mathbf{\Sigma}^0_\alpha(\mathbf{\Pi}^1_1)$, $\mathbf{\Pi}^0_\alpha(\mathbf{\Pi}^1_1)$, and $\mathbf{\Delta}^0_\alpha(\mathbf{\Pi}^1_1)$ in the obvious way. John Steel and the author found a proof of the determinacy of $\mathbf{\Delta}^0_2(\mathbf{\Pi}^1_1)$ games in any given T from the existence of a cardinal $\kappa > |T|$ of level κ^+ in the "measurable limit of" hierarchy, but they did not prove an implication in the other direction. There are larger classes for each of which has been proved half of what should be an equivalence theorem. One such class is $\mathcal{A}(\mathbf{\Pi}^1_1)$, which we now define.

For a class Γ of subsets of a set X, a subset A of X belongs to $\mathcal{A}(\Gamma)$ just in case there are sets B_s, $s \in {}^{<\omega}\omega$, such that each $B_s \in \Gamma$ and

$$(\forall x \in X)(x \in A \leftrightarrow (\exists y \in {}^\omega\omega)(\forall n \in \omega)\, x \in B_s).$$

Steel [1982] proves that, if there is a transitive model of ZFC + "there is a cardinal κ of Mitchell order κ^{++}," then there is such a model in which not all $\mathcal{A}(\mathbf{\Pi}^1_1)$ games in countable trees are determined. "Mitchell order" is the order introduced in Mitchell [1974]. The hint to Exercise 5.4.4 indicates briefly the technique Steel uses to prove this and related results. From a combination Steel's theorem and a generalization in Mitchell [1992] of a result of Ronald Jensen, it follows that the determinacy of all $\mathcal{A}(\mathbf{\Pi}^1_1)$ games in countable trees implies the existence of a proper class of indiscernibles for a class model of ZFC + "there is a cardinal κ of Mitchell order κ^{++}." One suspects that the converse of this implication should also hold.

This conjecture about the determinacy of $\mathcal{A}(\mathbf{\Pi}^1_1)$ games is related to the question of whether every $\mathbf{\Pi}^1_1$ subset of $\lceil \mathbf{T} \rceil$ can be unraveled by a covering of $\mathbf{T}$. Suppose that, from some hypothesis H, one could prove such an unraveling theorem for arbitrary trees or that one could prove, for a some fixed tree $\mathbf{T}$, that any countable family of $\mathbf{\Pi}^1_1$ sets can be simultaneously unraveled by a covering of $\mathbf{T}$. Then one could prove from H that all games in the σ-algebra generated by the $\mathbf{\Pi}^1_1$ subsets of $\lceil \mathbf{T} \rceil$ are determined. Combining the unraveling proof with the ideas introduced in §2.2, one might in fact be able to prove the determinacy of $G(A; \mathbf{T})$ for all A such that both A and $\neg A$ are $\mathcal{A}(\mathbf{\Pi}^1_1)$. Should such unraveling theorems be expected? Mixing the techniques of Steel [1982] with those of Friedman [1971], Steel proved that, for positive integers n, if there is a transitive model of ZF $-$ Power Set + "there is a κ of Mitchell order κ^{++} such that $\mathcal{P}^n(\kappa)$ exists," then there is such a model in which not all all $\Sigma^0_{n+3}(\mathbf{\Pi}^1_1)$ games in countable trees are determined. This result has a generalization to the transfinite. Thus the situation looks similar to that for Borel games, and this makes one anticipate that unraveling theorems for $\mathbf{\Pi}^1_1$ subsets of $\lceil \mathbf{T} \rceil$ will be proved in a weak set theory from the assumption of the existence of a cardinal $\kappa > |T|$ of Mitchell order κ^{++}.

Exercise 5.4.1. Prove the converses of Theorems 5.4.9 and 5.4.10.

Hint. We consider the case of Theorem 5.4.10. The other case is similar, except that the method of Exercise 4.4.2 has to be used when λ is uncountable. We sketch Simms' original method of proof. The hint to Exercise 5.4.4 indicates a very different method for proving a stronger result.

Let $\varphi(v_1, \ldots, v_{\bar{n}})$ be a formula of the language of set theory. Consider the following game G in ${}^{<\omega}\omega$.

Let I's part of a play z give a relation r^z in ω, elements m^z_β, $\beta < \omega\bar{n}$, of ω, and elements $c^z_{i,\beta}$ $i \in \omega$ and $\beta < \omega^2$, of ω. Similarly, let II's part of the play give a relation s^z in ω, numbers n^z_β, $\beta < \omega\bar{n}$, of ω, and numbers $d^z_{i,\beta}$ $i \in \omega$ and $\beta < \omega^2$.

For $\beta < \omega\bar{n}$, if r^z wellorders the set of numbers that bear r^z to m^z_β, then let η^z_β be the order type of this wellordering. If r^z wellorders ω, let $\eta^z_{\omega\bar{n}}$ be the order type of this wellordering. For $i \in \omega$ and $\beta < \omega^2$, if r^z wellorders the set of numbers that bear r^z to $c^z_{i\beta}$, then let $\mu^z_{i,\beta}$ be the order type of this wellordering. Similarly define ξ^z_β, $\beta \leq \omega\bar{n}$, and $\nu^z_{i,\beta}$, $i \in \omega$ and $\beta < \omega^2$, from s^z and the n^z_β and the $d^z_{i,\beta}$ respectively.

For each i such that r^z wellorders the set of r^z-predecessors of i, let $f(i)$ be order type of this wellordering. Similarly define $g(i)$ from s^z and i.

The winning conditions for G are as follows:

(1) I loses unless all of the following hold.

(a) r^z is a linear ordering of ω.

(b) $(\forall\beta < \omega\bar{n})(\forall\gamma < \omega\bar{n})(\beta < \gamma \to m^z_\beta\, r^z\, m^z_\gamma)$.

(c) $(\forall i \in \omega)(\forall j \in \omega)(\forall\beta < \omega^2)(i\, r^z\, j \to c^z_{i,\beta}\, r^z\, c^z_{j,0})$.

(d) $(\forall i \in \omega)(\forall\beta < \omega^2)(\forall\gamma < \omega^2)(\beta < \gamma \to c^z_{i,\beta}\, r^z\, c^z_{i,\gamma})$.

(2) If I does not lose becouse of (1), then II loses unless all of the following hold.

(a) s^z is a linear ordering of ω.

(b) $(\forall\beta < \omega\bar{n})(\forall\gamma < \omega\bar{n})(\beta < \gamma \to n^z_\beta\, r^z\, n^z_\gamma)$.

(c) $(\forall i \in \omega)(\forall j \in \omega)(\forall\beta < \omega^2)(i\, r^z\, j \to d^z_{i,\beta}\, r^z\, d^z_{j,0})$.

(d) $(\forall i \in \omega)(\forall\beta < \omega^2)(\forall\gamma < \omega^2)(\beta < \gamma \to d^z_{i,\beta}\, r^z\, d^z_{i,\gamma})$.

(3) Assume that no one loses because of (1) or (2). Assume also that there is an ordinal α and there are numbers i and j such that both the set of r^z-predecessors of i and the set of s^z-predecessors of j have order type α and such that, for some $\beta < \omega^2$, at least one of $\mu^z_{i,\beta}$ and $\nu^z_{i,\beta}$ is undefined. Consider the least such α, along with the corresponding i and j. Consider the least β for this value of α. I loses if $\mu^z_{i,\beta}$ is undefined and II loses otherwise.

(4) Assume that no one loses because of (1), (2), or (3). Assume also that at least one of (ω, r^z) or (ω, s^z) is not a wellordering. Let $\rho \leq \omega\bar{n}$ be least such that one or the other of η^z_ρ and ξ^z_ρ is undefined. I loses if η^z_ρ is undefined, and II loses otherwise.

(5) Assume that no one loses because of (1)–(4). Let $f^z : (\gamma^z; <) \cong (\omega, r^z)$ and $g^z : (\delta^z; <) \cong (\omega; s^z)$ be isomorphisms. For $\alpha < \max\{\gamma^z, \delta^z\}$ and for $m \in \omega$, let

$$\rho^z_{\alpha,m} = \sup_{k\in\omega}\max\{f^z(\alpha), g^z(\alpha)\},$$

where we take undefined values to be 0. For such α, let $\kappa^z_\alpha = \sup_{\mathrm{m}}\rho^z_{\alpha,m}$ and let $\mathcal{V}^z_\alpha$ be the filter on κ^z_α generated by the tails of the sequence $\langle\rho^z_{\alpha,m} \mid m \in \omega\rangle$.

If there is an $\alpha < \max\{\gamma^z, \delta^z\}$ such that $\mathcal{V}^z_\alpha \cap L_{\max\{\gamma^z,\delta^z\}}[\langle\!\langle\mathcal{V}^z_\alpha \mid \alpha < \max\{\gamma^z, \delta^z\}\rangle\!\rangle]$ is not a normal ultrafilters $L_{\max\{\gamma^z,\delta^z\}}[\langle\!\langle\mathcal{V}^z_\alpha \mid \alpha < \max\{\gamma^z, \delta^z\}\rangle\!\rangle]$, then let α be least witnessing this and let $h : \kappa^z_\alpha \to \kappa^z_\alpha$ be the $L_{\max\{\gamma^z,\delta^z\}}[\langle\!\langle\mathcal{V}^z_\alpha \mid$

$\alpha < \max\{\gamma^z, \delta^z\}\rangle]$-least $h' : \kappa^z_\alpha \to \kappa^z_\alpha$ such that $\{\alpha \mid h'(\alpha) < \alpha\} \in \mathcal{V}^z_\alpha$ but h' is not constant on any set in $\mathcal{V}^z_\alpha$. I wins just in case $h(\rho^z_{\alpha,1}) > h(\rho^z_{\alpha,0})$.

For $j < n$, let $\zeta^z_j = \sup_{k\in\omega}\max\{\eta^z_{\omega j+k}, \xi^z_{\omega j+k}\}$. If all $\mathcal{V}^z_\alpha$ are normal ultrafilters, then I wins just in case

$$L_{\max\{\gamma^z,\delta^z\}}[\langle\mathcal{V}^z_\alpha \mid \alpha < \max\{\gamma^z, \delta^z\}\rangle] \models \varphi[\zeta^z_0, \ldots, \zeta^z_{n-1}].$$

Show that G is a $\Sigma^0_1(\Pi^1_1)$–$\omega\bar{n} + 1$ game. (This is a bit of work, because we formulated our definition of the hierarchy so as to facilitate the proof of Theorem 5.4.9. Other formulations would make that proof harder and the present computation easier.)

Let σ be a winning strategy for one of the players for G. Let $\langle\rho_\alpha \mid \alpha < \omega_1\rangle$ be any strictly increasing sequence of countable ordinals containing none of its own limit points and such that each ρ_α is admissible relative to $\langle\sigma, \langle\rho_\beta \mid \beta < \alpha\rangle\rangle$. Let $\langle\zeta_j \mid j \leq n\rangle$ be a strictly increasing sequence of ordinals such that each ζ_n is closed under $\alpha \mapsto \rho_\alpha$ and is admissbile relative to $\langle\sigma, \langle\rho_\alpha \mid \alpha < \zeta_n\rangle\rangle$. Use boundedness to show that there is a play z, consistent with σ such that $\zeta^z_j = \zeta_j$ for all $j < n$, $\max\{\gamma^z, \delta^z\} = \zeta_n$, and $\rho^z_\alpha = \rho_\alpha$ for all $\alpha < \zeta_n$. Now consider the model $L[\langle\mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\rangle]$, with each $\mathcal{V}_\alpha$ being the filter generated by the tails of $\langle\rho_{\omega\alpha+k} \mid k \in \omega\rangle$, where ρ_β is the βth σ-admissible. Use techniques from the hints to exercises in §3.3 to show that all $\mathcal{V}_\alpha$ are normal ultrafilters in this model and that the σ-admissible limits of σ-admissibles are indiscernible in the model with respect to φ. Since φ was arbitrary, this completes the proof.

Exercise 5.4.2. The results of this and the next exercise are due to Simms [1979].

For any ordinal number $\alpha > 0$, say that a cardinal κ is *measurable*$_\alpha$ if κ is measurable and, for each $\beta < \alpha$, the set of $\lambda < \kappa$ such that λ is measurable$_\beta$ is unbounded in κ.

Let λ be an infinite cardinal number. Let α be a countable ordinal. Assume that for every $a \subseteq \lambda$ there is a transitive class model M of ZFC such that

(i) for all $\beta < 1+\alpha$, the class of $\kappa > \lambda$ such that $M \models$ "κ is measurable$_\beta$" is unbounded in the ordinals;

(ii) $a \in M$ and there is a proper class of indiscernibles for M, a.

(a) Prove that all $(\alpha+1)$–$(\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1))$ games in trees of size λ are determined.

(b) Formulate and prove a theorem that is related to part (a) of the exercise as Theorem 5.4.9 is related to Theorem 5.4.7.

(c) Similarly formulate and prove the analogue of Theorem 5.4.10.

Hint. For (a), begin by proving the following normal form theorem. If $\alpha > 0$, T is a game tree without terminal positions, and $A \subseteq [T]$, then $A \in \alpha$–$\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1)$ if and only if there is a game tree $S \subseteq {}^{<\omega}\omega$ such that

(i) $\|S\| = 1+\alpha$;

(ii) for all $t \in S$ and all $e \in \omega$, $t^\frown\langle e\rangle \in S$ and $\|t^\frown\langle e\rangle\|^S$ is odd if and only if $\|t\|^S$ is even;

and, for every $t \in S$ such that t is not terminal in S, there is a function $p \mapsto <^t_p$ with domain T such that

(1) $<^t_\emptyset = \emptyset$ and, for all $p \in T \setminus \{\emptyset\}$, $<^t_p$ is a linear ordering of $\ell\mathrm{h}(p)$ with greatest element 0;

(2) for elements p and p' of T, if $p \subseteq p'$ then $<^t_p \subseteq <^t_{p'}$;

(3) for all $x \in [T]$, every non-zero member of ω has an immediate successor with respect to $<^t_x$, where

$$<^t_x = \bigcup_{n\in\omega} <_{x\restriction n};$$

(4) for all $x \in [T]$, whenever both $t^\frown\langle e\rangle$ and $t^\frown\langle e'\rangle$ belong to S and $\|t^\frown\langle e\rangle\|^S < \|t^\frown\langle e'\rangle\|^S$, then $e <^t_x e'$;

and such that, for all $x \in [T]$, $x \in A$ if and only if β_x is odd, where β_x is the least β such that either $\beta = \alpha$ or else there is a $t \in S$ with $\|t\|^S = \beta+1$ such that $<^t_x$ is not a wellordering but, for all n with $n < \ell\mathrm{h}(t)$, the (obvious) Brouwer-Kleene ordering restricted to $\{s \in S \mid s(n) < t(n)\}$ is a wellordering.

Now let $A \in (\alpha+1)$–$\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1)$. Let S and the $p \mapsto <^t_p$ be given by the normal form theorem. Consider an auxiliary game where I tries to embed $<^t_x$ into the ordinals for $\|t\|^S$ even and II for $\|t\|^S$ odd. Require that the ordinal assigned to e' in trying to embed $<^{t^\frown\langle e\rangle}_x$ be less than the ηth measurable$_{\|t^\frown\langle e'\rangle\|^S-1}$, where η is the ordinal the opponent has assigned to e in trying to embed $<^t_x$.

Exercise 5.4.3. Prove the converse of the result of part (c) of the result of Exercise 5.4.2.

Hint. The proof is analogous to that for Exercise 5.4.5, except that instead of just the $c^z_{i,\beta}$ and the $d^z_{i,\beta}$, one needs $c^z_{\gamma,i,\beta}$ and $d^z_{\gamma,i\beta}$ for each $\gamma < 1+\alpha$. When both players meet all wellfoundedness conditions, then define ordinals $\rho^z_{\gamma,\zeta,m}$ and the $\kappa^z_{\gamma,\zeta}$ in the obvious way. In the good case, $\kappa^z_{\gamma,\zeta}$ will be in the model the $1+\zeta$th measurable$_\gamma$ that is not measurable$_{\gamma+1}$. (What we are omitting from this hint is the winning conditions when wellfoundedness fails. This omission leaves the reader a genuine problem, though not a terribly difficult one.)

Exercise 5.4.4. (a) Prove the converse of Theorem 5.4.7.

(b) Prove the converse of the result of part (a) of Exercise 5.4.2.

Hint. (a) We drop the set a and assume that $\lambda = \omega$ in order to indicate the sense in which the method of proof we sketch gives a partial converse to Theorem 5.4.7. When λ is uncountable, use the methods of Exercise 4.4.2.

The plan is as follows.

(1) Assume that $\tilde{L} = L[\langle\mathcal{U}_\gamma \mid \gamma < \rho\rangle]$, where $\rho \leq \mathrm{Ord}$. Assume that $\langle\mathcal{U}_\gamma \mid \gamma < \rho\rangle$ is *coherent* in the sense of Mitchell [1974]. Assume finally that in $\tilde{L}$ there is no measurable limit of measurable cardinals. Show that the canonical wellordering $<_{\tilde{L}}$ of ${}^\omega\omega$ in $\tilde{L}$ is $\mathcal{G}$-$\Sigma_1^0(\Pi_1^1)$. That is, show that with there is a $\Sigma_1^0(\Pi_1^1)$ subset A of $({}^\omega\omega)^3$ such that, for all elements x and y of ${}^{<\omega}\omega$,

$$x <_{\tilde{L}} y \;\leftrightarrow\; I \text{ has a winning strategy for } G(\{z \mid \langle x, y, z\rangle \in A\}; {}^{<\omega}\omega).$$

Do this using a set A such that, for x and y belonging to $\tilde{L}$, the game $G(\{z \mid \langle x, y, z\rangle \in A\}; {}^{<\omega}\omega)$ has a winning strategy that belongs to $\tilde{L}$.

(2) Use the following result of Kechris [1978] to deduce that no $\tilde{L}$ as in (1) satisfies "All $\Sigma_1^0(\Pi_1^1)$ games are determined": *Let Γ be any class of subsets of ${}^\omega\omega$ closed under recursive preimages. If all Γ games are determined, then there is no $\mathcal{G}$-Γ wellordering of ${}^\omega\omega$.*

(3) Use the theorem of Mitchell [1992] on the $\mathbf{\Sigma}_3^1$ correctness of the core model do deduce the converse of Theorem 5.4.7.

Remark. Steel [1982] carries out steps (1) and (2). Step (3) had to await Mitchell's generalization of Ronald Jensen's Σ_3^1 correctness theorem for the classical core model (Jensen [1981]).

To accomplish (1), fix elements x and y of ${}^\omega\omega \cap M$ Consider the following game $G_{x,y}$ in ${}^{<\omega}\omega$.

To specify the winning conditions for $G_{x,y}$, let z be any play. Let I's part of z code a relation r^z in ω and let II's part code a relation s^z in ω. Player I loses unless $(\omega; r^z)$ is isomorphic to a transitive model $M^z = L[\langle\!\langle \mathcal{V}_\alpha \mid \alpha \leq \mu \rangle\!\rangle]$ of ZFC$^-$ + "$\langle \mathcal{V}_\alpha \mid \alpha < \mu\rangle$ is coherent" + "there is no measurable limit of measurable cardinals" such that $x <_{M^z} y$. If such an M^z exists, then II loses unless $(\omega; s^z)$ is isomorphic to a transitive model $N^z = L[\langle\!\langle \mathcal{W}_\alpha \mid \alpha \leq \nu \rangle\!\rangle]$ of the analogous theory and such that $y \leq_{N^z} x$.

Suppose both M^z and N^z exist. Then define $\langle M_\alpha, i_{\alpha\beta} \mid \alpha \leq \beta < \eta_I\rangle$ and $\langle N_\alpha, j_{\alpha\beta} \mid \alpha \leq \beta < \eta_{II}\rangle$ as follows. Assume that $\langle M_\alpha, i_{\alpha\beta} \mid \alpha \leq \beta < \xi\rangle$ and $\langle N_\alpha, j_{\alpha\beta} \mid \alpha \leq \beta < \xi\rangle$ have been defined and are such that

(i) $M_0 = M^z$ and $N_0 = N^z$;

(ii) all M_α and N_α are transitive;

(iii) each $i_{\alpha,\beta} : M_\alpha \prec M_\beta$ and each $j_{\alpha,\beta} : N_\alpha \prec N_\beta$;

(iv) for $\alpha \leq \beta \leq \gamma < \xi$, $i_{\alpha,\gamma} = i_{\beta,\gamma} \circ i_{\alpha,\beta}$ and $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$.

If ξ is a limit ordinal and the direct limit model of $\langle M_\alpha, i_\alpha\beta \mid \alpha \leq \beta < \xi\rangle$ is wellfounded, then let M_ξ be the transitive set isomorphic to it and define the $i_{\alpha,\xi}$ so as to preserve (iii) and (iv). Otherwise let $\xi = \eta_I$. Similarly define N_ξ and the $j_{\alpha,\xi}$ or let $\xi = \eta_{II}$.

Suppose $\xi = \alpha + 1$. Let $\langle \mathcal{V}^\alpha_\gamma \mid \gamma < i_{0,\alpha}(\mu)\rangle = i_{0,\alpha}(\langle \mathcal{V}_\gamma \mid \gamma < \mu\rangle)$ and similarly define $\langle \mathcal{W}^\alpha_\gamma \mid \gamma < j_{0,\alpha}(\nu)\rangle$. For $\gamma < i_{0,\alpha}(\mu)$, let κ^α_γ be the cardinal on which $\mathcal{V}^\alpha_\gamma$ is a normal ultrafilter. Similarly define λ^α_γ from $\mathcal{W}^\alpha_\gamma$ for $\gamma < j_{0,\alpha}(\nu)$. Let γ_α be the least γ such that $\mathcal{V}^\alpha_\gamma$ and $\mathcal{W}^\alpha_\gamma$ exist and

$$\mathcal{V}^\alpha_\gamma \cap M_\alpha \cap N_\alpha \neq \mathcal{W}^\alpha_\gamma \cap M_\alpha \cap N_\alpha.$$

(Show that the existence of such a γ is implied by the fact that $x <_{M_\alpha} y$ and $y \leq_{N_\alpha} x$.) If $\kappa^\alpha_{\gamma_\alpha} < \lambda^\alpha_{\gamma_\alpha}$, then let $M_\xi = \mathrm{Ult}(M_\xi; \mathcal{V}^\alpha_{\gamma_\alpha})$ and $N_\xi = N_\alpha$. If $\lambda^\alpha_{\gamma_\alpha} < \kappa^\alpha_{\gamma_\alpha}$, then let $M_\xi = M_\alpha$ and let $N_\xi = \mathrm{Ult}(N_\alpha; \mathcal{W}^\alpha_{\gamma_\alpha})$. If $\kappa^\alpha_{\gamma_\alpha} = \lambda^\alpha_{\gamma_\alpha}$, then let $M_\xi = \mathrm{Ult}(M_\xi; \mathcal{V}^\alpha_{\gamma_\alpha})$ and let $N_\xi = \mathrm{Ult}(N_\alpha; \mathcal{W}^\alpha_{\gamma_\alpha})$.

I wins if and only if there is a $\xi < \omega_1^{\mathrm{CK}}(z)$ such that M_α exists but N_α does not. In other words, I wins just in case $\eta_{II} < \min\{\eta_I, \omega_1^{CK}(z)\}$.

Verify that $\{\langle x, y, z\rangle \mid z$ is won by I in $G_{x,y}\}$ belongs to $\Sigma^0_1(\Pi^1_1)$.

Prove that I has a winning strategy for $G_{x,y}$ if $x <_{\tilde{L}} y$ and that II has a winning strategy for $G_{x,y}$ if $y \leq_{\tilde{L}} x$. Indeed, show that in each case it is a winning strategy to play a model isomorphic to an initial segment of $\tilde{L}$ to which both x and y belong. In the case that $x <_{\tilde{L}} y$, use the fact that $\tilde{L}$ has no measurable limits of measurables to show that no play against I's strategy has $\eta_{II} \geq \omega_1^{CK}(z)$.

Step (2) of the plan comes immediately from step (1) and Kechris' theorem.

For step (3), assume that there is no indiscernibles exist for any model of ZFC + "There is a proper class of measurable cardinals." Then Theorem 1.2 of Mitchell [1992] implies that if

$$(\forall x \in {}^{\omega}\omega)\, x^{\#} \text{ exists,}$$

then there is a class model like our $\tilde{L}$ above that is Σ_3^1 correct. By step (2), either some $x \in {}^{\omega}\omega$ has no $\#$ or else there is a $\Sigma_1^0(\Pi_1^1)$ game that is not determined in this class model and hence is not determined in V. Since the determinacy of just all $\mathbf{\Pi}_1^1$ games implies $(\forall x \in {}^{\omega}\omega)\, x^{\#}$ exists, our assumption gives the existence of an undetermined $\mathbf{\Sigma}_1^0(\mathbf{\Pi}_1^1)$ game.

(b) Proceed with steps (1)–(3) in the hint for (a), with the following changes. Instead of assuming that in $\tilde{L}$ there is no cardinal that is measurable$_1$, assume that in $\tilde{L}$ there is no cardinal that is measurable$_{1+\alpha}$. Assume that α is countable in M. Replace $\Sigma_1^0(\Pi_1^1)$ (in the general case, $\mathbf{\Sigma}_1^0(\mathbf{\Pi}_1^1)$) by $(\alpha+1)$–$\mathbf{\Pi}_1^0(\mathbf{\Pi}_1^1)$. Change the winning conditions for $G_{x,y}$ by making I's winning set be the $(\alpha+1)$–$\mathbf{\Pi}_1^0(\mathbf{\Pi}_1^1)$ set given by $\langle A_\beta \mid \beta \leq \alpha \rangle$, where the A_β are defined as follows. A play z belongs to A_0 just in case

$$\eta_I < \omega_1^{\mathrm{CK}}(z) \rightarrow \eta_{II} < \eta_I.$$

For odd β, $z \notin A_\beta$ just in case $\min\{\eta_{\mathrm{I}}, \eta_{\mathrm{II}}\} < \omega_1^{\mathrm{CK}}(\mathrm{z})$ or else there are β', ξ, and κ such that

(i) $\beta' \leq \beta$;

(ii) $\xi < \omega_1^{\mathrm{CK}}(z)$;

(iii) κ is an cardinal of N_ξ;

(iv) in N_ξ, κ is measurable$_{\beta'}$ but not measurable$_{\beta'+1}$;

(v) for all ξ' with $\xi < \xi' < \omega_1^{\mathrm{CK}}(z)$, $\lambda^{\xi'}_{\gamma_{\xi'}} < j_{\xi,\xi'}(\kappa)$;

(vi) In N_ξ, κ is the least measurable$_{\beta'}$ that is $\geq \lambda^{\xi}_{\gamma_\xi}$.

The definition for the case of even $\beta > 0$ is analogous, with the obvious replacements.

Remark. As with part (a) of the exercise, steps (1) and (2) were known to Steel at the time of Steel [1982].

Exercise 5.4.5. Prove the converses of Theorems 5.4.9 and 5.4.10.

Hint. We consider the case of Theorem 5.4.10. The other case is similar, except that the method of Exercise 4.4.2 has to be used when λ is uncountable. We sketch Simms' original method of proof. The hint to Exercise 5.4.4 indicates a very different method for proving a stronger result.

Let $\varphi(v_1, \ldots, v_{\bar{n}})$ be a formula of the language of set theory. Consider the following game G in ${}^{<\omega}\omega$.

Let I's part of a play z give $z(0)$, together with a relation r^z in ω and with elements $c^z_{i,\beta}$, $i \in \omega$ and $\beta < \omega^2$, of ω. Let II's part of the play give relations E^z and s^z in ω, together with numbers $d^z_{i,\beta}$, $i \in \omega$ and $\beta < \omega^2$.

For $i \in \omega$ and $\beta < \omega^2$, if r^z wellorders the set of numbers that bear r^z to $c^z_{i\beta}$, then let $\mu^z_{i,\beta}$ be the order type of this wellordering. For $i \in \omega$ and $\beta < \omega^2$, if s^z wellorders the set of numbers that bear s^z to $d^z_{i\beta}$, then let $\nu^z_{i,\beta}$ be the order type of this wellordering.

For each i such that r^z wellorders the set of r^z-predecessors of i, let $f^z(i)$ be order type of this wellordering. Similarly define $g^z(i)$ from s^z and i.

The winning conditions for G are as follows:

(1) I loses unless all of the following hold.

(a) r^z is a linear ordering of ω.

(b) $(\forall i \in \omega)(\forall j \in \omega)(\forall \beta < \omega^2)(i\, r^z\, j \rightarrow c^z_{i,\beta}\, r^z\, c^z_{j,0})$.

(c) $(\forall i \in \omega)(\forall \beta < \omega^2)(\forall \gamma < \omega^2)(\beta < \gamma \rightarrow c^z_{i,\beta}\, r^z\, c^z_{i,\gamma})$.

(2) If I does not lose because of (1), then II loses unless all of the following hold.

(a) s^z is a linear ordering of ω.

(b) $(\forall i \in \omega)(\forall j \in \omega)(\forall \beta < \omega^2)(i\, s^z\, j \rightarrow d^z_{i,\beta}\, s^z\, d^z_{j,0})$.

(c) $(\forall i \in \omega)(\forall \beta < \omega^2)(\forall \gamma < \omega^2)(\beta < \gamma \rightarrow d^z_{i,\beta}\, s^z\, d^z_{i,\gamma})$.

(3) Assume that no one loses because of (1) or (2). Assume also that there is an ordinal α and there are numbers i and j such that $f^z(i) = g^z(j) = \alpha$ and such that, for some $\beta < \omega^2$, at least one of $\mu^z_{i,\beta}$ and $\nu^z_{i,\beta}$ is undefined. Consider the least such α, along with the corresponding i and j. Consider the least β for this value of α. I loses if $\mu^z_{i,\beta}$ is undefined, and II loses otherwise.

(4) Assume that no one loses because of (1), (2), or (3). If $(\omega; r^z)$ is not a wellordering, then I loses.

(5) Assume that no one loses because of (1)–(4). If $(\omega; s^z)$ is not a wellordering whose order type is $\geq$ the order type of $(\omega; r^z)$, then II loses.

(6) Assume that no one loses because of (1)–(5). Then there are γ^z and δ^z such that $f^z : (\omega; r^z) \cong (\gamma^z; <)$ and $g^z : (\omega; s^z) \cong (\delta^z; s^z)$. For $\alpha < \gamma^z$ and for $m \in \omega$, let

$$\rho^z_{\alpha,m} = \sup_{k\in\omega}\max\{\mu^z_{(f^z)^{-1}(\alpha),\omega m+k}, \nu^z_{(g^z)^{-1}(\alpha),\omega m+k}\}.$$

For $\alpha < \gamma^z$, let $\kappa^z_\alpha = \sup_{m\in\omega}\rho^z_{\alpha,m}$ and let $\mathcal{V}^z_\alpha$ be the filter on κ^z_α generated by the tails of the sequence $\langle \rho^z_{\alpha,m} \mid m \in \omega\rangle$.

If there is an $\alpha < \gamma^z$ such that $\mathcal{V}^z_\alpha \cap L_{\gamma^z}[\prec\mathcal{V}^z_\alpha \mid \alpha < \gamma^z\succ]$ is not a normal ultrafilter in $L_{\gamma^z}[\prec\mathcal{V}^z_\alpha \mid \alpha < \gamma^z\succ]$, then let α^z be least witnessing this and let $h^z : \kappa^z_{\alpha^z} \to \kappa^z_{\alpha^z}$ be the $<_{L_{\gamma^z}[\prec\mathcal{V}^z_\alpha|\alpha<\gamma^z\succ]}$-least $h : \kappa^z_{\alpha^z} \to \kappa^z_{\alpha^z}$ such that $\{\gamma \mid h(\gamma) < \gamma\} \in \mathcal{V}^z_{\alpha^z}$ but h is not constant on any set in $\mathcal{V}^z_{\alpha^z}$. I wins just in case both

(a) $(\forall n \in \omega)\, h^z(\rho^z_{\alpha,n}) < \rho^z_{\alpha,n}$;

(b) $h(\rho^z_{\alpha,1}) > h(\rho^z_{\alpha,0}) \leftrightarrow z(0) > 0$.

(7) Assume that no one loses because of (1)–(6). Then II wins if and only if $(\omega; E)$ is a model of Extensionality and there is a

$$k : L_{\gamma^z}[\prec\mathcal{V}^z_\alpha \mid \alpha < \gamma^z\succ] \to \omega$$

that embeds $L_{\gamma^z}[\prec\mathcal{V}^z_\alpha \mid \alpha < \gamma^z\succ]$ into $(\omega; E^z)$ as an initial segment.

Show that G is a $\Sigma^0_1(\Pi^1_1)$ game.

Assume that there is a winning strategy σ for I for G. Let $\langle \xi_\alpha \mid \alpha < \omega_1\rangle$ be any strictly increasing sequence of countable ordinals containing none of its own limit points and such that each ξ_α is admissible relative to $\langle \sigma, \langle \xi_\beta \mid \beta < \alpha\rangle\rangle$. Let $\zeta < \omega_1$ be any countable limit point of $\{\xi_\alpha \mid \alpha < \omega_1\}$ such that ζ is admissible relative to $\langle \sigma, \langle \xi_\alpha \mid \alpha < \zeta\rangle\rangle$. Use boundedness to show that there is a play z consistent with σ such that no one loses because of

(1)–(5), $\gamma^z < \delta^z = \zeta$, $\rho^z_{\alpha,m} = \xi_{\omega\alpha+m}$ for all $\alpha < \gamma^z$ and all $m \in \omega$, and $(\omega; E^z) \cong L_{\gamma^z}[\{\mathcal{V}^z_\alpha \mid \alpha < \gamma^z\}]$. Use techniques from the hint to Exercise 5.3.4 to derive a contradiction.

By the determinacy hypothesis, let τ be a winning strategy for II for G. Let $a \in \omega\omega$ code τ.

Let $\langle \xi_\alpha \mid \alpha < \omega_1 \rangle$ enumerate in order of magnitude the countable a-admissibles that are not limits of a-admissibles. Let ζ be any a-admissible limit of a-admissibles. Show that there is a play z consistent with τ such that no one loses because of (1)–(5), $\gamma^z = \zeta$, and $\rho^z_{\alpha,m} = \xi_{\omega\alpha+m}$ for all $\alpha < \gamma^z$ and all $m \in \omega$. For $\alpha \in \mathrm{Ord}$, let $\kappa_\alpha = \sup_{m\in\omega}\xi_{\omega\alpha+m}$ and let $\mathcal{V}_\alpha$ be the filter on κ_α generated by the tails of $\langle \xi_{\omega\alpha+n} \mid m \in \omega\rangle$.

Use techniques from the hint to Exercise 5.3.4 to argue that $L_{\omega_1}[\{\mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\}] \models$ "$\mathcal{V}_\alpha$ is a normal ultrafilter on κ_α" for all $\alpha < \omega_1$ and so that $L[\{\mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\}] \models$ "$\mathcal{V}_\alpha$ is a normal ultrafilter on κ_α" for all $\alpha \in \mathrm{Ord}$.

The rest of the proof is an adaptation of the proof outlined in the hint for Exercise 4.4.1.

First show that that the existence of indiscernibles for $L[\{\mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\}]$ is implied by the assertion that, if λ is any a-admissible limit of a-admissibles and $\gamma < \lambda$, then every subset of γ that belongs to $L[\{\mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\}]$ belongs to $L_\lambda[a]$. To do this, follow closely the first part of the hint for Exercise 4.4.1 to get an uncountable set of indiscernibles for $L[\{\mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\}]$. To show that this gives a proper class of indiscernibles for the model, use techniques from §1.4 and from Mitchell [1974].

For $\beta < \omega_1$, say that a play z of G is *β-good* if (a) z is consistent with τ, (b) clause (1) does not cause z to be a loss for I, (c) the relation r^z is a wellordering of ω of order type β, and (d) the sequence $\langle \mu^z_{\alpha,\omega m+k} \mid k \in \omega\rangle$ has limit $\xi_{\omega\alpha+m}$ for every $\alpha < \beta$ and every $m \in \omega$.

Show that if $\gamma < \beta < \omega_1$, if b is a subset of γ belonging to L_β, and if z is a β-good play of G, then $b \in L_{\gamma+\omega}[z]$.

For each countable ordinal α let the partial ordering $(\mathbf{Q}(\alpha); \leq_\alpha)$ be defined as in the hint to Exercise 4.4.1.

For each countable ordinal α, let $(\mathbf{Q}^*(\alpha); \leq^*_\alpha)$ be the following partial ordering: The members of $\mathbf{Q}^*(\alpha)$ are those triples $\langle t, h, k\rangle$ such that

(i) $\langle t, h\rangle \in \mathbf{Q}(\alpha)$;

(ii) k is a function with domain t;

(iii) for each $s \in t$ such that $h(s) = \infty$, $k(s) \in \mathbf{Q}(\xi_{\omega\alpha})$;

(iv) for each $s \in t$ such that $h(s) < \omega\xi$, $k(s)$ is a pair $\langle \bar{t}, \bar{h}\rangle$ that satisfies the conditions for belonging to $\mathbf{Q}(\xi_{\omega\alpha})$ with condition (iii) replaced by the requirement that $\bar{h}(\emptyset) = \xi_{h(s)}$.

Let $\langle t, h, k\rangle \leq^*_\alpha \langle t', h', k'\rangle$ hold if and only if $\langle t, h\rangle \leq_\alpha \langle t, h'\rangle$ and, for all $s \in t'$, if $k(s)$ is $\langle \bar{t}, \bar{h}\rangle$ and $k'(s)$ is $\langle \bar{t}', \bar{h}'\rangle$, then $\bar{t}' \subseteq \bar{t} \wedge \bar{h} \restriction \bar{t}' = \bar{h}'$.

The definition in the hint to Exercise 4.4.1 of the operation $\langle p, \xi\rangle \mapsto p(\xi)$ makes sense whenever $p = \langle t, h\rangle$ satisfies conditions (i), (ii), and (iv) for membership in $\mathbf{Q}(\alpha)$ for some ordinal $\alpha \geq \xi$. Let us regard the definition as applying to all such p and ξ.

If $p = \langle t, h, k\rangle \in \mathbf{Q}^*(\alpha)$ and $\alpha' \leq \alpha$, define $p[\alpha'] \in \mathbf{Q}^*(\alpha')$ by setting $p[\alpha'] = \langle t, h', k'\rangle$, where $\langle t, h'\rangle = \langle t, h\rangle(\alpha')$ and where $k'(s)$ is $(k(s))(\xi_{\omega\alpha'})$ for each $s \in t$.

Prove the analogue of the assertion $(*)$ in the hint to Exercise 4.4.1.

Define a class $\mathcal{S}^*$ of *ranked sentences* by imitating the definition of the class $\mathcal{S}^*$ in the hint to Exercise 4.4.1, except for clause (a), which should now be

(a) If $s \in {}^{<\omega}\omega$ then $\boldsymbol{s} \in \boldsymbol{T}$ is a ranked sentence of rank 1 and, if $s' \in {}^{<\omega}\omega$, then $\boldsymbol{s'} \in \boldsymbol{T^s}$ is a ranked sentence of rank 1.

Define a *forcing relation* $\Vdash^*_\alpha$ between elements of $\mathbf{Q}^*(\alpha)$ and elements of $\mathcal{S}^*$ by imitating the definition of $\Vdash_\alpha$ in the hint to Exercise 4.4.1, with the necessary modification of clause (a).

Prove the analogue of the assertion (†) in the hint to Exercise 4.4.1.

Let $\lambda < \omega_1$ be a-admissible and a limit of a-admissibles. Assume for a contradiction that there is are $\gamma < \lambda$ and $\beta < \omega_1$ and there is a set $b \in L_\beta[\langle \mathcal{V}_\alpha \mid \alpha \in \mathrm{Ord}\rangle]$ such that $b \subseteq \gamma$ and $b \notin L_\lambda[a]$. Let $\mathbf{G}$ be $\mathbf{Q}^*(\beta+1)$ generic over $L_{\beta+\omega}[a]$ $(= L_{\omega\beta+\omega}[a])$. Let $\langle T, H, K\rangle$ be the obvious triple given by $\mathbf{G}$. If K is the function $s \mapsto \langle T^s, H^s\rangle$, let K_0 be the function $s \mapsto T^s$. Show that there is a β-good play of G that is recursive in $\langle a, T, K_0\rangle$. From this it follows that $b \in L_{\gamma+\omega}[a, T, K_0]$. Imitate the last part of the hint to Exercise 4.4.1 and derive the contradiction that $b \in L_{\gamma+\omega}[a]$.

Let $\alpha < \omega_1$ be a-admissible and a limit of a-admissibles. Assume for a contradiction that α is not a cardinal in $L[\langle \mathcal{V}_\xi \mid \xi \in \mathrm{Ord}\rangle]$. Then there are ordinals $\gamma < \alpha$ and $\beta < \omega_1$ and there is a set $b \in L_\beta[\langle \mathcal{V}_\xi \mid \xi \in \mathrm{Ord}\rangle]$ such that $b \subseteq \gamma$ and b codes a wellordering of γ of order type α. Let $\mathbf{G}$ be $\mathbf{Q}(\beta+1)$-generic over $L_{\omega\beta+\omega}[a]$. Let $\langle T, H\rangle$ be given by $\mathbf{G}$. There is an $s \in T$ such that $\|s\|^T = \beta$. Thus there is an $s' \in T$ such that $<^{\mathrm{BK}} \restriction s'$ has order type β.

Let z be the play consistent with τ in which $r^z =<^{\mathrm{BK}} \restriction s'$,

$$c^z_{i,v} = \text{the unique } j \text{ such that} h(j) = \delta^z_{\omega^2 h(i)+\beta},$$

and $z(1) =$, say, 0.

As in Exercise 4.4.1, $b \in L_{\gamma+\omega}[z]$. Hence $b \in$

Chapter 6

Woodin Cardinals

The main goal of this chapter is to introduce Woodin cardinals and prove the consequences of their existence that will be used in the determinacy proofs of Chapter 8. Along the way we give a general survey of large cardinal properties stronger than measurability. The chapter can be read by anyone who has read the first three sections of Chapter 3.

Woodin cardinals and certain other large cardinals cannot be characterized in terms of individual ultrafilters but only in terms of systems of ultrafilters. These systems are called *extenders*, and we will introduce and study them in §1. Extenders will play a central role in Chapters 7 and 8. In §2 we introduce a variety of strong large cardinal axioms and we relate them to one another and to ultrafilters and extenders. We also prove Kunen's results on the limits of large cardinal axioms. Woodin cardinals are introduced in §2, but they are not singled out for special attention. Section 3 is devoted to some of the basic theory of Woodin cardinals. It ends with a technical result that will be an important tool in Chapter 8.

It is possible to proceed directly from §6.1 to Chapter 7. In a sense this is a more logical order than the order of the book. The concepts and theorems of Chapter 7 do not depend on the material in §6.2–3, and the technical result of §6.3 mentioned above will be used only to construct iteration trees, which are the subject matter of Chapter 7. We chose the actual order only because it put what seemed less technical material first.

6.1 Extenders

Suppose that $j : V \prec M$ with M transitive and $\text{crit}\,(j) = \kappa$. In the proof of the (c) $\Rightarrow$ (a) part of Theorem 3.2.12, it is shown that if

$$\mathcal{U} = \{X \subseteq \kappa \mid \kappa \in j(X)\},$$

then $\mathcal{U}$ is a κ-complete non-principal ultrafilter on κ (and the proof of Lemma 3.2.13 shows further that $\mathcal{U}$ is normal). The following lemma gives a general version of this construction of an ultrafilter from such an embedding j.

Lemma 6.1.1. *Let $j : V \prec M$ with M transitive and* $\text{crit}\,(j) = \kappa$. *Let $y \in M$. Let A be any set such that $y \in j(A)$.*

$$\mathcal{U} = \{X \subseteq A \mid y \in j(X)\}.$$

Then

(i) $\mathcal{U}$ is a κ-complete ultrafilter on A;

(ii) $\mathcal{U}$ is principal if and only if $y \in$ range (j).

Proof. (i). The proof is very much like that of the (c) $\Rightarrow$ (a) part of Theorem 3.2.12. Since $y \in j(A)$, the definition of $\mathcal{U}$ gives that $A \in \mathcal{U}$. By the elementarity of j, we have that $j(\emptyset) = \emptyset$ and so that $y \notin j(\emptyset)$. Thus $\mathcal{U}$ satisfies clause (a) in the definition of a filter. The elementarity of j also gives that $j(X \cap Y) = j(X) \cap j(Y)$, that $X \subseteq Y \rightarrow j(X) \subseteq j(Y)$, and that $j(A \setminus X) = j(A) \setminus j(X)$; therefore $\mathcal{U}$ satisfies clauses (b), (c), and (d) in the definition of an ultrafilter. To verify the κ-completeness of $\mathcal{U}$, let $\delta < \kappa$ and let $X = \langle X_\gamma \mid \gamma < \delta\rangle$ be a sequence of elements of $\mathcal{U}$. The elementarity of j and the fact that $\delta < \text{crit}\,(j)$ yield that

$$j(\bigcap_{\gamma<\delta} X_\gamma) = \bigcap_{\gamma<j(\delta)} (j(X))_\gamma = \bigcap_{\gamma<\delta} j(X_\gamma).$$

But $y \in \bigcap_{\gamma<\delta} j(X_\gamma)$, so $\bigcap_{\gamma<\delta} X_\gamma \in \mathcal{U}$.

(ii). The ultrafilter $\mathcal{U}$ is principal if and only if there is an $a \in A$ such that $\{a\} \in \mathcal{U}$. But $\{a\} \in \mathcal{U}$ if and only if $y \in j(\{a\}) = \{j(a)\}$, i.e. if and only if $y = j(a)$. $\square$

Many large cardinal properties are like measurability in that they can be formulated in two basic ways: in terms of ultrafilters and in terms of elementary embeddings. In some cases an elementary embedding corresponds to a single ultrafilter, as is the case for measurabilty, but in other important cases an elementary embedding corresponds to whole system of ultrafilters. Such systems of ultrafilters were first studied in [Mitchell, 1979], and a refinement of Mitchell's concept was formulated by Dodd and Jensen. (See [Dodd, 1982].) This refinement is the notion of an *extender*, to which we now turn. All the results of this section were known to Dodd and Jensen and, in a different form, to Mitchell.

First we introduce a standard item of notation, related to the notation $[z]^\gamma$. For sets z and cardinals γ, define

$$[z]^{<\gamma} = \{x \subseteq z \mid |x| < \gamma\}.$$

Let $j : V \prec M$ with M transitive and $\operatorname{crit}(j) = \kappa$. Let λ be an ordinal number with $\kappa < \lambda \leq j(\kappa)$. The (κ, λ)-*extender derived from* j is the system

$$\langle E_a \mid a \in [\lambda]^{<\omega}\rangle,$$

where the E_a are defined by

$$E_a = \{X \subseteq [\kappa]^{|a|} \mid a \in j(X)\}.$$

To state the next lemma, we state a useful convention and a few definitions. First the convention: If $n \in \omega$ and $z \in [\mathrm{Ord}]^n$ then we write z_i for the ith member of z in order of magnitude, that is, $z = \{z_1, \dots, z_n\}$ with $z_1 < \cdots < z_n$. To give the definitions, let us fix $n \in \omega$, $b \in [\mathrm{Ord}]^n$, and $a \subseteq b$. Let $a = \{b_{i_1}, \dots, b_{i_k}\}$, with $i_1 < \cdots < i_k$. For $z \in [\mathrm{Ord}]^n$ set

$$z_{a,b} = \{z_{i_1}, \dots, z_{i_k}\}.$$

For $\alpha \in \mathrm{Ord}$ and $X \subseteq [\alpha]^k$, define $X^{a,b}_\alpha \subseteq [\alpha]^n$ by

$$X^{a,b}_\alpha = \{z \mid z_{a,b} \in X\}.$$

Similarly, for $\alpha \in \mathrm{Ord}$ and $f : [\alpha]^k \to V$, define $f^{a,b} : [\alpha]^n \to V$ by

$$f^{a,b}(z) = f(z_{a,b}).$$

Before proceeding to the lemma, let us introduce one more piece of notation. For any function f and any set $x \subseteq \text{domain}(f)$ let

$$f''x = \text{range}(f \upharpoonright x).$$

We have earlier in the book written "$f(x)$" for $f''x$, but from now on we will reserve "$f(x)$" for the value of f on the argument x.

Lemma 6.1.2. *Let $j : V \prec M$ with M transitive and $\text{crit}(j) = \kappa$. Let $\kappa < \lambda \leq j(\kappa)$ and let $E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ be the (κ, λ)-extender derived from j. Then E has the following properties:*

(1) For each $a \in [\lambda]^{<\omega}$, E_a is a κ-complete ultrafilter on $[\kappa]^{|a|}$, and E_a is principal if and only if $a \subseteq \kappa$.

(2) (Compatibility) *If $a \subseteq b \in [\lambda]^{<\omega}$ and $X \in E_a$, then $X_\kappa^{a,b} \in E_b$.*

(3) (Normality) *Let $a \in [\lambda]^{<\omega}$. Let $f : [\kappa]^{|a|} \to \kappa$ and $i \leq |a|$ be such that*

$$\{z \mid f(z) < z_i\} \in E_a.$$

Then there is a $\beta < a_i$ such that

$$\{z \in [\kappa]^{|a \cup \{\beta\}|} \mid f(z_{a, a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}},$$

where $\beta = (a \cup \{\beta\})_k$.

(4) (Countable Completeness) *Let $\langle a_i \mid i \in \omega \rangle$ be such that each $a_i \in [\lambda]^{<\omega}$. Let $X_i \in E_{a_i}$ for each $i \in \omega$. Then there is an order preserving $h : \bigcup_{i\in\omega} a_i \to \kappa$ such that $h''a_i \in X_i$ for all $i \in \omega$.*

Proof. The first assertion of (1) follows directly from part (i) of Lemma 6.1.1. Since if $a \subseteq j(\kappa)$ but $a \not\subseteq \kappa$ then $a \notin \text{range}(j)$, the second assertion of (1) follows from part (ii) of Lemma 6.1.1.

For (2), let $n \in \omega$, let $a \subseteq b \in [\lambda]^n$, and let $X \in E_a$. By definition, we have that $a \in j(X)$. Now $a = b_{a,b}$, so $b_{a,b} \in j(X)$. But this just means that $b \in (j(X))_{j(\kappa)}^{a,b} = j(X_\kappa^{a,b})$ and so that $X_\kappa^{a,b} \in E_b$.

Let a, i, and f be as in the hypothesis of (3). By the definition of E_a, we get that $a \in \{z \in [j(\kappa)]^{|a|} \mid (j(f))(z) < z_i\}$. Hence $(j(f))(a) < a_i$. Let $\beta = (j(f))(a)$. Let k be such that $(a \cup \{\beta\})_k = \beta$. Then

$$(j(f))(a) = \beta = (a \cup \{\beta\})_k.$$

By the definition of $E_{a\cup\{\beta\}}$, we have that

$$\{z \in [\kappa]^{|a\cup\{\beta\}|} \mid f(z_{a,a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}}.$$

Let $a = \langle a_i \mid i \in \omega\rangle$ and $X = \langle X_i \mid i \in \omega\rangle$ be as in the hypotheses of (4). Let $b = \bigcup_{i\in\omega} a_i$. If b is finite, then $(j \restriction b)^{-1} : j(b) \to b$ witnesses that

$$\begin{aligned} M \ \models\ & (\exists h)(h : j(b) \to j(\kappa) \wedge h \text{ is order preserving} \\ & \qquad \wedge (\forall i \in \omega)\, h''(j(a))_i \in (j(X))_i). \end{aligned}$$

The desired conclusion follows by the elementarity of j. If b is infinite, then we cannot assume that $j \restriction b \in M$, and so we do not know that its inverse belongs to M. Instead we let

$$\begin{aligned} U = \{s \mid (\exists k \in \omega)(s : {\textstyle\bigcup_{i\leq k}} a_i \to \kappa \wedge s \text{ is order preserving} \\ \wedge (\forall i \leq k)\, s''a_i \in X_i)\}. \end{aligned}$$

If s and t belong to U define

$$s \prec t \leftrightarrow s \supsetneq t.$$

The inverse of $j \restriction b$ witnesses that $j(\prec)$ is not wellfounded in V. The absoluteness of wellfoundedness implies that $j(\prec)$ is not wellfounded in M. The elementarity of j then implies that $\prec$ is not wellfounded. If $\langle s_i \mid i \in \omega\rangle$ is an infinite descending sequence in $\prec$, then $\bigcup_{i\in\omega} s_i$ is our desired h. $\square$

If κ is an uncountable cardinal number and $\lambda > \kappa$ is an ordinal number, then a (κ, λ)-*extender* is a system $\langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ that satisfies clauses (1)–(4) of Lemma 6.1.2. An *extender* is anything that is a (κ, λ)-extender for some pair $\langle \kappa, \lambda\rangle$.

Remarks.

(1) There can be a (κ, λ)-extender only if κ is a measurable cardinal. A $(\kappa, \kappa + 1)$-extender is essentially a uniform normal ultrafilter on κ. See Exercise 6.1.1.

(2) There is no real reason for the demand that $\lambda \leq j(\kappa)$ in order for the (κ, λ)-extender derived from j to be defined. Removing this requirement would give us ultrafilters E_a that are not all on $[\kappa]^{<\omega}$. See Exercise 6.1.2.

Let $E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ be a (κ, λ)-extender. We define

$$\prod_E (V; \in),$$

which we call the *ultrapower* of $(V; \in)$ with respect to E, even though it is not literally an ultrapower.

Let

$$\mathcal{D}_E = \{\langle a, f\rangle \mid a \in [\lambda]^{<\omega} \wedge f : [\kappa]^{|a|} \to V\}.$$

If $\langle a, f\rangle$ and $\langle b, g\rangle$ are elements of $\mathcal{D}_E$, define

$$\langle a, f\rangle \sim_E \langle b, g\rangle \leftrightarrow \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) = g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

It is easily verified that $\sim_E$ is an equivalence relation on the class $\mathcal{D}_E$. For $\langle a, f\rangle \in \mathcal{D}_E$, let $[\![a, f]\!]_E$ be the set of all elements of minimal rank belonging to the equivalence class of $\langle a, f\rangle$. (We will omit the subscript "E" when there is no danger of confusion.) The universe of our ultrapower $\prod_E(V; \in)$ is the class of all the $[\![a, f]\!]$ for $\langle a, f\rangle \in \mathcal{D}_E$. The relation, which we write $\in_E$, is given by

$$[\![a, f]\!]_E \in_E [\![b, g]\!]_E \leftrightarrow \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) \in g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

Remark. An alternative way of defining $\prod_E(V; \in)$ is as a direct limit of the ordinary ultrapowers $\prod_{E_a}(V; \in)$. See Exercise 6.1.3.

Theorem 6.1.3. *Let $E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ be a (κ, λ)-extender. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1\rangle, \ldots, \langle a_n, f_n\rangle$ be elements of $\mathcal{D}_E$. Let $b = \bigcup_{1\leq i\leq n} a_i$. Then*

$$\begin{aligned} &\textstyle\prod_E(V; \in) \models \varphi[[\![a_1, f_1]\!], \ldots, [\![a_n, f_n]\!]] \leftrightarrow \\ &\quad \{z \in [\kappa]^{|b|} \mid (V; \in) \models \varphi[f_1(z_{a_1,b}), \ldots, f_n(z_{a_n,b})]\} \in E_b. \end{aligned}$$

Proof. The proof is similar to those of Theorem 3.2.1 and Theorem 3.2.5, and we omit it.

As with ordinary ultrapowers, we get a canonical elementary embedding which we call i'_E of $(V; \in)$ into $\prod_E(V; E)$, where i'_E is defined by, e.g.,

$$i'_E(x) = [\![\emptyset, c_x]\!]_E.$$

Here $\emptyset$ could be replaced by any other $a \in [\lambda]^{<\omega}$ without affecting the definition.

Lemma 6.1.4. *If E is an extender then $\prod_E(V;\in)$ is wellfounded.*

Proof. Let $E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ be an extender and suppose for a contradiction that

$$\cdots \in_E [\![a_2, f_2]\!] \in_E [\![a_1, f_1]\!] \in_E [\![a_0, f_0]\!].$$

Replacing each a_i by $\bigcup_{i' \leq i} a_{i'}$ and each f_i by $f_i^{a_i, \cup_{i' \leq i} a_i}$, we may assume that $a_0 \subseteq a_1 \subseteq \ldots$. Let $X_0 = [\kappa]^{|a_0|}$ and for each i let

$$X_{i+1} = \{z \in [\kappa]^{|a_{i+1}|} \mid f_{i+1}(z) \in f_i(z_{a_i, a_{i+1}})\}.$$

For each $i \in \omega$, we have that $X_i \in E_{a_i}$. By countable completeness (property (4) of extenders), let $h : \bigcup_{i\in\omega} a_i \to \kappa$ be order preserving and such that $h''a_i \in X_i$ for all i. We get the contradiction that

$$(\forall i \in \omega)\, f_{i+1}(h''a_{i+1}) \in f_i(h''a_i). \qquad \square$$

The wellfoundedness of $\prod_E(V;\in)$ is actually equivalent with the countable completeness of E:

Lemma 6.1.5. *If $E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ has properties (1)–(3) of (κ,λ)-extenders, then E is countably complete (i.e., E is a an extender) if and only if $\prod_E(V;\in)$ is wellfounded.*

Proof. The "only if" part of the corollary is Lemma 6.1.4.

For the "if" part, assume that $\langle a_i \mid i \in \omega\rangle$ and $\langle X_i \mid i \in \omega\rangle$ are a counterexample to the countable completeness of E. Replacing, if necessary, each a_k by $\bigcup_{i\leq k} a_i$ and X_k by $(X_k)_\kappa^{a_k, \cup_{i\leq k} a_i}$, we may assume that $a_i \subseteq a_k$ for all $i \leq k \in \omega$. Next replacing, if necessary, each X_k by $\bigcap_{i\leq k}(X_k)_\kappa^{a_i,a_k}$, we may assume that

$$(\forall z \in X_k)(\forall i \leq k)\, z_{a_i,a_k} \in X_i.$$

As in the proof of Lemma 6.1.2, let

$$\begin{aligned} U = \{s \mid (\exists k \in \omega)(s : a_k \to \kappa \wedge s \text{ is order preserving} \\ \wedge\, s''a_k \in X_k)\} \end{aligned}$$

and define, for elements s and t of U,

$$s \prec t \leftrightarrow s \supsetneq t.$$

To say that $\langle a_i \mid i \in \omega\rangle$ and $\langle X_i \mid i \in \omega\rangle$ are a counterexample to the countable completeness of E is just to say that $\prec$ is wellfounded. Define $\| \; \|^{\prec} : U \to \mathrm{Ord}$ by induction on $\prec$ as follows.

$$\|s\|^{\prec} = \sup\{\|t\|^{\prec} + 1 \mid t \prec s\}.$$

For each $k \in \omega$ and each $z \in X_k$, there is a unique $s_z \in U$ such that $z = s_z{}''a_k$. For $k \in \omega$, define

$$f_k : X_k \to \mathrm{Ord}$$

by

$$f_k(z) = \|s_z\|^{\prec}.$$

For each k and each $z \in X_{k+1}$,

$$f_{k+1}(z) = \|s_z\|^{\prec} > \|s_z \restriction a_k\|^{\prec} = \|s_{z_{a_k,a_{k+1}}}\|^{\prec} = f_k(z_{a_k,a_{k+1}}).$$

Hence the $[\![a_k, f_k]\!]$ witness that $\prod_E(V;\in)$ is not wellfounded. $\square$

Lemma 6.1.6. *If E is an extender then $\prod_E(V;\in)$ is set-like.*

We omit the proof, which is similar to that of Lemma 3.2.9.

If E is an extender then Lemmas 6.1.4, 6.1.6, and 3.2.8 give us a unique

$$\pi_E : \prod_E (V;\in) \cong (\mathrm{Ult}(V;E);\in),$$

with $\mathrm{Ult}(V;E)$ transitive. Let $i_E : V \prec \mathrm{Ult}(V;E)$ be given by $i_E = \pi_E \circ i'_E$. (Note that we continue the convention whereby we may write, for example, "V" instead of "$(V;\in)$.")

Lemma 6.1.7. *Let E be an extender. Then i_E is the identity on V_κ, and $\kappa = \mathrm{crit}\,(i_E)$.*

Proof. The proof is like that of Lemma 3.2.10.

We first show that i_E is the identity on κ. To do this we prove by induction that $i_E(\alpha) = \alpha$ for all $\alpha < \kappa$. Suppose then that $\alpha < \kappa$ and that $i_E(\beta) = \beta$ for all $\beta < \alpha$. For each $\beta < \alpha$, $i_E(\beta) \in i_E(\alpha)$, by the elementarity of i_E. Suppose that $\pi([\![a, f]\!]) \in i_E(\alpha)$, where $\pi = \pi_E : \prod_E(V;\in) \cong (\mathrm{Ult}(V;E);\in)$.

Then $\{z \in [\kappa]^{|a|} \mid f(z) \in \alpha\} \in E_a$. But then the κ-completeness of E_a implies that there is a $\beta < \alpha$ such that $\{z \in [\kappa]^{|a|} \mid f(z) = \beta\} \in E_a$. This means that

$$\pi([\![a, f]\!]) = \pi([\![\emptyset, c_\beta]\!]) = i_E(\beta) = \beta.$$

This completes the inductive proof that i_E is the identity on κ. From this fact it follows exactly as in the proof of Lemma 3.2.10 that i_E is the identity on V_κ.

To see that $i_E(\kappa) > \kappa$, consider $[\![\{\kappa\}, f]\!]$, where $f = \bigcup \restriction [\kappa]^1$. Note that $\bigcup(\{\alpha\}) = \alpha$ for each $\{\alpha\} \in [\kappa]^1$. Since $E_{\{\kappa\}}$ is non-principal and κ-complete by part (1) of Lemma 6.1.2, we have that

$$(\forall \beta < \kappa)\, \{\{\alpha\} \mid \beta < \alpha\} \in E_{\{\kappa\}}.$$

Thus $\pi([\![\{\kappa\}, f]\!]) > \beta$ for all $\beta < \kappa$; hence $\pi([\![\{\kappa\}, f]\!]) \geq \kappa$. (It follows from Lemma 6.1.8 below that $\pi([\![\{\kappa\}, f]\!]) = \kappa$.) But $f : [\kappa]^1 \to \kappa$, so $\pi([\![\{\kappa\}, f]\!]) < \pi([\![\emptyset, c_\kappa]\!]) = i_E(\kappa)$. □

If $\mathcal{U}$ is a uniform normal ultrafilter on κ, then (Exercise 3.2.2) $\pi_{\mathcal{U}}([\![\mathrm{id}]\!]_{\mathcal{U}}) = \kappa$. The next lemma is the version of this fact for extenders, and it is proved using the normality of extenders (property (3)).

Lemma 6.1.8. *Let $E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ be a (κ, λ)-extender.*

(i) For each $a \in [\lambda]^{<\omega}$ and each i, $1 \leq i \leq |a|$, $\pi_E([\![a, z \mapsto z_i]\!]) = a_i$.

(ii) For each $a \in [\lambda]^{<\omega}$, $\pi_E([\![a, \mathrm{id}]\!]) = a$.

Proof. Let $\pi = \pi_E$. We prove (i) by induction on the ordinal a_i, simultaneously for all $a \in [\lambda]^{<\omega}$. Assume then that, for all $b \in [\lambda]^{<\omega}$ and all k, $1 \leq k \leq |b|$, if $b_k < a_i$, then $\pi([\![b, z \mapsto z_k]\!]) = b_k$.

Let $\beta < a_i$. Let $\beta = (a \cup \{\beta\})_k$. Using our induction hypothesis, we get that $\beta = \pi([\![a \cup \{\beta\}, z \mapsto z_k]\!])$. But $z_k < (z_{a, a\cup\{\beta\}})_i$ for all $z \in [\kappa]^{|a \cup \{\beta\}|}$, and so Theorem 6.1.3 implies that $\pi([\![a \cup \{\beta\}, z \mapsto z_k]\!]) < \pi([\![a, z \mapsto z_i]\!])$, and so that $\beta < \pi([\![a, z \mapsto z_i]\!])$.

Assume now that $\pi([\![c, f]\!])$ is an ordinal smaller than $\pi([\![a, z \mapsto z_i]\!])$. We may assume that range $(f) \subseteq$ Ord. Let $b = a \cup c$. We have by Theorem 6.1.3 that

$$\{z \in [\kappa]^{|b|} \mid f(z_{c,b}) < (z_{a,b})_i\} \in E_b.$$

By normality this gives us a $\beta < a_i$ such that

$$\{z \in [\kappa]^{|b\cup\{\beta\}|} \mid f(z_{c,b\cup\{\beta\}}) = z_k\} \in E_{b\cup\{\beta\}},$$

where $\beta = (b \cup \{\beta\})_k$. Thus $\pi(\llbracket c, f\rrbracket) = \pi(\llbracket b \cup \{\beta\}, z \mapsto z_k\rrbracket)$. Our induction hypothesis then gives us that $\pi(\llbracket c, f\rrbracket) = (b \cup \{\beta\})_k = \beta$.

We have shown that the ordinals smaller than $\pi(\llbracket a, z \mapsto z_i\rrbracket)$ are precisely the ordinals smaller than a_i; hence $\pi(\llbracket a, z \mapsto z_i\rrbracket) = a_i$.

It is easy to see that (i) implies (ii). $\square$

Corollary 6.1.9. *If E is a (κ, λ)-extender, then $\lambda \leq i_E(\kappa)$, and E is the (κ, λ)-extender derived from i_E.*

Proof. Let $\alpha < \lambda$. By Lemma 6.1.8, $\alpha = \pi_E(\llbracket\{\alpha\}, z \mapsto z_1\rrbracket)$. Since $\{z \in [\kappa]^1 \mid z_1 < \kappa\} \in E_{\{\alpha\}}$, Theorem 6.1.3 gives that $\pi_E(\llbracket\{\alpha\}, z \mapsto z_1\rrbracket) < \pi_E(\llbracket\emptyset, \kappa\rrbracket) = i_E(\kappa)$.

Let $a \in [\lambda]^{<\omega}$ and let $X \in [\kappa]^{|a|}$. By Lemma 6.1.8, $a \in i_E(X)$ if and only if $\pi_E(\llbracket a, \mathrm{id}\rrbracket) \in \pi_E(\llbracket\emptyset, c_X\rrbracket)$. By Theorem 6.1.3, this holds if and only if $\{z \in [\kappa]^{|a|} \mid z \in X\} \in E_a$, i.e., if and only if $X \in E_a$. $\square$

Now let us return to the topic with which we began this section. Let $j : V \prec M$ with M transitive and $\mathrm{crit}\,(j) = \kappa$. Let $\lambda \leq j(\kappa)$. Let $\langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ be the (κ, λ)-extender derived from j. The next two lemmas will show that $i_E : V \prec \mathrm{Ult}(V; E)$ is an approximation of $j : V \prec M$. Define $k : \mathrm{Ult}(V; E) \to M$ by

$$k(\pi_E(\llbracket a, f\rrbracket_E)) = (j(f))(a).$$

The function k is well-defined, since

$$\begin{array}{l} \llbracket a, f\rrbracket = \llbracket b, g\rrbracket \ \to \\ \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) = g(z_{b,a\cup b})\} \in E_{a\cup b} \ \to \\ (j(f))(a) = (j(g))(b). \end{array}$$

Lemma 6.1.10. *Let j, M, λ, E, and k be as in the preceding paragraph. Then*

(a) $k : \mathrm{Ult}(V; E) \prec M$;

(b) $k \circ i_E = j$;

(c) $k \restriction \lambda$ is the identity.

Proof. The proof of (a) is similar to the proof that k is well-defined. Let $\varphi(v_1, \dots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1 \rangle, \dots, \langle a_n, f_n \rangle$ be elements of $\mathcal{D}_E$. Let $b = \bigcup_{1 \le i \le n} a_i$. Then

$$\begin{array}{l}\prod_E(V;\in) \models \varphi[[\![a_1, f_1]\!], \dots, [\![a_n, f_n]\!]] \leftrightarrow \text{(by Theorem 6.1.3)}\\ \{z \in [\kappa]^{|b|} \mid (V;\in) \models \varphi[f_1(z_{a_1,b}), \dots, f_n(z_{a_n,b})]\} \in E_b \leftrightarrow \\ (M;\in) \models \varphi[(j(f_1))(a_1), \dots, (j(f_n))(a_n)] \leftrightarrow \\ (M;\in) \models \varphi[k(\pi_E([\![a_1, f_1]\!])), \dots, k(\pi_E([\![a_n, f_n]\!]))].\end{array}$$

To see that $k \circ i_E = j$, observe that

$$k(i_E(x)) = k(\pi_E([\![\emptyset, c_x]\!])) = c_{j(x)}(\emptyset) = j(x).$$

We finish the proof of the lemma by showing that $k \restriction [\lambda]^{<\omega}$ is the identity. This is clearly equivalent with (c). Let $a \in [\lambda]^{<\omega}$. Lemma 6.1.8 implies that $k(a) = k(\pi_E([\![a, \mathrm{id}]\!])) = (j(\mathrm{id}))(a) = a$. □

Lemma 6.1.11. *Let j, M, λ, E, and k be as in the paragraph preceding Lemma 6.1.10. Let $\eta < \lambda$ be such that*

$$|V_\eta^M|^M \le \lambda.$$

Then $V_\eta^{\mathrm{Ult}(V;E)} = V_\eta^M$ and $k \restriction V_\eta^{\mathrm{Ult}(V;E)}$ is the identity.

Proof. Let $\gamma = |V_\eta^{\mathrm{Ult}(V;E)}|^{\mathrm{Ult}(V;E)}$. Since

$$\gamma \le k(\gamma) = |V_\eta^M|^M \le \lambda,$$

we must have $k(\gamma) = \gamma$.

Let $\langle X_\beta \mid \beta < \gamma \rangle$ be an enumeration of all elements of $V_\eta^{\mathrm{Ult}(V;E)}$. Then $\langle k(X_\beta) \mid \beta < \gamma \rangle$ is an enumeration of all elements of V_η^M. By the elementarity of k, this means that

$$k \restriction V_\eta^{\mathrm{Ult}(V;E)} : (V_\eta^{\mathrm{Ult}(V;E)}; \in) \cong (V_\eta^M; \in).$$

But an isomorphism between transitive sets must be the identity. □

For the case $j = i_E$, the elementary embedding k of the preceding lemmas is the identity:

Lemma 6.1.12. *For any extender E and any $[\![a, f]\!]$,*

$$(i_E(f))(a) = \pi_E([\![a, f]\!]).$$

Proof. For E, a, and f, as in the statement of the lemma,

$$\begin{aligned}(i_E(f))(a) &= (\pi_E([\![\emptyset, c_f]\!]))(\pi_E([\![a, \mathrm{id}]\!])) \\ &= \pi_E([\![a, f]\!]),\end{aligned}$$

where the last equality is by Theorem 6.1.3, with "v_1 is a function and $v_3 = v_1(v_2)$" as the formula φ, with $\emptyset$ as a_1 and a as a_2 and a_3, with c_f as f_1, with id as f_2, and with f as f_3. □

Exercise 6.1.1. (a) Let $\langle E_a \mid a \subseteq [\kappa+1]^{<\omega}\rangle$ be a $(\kappa, \kappa+1)$-extender. Show that

$$\{X \subseteq \kappa \mid (\exists Y \in E_{\{\kappa\}})(\forall \alpha < \kappa)(\alpha \in X \leftrightarrow \{\alpha\} \in Y)\}$$

is a uniform normal ultrafilter on κ.

(b) Prove that a cardinal κ is measurable if and only if there exists a $(\kappa, \kappa+1)$-extender.

Exercise 6.1.2. Let us generalize the notion of extender as follows. Let $j : V \prec M$ with M transitive and $\mathrm{crit}\,(j) = \kappa$. Let λ be any ordinal number such that $\kappa < \lambda$. For $a \in [\lambda]^{<\omega}$, let γ_a be the least ordinal $\gamma \geq \kappa$ such that $a \in [j(\gamma)]^{<\omega}$. The (κ, λ)-*extender derived from* j is the system

$$\langle E_a \mid a \in [\lambda]^{<\omega}\rangle,$$

where the E_a are defined by

$$E_a = \{X \subseteq [\gamma_a]^{|a|} \mid a \in j(X)\}.$$

Remark. The requirement that $\gamma_a \geq \kappa$ has no purpose other than to make ordinary extenders be extenders in the generalized sense.

Prove that Lemma 6.1.2 becomes true for derived extenders in this generalized sense when (1)–(4) are replaced by the following clauses (1′)–(4′).

(1′) For each $a \in [\lambda]^{<\omega}$, E_a is a κ-complete ultrafilter on $[\gamma_a]^{|a|}$, and E_a is principal if and only if $a \in \mathrm{range}\,(j)$.

(2′) (Compatibility) If $a \subseteq b \in [\lambda]^{<\omega}$ and $X \in E_a$, then $X^{a,b}_{\gamma_b} \in E_b$.

(3′) (Normality) Let $a \in [\lambda]^{<\omega}$. Let $f : [\gamma_a]^{|a|} \to \gamma_a$ and $i \le |a|$ be such that

$$\{z \mid f(z) < z_i\} \in E_a.$$

Then there is a $\beta < a_i$ such that

$$\{z \in [\gamma_a]^{|a \cup \{\beta\}|} \mid f(z_{a,a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}},$$

where $\beta = (a \cup \{\beta\})_k$.

(4′) (Countable Completeness) Let $\langle a_i \mid i \in \omega\rangle$ be such that each $a_i \in [\lambda]^{<\omega}$. Let $X_i \in E_{a_i}$ for each $i \in \omega$. Then there is an order preserving $h : \bigcup_{i\in\omega} a_i \to \bigcup_{i\in\omega} \gamma_{a_i}$ such that $h''a_i \in X_i$ for all $i \in \omega$.

A (κ, λ)-*extender* in the generalized sense is defined using (1′)–(4′) and (if one wants) the condition that no bounded subset of γ_a belongs to E_a unless $\gamma_a = \kappa$.

Exercise 6.1.3. Let $E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ be a (κ, λ)-extender.

(a) For $a \subseteq b \in [\lambda]^{<\omega}$, show that

$$f \mapsto f^{a,b}$$

induces an elementary embedding

$$i_{E_a,E_b} : \mathrm{Ult}(V; E_a) \prec \mathrm{Ult}(V; E_b).$$

(b) Prove that

$$(\langle \mathrm{Ult}(V; E_a) \mid a \in [\lambda]^{<\omega}\rangle, \langle i_{E_a,E_b} \mid a \subseteq b \in [\lambda]^{<\omega}\rangle)$$

is a directed system of elementary embeddings.

(c) Let

$$(\tilde{\mathcal{M}}, \langle \tilde{\imath}_{E_a} \mid a \in [\lambda]^{<\omega}\rangle)$$

be the direct limit of the directed system of (b). Prove that there is a (unique) $\pi : \tilde{\mathcal{M}} \cong (\mathrm{Ult}(V; E); \in)$ and that $i_E = \pi \circ \tilde{\imath}_{E_a} \circ i_{E_a}$.

6.2 Large Large Cardinals

By Lemma 3.2.11 and Theorem 3.2.12, the measurability of a cardinal κ is equivalent with each of the following:

(a) There are a transitive class M and an embedding $j : V \prec M$ such that $\mathrm{crit}\,(j) = \kappa$ and such that $V_{\kappa+1} \subseteq M$.

(b) There are a transitive class M and an embedding $j : V \prec M$ such that $\mathrm{crit}\,(j) = \kappa$ and such that ${}^{\kappa}M \subseteq M$.

These two equivalents of measurability lead to two different ways to generalize the notion of a measurable cardinal. The one corresponding to (a) was considered in [Gaifman, 1974], but became prominent only through work of [Mitchell, 1979] and of Anthony Dodd and Ronald Jensen. (See [Dodd, 1982] and [Dodd,].) The one corresponding to (b) is was pursued earlier, by William Reinhardt and Robert Solovay. (See [Solovay et al., 1978].)

If κ is a cardinal number and η is an ordinal number greater than κ, then κ is *η-strong* if there are a transitive class M and an embedding $j : V \prec M$ such that $\mathrm{crit}\,(j) = \kappa$, $\eta < j(\kappa)$, and $V_\eta \subseteq M$. A cardinal κ is *strong* if κ is η-strong for every ordinal $\eta > \kappa$.

If κ and $\lambda \geq \kappa$ are cardinal numbers, then κ is *λ-supercompact* if there are a transitive class M and an embedding $j : V \prec M$ such that $\mathrm{crit}\,(j) = \kappa$, $\lambda < j(\kappa)$, and ${}^{\lambda}M \subseteq M$. A cardinal κ is *supercompact* if κ is λ-supercompact for every cardinal $\lambda \geq \kappa$.

The condition $\eta < j(\kappa)$ can be dropped from the definition of η-strong without changing the concept, and the condition $\lambda < j(\kappa)$ can similarly be dropped from the definition of λ-supercompact. This will be proved later (Theorem 6.2.15).

A cardinal κ is measurable if and only if κ is $(\kappa+1)$-strong if and only if κ is κ-supercompact. It is clear that if κ is 2^κ-supercompact then κ is $(\kappa+2)$-strong. But the converse fails: If κ is 2^κ-supercompact then, as we will see below, κ is the κth cardinal γ such that γ is $(\gamma+2)$-strong—indeed there are κ cardinals $\gamma < \kappa$ such that γ is κ-strong.

There is an equivalent definition of λ-supercompactness that generalizes our basic definition of measurability in terms of ultrafilters. To state this definition, we need to make some preliminary definitions.

For cardinals κ and λ,

$$\mathcal{P}_\kappa(\lambda) = \{x \subseteq \lambda \mid |x| < \kappa\}.$$

In other notation, $\mathcal{P}_\kappa(\lambda) = [\lambda]^{<\kappa}$. An ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$ (i) is *fine* if

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x\} \in \mathcal{U}$$

for each $\alpha < \lambda$ and (ii) is *normal* if, for every $f : \mathcal{P}_\kappa(\lambda) \to \lambda$, if

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) \in x\} \in \mathcal{U}$$

then there is an $\alpha < \lambda$ such that

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) = \alpha\} \in \mathcal{U}.$$

Note that an ultrafilter $\mathcal{V}$ on an infinite cardinal κ generates an ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\kappa)$: $X \in \mathcal{U} \leftrightarrow X \cap \kappa \in \mathcal{V}$ ($\leftrightarrow X \cap \mathrm{Ord} \in \mathcal{V}$). The ultrafilter $\mathcal{U}$ is κ-complete if and only if $\mathcal{V}$ is κ-complete; $\mathcal{U}$ is fine if and only if $\mathcal{V}$ is uniform; $\mathcal{U}$ is normal if and only if $\mathcal{V}$ is normal.

Theorem 6.2.1. (Reinhardt, Solovay; see [Solovay et al., 1978]) *If κ and $\lambda \geq \kappa$ are cardinals, then the following are equivalent:*

(1) κ is λ-supercompact.

(2) There is a κ-complete fine normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

Proof. First suppose that $j : V \prec M$ witnesses that κ is λ-supercompact. Let

$$\mathcal{U} = \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid j''\lambda \in j(X)\}.$$

(Recall that $j''\lambda = \mathrm{range}\,(j \restriction \lambda)$.) This definition is legitimate, for $|j''\lambda| = \lambda$ and so $j''\lambda \in M$. Since $\lambda < j(\kappa)$, we have that $j''\lambda \in j(\mathcal{P}_\kappa(\lambda))$. Thus Lemma 6.1.1 implies that $\mathcal{U}$ is a κ-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

To see that $\mathcal{U}$ is fine, let $\alpha < \lambda$. Since $j(\alpha) \in j''\lambda$, we have that

$$j''\lambda \in j(\{x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x\}).$$

Hence $\{x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x\} \in \mathcal{U}$.

To verify the normality of $\mathcal{U}$, let $f : \mathcal{P}_\kappa(\lambda) \to \lambda$ be such that

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) \in x\} \in \mathcal{U}.$$

By the definition of $\mathcal{U}$, we have that

$$(j(f))(j''\lambda) \in j''\lambda.$$

But this means that there is an $\alpha < \lambda$ such that

$$(j(f))(j''\lambda) = j(\alpha).$$

By the definition of $\mathcal{U}$,

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) = \alpha\} \in \mathcal{U}.$$

Now suppose that $\mathcal{U}$ is a κ-complete fine normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Since $i_\mathcal{U} : V \prec \mathrm{Ult}(V;\mathcal{U})$, it is enough to show

(i) ${}^\lambda(\mathrm{Ult}(V;\mathcal{U})) \subseteq \mathrm{Ult}(V;\mathcal{U})$;

(ii) $\lambda < i_\mathcal{U}(\kappa)$;

(iii) $\mathrm{crit}\,(i_\mathcal{U}) = \kappa$.

In order to prove (i) we will first show that

$$i_\mathcal{U}''\lambda \in \mathrm{Ult}(V;\mathcal{U}).$$

Let $\pi = \pi_\mathcal{U} : \prod_\mathcal{U}(V;\in) \cong (\mathrm{Ult}(V;\mathcal{U});\in)$. We show that $i_\mathcal{U}''\lambda = \pi(\llbracket\mathrm{id}\rrbracket)$. If $\alpha < \lambda$ then, by fineness of $\mathcal{U}$,

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x\} \in \mathcal{U};$$

hence

$$\llbracket c_\alpha \rrbracket \;\in_\mathcal{U}\; \llbracket\mathrm{id}\rrbracket$$

and so

$$i_\mathcal{U}(\alpha) \in \pi(\llbracket\mathrm{id}\rrbracket).$$

Thus $i_\mathcal{U}''\lambda \subseteq \pi(\llbracket\mathrm{id}\rrbracket)$. To establish the reverse inclusion, let

$$\llbracket f \rrbracket \;\in_\mathcal{U}\; \llbracket\mathrm{id}\rrbracket.$$

Then

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) \in x\} \in \mathcal{U}.$$

By the normality of $\mathcal{U}$, there is an $\alpha < \lambda$ such that

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) = \alpha\} \in \mathcal{U}.$$

But then $\llbracket f \rrbracket = \llbracket c_\alpha \rrbracket$ and so

$$\pi(\llbracket f \rrbracket) = i_\mathcal{U}(\alpha).$$

This shows that $i_{\mathcal{U}}''\lambda \supseteq \pi([\![\mathrm{id}]\!])$ and so that $i_{\mathcal{U}}''\lambda = \pi([\![\mathrm{id}]\!])$.

Now suppose that $h : \lambda \to \mathrm{Ult}(V;\mathcal{U})$. For each $\alpha < \lambda$, let $h(\alpha) = \pi([\![g_\alpha]\!])$. Define $\tilde{g} : \mathcal{P}_\kappa(\lambda) \to {}^\lambda V$ by

$$(\tilde{g}(x))(\alpha) = g_\alpha(x).$$

The function $\pi([\![\tilde{g}]\!])$ has domain $i_{\mathcal{U}}(\lambda)$. For each $\alpha < \lambda$,

$$(\pi([\![\tilde{g}]\!]))(i_{\mathcal{U}}(\alpha)) = \pi([\![g_\alpha]\!]) = h(\alpha).$$

Thus

$$h = \pi([\![\tilde{g}]\!]) \circ (i_{\mathcal{U}} \restriction \lambda) \ \in \mathrm{Ult}(V;\mathcal{U}).$$

For (ii), note that

$$\begin{aligned} \lambda &= \text{the order type of } i_{\mathcal{U}}''\lambda \\ &= \text{the order type of } \pi([\![\mathrm{id}]\!]) \\ &< \pi([\![c_\kappa]\!]) \\ &= i_{\mathcal{U}}(\kappa). \end{aligned}$$

Now by Lemma 3.2.10, crit $(i_{\mathcal{U}})$ is the completeness of $\mathcal{U}$, which is $\geq \kappa$. If crit $(i_{\mathcal{U}}) > \kappa$, then we have the contradiction that $\lambda \geq \kappa = i_{\mathcal{U}}(\kappa) > \lambda$. Thus (iii) is proved. □

There is no analogue of Theorem 6.2.1 that characterizes η-strength in terms of the existence of an ultrafilter. To get an analogue of Theorem 6.2.1 we need to use extenders instead of ultrafilters.

Lemma 6.2.2. *Let κ be an η-strong cardinal. Then there is an extender E such that i_E witnesses that κ is η-strong.*

Proof. Let $j : V \prec M$ witness that κ is η-strong. Thus $V_\eta^M = V_\eta$. Let $\lambda = |V_\eta|^M$. Let E be the (κ, λ)-extender derived from j. By Lemma 6.1.11, $V_\eta^{\mathrm{Ult}(V;E)} = V_\eta$. Hence i_E witnesses that κ is η-strong.

□

For any extender E, let strength (E) be the largest ordinal η such that $V_\eta \subseteq \mathrm{Ult}(V;E)$. The next theorem is a direct consequence of Lemma 6.2.2.

Theorem 6.2.3. (Mitchell; Dodd and Jensen) *For cardinals κ and $\eta > \kappa$, the following are equivalent:*

(a) κ *is* η*-strong.*

(b) *There is an extender* E *with* $\operatorname{crit}(i_E) = \kappa$*,* $\operatorname{strength}(E) \geq \eta$*, and* $\eta < i_E(\kappa)$*.*

Remark. In [Martin and Steel, 1989], the word *extender* is used for a wider class than that of (κ, λ)-extenders. The extenders of that paper do not necessarily have the form $\langle E_a \mid a \in [\lambda]^{<\omega}\rangle$. They can have the more general form $\langle E_a \mid a \in [Y]^{<\omega}\rangle$, where Y is required only to be a transitive set. If $j : V \prec M$ and $Y \subseteq V^M_{j(\kappa)}$ is transitive, then we can get such an extender by setting $E_a = \{X \subseteq [Y]^{|a|} \mid a \in j(X)\}$. Y is called the *support* of E. A cardinal κ is η-strong if and only if there is an extender E in the sense of [Martin and Steel, 1989] such that $\operatorname{crit}(i_E) = \kappa$ and the support of E contains V_η.

Let us begin to show that supercompactness is a much stronger property than strength. To do this we introduce three classes of large cardinals that lie between strong cardinals and supercompact cardinals. Among these will be *Woodin cardinals*, the cardinals we will use in determinacy proofs.

A cardinal κ is called *superstrong* if there is an elementary embedding $j : V \prec M$ such that $\operatorname{crit}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$. Superstrong cardinals, like strong cardinals, can be characterized in terms of extenders:

Theorem 6.2.4. ([Dodd, 1982]) *For cardinals* κ *the following are equivalent:*

(a) κ *is superstrong.*

(b) *There is a* $\lambda > \kappa$ *and a* (κ, λ)*-extender* E *such that* $\operatorname{strength}(E) \geq \lambda = i_E(\kappa)$*.*

Proof. If E witnesses that (b) holds, then clearly i_E witnesses that κ is superstrong.

Suppose that $j : V \prec M$ witnesses that κ is superstrong. Let E be the $(\kappa, j(\kappa))$-extender derived from j. Let $k : \operatorname{Ult}(V; E) \prec M$ be defined as on page 330. By Lemma 6.1.10, $k \restriction j(\kappa)$ is the identity.

We now apply Lemma 6.1.11 with $\lambda = j(\kappa)$. Since $j(\kappa)$ is a strong limit cardinal in M, the hypotheses of Lemma 6.1.11 hold for every $\eta < j(\kappa)$. The lemma thus yields for every $\eta < j(\kappa)$ that $V^M_\eta = V^{\operatorname{Ult}(V;E)}_\eta$. It follows that $V_{j(\kappa)} = V^M_{j(\kappa)} = V^{\operatorname{Ult}(V;E)}_{j(\kappa)}$.

To finish the proof, we need only show that $i_E(\kappa) = j(\kappa)$. Part (b) of Lemma 6.1.10 gives that $k(i_E(\kappa)) = j(\kappa)$. Since $i_E(\kappa) \leq k(i_E(\kappa))$ and since $k(\alpha) = \alpha$ for all $\alpha < j(\kappa)$, this implies that $i_E(\kappa) = j(\kappa)$. $\square$

The next lemma will be used in proving that supercompactness is essentially a stronger property than superstrength. But it—and variants of it—will also be useful on other occasions.

To state the lemma, we need to introduce the analogue for extenders of the $\mathrm{Ult}(M;\mathcal{U})$ of Chapter 3. Suppose that M is a transitive class model of ZFC and that E is a (κ, λ)-extender in M, i.e. that $E \in M$ and $M \models$ "E is a (κ, λ)-extender." Then we can form what is in M the ultrapower of M with respect to E. The universe of this ultrapower consists of equivalence classes (modified *á la* Scott) of pairs $\langle a, f\rangle$, where $a \in [\lambda]^{<\omega}$ and $f \in M$ is such that $f : [\kappa]^{|a|} \to M$. Let us denote the class of all such pairs by $\mathcal{D}_E^M$, and let us denote the equivalence class of $\langle a, f\rangle$ by

$$[\![a, f]\!]_E^M .$$

The relation of the ultrapower, which we call $\in_E^M$, is given by

$$[\![a, f]\!]_E^M \in_E^M [\![b, g]\!]_E^M \leftrightarrow \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) \in g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

The ultrapower we will denote by

$$\prod_E^M (M; \in).$$

By ZFC in M, this ultrapower is well-founded and set-like, and so we have a unique $\pi_E^M : \prod_E^M (M; \in) \prec (\mathrm{Ult}(M; E); \in)$. We also have the canonical elementary embedding $i_E^M : M \prec \mathrm{Ult}(M; E)$.

Lemma 6.2.5. *Let M be a transitive class model of* ZFC*. Let E be a (κ, λ)-extender such that $E \in M$. (This implies in particular that $V_{\kappa+1}^M = V_{\kappa+1}$.) Let $\zeta \geq \kappa$ be such that $V_{\zeta+1}^M = V_{\zeta+1}$. Then*

(i) $M \models$ "E is an extender"*;*

(ii) $(\forall \alpha \leq \zeta^+)\, i_E^M(\alpha) = i_E(\alpha)$*; in particular,* $i_E^M(\kappa) = i_E(\kappa)$*;*

(iii) $V_{i_E(\zeta)+1}^{\mathrm{Ult}(M;E)} = V_{i_E(\zeta)+1}^{\mathrm{Ult}(V;E)}$*; hence* $V_{i_E(\kappa)+1}^{\mathrm{Ult}(M;E)} = V_{i_E(\kappa)+1}^{\mathrm{Ult}(V;E)}$.

Proof. That clauses (1)–(4) in the definition of an extender hold for E in M follows from the facts that $[\lambda]^{<\omega} \subseteq M$ and $V_{\kappa+1} \subseteq M$. Clause (4) can be proved either directly, using the absoluteness of wellfoundedness of trees, or indirectly, using Lemma 6.1.5.

(ii) and (iii) follow from the fact that, for $n \in \omega$, V and M have exactly the same functions $f : [\kappa]^n \to \zeta^+$ and $g : [\kappa]^n \to V_{\zeta+1}$. (Such functions f can be coded by a wellordering R of ζ of order type $\sup(\text{range}(f))$ and a $\tilde{g} : [\kappa]^n \to \zeta$. The pair $\langle R, \tilde{g}\rangle$ can be coded by a $g : [\kappa]^n \to V_{\zeta+1}$. Such a g can in turn be coded by an element of $V_{\zeta+1} = V^M_{\zeta+1}$.) $\square$

The following lemma is possibly due to Dodd.

Theorem 6.2.6. *Let κ be 2^κ-supercompact. Then there is a uniform normal ultrafilter $\mathcal{U}$ on κ such that*

$$\{\alpha < \kappa \mid \alpha \text{ is superstrong}\} \in \mathcal{U}.$$

Proof. Let $j : V \prec M$ witness that κ is 2^κ-supercompact. Let E be the $(\kappa, j(\kappa))$-extender derived from j.

For each $a \in [j(\kappa)]^{<\omega}$,

$$E_a = \{X \subseteq [\kappa]^{|a|} \mid a \in j(X)\}.$$

Now $j \restriction \bigcup_{n\in\omega} \mathcal{P}([\kappa]^n)$ is a subset of M of size 2^κ and is therefore a member of M. It follows that $E \in M$.

As in the proof of Theorem 6.2.4, we get that $i_E(\kappa) = j(\kappa)$ and that $V^M_{j(\kappa)} = V^{\text{Ult}(V;E)}_{j(\kappa)}$. Lemma 6.2.5 gives that E is an extender in M, that $i_E(\kappa) = i^M_E(\kappa)$, and that $V^{\text{Ult}(M;E)}_{i_E(\kappa)} = V^{\text{Ult}(V;E)}_{i_E(\kappa)}$. Putting these facts together, we get that

$$V^{\text{Ult}(M;E)}_{i^M_E(\kappa)} = V^M_{i^M_E(\kappa)}.$$

But this means that

$$M \models \ \kappa \text{ is superstrong}.$$

Let $\mathcal{U} = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. By Lemma 3.2.13 we know that $\mathcal{U}$ is a uniform normal ultrafilter on κ. For $X = \{\alpha < \kappa \mid \alpha \text{ is superstrong}\}$, we have shown that $\kappa \in j(X)$; hence $X \in \mathcal{U}$. $\square$

Remark. One thing the theorem does not show is that if κ is 2^κ-supercompact then κ is superstrong. Assuming that the existence of supercompact cardinals is consistent with ZFC, one can show that it is also consistent with ZFC that there is a supercompact cardinal that is not superstrong. (See exercise 6.2.1,) One the other hand, it is trivial that every supercompact cardinal is strong.

For some time, nothing interesting was known between strong and superstrong cardinals. Then Saharon Shelah, in weakening the hypothesis of results of [Foreman et al., 1988] and of related theorems (see [Shelah and Woodin, 1990]), discovered a significant intermediate class of large cardinals.

For any cardinal κ and any $f : \kappa \to \kappa$, let us say that κ is *Shelah for* f if there is a $j : V \prec M$ such that M is transitive, $\operatorname{crit}(j) = \kappa$, and $V_{(j(f))(\kappa)} \subseteq M$. A cardinal κ is *Shelah* if, for every $f : \kappa \to \kappa$, κ is Shelah for f.

A routine argument shows that supercompactness is a stronger property than that of being a Shelah cardinal:

Theorem 6.2.7. *Let κ be superstrong. Then κ is Shelah and there is a uniform normal ultrafilter $\mathcal{U}$ on κ such that*

$$\{\alpha < \kappa \mid \alpha \text{ is Shelah}\} \in \mathcal{U}.$$

Proof. Let j witness that κ is superstrong.

To prove that κ is Shelah, let $f : \kappa \to \kappa$. Since $(j(f))(\kappa) < j(\kappa)$, we have that

$$V_{(j(f))(\kappa)} \subseteq V_{j(\kappa)} \subseteq M.$$

For the second assertion of the theorem, we proceed as in the proof of Theorem 6.2.6. We show that $M \models$ "κ is Shelah." Just as in the proof of Theorem 6.2.6, this suffices. Let $f : \kappa \to \kappa$ and set

$$\lambda = \max\{\kappa + 1,\ (j(f))(\kappa) + 1,\ |V_{(j(f))(\kappa)}|\}.$$

(Note that $|V_{(j(f))(\kappa)}| = |V^M_{(j(f))(\kappa)}|^M$.) Let E be the (κ, λ)-extender derived from j. Let $k : \mathrm{Ult}(V; E) \prec M$ be the canonical embedding, i.e. let k be defined as on page 330. By Lemma 6.1.10, $k \restriction \lambda$ is the identity. This implies that $k((j(f))(\kappa)) = (j(f))(\kappa)$. But

$$k((i_E(f))(\kappa)) = (k(i_E(f)))(k(\kappa)) = (j(f))(k(\kappa)) = (j(f))(\kappa),$$

and so $(i_E(f))(\kappa) = (j(f))(\kappa)$. By Lemma 6.1.11, $V^{\mathrm{Ult}(V;E)}_{(j(f))(\kappa)} = V^M_{(j(f))(\kappa)}$. By Lemma 6.2.5, we have that E is an extender in M, that $i^M_E(f) = i_E(f)$ (and so these functions agree on the argument κ), and that $V^{\mathrm{Ult}(M;E)}_{(j(f))(\kappa)} = V^{\mathrm{Ult}(V;E)}_{(j(f))(\kappa)}$. Combining these facts we get that

$$\begin{aligned} (i^M_E(f))(\kappa) &= (j(f))(\kappa); \\ V^{\mathrm{Ult}(M;E)}_{(i^M_E(f))(\kappa)} &= V^M_{(i^M_E(f))(\kappa)}. \end{aligned}$$

Thus i^M_E witnesses in M that κ is Shelah for f. □

Hugh Woodin discovered a weakening of the concept of Shelah cardinals that has turned out to be extremely important. For any cardinal κ and any $f : \kappa \to \kappa$, κ is *Woodin for* f if there are $\delta < \kappa$ and $j : V \prec M$ such that M is transitive, δ is closed under f, $\mathrm{crit}\,(j) = \delta$, and $V_{(j(f))(\delta)} \subseteq M$. A cardinal κ is *Woodin* if, for every function $f : \kappa \to \kappa$, κ is Woodin for f.

The next two theorems, both known to Woodin, show how Woodin cardinals sit within the large cardinal hierarchy.

Theorem 6.2.8. *Let κ be Shelah. Then κ is Woodin and there is a uniform normal ultrafilter $\mathcal{U}$ on κ such that*

$$\{\alpha < \kappa \mid \alpha \text{ is Woodin}\} \in \mathcal{U}.$$

Proof. Let $f : \kappa \to \kappa$. Define $g : \kappa \to \kappa$ by

$$g(\alpha) = \max\{\alpha + 2,\ f(\alpha) + 1,\ |V_{f(\alpha)}|\} + 1.$$

Let $j : V \prec M$ witness that κ is Shelah for g. Let

$$\lambda = \max\{\kappa + 1,\ (j(f))(\kappa) + 1,\ |V_{(j(f))(\kappa)}|\}.$$

Let E be the (κ, λ)-extender derived from j. Since $E : [\lambda]^{<\omega} \to V_{\kappa+2}$, it is easy to see that E can be coded by an element of $V_{\max\{\lambda,\kappa+2\}+1} \subseteq V_{(j(g))(\kappa)}$. Thus E belongs to M. Using Lemmas 6.1.10, 6.1.11, and 6.2.5 as in the preceding two proofs, we get that E is an extender in M, that $(i^M_E(f))(\kappa) = (j(f))(\kappa)$, and that $V^{\mathrm{Ult}(M;E)}_{(i^M_E(f))(\kappa)} = V^M_{(i^M_E(f))(\kappa)}$. Since $(j(f)) \restriction \kappa = f$, we have that κ is closed under $j(f)$ and that $(i^M_E(j(f)))(\kappa) = (i^M_E(f))(\kappa)$. The latter of these facts gives, since $V^M_{(i^M_E(f))(\kappa)} \subseteq \mathrm{Ult}(M;E)$, that $V^M_{(i^M_E(j(f)))(\kappa)} \subseteq \mathrm{Ult}(M;E)$. Thus κ and i^M_E witness in M that $j(\kappa)$ is Woodin for $j(f)$. By the elementarity of j, we get that in V there is an extender F such that $\mathrm{crit}\,(i_F)$ and i_F witness that κ is Woodin for f. Since f was arbitrary, we have shown that κ is Woodin.

Now $E \in V^M_{j(\kappa)}$, so the elementarity of j gives the stronger fact that there is an extender $F \in V_\kappa$ such that $\text{crit}\,(i_F)$ and i_F witness that κ is Woodin for f. But such an F belongs to M. Moreover Lemma 6.2.5 implies that $i^M_F(f) = i_F(f)$ and $V_\kappa^{\text{Ult}(M;F)} = V_\kappa^{\text{Ult}(V;F)}$. Hence $\text{crit}\,(i^M_F)$ and i^M_F witness in M that κ is Woodin for f. The second assertion of the Theorem follows as in the two preceding proofs. □

Woodinness is different from the other large cardinal properties we have studied in this chapter in that it is not characterized in terms of elementary embeddings whose critical point is the cardinal itself. Indeed a Woodin cardinal need not be measurable. (See Exercise 6.3.2.) Nevertheless we have the following result, which shows that Woodinness is a stronger property than strength.

Theorem 6.2.9. *Let κ be Woodin. Then*

(1) κ is inaccessible;

(2) The set of cardinals $\delta < \kappa$ such that

$$(\forall \eta)(\delta < \eta < \kappa \;\rightarrow\; \delta \text{ is } \eta\text{-strong})$$

is unbounded in κ.

Proof. (1) To show that κ is regular, suppose that $\gamma < \kappa$ and that $f : \gamma \to \kappa$. Set

$$\begin{aligned} &g(0) = \gamma; \\ &g(1+\alpha) = f(\alpha) \text{ for } \alpha < \gamma; \\ &g(\alpha) = 0 \text{ for } \gamma \leq \alpha. \end{aligned}$$

Since κ is Woodin, there must in particular be a non-zero ordinal $\beta < \kappa$ that is closed under g. But any such β must be larger than every element of the range of f.

To show that κ is a strong limit cardinal, let $\gamma < \kappa$ be a cardinal number. Let $f : \kappa \to \kappa$ be such that $f(0) = \gamma$. Let $\delta < \kappa$ and $j : V \prec M$ witness that κ is Woodin for f. Then $\delta > \gamma$ and δ is measurable. Hence $\delta > 2^\gamma$.

(2) Assume for a contradiction that there is a $\beta < \kappa$ such that

$$(\forall \delta)(\beta \leq \delta < \kappa \;\rightarrow\; (\exists \eta)(\delta < \eta < \kappa \wedge \delta \text{ is not } \eta\text{-strong})).$$

Without loss of generality we may take β to be a limit ordinal. For ordinals α such that $\beta \leq \alpha < \kappa$, let $\eta(\alpha)$ be the least $\eta > \alpha$ such that α is not η-strong. Let

$$g(\alpha) = \begin{cases} \beta & \text{if } \alpha < \beta; \\ \max\{\eta(\alpha)+1, |V_{\eta(\alpha)}|\}+1 & \text{if } \beta \leq \alpha < \kappa. \end{cases}$$

Note that $g(\alpha) \geq \alpha + 2$ for all α. Since κ is inaccessible by (1), we have that $g : \kappa \to \kappa$. Let δ and $j : V \prec M$ witness that κ is Woodin for g. Clearly $\delta > \beta$. Let

$$\lambda = \max\{(j(\eta))(\delta)+1, |V_{(j(\eta))(\delta)}|\}.$$

Let E be the (δ, λ)-extender derived from j. Arguing just as in the proof of Theorem 6.2.8, we get that $E \in M$ and so, by Lemma 6.2.5, that E is an extender in M. By Lemmas 6.2.5 and 6.1.11,

$$V^{\mathrm{Ult}(M;E)}_{(j(\eta))(\delta)} = V^{\mathrm{Ult}(V;E)}_{(j(\eta))(\delta)} = V^{M}_{(j(\eta))(\delta)}.$$

Thus i^M_E witnesses in M that δ is $(j(\eta))(\delta)$-strong. This contradicts the elementarity of j. $\square$

Remark. Theorem 6.2.9 implies that if κ is Woodin then $V_\kappa \models$ ZFC + "There is a proper class of strong cardinals." Theorem 6.3.1 will shed more light on the relation between strong cardinals and Woodin cardinals.

We will develop the theory of Woodin cardinals in the next section. In the rest of this section, we will briefly discuss some very strong large cardinal properties and in doing so prove that, for example, the condition $\lambda < j(\kappa)$ in the definition of λ-supercompactness is unnecessary.

The following definitions and theorem appear in [Solovay et al., 1978]. For $n \in \omega$, a cardinal κ is said to be *n-huge* if there is a $j : V \prec M$ such that M is transitive, crit $(j) = \kappa$, and

$${}^{\kappa_n}M \subseteq M,$$

where $\kappa_n = j_{0,n}(\kappa)$. Being 0-huge is the same as being measurable. Cardinals that are 1-huge are simply called *huge.* Huge cardinals were introduced in the early 1970's by Kenneth Kunen.

As with λ-supercompactness, n-hugeness can be characterized in terms of ultrafilters. If A is any set, an ultrafilter $\mathcal{U}$ on $\mathcal{P}(A)$ is *fine* if

$$(\forall a \in A)\{x \subseteq A \mid a \in x\} \in \mathcal{U}$$

and is *normal* if, for all $f : \mathcal{P}(A) \to A$, if

$$\{x \subseteq A \mid f(x) \in x\} \in \mathcal{U}$$

then f is constant on a set in $\mathcal{U}$.

If x is a set of ordinals, let $\mathrm{ot}(x)$ be the order type of x.

Theorem 6.2.10. ([Solovay et al., 1978]) *If $n \in \omega$ and κ is an infinite cardinal, then the following are equivalent:*

(1) κ is n-huge.

(2) There are cardinals $\kappa = \lambda_0 < \cdots < \lambda_n = \lambda$ and there is a κ-complete fine normal ultrafilter $\mathcal{U}$ on $\mathcal{P}(\lambda)$ such that

$$(\forall i < n)\, \{x \subseteq \lambda \mid \mathrm{ot}(x \cap \lambda_{i+1}) = \lambda_i\} \in \mathcal{U}.$$

Proof. Let $j : V \prec M$ witness that κ is n-huge. Let $\kappa_i = j_{0,i}(\kappa)$ for $i \leq n+1$. Let

$$\mathcal{U} = \{X \subseteq \mathcal{P}(\kappa_n) \mid j''\kappa_n \in j(X)\}.$$

Evidently $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\kappa_n)$. The proofs that $\mathcal{U}$ is κ-complete, fine, and normal are exactly like the corresponding parts of the proof of Theorem 6.2.1. Fix $i < n$. We have that

$$\begin{array}{l} j''\kappa_n \cap j(\kappa_{i+1}) = j''\kappa_n \cap \kappa_{i+2} = j''\kappa_{i+1}; \\ \mathrm{ot}(j''\kappa_{i+1}) = \kappa_{i+1} = j(\kappa_i). \end{array}$$

Thus

$$j''\kappa_n \in j(\{x \subseteq \kappa_n \mid \mathrm{ot}(x \cap \kappa_{i+1}) = \kappa_i\}).$$

But this means that

$$\{x \subseteq \kappa_n \mid \mathrm{ot}(x \cap \kappa_{i+1}) = \kappa_i\} \in \mathcal{U}.$$

Thus we can set $\lambda_i = \kappa_i$ for each $i < n$ and satisfy all the clauses of condition (2).

Now suppose that $\kappa = \lambda_0 < \ldots < \lambda_n = \lambda$ and $\mathcal{U}$ satisfy (2). We will show that $i_{\mathcal{U}} : V \prec \mathrm{Ult}(V;\mathcal{U})$ witnesses that κ is n-huge. By Lemma 3.2.10, $\mathrm{crit}\,(i_{\mathcal{U}}) \geq \kappa$. For $\gamma \leq \lambda$ let $\mathrm{id}_\gamma : \mathcal{P}(\lambda) \to V$ be given by

$$\mathrm{id}_\gamma(x) = x \cap \gamma.$$

By an argument like the one in the corresponding part of the proof of Theorem 6.2.1, we can show that

$$(\forall i < n)\, i_{\mathcal{U}}{''}\gamma = \pi_{\mathcal{U}}(\llbracket \mathrm{id}_\gamma \rrbracket).$$

One consequence of this is that $i_{\mathcal{U}}{''}\lambda \in \mathrm{Ult}(V;\mathcal{U})$; by another argument like one in the proof of Theorem 6.2.1, this implies that ${}^{\lambda}(\mathrm{Ult}(V;\mathcal{U})) \subseteq \mathrm{Ult}(V;\mathcal{U})$. Another consequence is that $i_{\mathcal{U}}(\lambda_i) = \lambda_{i+1}$ for all $i < n$. This follows by Theorem 3.2.5, the elementarity of $\pi_{\mathcal{U}}$, the hypothesis that $\{x \subseteq \lambda \mid \mathrm{ot}(x \cap \lambda_{i+1}) = \lambda_i\} \in \mathcal{U}$, and the fact that $\mathrm{ot}(i_{\mathcal{U}}{''}\lambda_{i+1}) = \lambda_{i+1}$. Since, in particular, $i_{\mathcal{U}}(\kappa) = \lambda_1 > \kappa$, the proof that $\mathrm{crit}\,(i_{\mathcal{U}}) = \kappa$ is now complete. Moreover we have that $\lambda_i = (i_{\mathcal{U}})_{0,i}(\kappa)$, so the proof of the theorem is complete. □

The property n-hugeness is related to supercompactness rather than to strength. The large cardinal property that bears an analogous relation to strength can be defined by replacing the condition ${}^{\kappa_n}M \subseteq M$ in the definition of n-hugeness by the weaker condition $V_{\kappa_n} \subseteq M$. This property, which has no standard name, is weaker than n-hugeness. On the other hand, it implies $(n-1)$-hugeness when $n > 0$. (See Exercise 6.2.4.)

The following observation in [Kunen, 1978] shows that hugeness is a more powerful large cardinal property than supercompactness.

Theorem 6.2.11. *Let κ be huge. Then there is a uniform normal ultrafilter $\mathcal{U}$ on κ such that $\{\alpha < \kappa \mid (\forall \beta < \kappa)\, \alpha$ is β-supercompact$\} \in \mathcal{U}$.*

Proof. Let $j : V \prec M$ witness that κ is huge. Let $\lambda < j(\kappa)$. Then j witnesses that κ is λ-supercompact. By Theorem 6.2.1, let $\mathcal{V}$ be a κ-complete fine normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$. It is clear that $\mathcal{V} \in M$ and that $M \models$ "$\mathcal{V}$ is a κ-complete fine normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$." Hence κ is λ-supercompact in M. The conclusion of the theorem follows as in the proof of Theorem 6.2.6. □

Remark. The proof of the theorem would go through unchanged if we weakened the ${}^{\kappa_1}M \subseteq M$ part of the hugeness hypothesis to $V_{\kappa_1} \subseteq M$. A number of other large cardinal properties have been studied that lie between hugeness and supercompactness. See [Solovay et al., 1978].

The notion of n-huge cardinals cries out for generalization to the transfinite. One could define κ to be *α-huge*, for α an arbitrary ordinal, if there is a $j : V \prec M$ such that M is transitive, $\mathrm{crit}(j) = \kappa$, and ${}^{\kappa_\alpha}M \subseteq M$, where $\kappa_\alpha = j_{0,\alpha}(\kappa)$. Unfortunately [Kunen, 1971] shows that even ω-huge cardinals in this sense do not exist. Kunen's proof uses the following result of [Erdös and Hajnal, 1966]:

Theorem 6.2.12. *Let λ be an infinite cardinal. There is a function $f : [\lambda]^\omega \to \lambda$ such that*

$$(\forall X \subseteq \lambda)(|X| = \lambda \ \rightarrow \ f''[X]^\omega = \lambda).$$

($[X]^\omega = [X]^{\aleph_0}$ = the set of all countably infinite subsets of X.)

Proof. The proof we give is from [Galvin and Prikry, 1976]. Let $\mathcal{E}$ be the set of all elements x of $[\lambda]^\omega$ such that $\mathrm{ot}(x) = \omega$. For x and y belonging to $\mathcal{E}$, say that $x \sim y$ if the symmetric difference of x and y is finite. For $x \in \mathcal{E}$, let $[x]$ be the equivalence class of x. Let g be a choice function for the set of all equivalence classes, i.e. let $g([x]) \in [x]$ for each $x \in \mathcal{E}$. Let $h : \mathcal{E} \to \lambda$ be given by

$$h(x) = \begin{cases} \text{the greatest element of } g([x]) \setminus x \text{ if } g([x]) \not\subseteq x; \\ 0 \text{ otherwise.} \end{cases}$$

We will show that there is an $A \subseteq \lambda$ such that $|A| = \lambda$ and such that

$$(\forall X \subseteq A)(|X| = \lambda \ \rightarrow \ h''([X]^\omega \cap \mathcal{E}) \supseteq A).$$

Given such an A, one can easily construct an f with the required properties.

Suppose that no such A exists. We construct a strictly increasing sequence $\langle \alpha_i \mid i \in \omega \rangle$ of elements of λ and a sequence $\langle B_i \mid i \in \omega \rangle$ of subsets of λ of cardinality λ. Let $B_0 = \lambda$. Given B_i, let $\alpha_i \in B_i$ and $B_{i+1} \subseteq B_i \setminus \alpha_i + 1$ be such that $|B_{i+1}| = \lambda$ and

$$\alpha_i \notin h''([B_{i+1}]^\omega \cap \mathcal{E}).$$

The existence of such a pair follows easily from the nonexistence of A. Now let $x = \{\alpha_i \mid i \in \omega\}$. Let α_n be the least element of x that is larger than every element of the symmetric difference of $g([x])$ and x. Let $y = \{\alpha_i \mid i > n\}$. Now $h(y) = \alpha_n$, since α_n is the greatest element of $g([x]) \setminus y$. But this is a contradiction, for $y \subseteq B_{n+1}$. $\square$

Now we are ready to prove Kunen's theorem.

Theorem 6.2.13. ([Kunen, 1971]) *Let $j : V \prec M$ with M transitive. Let $\kappa = \mathrm{crit}\,(j)$. Let $\lambda = j_{0,\omega}(\kappa)$. ($\lambda$ can also be characterized as the least fixed point of j that is greater than κ.) Then $j''\lambda \notin M$.*

Proof. Let $f : [\lambda]^\omega \to \lambda$ be given by Theorem 6.2.12. By the elementarity of j, if $X \subseteq \lambda$ belongs to M and if $|X| = \lambda$, then $j(f)''[X]^\omega = \lambda$. We will prove that $j''\lambda \notin M$ by showing that $j(f)''[j''\lambda]^\omega \neq \lambda$.

Since $\omega < \kappa = \mathrm{crit}\,(j)$, it is easy to see that

$$(\forall x \in [\lambda]^\omega)\, j(x) = \{j(\alpha) \mid \alpha \in x\}.$$

In particular this means that every $y \in [j''\lambda]^\omega$ belongs to the range of $j \restriction [\lambda]^\omega$. If $y \in [j''\lambda]^\omega$ and $y = j(x)$, then $(j(f))(y) = (j(f))(j(x)) = j(f(x))$. This shows that

$$j(f)''[j''\lambda]^\omega \subseteq j''\lambda \neq \lambda.$$

(That $j''\lambda \neq \lambda$ follows from the fact that $\kappa \in \lambda \setminus j''\lambda$.) □

For other proofs of Theorem 6.2.13, see §23 of [Kanamori, 1994].

Kunen's theorem and its proof give some more negative results. For $j : V \prec M$ or $j : V_\eta \prec M$ with M transitive, let us for the moment denote by λ the first fixed point of j greater than $\mathrm{crit}\,(j)$, if it exists.

Theorem 6.2.14. *(1) If $j : V \prec M$, j is not the identity, and M is transitive, then (a) $V_{\lambda+1} \notin M$ and (b) ${}^\omega(V_\lambda) \not\subseteq M$. (2) There is no $j : V_{\lambda+2} \prec V_{\lambda+2}$.*

Proof. (1)(a) follows immediately from Theorem 6.2.13. (1)(b) follows from the fact that, since $\mathrm{cf}(\lambda) = \omega$, every subset of λ is the union of countably many elements of V_λ. (Of course, (1)(b) implies (1)(a).) To verify (2), note that the f of the proof of Theorem 6.2.13 belongs to $V_{\lambda+2}$. □

No inconsistency has been derived from any of the following (where we continue to use "λ" as above, so that the embeddings are implicitly asserted to be non-trivial):

(a) There is a $j : V_\lambda \prec V_\lambda$.

(b) There is a $j : V \prec M$ with M transitive and $V_\lambda \subseteq M$.

(c) There is a $j : V_{\lambda+1} \prec V_{\lambda+1}$.

(d) There is a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$.

These assertions are listed in order of (strictly) increasing strength. [Martin, 1980] proved the determinacy of all $\mathbf{\Pi}^1_2$ games from a hypothesis intermediate between (a) and (b). Hugh Woodin (unpublished) subsequently proved $\mathrm{AD}^{L(\mathcal{R})}$ and much more from (d). The determinacy proofs we will give in Chapters 8 and 9 will have, of course, much weaker hypotheses.

No inconsistency with ZFC minus Choice is known for the existence of an elementary embedding of the whole universe into itself.

Kunen's results, as he noted, make it possible to simplify the definitions of η-strong and λ-supercompact in the way mentioned earlier:

Theorem 6.2.15. *Let κ be a cardinal number.*

(1) For cardinal numbers $\lambda \geq \kappa$, κ is λ-supercompact if and only if there is a $j : V \prec M$ such that M is transitive, $\mathrm{crit}\,(j) = \kappa$, *and* ${}^{\lambda}M \subseteq M$.

(2) For ordinal numbers $\eta > \kappa$, κ is η-strong if and only if there is a $j : V \prec M$ such that M is transitive, $\mathrm{crit}\,(j) = \kappa$, *and $V_\eta \subseteq M$.*

Proof. (1) Let $\lambda \geq \kappa$. Clearly we need only prove the "if" part. Let $j : V \prec M$ be as in the statement of (1). For ordinals α, let $\kappa_\alpha = j_{0,\alpha}(\kappa)$ and let $M_\alpha = j_{0,\alpha}(V)$. (See §3.3.) By Theorem 6.2.13, $\lambda < \kappa_\omega$. Let n be the least number such that $\lambda < \kappa_n$. Since $j_{i,i+1} = j_{0,i}(j)$ for each i, the elementarity of $j_{0,i}$ implies that $(M_i \cap {}^{j_{0,i}(\lambda)}M_{i+1}) \subseteq M_{i+1}$ and so that $(M_i \cap {}^{\lambda}M_{i+1}) \subseteq M_{i+1}$. By induction we then get that ${}^{\lambda}M_n \subseteq M_n$. Thus $j_{0,n} : V \prec M_n$ witnesses that κ is λ-supercompact.

(2) Let $\eta > \kappa$. As for (1) we need only proof the "if" part. Let $j : V \prec M$ be as in the statement of (2). Define κ_α and M_α, $\alpha \in \mathrm{Ord}$, as in the proof of (1). By Theorem 6.2.14, we know that $\eta \leq \kappa_\omega$. An argument as in the proof of (1) shows that $V_\eta \subseteq M_i$ for each $i \in \omega$. Since $\eta \leq \kappa_\omega$, this implies that $V_\eta \subseteq M_\omega$. Since $\kappa_\omega = \mathrm{crit}\,(j_{\omega,\omega+1})$, we finally get that $V_\eta \subseteq M_{\omega+1}$. Thus $j_{0,\omega+1} : V \prec M_{\omega+1}$ witnesses that κ is η-strong. (If $\eta < \kappa_\omega$ then $j_{0,n}$ will also work for any n such that $\eta < \kappa_n$.) $\square$

Exercise 6.2.1. (a) Show that if κ is superstrong then there are measurable cardinals larger than κ.

(b) Let κ be any cardinal number. Show that either κ is not superstrong or else there is an inaccessible $\delta > \kappa$ such that $V_\delta \models$ "κ is not superstrong."

(c) Show that if κ is supercompact and $\delta > \kappa$ is inaccessible then $V_\delta \models$ "κ is supercompact."

(d) Prove that if ZFC + "There is a supercompact cardinal" is consistent then so is ZFC + "There is a supercompact cardinal that is not superstrong."

Hint. For (a) first show that κ is measurable in M, where $j : V \prec M$ witnesses that κ is superstrong. Use this to show that κ, and hence, $j(\kappa)$, is a limit of measurable cardinals. For (c) use Theorem 6.2.1.

Exercise 6.2.2. Let κ be a regular cardinal. Assume that the set of Woodin cardinals smaller than κ is stationary in κ. (See Exercise 3.2.7 for the definition of *stationary*.) Prove that κ is Woodin.

Exercise 6.2.3. Prove that every Woodin cardinal is Mahlo. (See Exercise 3.2.7.)

Exercise 6.2.4. Let $n \in \omega$ and let κ be a cardinal number. Assume that there is a $j : V \prec M$ such that M is transitive, $\operatorname{crit}(j) = \kappa$, and $V_{j_{0,n+1}(\kappa)} \subseteq M$. Prove that κ is n-huge.

Hint. The proof of the (1) $\Rightarrow$ (2) part of Theorem 6.2.10 goes through under our present hypotheses.

6.3 Equivalents of Woodinness

The main aim of this section is to prove a property of Woodin cardinals (actually an equivalent of Woodinness) that will be the basis for our constructions in the determinacy proofs of Chapter 8 and to prove the technical consequence of this property that is actually used in the constructions.

We begin by giving an equivalent of Woodinness that is very useful in applications and that throws into clear relief the relation between Woodin cardinals and strong cardinals.

If A is any class, κ is a cardinal, and $\eta > \kappa$ is an ordinal, then κ is *η-strong in A* if there is a $j : V \prec M$ such that

(i) j witnesses that κ is η-strong;

(ii) $j(A) \cap V_\eta = A \cap V_\eta$.

The following fact was surely first noticed by Woodin.

Theorem 6.3.1. *Let κ be any infinite cardinal number. The following are equivalent:*

(1) κ is Woodin.

(2) $(\forall A \subseteq V_\kappa)(\exists \delta < \kappa)(\forall \eta)(\delta < \eta < \kappa \;\rightarrow\; \delta$ *is* η*-strong in* A*).*

Proof. First suppose that κ satisfies (2). Let $f : \kappa \to \kappa$. By (2) let $\delta < \kappa$ be such that δ is η-strong in f (i.e. in graph (f)) for all η, $\delta < \eta < \kappa$. For $\beta \leq \delta$, let

$$\eta_\beta = \max\{\beta, f(\beta)\} + 3$$

and let $j_\beta : V \prec M_\beta$ witness that δ is η_β-strong in f. For any $\beta \leq \delta$, we have that $\langle \beta, f(\beta)\rangle \in f \cap V_{\eta_\beta}$ and so that $\langle \beta, f(\beta)\rangle \in j_\beta(f) \cap V_{\eta_\beta}$. But this means that

$$(\forall \beta \leq \delta)(j_\beta(f))(\beta) = f(\beta) < \eta_\beta.$$

Taking $\beta = \delta$, we deduce that $(j_\delta(f))(\delta) < \eta_\delta$. Hence

$$V_{(j_\delta(f))(\delta)} \subseteq V_{\eta_\delta} \subseteq M_{\eta_\delta}.$$

To show that δ and j_δ witness that κ is Woodin for f, we need only prove that δ is closed under f. For this, assume that $\beta < \delta$. Then

$$f(\beta) = (j_\beta(f))(\beta) = (j_\beta(f))(j_\beta(\beta)) = j_\beta(f(\beta)).$$

But this implies that $f(\beta) < \delta$, for if $f(\beta) \geq \delta$ then $j_\beta(f(\beta)) \geq j_\beta(\delta) > \eta_\beta > f(\beta)$.

Now we turn to the proof that (1) implies (2). This will be similar to the proof of part (2) of Theorem 6.2.9, with one ingredient missing (the ordinal β) and another ingredient added. Suppose that κ is Woodin. Let $A \subseteq V_\kappa$. Assume that (2) fails for A. For $\alpha < \kappa$ let $\eta(\alpha)$ be the least $\eta > \alpha$ such that α is not η-strong in A. For $\alpha < \kappa$ let

$$g(\alpha) = \max\{\eta(\alpha) + 1,\, |V_{\eta(\alpha)}|\} + 1.$$

As with the analogous function in the proof of Theorem 6.2.9, we have that $g(\alpha) \geq \alpha + 2$ for all α and that $g : \kappa \to \kappa$. Let $\delta < \kappa$ and $j : V \prec M$ witness that κ is Woodin for g. Let

$$\lambda = \max\{(j(\eta))(\delta) + 1,\, |V_{(j(\eta))(\delta)}|\}.$$

Let E be the (δ, λ)-extender derived from j. As in the proof of Theorem 6.2.9, we get that E is an extender in M and that

$$V^{\mathrm{Ult}(M;E)}_{(j(\eta))(\delta)} = V^{\mathrm{Ult}(V;E)}_{(j(\eta))(\delta)} = V^{M}_{(j(\eta))(\delta)}.$$

To derive the contradiction that i_E witnesses in M that δ is $(j(\eta))(\delta)$-strong in $j(A)$, we need only show that

$$i_E^M(j(A)) \cap V^M_{(j(\eta))(\delta)} = j(A) \cap V^M_{(j(\eta))(\delta)}.$$

Let $k : \mathrm{Ult}(V;E) \prec M$ be defined as on page 330. Since $(j(\eta))(\delta) < \lambda$, Lemma 6.1.10 implies that $(i_E(\eta))(\delta) = (j(\eta))(\delta)$. (This is proved using k just as we showed, in the proof of Theorem 6.2.7, that $(i_E(f))(\kappa) = (j(f))(\kappa)$.) It follows that $(j(\eta))(\delta) < i_E(\delta) =$ (by Lemma 6.2.5) $i_E^M(\delta)$. Since $A \cap V_\delta = j(A) \cap V_\delta$, we have that $i_E^M(A) \cap V_{i_E^M(\delta)} = i_E^M(j(A)) \cap V_{i_E^M(\delta)}$. What we must prove is thus that

$$i_E^M(A) \cap V^M_{(j(\eta))(\delta)} = j(A) \cap V^M_{(j(\eta))(\delta)}.$$

By Lemma 6.1.10, $k \circ i_E = j$. By Lemma 6.1.11, $k \restriction V^{\mathrm{Ult}(V;E)}_{(j(\eta))(\delta)}$ is the identity. If $x \in V^M_{(j(\eta))(\delta)}$ then $x \in V^{\mathrm{Ult}(V;E)}_{(j(\eta))(\delta)}$ and

$$x \in i_E(A) \;\leftrightarrow\; k(x) \in k(i_E(A)) \;\leftrightarrow\; x \in j(A).$$

We are thus finally reduced to showing that

$$i_E^M(A) \cap V^M_{(j(\eta))(\delta)} = i_E(A) \cap V^M_{(j(\eta))(\delta)}.$$

But this follows from Lemma 6.2.5. □

The (1) ⇒ (2) half of Theorem 6.3.1 isn't the full "A-strong" analogue of part (2) of Theorem 6.2.9. The latter says that a certain set is unbounded in κ, while the former says only that the analogous set is non-empty. The full analogue is nevertheless true. The next theorem records this fact.

Theorem 6.3.2. *Let κ be any infinite cardinal number. The following are equivalent:*

(1) κ is Woodin.

(2) For all $A \subseteq V_\kappa$ the set of cardinals $\delta < \kappa$ such that

$$(\forall \eta)(\delta < \eta < \kappa \;\rightarrow\; \delta \text{ is } \eta\text{-strong in } A)$$

is unbounded in κ.

Proof. That (2) implies (1) follows from Theorem 6.3.1. That (1) implies (2) can be demonstrated by a routine combination of the proofs of the corresponding half of Theorem 6.3.1 and part (2) of Theorem 6.2.9. We leave this to the reader. □

Remark. Theorem 6.3.2 remains true if "unbounded" is replaced by "stationary" in its statement. See Exercise 6.3.1. The hint for that exercise also indicates a way to prove (2) of Theorem 6.3.2 directly from (2) of Theorem 6.3.1.

From now through Theorem 6.3.8, we will be showing that Woodinness of κ is witnessed by embeddings coming from extenders in V_κ, extenders whose ultrapowers may be taken to have certain closure properties. These results are pretty routine, but they should be attributed to Woodin if to anyone.

We begin with the following fact, which gives another useful strengthening of (2) of Theorem 6.3.1.

Theorem 6.3.3. *Let κ be a strong limit cardinal. Let $A \subseteq \kappa$. Let $\delta < \eta < \kappa$ be such that δ is η-strong in A. Then there is an extender $E \in V_\kappa$ such that i_E witnesses that δ is η-strong in A.*

Proof. Let $j : V \prec M$ witness that δ is η-strong in A. Let E be the $(\delta, |V_{\eta+1}|)$-extender derived from j. Lemma 6.1.11 implies that $V_\eta^{\mathrm{Ult}(V;E)} = V_\eta^M$ and gives the first and third equalities of the following chain.

$$\begin{aligned} i_E(A) \cap V_\eta &= k(i_E(A) \cap V_\eta) \\ &= k(i_E(A)) \cap k(V_\eta) \\ &= k(i_E(A)) \cap V_\eta \\ &= j(A) \cap V_\eta \\ &= A \cap V_\eta . \end{aligned}$$

Here k is as usual. □

Remark. The proof of the (1) ⇒ (2) part of Theorem 6.3.1 could have been simplified very slightly if we had, in analogy with the proof of Theorem 6.3.3, defined λ as $|V_{(j(\eta))(\delta)+1}|$. We chose instead to keep a closer correspondence with the proof of Theorem 6.2.9.

Theorem 6.3.4. *Let κ be Woodin and let $f : \kappa \to \kappa$. There is an extender $E \in V_\kappa$ such that* crit (i_E) *and i_E witness that κ is Woodin for f.*

Proof. As in the first half (the (2) $\Rightarrow$ (1) half) of the proof of Theorem 6.3.1, let $A = f$. Also let δ, η, and $j : V \prec M$ be as in the first half of the proof of Theorem 6.3.1. By Theorem 6.3.3, let E be an extender in V_κ such that i_E witnesses that δ is η-strong in A. The first half of the proof of Theorem 6.3.1 shows that $\operatorname{crit}(i_E)$ and i_E witness that κ is Woodin for f. □

Corollary 6.3.5. *Let κ be Woodin and let M be a transitive class model of* ZFC *such that $V_\kappa \subseteq M$. Then $M \models$ "κ is Woodin."*

Proof. The corollary follows easily from Theorem 6.3.4 or from Theorems 6.3.1 and 6.3.3. □

In Chapter 9 we will need to know that we can demand, of δ and $j : V \prec M$ witnessing Woodinness of a cardinal κ, that ${}^{<\delta}M \subseteq M$. In fact, we can demand even that ${}^{\delta}M \subseteq M$. In order to prove this, we need the following lemma.

Lemma 6.3.6. *Let E be a (δ, λ)-extender such that ${}^{\delta}\lambda \subseteq \mathrm{Ult}(V; E)$. Then ${}^{\delta}\mathrm{Ult}(V; E) \subseteq \mathrm{Ult}(V; E)$.*

Proof. Let $\langle x_\beta \mid \beta < \delta \rangle$ be elements of M. For $\beta < \delta$ let $\pi_E([\![a_\beta, f_\beta]\!]) = x_\beta$. The hypothesis that ${}^{\delta}\lambda \subseteq \mathrm{Ult}(V; E)$ implies that ${}^{\delta}([\lambda]^{<\omega}) \subseteq \mathrm{Ult}(V; E)$. Hence $\langle a_\beta \mid \beta < \delta \rangle \in \mathrm{Ult}(V; E)$. Define $g : {}^{<\delta}([\delta]^{<\omega}) \to V$ whose values are functions as follows. For $h \in {}^{<\delta}([\delta]^{<\omega})$, set

(i) $\operatorname{domain}(g(h)) = \operatorname{domain}(h)$;

(ii) $(g(h))(\beta) = f_\beta(h(\beta))$ for all $\beta \in \operatorname{domain}(h)$.

The function $\langle a_\beta \mid \beta < \delta \rangle$ belongs to $\operatorname{domain}(i_E(g))$. Thus

$$\operatorname{domain}((i_E(g))(\langle a_\beta \mid \beta < \delta \rangle)) = \delta$$

and, for all $\beta < \delta$, Lemma 6.1.12 gives that

$$\begin{aligned} ((i_E(g))(\langle a_\beta \mid \beta < \delta \rangle))(\beta) &= (i_E(f_\beta))(a_\beta) \\ &= \pi_E([\![a_\beta, f_\beta]\!]) \\ &= x_\beta. \end{aligned}$$

Thus $(i_E(g))(\langle a_\beta \mid \beta < \delta \rangle) = \langle x_\beta \mid \beta < \delta \rangle$. □

Theorem 6.3.7. *Let κ be inaccessible. Let $A \subseteq \kappa$. Let $\delta < \kappa$ be such that, for all η such that $\delta < \eta < \kappa$, δ is η-strong in A. Then, for all η such that $\delta < \eta < \kappa$, there is an extender $E \in V_\kappa$ such that i_E witnesses that δ is η-strong in A and such that ${}^\delta\mathrm{Ult}(V;E) \subseteq \mathrm{Ult}(V;E)$.*

Proof. Let $\delta < \eta < \kappa$. Let λ be a strong limit cardinal of cofinality $> \delta$ such that $\eta \leq \lambda < \kappa$. By Theorem 6.3.3, let $\hat{E} \in V_\kappa$ be an extender such that $i_{\hat{E}}$ witnesses that δ is λ-strong in A. Let E be the (δ, λ)-extender derived from $i_{\hat{E}}$. Lemma 6.1.11 implies that $V_\lambda \subseteq \mathrm{Ult}(V;E)$ and implies that i_E witnesses that δ is λ-strong, and therefore η-strong, in A. (See the proof of Lemma 6.3.3.) Since $\mathrm{cf}(\lambda) > \delta$, we have that ${}^\delta\lambda \subseteq V_\lambda \subseteq \mathrm{Ult}(V;E)$. By Lemma 6.3.6, we get that ${}^\delta\mathrm{Ult}(V;E) \subseteq \mathrm{Ult}(V;E)$. □

Theorem 6.3.8. *Let κ be a Woodin cardinal and let $f : \kappa \to \kappa$. There is an extender $E \in V_\kappa$ such that $\delta = \mathrm{crit}\,(i_E)$ and E witness that κ is Woodin for f and such that ${}^\delta\mathrm{Ult}(V;E) \subseteq \mathrm{Ult}(V;E)$.*

Proof. The proof is just like that of Theorem 6.3.4, except that we apply Theorem 6.3.7 instead of Theorem 6.3.3. □

We now turn to an equivalent of Woodinness that is rather technical but that will play a central role in the constructions of Chapter 8. The definitions and results that follow are, with minor changes, from [Martin and Steel, 1988] and [Martin and Steel, 1989].

For ordinals α, β, and $\gamma \geq \alpha$, let $\mathcal{L}^\alpha_{\gamma,\beta}$ be the result of adding to the language of set theory a constant c_a for each element a of V_α and, if $\beta > 0$, a constant d. Let $\mathcal{V}^\alpha_{\gamma,\beta}$ be the expansion of the model $(V_{\gamma+\beta}; \in)$ gotten by interpreting each c_a by a and, if $\beta > 0$, d by γ. For $z \in {}^{<\omega}(V_{\gamma+\beta})$, let $\mathrm{tp}^\alpha_{\gamma,\beta}(z)$ be the type realized by z in $\mathcal{V}^\alpha_{\gamma,\beta}$, i.e. let

$$\mathrm{tp}^\alpha_{\gamma,\beta}(z) = \{\varphi(v_1, \ldots, v_{\ell\mathrm{h}(z)}) \in \mathcal{L}^\alpha_{\gamma,\beta} \mid \mathcal{V}^\alpha_{\gamma,\beta} \models \varphi[z]\}.$$

We want to think of the objects $\mathrm{tp}^\alpha_{\gamma,\beta}(z)$ as sets, so let us choose a way of so representing them. Let the symbols of the language of set theory be the odd natural numbers in some reasonable order. Let the constant d be 0. For sets a let c_a be a itself unless $a \in \omega$; for $a \in \omega$ let c_a be $2a + 2$. It would be natural to take formulas simply to be finite sequences of symbols. We do not do so, for we would like to make the formulas of $\mathcal{L}^\alpha_{\gamma,\beta}$ be members of V_α for each infinite α, but ${}^{<\omega}(V_\alpha) \subseteq V_\alpha$ only for limit α. Instead we choose an injection $f : {}^{<\omega}V \to V$ with the property that $f''({}^{<\omega}(V_\alpha)) \subseteq V_\alpha$ for all infinite α, and we let the formula corresponding to a sequence s of symbols be $f(s)$. To be explicit, let $f \restriction {}^{<\omega}(V_\omega)$ be the identity and, for $\alpha \geq \omega$ and $s = \langle s_n \mid n < \ell\mathrm{h}(s)\rangle \in {}^{<\omega}(V_{\alpha+1}) \setminus {}^{<\omega}(V_\alpha)$, let

$$f(s) = \{f(\langle n, y\rangle) \mid n < \ell\mathrm{h}(s) \wedge y \in s_n\}.$$

It is easy to check that this f is one-one and that the rank of $f(s)$ is the maximum of the rank (s_n), $n < \ell\mathrm{h}(s)$, for all s of infinite rank. Thus we have, for all ordinals γ and β and all $z \in {}^{<\omega}(V_{\gamma+\beta})$,

$$\begin{array}{l}(\forall\alpha)(\omega \leq \alpha \leq \gamma \ \rightarrow\ \mathrm{tp}^{\alpha}_{\gamma,\beta}(z) \subseteq V_\alpha);\\ (\forall\alpha)(\forall\alpha^*)(\omega \leq \alpha \leq \alpha^* \leq \gamma \ \rightarrow\ \mathrm{tp}^{\alpha^*}_{\gamma,\beta}(z) \cap V_\alpha = \mathrm{tp}^{\alpha}_{\gamma,\beta}(z)).\end{array}$$

For ordinals β, limit ordinals γ, cardinals $\delta < \gamma$, and elements z of ${}^{<\omega}(V_{\gamma+\beta})$, we say that δ is *β-reflecting in z relative to γ* if

$$(\forall\eta)(\delta < \eta < \gamma \ \rightarrow\ \delta \text{ is } \eta\text{-strong in } \mathrm{tp}^{\gamma}_{\gamma,\beta}(z)).$$

Theorem 6.3.9. *Let κ be a cardinal. The following are equivalent.*

(1) κ is Woodin.

(2) For all ordinals β and for all $z \in {}^{<\omega}(V_{\kappa+\beta})$, the set of all $\delta < \kappa$ such that δ is β-reflecting in z relative to κ is unbounded in κ.

(3) For all $z \in {}^{<\omega}(V_{\kappa+1})$, there is a $\delta < \kappa$ such that δ is 1-reflecting in z relative to κ.

Proof. (1) implies (2) by Theorem 6.3.2. (2) trivially implies (3). Thus we need only show that (3) implies (1). Assume then that κ satisfies (3). Let $f : \kappa \to \kappa$. Let δ be 1-reflecting in $\langle f\rangle$ relative to κ. Let $\alpha < \kappa$ be such that

$$\alpha > \max\{\delta, \sup\{f(\xi) \mid \xi \leq \delta\}\}.$$

Let $j : V \prec M$ witness that δ is α-strong in $\mathrm{tp}^{\kappa}_{\kappa,1}(\langle f\rangle)$. Thus

$$\begin{aligned}(\mathrm{tp}^{\alpha}_{j(\kappa),1})^M(\langle j(f)\rangle) &= (\mathrm{tp}^{j(\kappa)}_{j(\kappa),1})^M(\langle j(f)\rangle) \cap V^M_\alpha\\ &= j(\mathrm{tp}^{\kappa}_{\kappa,1}(\langle f\rangle)) \cap V_\alpha\\ &= \mathrm{tp}^{\kappa}_{\kappa,1}(\langle f\rangle) \cap V_\alpha\\ &= \mathrm{tp}^{\alpha}_{\kappa,1}(\langle f\rangle).\end{aligned}$$

Let $\xi \leq \delta$ and let $\gamma = f(\xi)$. Then both ξ and $f(\xi)$ are smaller than α and so belong to V_α. The fact that $\gamma = f(\xi)$ is thus expressed by a member of $\mathrm{tp}^\alpha_{\kappa,1}(\langle f \rangle)$. Hence the same member of $\mathrm{tp}^\alpha_{j(\kappa),1}(\langle j(f) \rangle)$ expresses the fact that $(j(f))(\xi) = \gamma$. It follows that

$$(j(f))(\xi) = f(\xi) < \alpha.$$

For $\xi < \delta$ this gives us that $(j(f))(\xi) < \alpha < j(\delta)$ and so that $f(\xi) < \delta$. Thus δ is closed under f. For $\xi = \delta$ we get that $V_{(j(f))(\delta)} \subseteq V_\alpha \subseteq M$. Therefore j witnesses that κ is Woodin for f. □

Theorem 6.3.10. *Let κ be a strong limit cardinal, let $\delta < \kappa$ be a cardinal, let β be an ordinal, and let $z \in {}^{<\omega}(V_{\kappa+\beta})$. Then δ is β-reflecting in z relative to κ if and only if for all α such that $\delta < \alpha < \kappa$ there is an extender $E \in V_\kappa$ such that*

(a) $\mathrm{crit}\,(i_E) = \delta$*;*

(b) $\mathrm{strength}\,(E) \geq \alpha$ *(i.e. $V_\alpha \subseteq \mathrm{Ult}(V; E)$);*

(c) $\alpha < i_E(\delta)$*;*

(d) $\mathrm{tp}^\alpha_{\kappa,\beta}(z) = (\mathrm{tp}^\alpha_{i_E(\kappa), i_E(\beta)})^{\mathrm{Ult}(V;E)}(i_E(z))$.

Proof. The theorem follows easily from Theorem 6.3.3. □

If κ is inaccessible and E is an extender belonging to V_κ, then $i_E(\kappa) = \kappa$. Thus we have

Corollary 6.3.11. *Theorem 6.3.10 remains true if "κ is a strong limit cardinal" is replaced by "κ is inaccessible" and clause (d) is replaced by*

(e) $i_E(\kappa) = \kappa \;\wedge\; \mathrm{tp}^\alpha_{\kappa,\beta}(z) = (\mathrm{tp}^\alpha_{\kappa, i_E(\beta)})^{\mathrm{Ult}(V;E)}(i_E(z))$.

Lemma 6.3.12. *Let $n \in \omega$. There is a formula $\mathrm{TYPE}_n(v_1, \ldots, v_{n+4})$ of the language of set theory such that, for all α, β, γ, α', and β' with*

$$\omega \leq \alpha' < \alpha \leq \gamma \;\;\wedge\;\; \beta' < \beta,$$

for all $z \in {}^n(V_{\gamma+\beta'})$, and for all $a \in V_\alpha$,

$$\begin{aligned} & a = \mathrm{tp}^{\alpha'}_{\gamma,\beta'}(z) \\ & \leftrightarrow\; V_{\gamma+\beta} \models \mathrm{TYPE}_n[z^\frown\langle \beta', a, \alpha', \gamma\rangle] \\ & \leftrightarrow\; \mathrm{TYPE}_n(v_1, \ldots, v_{n+1}, c_a, c_{\alpha'}, d) \in \mathrm{tp}^\alpha_{\gamma,\beta}(z^\frown\langle \beta' \rangle). \end{aligned}$$

Thus $\mathrm{tp}^{\alpha'}_{\gamma,\beta'}(z)$ is identified by a single element of $\mathrm{tp}^\alpha_{\gamma,\beta}(z^\frown\langle \beta' \rangle)$.

The proof is routine, and we omit it.

Lemma 6.3.13. *Let $n \in \omega$. There is a formula* $\mathrm{REFL}_n(v_1, \ldots, v_{n+3})$ *of the language of set theory such that, for all κ, δ, β, β', and z such that κ is a strong limit cardinal, $\delta < \kappa$, $\beta' < \beta$, and $z \in {}^n(V_{\kappa+\beta'})$,*

$$\begin{aligned} &\delta \textit{ is } \beta'\textit{-reflecting in } z \textit{ relative to } \kappa \\ &\leftrightarrow\ V_{\kappa+\beta} \models \mathrm{REFL}_n[z^\frown\langle\beta', \delta, \kappa\rangle] \\ &\leftrightarrow\ \mathrm{REFL}_n(v_1, \ldots, v_{n+1}, c_\delta, d) \in \mathrm{tp}^{\delta+1}_{\kappa,\beta}(z^\frown\langle\beta'\rangle). \end{aligned}$$

Proof. The construction of REFL_n is a straightforward application of Theorem 6.3.10, except perhaps for the matter of clause (d) from that theorem. We have to say, for a (δ, λ)-extender $E \in V_\kappa$,

$$\mathrm{tp}^\alpha_{\kappa,\beta'}(z) = (\mathrm{tp}^\alpha_{i_E(\kappa),i_E(\beta')})^{\mathrm{Ult}(V;E)}(i_E(z))$$

by a formula of $\mathcal{L}^{\delta+1}_{\kappa,\beta}$, using the parameters β', z, κ, α, E, δ, and λ. What our formula must say can be rephrased as

> For all $a \in {}^{<\omega}(V_\alpha)$ and for every formula $\varphi(v_1, \ldots, v_{\ell\mathrm{h}(z)+\ell\mathrm{h}(a)+1})$ of the language of set theory,
>
> $$V_{\kappa+\beta'} \models \varphi[z^\frown a^\frown\langle\kappa\rangle] \ \leftrightarrow\ V^{\mathrm{Ult}(V;E)}_{i_E(\kappa)+i_E(\beta')} \models \varphi[i_E(z)^\frown a^\frown\langle i_E(\kappa)\rangle].$$

(For uniformity of notation, we are dealing only with the case $\beta' > 0$.) Here the only problem is with the second part of the biconditional, which can be rephrased as

> There are $b \in [\lambda]^{<\omega}$ and $f : [\delta]^{\ell\mathrm{h}(b)} \to V_\delta$ such that $a = \pi_E([\![b, f]\!]_E)$ and
>
> $$\{x \in [\delta]^{\ell\mathrm{h}(b)} \mid V_{\kappa+\beta'} \models \varphi[z^\frown f(x)^\frown\langle\kappa\rangle]\} \in E_b.$$

This is easily expressible by a formula of $\mathcal{L}^{\delta+1}_{\kappa,\beta}$. □

The property of being β-reflecting in z is preserved by "decreasing" z but not by decreasing β. Suppose that δ is β-reflecting in z relative to γ. If z' is a subsequence of the finite sequence z, then it clearly follows that δ is β-reflecting in z' relative to γ. On the other hand, it is not necessarily true that if $\beta' < \beta$ and $z \in {}^{<\omega}(V_{\gamma+\beta'})$ then δ is β'-reflecting in z relative to γ. (For γ Woodin, one can get a counterexample with $\beta = \kappa$ and $z = \emptyset$, using the fact that no δ can be δ-reflecting in $\emptyset$ relative to γ.) If, however, δ is also β-reflecting in $z^\frown\langle\beta'\rangle$ relative to γ, then it does follow easily that it is β'-reflecting in z. In particular, this is the case if β' is definable in $V_{\gamma+\beta}$ from elements of $V_\gamma \cup \{\gamma\}$, for then $\mathrm{tp}^\alpha_{\gamma,\beta}(z^\frown\langle\beta'\rangle)$ is, for all sufficiently large $\alpha < \gamma$, determined by $\mathrm{tp}^\alpha_{\gamma,\beta}(z)$. The following theorem gives an additional consequence of the assumption that δ is β-reflecting in $z^\frown\langle\beta'\rangle$ relative to γ, when γ is a strong limit cardinal.

Theorem 6.3.14. *Let κ be a strong limit cardinal, let β and $\beta' < \beta$ be ordinals, let $z \in {}^{<\omega}(V_{\kappa+\beta'})$, and let $\delta < \kappa$ be β-reflecting in $z^\frown\langle\beta'\rangle$ relative to κ. Then the set of $\delta' < \kappa$ such that δ' is β'-reflecting in z relative to κ is unbounded in κ.*

Proof. As we remarked above, the hypothesis of the theorem implies that δ is β'-reflecting in z relative to κ. Let $\delta < \alpha < \kappa$. Let $j : V \prec M$ witness that δ is $\alpha + 1$-strong in $\mathrm{tp}^\kappa_{\kappa,\beta}(z^\frown\langle\beta'\rangle)$. Then $j(\delta) > \alpha$ and

$$M \models \text{“}j(\delta) \text{ is } j(\beta')\text{-reflecting in } j(z) \text{ relative to } j(\kappa).\text{”}$$

Thus by Lemma 6.3.13,

$$V^M_{j(\kappa)+j(\beta)} \models \mathrm{REFL}_{\ell\mathrm{h}(z)}[j(z)^\frown\langle j(\beta'), j(\delta), j(\kappa)\rangle].$$

From this and the fact that $j(\kappa) > j(\delta) > \alpha$ it follows directly that $V^M_{j(\kappa)+j(\beta)}$ satisfies

$$(\exists v_{\ell\mathrm{h}(z)+2})\,(d > v_{\ell\mathrm{h}(z)+2} > v_{\ell\mathrm{h}(z)+4} \wedge \mathrm{REFL}_{\ell\mathrm{h}(z)})[j(z)^\frown\langle j(\beta'), j(\kappa), \alpha\rangle],$$

where α is assigned to $v_{\ell\mathrm{h}(z)+4}$. Hence

$$(\exists v_{\ell\mathrm{h}(z)+2})(d > v_{\ell\mathrm{h}(z)+2} > c_\alpha \wedge \mathrm{REFL}_{\ell\mathrm{h}(z)}(v_1, \ldots, v_{\ell\mathrm{h}(z)+2}, d))$$

belongs to $(\mathrm{tp}^{\alpha+1}_{j(\kappa),j(\beta)})^M(j(z)^\frown\langle j(\beta')\rangle)$. But then this formula also belongs to $\mathrm{tp}^{\alpha+1}_{\kappa,\beta}(z^\frown\langle\beta'\rangle)$, and so there is a δ' such that $\kappa > \delta' > \alpha$ and δ' is β'-reflecting in z relative to κ. $\square$

We will not make any direct use of Theorem 6.3.14. In our constructions κ will be Woodin, and therefore (2) of Theorem 6.3.9 will give us anything we could get from Theorem 6.3.14.

The remainder of this section will be devoted to establishing a technical lemma that *will* be directly used in Chapter 8.

If X and Y are classes and α is an ordinal, let us say that X and Y *agree through* α if $X \cap V_\alpha = Y \cap V_\alpha$.

Suppose that M and N are transitive class models of ZFC and that M and N agree through $\kappa + 1$. Suppose that $E \in M$ is a (κ, λ)-extender in M for some λ. Then we can define an ultrapower

$$\prod_E^N (N; \in)$$

of N with respect to E. The universe of this ultrapower is

$$\{[\![a, f]\!]_E^N \mid \langle a, f\rangle \in \mathcal{D}_E^N\},$$

where

$$\mathcal{D}_E^N = \{\langle a, f\rangle \mid a \in [\lambda]^{<\omega} \wedge f \in N \wedge f : [\kappa]^{|a|} \to N\}.$$

The relation of $\prod_E^N(N; \in)$, which we call $\in_E^N$, is given by

$$[\![a, f]\!]_E^N \in_E^N [\![b, g]\!]_E^N \leftrightarrow \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) \in g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

It is easy to check that Łoś' Theorem generalizes to such ultrapowers:

Theorem 6.3.15. *Let M, N, E, κ, and λ be as in the preceding two paragraphs. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1\rangle, \ldots, \langle a_n, f_n\rangle$ be elements of $\mathcal{D}_E^N$. Let $b = \bigcup_{1\le i\le n} a_i$. Then*

$$\begin{aligned}&\textstyle\prod_E^N(N; \in) \models \varphi[[\![a_1, f_1]\!]_E^N, \ldots, [\![a_n, f_n]\!]_E^N] \leftrightarrow \\ &\quad \{z \in [\kappa]^{|b|} \mid (N; \in) \models \varphi[f_1(z_{a_1,b}), \ldots, f_n(z_{a_n,b})]\} \in E_b.\end{aligned}$$

Thus we get a canonical $(i')_E^N : (N; \in) \prec \prod_E^N(N; \in)$.

It is also easy to prove that Lemma 3.2.9 generalizes to these ultrapowers:

Lemma 6.3.16. *Let M, N, and E be as above. Then $\prod_E^N(N; \in)$ is set-like.*

Unfortunately wellfoundedness does not in general hold. In the next section we will prove wellfoundedness for important special cases. If $\prod_E^N(N; \in)$ is wellfounded, let us denote by $\mathrm{Ult}(N; E)$ the unique transitive class N' such that $\prod_E^N(N; \in) \cong (N'; \in)$ and let us denote by π_E^N the unique isomorphism. In this case we get as usual the canonical

$$i_E^N : N \prec \mathrm{Ult}(N; E),$$

given by $i_E^N = \pi_E^N \circ (i')_E^N$.

Lemma 6.3.17. *Let M be a transitive class model of* ZFC*. Let E be a (κ,λ)-extender in M. Let $\zeta \geq \kappa$ be an ordinal of M. Let N be a transitive class model of* ZFC *such that M and N agree through $\zeta{+}1$. Assume that $\prod_E^N(N;\in)$ is wellfounded. Then*

(a) If $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to (\zeta^+)^M \cup V^M_{\zeta+1}$, then

$$\pi_E^M([\![a,f]\!]_E^M) = \pi_E^N([\![a,f]\!]_E^N);$$

(b) $(\forall\alpha)(\alpha \leq (\zeta^+)^M \to i_E^M(\alpha) = i_E^N(\alpha))$; in particular, $i_E^M(\kappa) = i_E^N(\kappa)$;

(c) $\mathrm{Ult}(M;E)$ *and* $\mathrm{Ult}(N;E)$ *agree through $i_E^M(\zeta){+}1$; in particular, they agree through $i_E^M(\kappa)+1$.*

Proof. For $n \in \omega$, the models M and N have exactly the same functions $f : [\kappa]^n \to (\zeta^+)^M \cup V^M_{\zeta+1}$ $(= (\zeta^+)^N \cup V^N_{\zeta+1})$. (See the proof of Lemma 6.2.5.) This implies (a), from which (b) and (c) follow. □

Remark. Parts (b) and (c) of Lemma 6.3.17 are analogous to parts (ii) and (iii) of Lemma 6.2.5. The analogue of (a) is true in the case of the earlier lemma; we simply didn't bother to state it as part of the lemma.

The technical lemma that follows is called the "One-Step Lemma" in [Martin and Steel, 1989], and we will use the same name for it here. It will be used in Chapter 8. Readers may want to skip it and return to it when its use is imminent. For those who do not skip it, the remarks after its proof may be of some help in understanding it.

Lemma 6.3.18 (One-Step-Lemma) *Let M and N be transitive class models of* ZFC*. Let $\kappa \in M \cap N$ be inaccessible in V and Woodin in M. Let δ and η be ordinals such that $\delta \leq \eta < \kappa$. Let β and $\xi < \beta$ be ordinals of M. Let β' be an ordinal of N. Let x and y belong to ${}^{<\omega}(V^M_{\kappa+\beta})$ and let x' belong to ${}^{<\omega}(V^N_{\kappa+\beta'})$ with $\ell\mathrm{h}(x') = \ell\mathrm{h}(x)$. Let $\chi(v)$ be a formula of the language of set theory. Suppose that*

(1) M and N agree through $\delta + 1$;

(2) $(\mathrm{tp}^\delta_{\kappa,\beta})^M(x) = (\mathrm{tp}^\delta_{\kappa,\beta'})^N(x')$;

(3) δ is β-reflecting in x relative to κ in M;

(4) $V^M_{\kappa+\beta} \models \chi[\xi]$.

Then there are a $\lambda < \kappa$ and an E such that E is a (δ, λ)-extender in M and such that either (a) $\prod_E^N(N; \in)$ is illfounded or (b) there are δ^*, ξ^*, and y^* such that $\eta < \delta^* < i_E^N(\delta) < \kappa$, $\xi^* < i_E^N(\beta')$, $i_E^N(x')$ and y^* both belong to ${}^{<\omega}(V_{\kappa+\xi^*}^{\mathrm{Ult}(N;E)})$, and

(1)* $\mathrm{Ult}(N;E)$ *and* M *agree through* $\delta^* + 1$*;*

(2)* $(\mathrm{tp}_{\kappa,\xi^*}^{\delta^*})^{\mathrm{Ult}(N;E)}(i_E^N(x')^\frown y^*) = (\mathrm{tp}_{\kappa,\xi}^{\delta^*})^M(x^\frown y)$*;*

(3)* δ^* *is* ξ^**-reflecting in* $i_E^N(x')^\frown y^*$ *relative to* κ *in* $\mathrm{Ult}(N;E)$*;*

(4)* $V_{\kappa+i_E^N(\beta')}^{\mathrm{Ult}(N;E)} \models \chi[\xi^*]$.

Furthermore, let α be any ordinal of $\mathrm{Ult}(N;E)$ *and let z be any element of ${}^{<\omega}(V_{\kappa+\alpha}^{\mathrm{Ult}(N;E)})$ such that*

$$(\mathrm{tp}_{\kappa,\alpha}^{\delta^*+1})^{\mathrm{Ult}(N;E)}(z) = (\mathrm{tp}_{\kappa,i_E^N(\beta')}^{\delta^*+1})^{\mathrm{Ult}(N;E)}(i_E^N(x')).$$

Then there are $\hat{\xi}$ and $\hat{y}$ such that $\hat{\xi} < \alpha$, $\hat{y} \in {}^{<\omega}(V_{\kappa+\hat{\xi}}^{\mathrm{Ult}(N;E)})$, and

($\hat{2}$) $(\mathrm{tp}_{\kappa,\hat{\xi}}^{\delta^*})^{\mathrm{Ult}(N;E)}(z^\frown \hat{y}) = (\mathrm{tp}_{\kappa,\xi}^{\delta^*})^M(x^\frown y)$*;*

($\hat{3}$) δ^* *is* $\hat{\xi}^*$*-reflecting in* $z^\frown \hat{y}$ *relative to* κ *in* $\mathrm{Ult}(N;E)$*;*

($\hat{4}$) $V_{\kappa+\alpha}^{\mathrm{Ult}(N;E)} \models \chi[\hat{\xi}]$.

Proof. By Theorem 6.3.9, let δ^* be such that $\eta < \delta^* < \kappa$ and δ^* is ξ-reflecting in $x^\frown y$ in M. By (3) and Corollary 6.3.11, let $\lambda < \kappa$ and E be such that E is a (δ, λ)-extender in M, $\mathrm{strength}^M(E) \geq \delta^* + 1$, $i_E^M(\kappa) = \kappa$, and

$$(\mathrm{tp}_{\kappa,i_E^M(\beta)}^{\delta^*+1})^{\mathrm{Ult}(M;E)}(i_E^M(x)) = (\mathrm{tp}_{\kappa,\beta}^{\delta^*+1})^M(x).$$

Assume that $\prod_E^N(N; \in)$ is wellfounded, since otherwise there is nothing to prove. Note that the inaccessibility of κ in V guarantees that $i_E^N(\kappa) = \kappa$.

$\mathrm{Ult}(M;E)$ and M agree through $\delta^* + 1$. By hypothesis (1) and part (c) of Lemma 6.3.17, $\mathrm{Ult}(N;E)$ and $\mathrm{Ult}(M;E)$ also agree through $\delta^* + 1$. Thus we have (1*).

Before choosing ξ^* and y^*, let us prove that

$$(\mathrm{tp}_{\kappa,i_E^N(\beta')}^{\delta^*+1})^{\mathrm{Ult}(N;E)}(i_E^N(x')) = (\mathrm{tp}_{\kappa,\beta}^{\delta^*+1})^M(x).$$

For this it is enough to show that

$$(\mathrm{tp}^{\delta^*+1}_{\kappa,i^N_E(\beta')})^{\mathrm{Ult}(N;E)}(i^N_E(x')) = (\mathrm{tp}^{\delta^*+1}_{\kappa,i^M_E(\beta)})^{\mathrm{Ult}(M;E)}(i^M_E(x)).$$

Let $\varphi(v_1,\ldots,v_{\ell\mathrm{h}(x)+n+1})$ be a formula of the language of set theory and let $b = \langle b_1,\ldots,b_n\rangle \in {}^n(V^{\mathrm{Ult}(M;E)}_{\delta^*+1})$. For $1 \le j \le n$ let $\langle a_j, f_j\rangle$ be such that $b_j = \pi^M_E([\![a_j, f_j]\!]^M_E)$. Letting $a = \bigcup_{1\le j\le n} a_j$ and replacing each f_j by $f_j^{a_j,a}$, we may assume that each $a_j = a$ and so that each $b_j = \pi^M_E([\![a, f_j]\!]^M_E)$. Since $\delta^* < \mathrm{strength}^M(E) \le \lambda < i^M_E(\delta)$, we may assume that each $f_j : [\delta]^{|a|} \to V_\delta$. By part (a) of Lemma 6.3.17, $b_j = \pi^N_E([\![a, f_j]\!]^N_E)$ for $1 \le j \le n$. By Theorem 6.3.15 and hypothesis (2),

$$\begin{aligned}
&\varphi(v_1,\ldots,v_{\ell\mathrm{h}(x)}, c_{b_1},\ldots,c_{b_n}, d) \in (\mathrm{tp}^{\delta^*+1}_{\kappa,i^N_E(\beta')})^{\mathrm{Ult}(N;E)}(i^N_E(x'))\\
&\leftrightarrow \{z \in [\kappa]^{|a|} \mid \varphi(v_1,\ldots,v_{\ell\mathrm{h}(x)}, c_{f_1(z)},\ldots,c_{f_n(z)}, d) \in (\mathrm{tp}^{\delta}_{\kappa,\beta'})^N(x')\} \in E_a\\
&\leftrightarrow \{z \in [\kappa]^{|a|} \mid \varphi(v_1,\ldots,v_{\ell\mathrm{h}(x)}, c_{f_1(z)},\ldots,c_{f_n(z)}, d) \in (\mathrm{tp}^{\delta}_{\kappa,\beta})^M(x)\} \in E_a\\
&\leftrightarrow \varphi(v_1,\ldots,v_{\ell\mathrm{h}(x)}, c_{b_1},\ldots,c_{b_n}, d) \in (\mathrm{tp}^{\delta^*+1}_{\kappa,i^M_E(\beta)})^{\mathrm{Ult}(M;E)}(i^M_E(x)).
\end{aligned}$$

Next we turn to the choice of ξ^* and y^*. Let

$$A = (\mathrm{tp}^{\delta^*}_{\kappa,\xi})^M(x^\frown y).$$

Let $\psi(v_1,\ldots,v_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)+1})$ be the formula of $(\mathcal{L}^{\delta^*+1}_{\kappa,\beta})^M$ given as follows:

$$\begin{aligned}
&v_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)+1} \in \mathrm{Ord}\\
&\quad\wedge \mathrm{TYPE}_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)}(v_1,\ldots,v_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)+1}, c_A, c_{\delta^*}, d)\\
&\quad\wedge \mathrm{REFL}_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)}(v_1,\ldots,v_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)+1}, c_{\delta^*}, d)\\
&\quad\wedge \chi(v_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)+1}).
\end{aligned}$$

The finite sequence y and the ordinal ξ witness that

$$(\exists v_{\ell\mathrm{h}(x)+1})\cdots(\exists v_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)+1})\,\psi(v_1,\ldots,v_{\ell\mathrm{h}(x)+\ell\mathrm{h}(y)+1})$$

belongs to $(\mathrm{tp}^{\delta^*+1}_{\kappa,\beta})^M(x)$. Thus this formula also belongs to

$$(\mathrm{tp}^{\delta^*+1}_{\kappa,i^N_E(\beta')})^{\mathrm{Ult}(N;E)}(i^N_E(x')).$$

If we let y^* and ξ^* witness this, then (2^*), (3^*), and (4^*) hold.

For the second part of the conclusion of the lemma, let α and z be as in the hypotheses. Then the formula above also belongs to $(\mathrm{tp}^{\delta^*+1}_{\kappa,\alpha})^{\mathrm{Ult}(N;E)}(z)$. Let $\hat{y}$ and $\hat{\xi}$ witness this. $\square$

Remarks:

(a) The lemma does not assert that $\prod_E^N(N;\in)$ is wellfounded; indeed the lemma is vacuously satisfied by an E such that this ultrapower is illfounded. In our applications, we will be able to prove wellfoundedness. Since a long-enough initial part of $\prod_E^N(N;\in)$ is always wellfounded, we could have formulated the lemma so that it would have had real content independently of full wellfoundedness. In [Martin and Steel, 1989] the problem of wellfoundedness is handled in a different way, by an assumption that M and N are countably closed.

(b) The hypothesis that κ is inaccessible in V was included for its notationally simplifying consequence that $i_E^N(\kappa) = \kappa$.

(c) It will be crucial in our applications of the lemma to know that κ is Woodin in $\mathrm{Ult}(N;E)$, for we will want to apply the lemma iteratively. In fact, it follows from the hypotheses of the Lemma that κ is Woodin in N. (See Exercise 6.3.5.) This implies that it is Woodin in $\mathrm{Ult}(N;E)$. In our applications κ will be a Woodin cardinal in V, and there will be an elementary embedding of V into N that fixes κ. Hence we will know immediately (without Exercise 6.3.5) that κ is Woodin in N and so in $\mathrm{Ult}(N;E)$.

(d) The ordinal δ^* is larger than the given δ. In fact, the arbitrariness of $\eta < \kappa$ means that it can be made as large as one wants, subject to being smaller than κ. On the other hand, the ordinal ξ is smaller than the given β. This will give us problems, though not insurmountable ones, in generating an infinite sequence of applications of the lemma. Of course ξ^* need not be smaller that β, but there is no obvious way to make use of this.

(e) The pair of clauses (4) and (4*) will be needed for technical reasons in the applications. The lemma could be strengthened by allowing the formula χ in (4) to be any element of $(\mathrm{tp}^\delta_{\kappa,\beta})^M(x^\frown y)$ and demanding in (4*) that $\chi \in (\mathrm{tp}^\delta_{\kappa,i_E^N(\beta')})^{\mathrm{Ult}(N;E)}(i_E^N(x')^\frown y^*)$.

(f) The second part of the lemma was not used in [Martin and Steel, 1989]. Using it will allow us to avoid a good deal of work that was done in that paper. (Nevertheless, we will do the work, in §8.3.)

Exercise 6.3.1. Let κ be Woodin. Prove that for all $A \subseteq V_\kappa$ the set of cardinals $\delta < \kappa$ such that

$$(\forall \eta)(\delta < \eta < \kappa \;\rightarrow\; \delta \text{ is } \eta\text{-strong in } A)$$

is stationary in κ. (See Exercise 3.2.7 for the definition of "stationary.")

Hint. One way to proceed is to modify the proof of the (1) $\Rightarrow$ (2) half of Theorem 6.3.1. Another way is to argue directly from (2) of Theorem 6.3.1, using the following fact: If $j : V \prec M$, $\mathrm{crit}\,(j) = \delta$, and $C \cap \delta$ is bounded in δ, then $j(C) \cap j(\delta) \subseteq \delta$.

Exercise 6.3.2. Use Corollary 6.3.5 to show that if there is a Woodin cardinal then the least Woodin cardinal is not measurable.

Exercise 6.3.3. Prove that a cardinal κ is Woodin if and only if for all $A \subseteq V_\kappa$ there is a $\delta < \kappa$ such that, for every η with $\delta < \eta < \kappa$,

$$(\exists j : V \prec M)(M \text{ is transitive } \wedge \ \mathrm{crit}\,(j) = \delta \ \wedge \ j(A) \cap V_\eta = A \cap V_\eta).$$

Note that the displayed statement would say that δ is η-strong in A if we added the condition that $V_\eta \subseteq M$.

Exercise 6.3.4. Suppose that κ is a strong limit cardinal and that $\delta < \kappa$ is 0-reflecting in $\emptyset$ relative to κ. Show that $V_\delta \prec V_\kappa$.

Hint. If $V_\kappa \models (\exists v)\, \psi(v)$, then there is an $\alpha < \kappa$ and an $x \in V_\alpha$ such that $V_\kappa \models \psi[x]$. Get $j : V \prec M$ from the hypothesis about δ. Note that $x \in V^M_{j(\delta)}$.

Exercise 6.3.5. Assume the hypotheses of Lemma 6.3.18. Prove that κ is Woodin in N. (The only hypotheses actually needed are that κ is Woodin in M, that $\beta > 0$, and the consequence of (2) that

$$(\mathrm{tp}^0_{\kappa,\beta})^M(\emptyset) = (\mathrm{tp}^0_{\kappa,\beta'})^N(\emptyset).)$$

Exercise 6.3.6. Call a (δ, λ)-extender E *strong* if $\delta + 1 < \mathrm{strength}\,(E) = \lambda < i_E(\delta)$. Let κ be Woodin. Say that a set $\mathcal{E}$ of extenders *strongly witnesses that κ is Woodin* if

(i) $\mathcal{E} \subseteq V_\kappa$;

(ii) each $E \in \mathcal{E}$ is strong;

(iii) for every $A \subseteq V_\kappa$ and for every $\eta < \kappa$, there are a $\delta < \kappa$ and an $E \in \mathcal{E}$ such that i_E witnesses that δ is η-strong in A.

Prove that there is an $\mathcal{E}$ strongly witnessing that κ is Woodin. (Of course, the set of all strong extenders that belong to V_κ works if any $\mathcal{E}$ does.)

Exercise 6.3.7. The following construction and the results of this and the next exercise are due to Woodin.

Let $\neg$, $\bigvee$, and $\mathbf{a}_n$, $n \in \omega$, be distinct sets, say natural numbers. The class of ∞-*Borel codes* is the smallest class satisfying the following conditions.

(a) For each $n \in \omega$, $\mathbf{a}_n$ is an ∞-Borel code.

(b) If $\mathbf{c}$ is a ∞-Borel code, then $\langle \neg, \mathbf{c} \rangle$ is an ∞-Borel code.

(c) If β is an ordinal and $\mathbf{c}_\alpha$, $\alpha < \beta$, are ∞-Borel codes, then $\langle \bigvee, \langle \mathbf{c}_\alpha \mid \alpha < \beta \rangle \rangle$ is an ∞-Borel code.

We write $\neg\, \mathbf{c}$ for $\langle \neg, \mathbf{c} \rangle$, and we write $\bigvee \langle \mathbf{c}_\alpha \mid \alpha < \beta \rangle$ for $\langle \bigvee, \langle \mathbf{c}_\alpha \mid \alpha < \beta \rangle \rangle$

We associate with each ∞-Borel code $\mathbf{c}$ a subset $B_\mathbf{c}$ of ${}^\omega 2$ inductively as follows:

$$\begin{aligned} B_{\mathbf{a}_n} &= \{x \in {}^\omega 2 \mid x(n) = 1\}; \\ B_{\neg \mathbf{c}} &= {}^\omega 2 \setminus B_\mathbf{c}; \\ B_{\bigvee \langle \mathbf{c}_\alpha \mid \alpha < \beta \rangle} &= \bigcup_{\alpha < \beta} B_{\mathbf{c}_\alpha}. \end{aligned}$$

Let $\mathbf{C}$ be the class of all ∞-Borel codes. If $\mathcal{I} \subseteq \mathbf{C} \times \mathbf{C}$ and $x \in {}^\omega 2$, then $\mathcal{I}$ is x-*consistent* if

$$(\forall \mathbf{c} \in \mathbf{C})(\forall \mathbf{c}' \in \mathbf{C})(\langle \mathbf{c}, \mathbf{c}' \rangle \in \mathcal{I} \rightarrow (x \in B_\mathbf{c} \leftrightarrow x \in B_{\mathbf{c}'})).$$

If $\mathbf{B}$ is a complete Boolean algebra and if $\tau : \{\mathbf{a}_n \mid n \in \omega\} \to \mathbf{B}$, then there is an obvious way to extend τ to $\tau^* : \mathbf{C} \to \mathbf{B}$ (as in the example above with $\mathbf{B} = \mathcal{P}({}^\omega 2)$). Say that $\tau : \{\mathbf{a}_n \mid n \in \omega\} \to \mathbf{B}$ *respects* $\mathcal{I}$ if whenever $\langle \mathbf{c}, \mathbf{c}' \rangle \in \mathcal{I}$ then $\tau^*(\mathbf{c}) = \tau^*(\mathbf{c}')$. (Thus $\mathbf{a}_n \mapsto \{x \mid x(n) = 1\}$ respects $\mathcal{I}$ if and only if $\mathcal{I}$ is x-consistent for every $x \in {}^\omega 2$.)

Let $\mathcal{I} \subseteq \mathbf{C} \times \mathbf{C}$. For ∞-Borel codes $\mathbf{c}$ and $\mathbf{c}'$, define $\mathbf{c} \sim_\mathcal{I} \mathbf{c}'$ to hold if, for every complete Boolean algebra $\mathbf{B}$ and every $\tau : \{\mathbf{a}_n \mid n \in \omega\} \to \mathbf{B}$, if τ respects $\mathcal{I}$ then $\tau^*(\mathbf{c}) = \tau^*(\mathbf{c}')$. (One can also define $\sim_\mathcal{I}$ by transfinite induction, using the laws of complete Boolean algebras.) It is evident that $\sim_\mathcal{I}$ is an equivalence relation. For $\mathbf{c} \in \mathbf{C}$, let $[\![\mathbf{c}]\!]_\mathcal{I}$ be the equivalence class of $\mathbf{c}$ with respect to $\mathcal{I}$, fixed up *à la* Scott to make it a set. Let $\mathbf{C}(\mathcal{I})$ be the class of all $[\![\mathbf{c}]\!]_\mathcal{I}$. If $\mathbf{C}(\mathcal{I})$ has more than one element (as it will if $\mathcal{I}$ is x-consistent for some $x \in {}^\omega 2$), then $\mathbf{C}(\mathcal{I})$ is a complete Boolean (class) algebra under the obvious complement and join operations. If $\mathcal{I}$ is x-consistent for every $x \in {}^\omega 2$, then $[\![\mathbf{c}]\!]_\mathcal{I} \mapsto B_\mathbf{c}$ is a complete homomorphism of $\mathbf{C}(\mathcal{I})$ onto $\mathcal{P}({}^\omega 2)$.

For any set $\mathcal{E}$ of extenders, we define a set $\mathcal{I}_{\mathcal{E}}$ of pairs of elements of $\mathbf{C}$. A pair belongs to $\mathcal{I}_{\mathcal{E}}$ only if this is required by the following. Let $E \in \mathcal{E}$. Let E be a (δ, λ)-extender. Let $\mathbf{c}_\alpha$, $\alpha < \delta$, be ∞-Borel codes belonging to V_δ. Let

$$i_E(\langle \mathbf{c}_\alpha \mid \alpha < \delta \rangle) = \langle \hat{\mathbf{c}}_\alpha \mid \alpha < i_E(\delta) \rangle.$$

(Note that $\hat{\mathbf{c}}_\alpha = \mathbf{c}_\alpha$ for $\alpha < \delta$.) All the $\hat{\mathbf{c}}_\alpha$ are ∞-Borel codes. If $\hat{\mathbf{c}}_\delta \in V_\lambda$, then

$$\left\langle \bigvee \langle \mathbf{c}_\alpha \mid \alpha < \delta \rangle, \bigvee \langle \hat{\mathbf{c}}_\alpha \mid \alpha \leq \delta \rangle \right\rangle \in \mathcal{I}_{\mathcal{E}}.$$

(a) Prove that $\mathcal{I}_{\mathcal{E}}$ is x-consistent for every $x \in {}^\omega 2$.

Suppose κ is Woodin. Let $\mathcal{E}$ be a collection of extenders strongly witnessing that κ is Woodin. (See exercise 6.3.6.)

(b) Prove that $\mathbf{C}(\mathcal{I}_{\mathcal{E}})$ is a set of size κ and, as a Boolean algebra, has the κ-chain condition.

Hint. Let $\mathbf{C}_\kappa = \mathbf{C} \cap V_\kappa$. Let $\mathbf{C}_\kappa(\mathcal{I}_{\mathcal{E}})$ be the corresponding Boolean subalgebra of $\mathbf{C}(\mathcal{I}_{\mathcal{E}})$. It suffices to prove that $\mathbf{C}_\kappa(\mathcal{I}_{\mathcal{E}})$ has the κ-chain condition. To prove this, use the fact that $\mathcal{E}$ strongly witnesses that κ is Woodin.

Exercise 6.3.8. Let $\mathcal{E}$ be a set of extenders. Let M be a transitive class model of ZFC with $\mathcal{E} \in V_{\mathrm{Ord}^M}$. Note that $\mathbf{C}^M = \mathbf{C} \cap M$. For $E \in \mathcal{E}$ with $E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$, let $E \restriction M = \langle E_a \cap M \mid a \in [\lambda]^{<\omega} \rangle$. Let $\mathcal{E} \restriction M = \{E \restriction M \mid E \in \mathcal{E}\}$. Suppose that $\mathcal{E} \restriction M \in M$ and that $\mathcal{E} \restriction M$ strongly witnesses in M that κ is Woodin. In M we have the algebra $(\mathbf{C}^M(\mathcal{I}^M_{\mathcal{E} \restriction M}))^M$. Let $\mathbf{P}^M_{\mathcal{E} \restriction M}$ be the partially ordered set $(\mathbf{C}^M(\mathcal{I}^M_{\mathcal{E} \restriction M}))^M \setminus \{\mathbf{0}\}$. Let $x \in {}^\omega 2$. Attempt to define $\mathbf{G}_x \subseteq \mathbf{P}^M_{\mathcal{E} \restriction M}$ by

$$[\![\mathbf{c}]\!]^M_{(\sim_{\mathcal{I}^M_{\mathcal{E} \restriction M}})^M} \in \mathbf{G}_x \leftrightarrow x \in B_{\mathbf{c}},$$

where $B_{\mathbf{c}}$ is as in Exercise 6.3.7. Prove that $\mathbf{G}_x$ is well-defined and is $\mathbf{P}^M_{\mathcal{E} \restriction M}$-generic over M and that $M[\mathbf{G}_x] = M[x]$.

Hint. Use an absoluteness argument to show that

$$(\sim_{\mathcal{I}^M_{\mathcal{E} \restriction M}})^M = \sim_{\mathcal{I}^M_{\mathcal{E} \restriction M}} \restriction \mathbf{C}^M.$$

(This is perhaps easier using the inductive definition of $\sim_{\mathcal{I}}$.) It follows that $(\mathbf{C}^M(\mathcal{I}^M_{\mathcal{E} \restriction M}))^M$ is an M-complete subalgebra of $\mathbf{C}(\mathcal{I}^M_{\mathcal{E} \restriction M})$.

Next observe that

$$\mathcal{I}^M_{\mathcal{E}\restriction M} = \mathcal{I}_{\mathcal{E}} \cap M.$$

This implies, by part (a) of Exercise 6.3.7, that $\mathcal{I}^M_{\mathcal{E}\restriction M}$ is y-consistent in V for every $y \in {}^{\omega}2$. In particular, $\mathcal{I}^M_{\mathcal{E}\restriction M}$ is x-consistent in V. Thus, in V, a complete homomorphism $\sigma : \mathbf{C}(\mathcal{I}^M_{\mathcal{E}\restriction M}) \to \{\mathbf{0}, \mathbf{1}\}$ is given by

$$\sigma(\llbracket \mathbf{c} \rrbracket_{\sim_{\mathcal{I}^M_{\mathcal{E}\restriction M}}}) = \mathbf{1} \;\leftrightarrow\; x \in B_{\mathbf{c}}.$$

The restriction of σ to $(\mathbf{C}^M(\mathcal{I}^M_{\mathcal{E}\restriction M}))^M$ is thus an M-complete homomorphism. The preimage of $\{\mathbf{1}\}$ is just $\mathbf{G}_x$.

Chapter 7

Iteration Trees

In this chapter, which can be read immediately after §6.1 if one is willing to refer back to Chapter 6 for one or two definitions, we introduce and prove some basic results about the main technical tool of the determinacy proofs of Chapter 8. This material comes from Martin–Steel [1988], [Martin and Steel, 1989], and [Martin and Steel, 1994]. Our treatment will follow that of [Martin and Steel, 1994], which is a bit more general than that of the other two papers.

Iteration trees are a generalization of iterated ultrapowers, which we studied in §3.3. They are more general in three ways: (1) The individual ultrapowers are with respect to extenders and not just ultrafilters. (2) The individual ultrapowers are not all with respect to images of the same ultrafilter or extender. (3) The iteration is not linear, but has a tree structure, and the individual ultrapowers are of models at one node of the tree but with respect to extenders in models at possibly different nodes.

In fact we have already introduced an even wider generalization of type (1). In §3.3 we defined transfinite iterations of an arbitrary elementary embedding $j : V \prec M$.

Before considering iterations with all three properties, we will consider, in §1, those with properties (1) and (2). Iterations with property (2) were first used in [Kunen, 1970], and [Mitchell, 1979] introduced iterations which essentially also had property (1)

In §2 we introduce iteration trees. In §3 we study finite iteration trees, and in §4 we study those of length ω. The definitions and results in the text of §2–§4 are almost all from [Martin and Steel, 1994], though—in order to avoid continually citing the paper—we will mostly omit explicitly citation.

7.1 Internal Iterations

Most of the business of this chapter will be proving that the direct limit models of various iterations are wellfounded. Even when the initial model of the iteration is V, we will have to deal in the proofs with iterations whose models are sets that may not satisfy full ZFC. To handle this, we introduce a concept from [Martin and Steel, 1994] that is general enough to cover all the models that will arise in our proofs.

First we need two preliminary definitions.

The Lévy hierarchy of formulas defined on page 19 can be defined for any language extending the language of set theory. The clauses in the definition are exactly the same as those on page 19, but now "atomic formula" means atomic formula of the extended language. Let $\mathcal{L}(P)$ be the result of adding to the language of set theory a one-place function symbol P. A class model $(M; E)$ *satisfies* $\Sigma_1(\mathcal{P})$ *Replacement* if M satisfies the Power Set Axiom and the expansion of $(M; \in)$ in which P is interpreted as the power set operation satisfies Replacement for Σ_1 formulas of $\mathcal{L}(P)$.

If $(M; E)$ is a class model and $u \in M$, then $(M; E)$ satisfies *Replacement for the domain* u if, for every formula $\varphi(u, x, y, z_1, \ldots, z_n)$ of the language of set theory,

$$(M; E) \models (\forall z_1) \cdots (\forall z_n)\,((\forall x \in u)(\exists! y)\,\varphi \;\rightarrow\; (\exists v)(\forall x \in u)(\exists y \in v)\,\varphi)\,.$$

Replacement for domain u implies that the range of any class function whose domain is u is a set, and so that the class function is a set function.

Now we turn to the concept from [Martin and Steel, 1994] that was mentioned above. A class model $\mathcal{M} = (M; \in, \delta)$ is a *premouse* if

(a) M is transitive;

(b) δ is an ordinal belonging to M;

(c) $(M; \in) \models$ ZC + $\Sigma_1(\mathcal{P})$ Replacement + Replacement for the domain V_δ.

For premice $\mathcal{M} = (M; \in, \delta)$ we write $\delta = \delta^{\mathcal{M}}$.

It is easy to show that if $(M; \in)$ satisfies ZC + $\Sigma_1(\mathcal{P})$ Replacement, then

$$(M; \in) \;\models\; (\forall \alpha \in \mathrm{Ord})\, V_\alpha \text{ exists.}$$

Thus clauses (a), (b), and the first two parts of clause (c) imply that M saitisfies "V_δ exists," and so that the third part of clause (c) makes sense.

Remark. The name "premouse" may seem an odd one. *Premice* and *mice* were introduced in [Dodd and Jensen, 1981] and generalizations have been defined by various authors. In all these versions, including ours, a premouse is required to satisfy some fragment of ZFC. In most versions—though not in ours, i.e., not in that of [Martin and Steel, 1994]—it is required also that a premouse have some specific structure, such as being $L[\mathbf{E}]$ for $\mathbf{E}$ a "coherent" sequence of extenders. In all versions, what makes a premouse a mouse is iterability, the existence of wellfounded limit models of any appropriate iteration whose initial model is the premouse. We will not define or discuss mice, but the main results of this chapter can be thought of as establishing mouse-like properties for certain classes of premice.

We have already noted that every premouse $\mathcal{M}$ satisfies the sentence "$(\forall\alpha \in \mathrm{Ord})\, V_\alpha$ exists." It is also easy to show that, for premice $\mathcal{M}$,

$$\mathcal{M} \models (\forall x)(\exists\alpha \in \mathrm{Ord})\, x \in V_\alpha.$$

If $\mathcal{M} = (M; \in, \delta)$ is a premouse and E is an extender in M, i.e. $E \in M$ and $\mathcal{M} \models$ "E is an extender," then we can define $\prod_E^{\mathcal{M}} \mathcal{M}$ just as we defined, on page 339, $\prod_E^M (M; \in)$ for class models M of ZFC. Our notation for such ultrapowers will be like that for the earlier ones. Since there seems no reason for preferring, e.g., one of the notations $[\![a, f]\!]_E^M$ and $[\![a, f]\!]_E^{\mathcal{M}}$ over the other, we will in this and other cases indiscriminately use both notations. Łoś' Theorem generalizes to these ultrapowers, except that we must restrict E to be a (κ, λ)-extender with $\kappa \leq \delta$:

Theorem 7.1.1. *Let $\mathcal{M} = (M; \in, \delta)$ be a premouse. Let κ and $\lambda > \kappa$ be ordinals of M with $\kappa \leq \delta$. Let E be an (κ, λ)-extender in $\mathcal{M}$. Let $\varphi(v_1, \dots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1\rangle, \dots, \langle a_n, f_n\rangle$ be elements of $\mathcal{D}_E^M$. Let $b = \bigcup_{1\leq i \leq n} a_i$. Then*

$$\begin{aligned}&\textstyle\prod_E^{\mathcal{M}} \mathcal{M} \models \varphi[[\![a_1, f_1]\!]_E^M, \dots, [\![a_n, f_n]\!]_E^M] \;\leftrightarrow\\ &\quad \{z \in [\kappa]^{|b|} \mid \mathcal{M} \models \varphi[f_1(z_{a_1,b}), \dots, f_n(z_{a_n,b})]\} \in E_b.\end{aligned}$$

Proof. The proof is by an induction on φ as usual. We sketch the case φ is $(\exists v_0)\psi$ to indicate how the axioms that hold in premice are used. In that case we have, suppressing some subscripts and superscripts,

$$\begin{aligned}
&\textstyle\prod_E^{\mathcal{M}} \mathcal{M} \models \varphi[[\![a_1, f_1]\!], \ldots, [\![a_n, f_n]\!]] \\
\leftrightarrow\ & (\exists a_0 \in [\lambda]^{<\omega})(\exists f_0 \in {}^{[\kappa]^{|a_0|}} M \cap M) \\
& \quad (\textstyle\prod_E^{\mathcal{M}} \mathcal{M} \models \psi[[\![a_0, f_0]\!], \ldots, [\![a_n, f_n]\!]]) \\
\leftrightarrow\ & (\exists a_0 \in [\lambda]^{<\omega})(\exists f_0 \in {}^{[\kappa]^{|a_0|}} M \cap M) \\
& \quad \big(\{z \in [\kappa]^{|a_0 \cup b|} \mid \mathcal{M} \models \psi[f_0(z_{a_0, a_0 \cup b}), \ldots, f_n(z_{a_n, a_0 \cup b})]\} \in E_{a_0 \cup b}\big) \\
\leftrightarrow\ & \{z \in [\kappa]^{|b|} \mid (\exists x \in M)\, \mathcal{M} \models \psi[x, f_1(z_{a_1, b}), \ldots, f_n(z_{a_n, b})]\} \in E_b \\
\leftrightarrow\ & \{z \in [\kappa]^{|b|} \mid \mathcal{M} \models \varphi[f_1(z_{a_1, b}), \ldots, f_n(z_{a_n, b})]\} \in E_b.
\end{aligned}$$

To show that the fourth line implies the third, one argues as follows: Because Replacement for the domain V_δ holds in $\mathcal{M}$, the fourth line implies that there is an ordinal $\alpha \in M$ such that

$$\{z \in [\kappa]^{|b|} \mid (\exists x \in V_\alpha^M)\, \mathcal{M} \models \psi[x, f_1(z_{a_1, b}), \ldots, f_n(z_{a_n, b})]\} \in E_b.$$

Since this set belongs to M as well as to E_b, Choice in M yields the third line. □

Theorem 7.1.1 gives us the canonical $(i')_E^{\mathcal{M}} : \mathcal{M} \prec \prod_E^{\mathcal{M}} \mathcal{M}$. The usual proofs give the following two results.

Lemma 7.1.2. *Let $\mathcal{M}$ be a premouse, let $\kappa \leq \delta^{\mathcal{M}}$, and let E be a (κ, λ)-extender in $\mathcal{M}$ for some λ. Then $\mathcal{M} \models$ "$\prod_E^{\mathcal{M}} \mathcal{M}$ is set-like," and so $\prod_E^{\mathcal{M}} \mathcal{M}$ is set-like.*

Lemma 7.1.3. *Let $\mathcal{M}$ be a premouse, let $\kappa \leq \delta^{\mathcal{M}}$, and let E be a (κ, λ)-extender in $\mathcal{M}$ for some λ. Then $\mathcal{M} \models$ "$\prod_E^{\mathcal{M}} \mathcal{M}$ is wellfounded," and so $\prod_E^{\mathcal{M}} \mathcal{M}$ is wellfounded.*

Thus, if $\mathcal{M} = (M; \in, \delta)$, we get a unique

$$\pi_E^{\mathcal{M}} : \prod_E^{\mathcal{M}} \mathcal{M} \cong \mathrm{Ult}(\mathcal{M}; E) = (\mathrm{Ult}(M; E); \in, \delta^{\mathrm{Ult}(\mathcal{M}; E)})$$

with $\mathrm{Ult}(M; E)$ transitive, and we define

$$i_E^{\mathcal{M}} = \pi_E^{\mathcal{M}} \circ (i')_E^{\mathcal{M}} : \mathcal{M} \prec \mathrm{Ult}(\mathcal{M}; E).$$

Note that all of the classes $\prod_E^{\mathcal{M}} \mathcal{M}$, $(i')_E^{\mathcal{M}}$, $\pi_E^{\mathcal{M}}$, $\mathrm{Ult}(M; E)$, $\mathrm{Ult}(\mathcal{M}; E)$, and $i_E^{\mathcal{M}}$ are classes in M. Note also that $\mathrm{Ult}(\mathcal{M}; E)$ is a premouse.

We now begin our study of iterations that have the first two of the properties mentioned in the introduction to this Chapter. We will define such iterations for transitive class models of ZFC and then for premice.

First we need some more terminology for talking about direct limits. Suppose that M_d is a transitive class for each $d \in D$ and that

$$(\langle (M_d; \in) \mid d \in D\rangle, \langle j_{d,d'} \mid d \in D \wedge d' \in D \wedge d\, R\, d'\rangle)$$

is a directed system of elementary embeddings. Let $(\tilde{\mathcal{M}}, \langle \tilde{j}_d \mid d \in D\rangle)$ be the direct limit of this directed system. We say that $\tilde{\mathcal{M}}$ is the *direct limit model* of the directed system. If $\tilde{\mathcal{M}}$ is wellfounded and set-like, let $\pi : \tilde{\mathcal{M}} \cong (N; \in)$, with N transitive. We say that

$$((N; \in), \langle \pi \circ \tilde{j}_d \mid d \in D\rangle)$$

is the *canonical limit* of the directed system and that $(N; \in)$ is the *canonical limit model* of the directed system. If either wellfoundedness or set-likeness fails, then there is no canonical limit and no canonical limit model. Note that if there is an R-maximal element d of D, then the direct limit model is isomorphic to M_d, the canonical limit is $(M_d, \langle j_{d',d} \mid d' \in D\rangle)$, and the canonical limit model is M_d.

Similarly define the *direct limit model*, the *canonical limit*, and the *canonical limit model* when the individual models of the directed system have additional structure, e.g., when they are premice $(M_d; \in, \delta_d)$.

If M is a transitive class model of ZFC and θ is a non-zero ordinal number, an *internal iteration* of M of length θ is a sequence $\langle E_\alpha \mid \alpha + 1 < \theta\rangle$ such that there are transitive classes M_α, $\alpha < \theta$, and embeddings $j_{\alpha,\beta}$, $\alpha \leq \beta < \theta$, satisfying

(a) $M_0 = M$;

(b) each E_α is an extender in M_α;

(c) for $\alpha \leq \beta \leq \gamma < \theta$, $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$;

(d) for each α such that $\alpha + 1 < \theta$, $M_{\alpha+1} = \mathrm{Ult}(M_\alpha; E_\alpha)$ and $j_{\alpha,\alpha+1} = i_{E_\alpha}^{M_\alpha}$;

(e) for each limit $\lambda < \theta$, $(M_\lambda, \langle j_{\alpha,\lambda} \mid \alpha < \lambda\rangle)$ is the canonical limit of $(\langle M_\alpha \mid \alpha < \lambda\rangle, \langle j_{\alpha,\beta} \mid \alpha \leq \beta < \lambda\rangle)$.

Note that the M_α and the $j_{\alpha,\beta}$ are uniquely determined by M and the E_α.

Remark. Our notion of internal iteration is in one sense less general and in another sense more general than the name suggests. A broader notion would permit the E_α to be extenders in the general sense of Exercise 6.1.2; an even broader notion would replace the $i^{M_\alpha}_{E_\alpha}$ by embeddings $j_\alpha : M_\alpha \prec M_{\alpha+1}$, requiring only that each j_α be a class in M_α. A narrower notion would require the iteration $\langle E_\alpha \mid \alpha + 1 < \theta\rangle$ to belong to M.

If $\mathcal{M}$ is a premouse and θ is a non-zero ordinal number, an *internal iteration of* $\mathcal{M}$ *of length* θ is a sequence $\langle E_\alpha \mid \alpha + 1 < \theta\rangle$ such that there are premice $\mathcal{M}_\alpha$, $\alpha < \theta$, and embeddings $j_{\alpha,\beta}$, $\alpha \leq \beta < \theta$, satisfying

(a) $\mathcal{M}_0 = \mathcal{M}$;

(b) each E_α is an extender in $\mathcal{M}_\alpha$ with $E_\alpha \in V^{\mathcal{M}_\alpha}_{\delta^{\mathcal{M}_\alpha}}$;

(c) for $\alpha \leq \beta \leq \gamma < \theta$, $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$;

(d) for each α such that $\alpha+1 < \theta$, $\mathcal{M}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}_\alpha; E_\alpha)$ and $j_{\alpha,\alpha+1} = i^{\mathcal{M}_\alpha}_{E_\alpha}$;

(e) for each limit $\lambda < \theta$, $(\mathcal{M}_\lambda, \langle j_{\alpha,\lambda} \mid \alpha < \lambda\rangle)$ is the canonical limit of $(\langle \mathcal{M}_\alpha \mid \alpha < \lambda\rangle, \langle j_{\alpha,\beta} \mid \alpha \leq \beta < \lambda\rangle)$.

Remark. The condition that $E_\alpha \in V^{\mathcal{M}_\alpha}_{\delta^{\mathcal{M}_\alpha}}$ is stronger than necessary. One could simply require that E_α be a (κ, λ)-extender in $\mathcal{M}_\alpha$ for some κ and λ with $\kappa \leq \delta^{\mathcal{M}_\alpha}$.

We would like to think of transitive class models M of ZFC as giving "premice" $(M; \in, \mathrm{Ord} \cap M)$, so that, for example, we can think of the internal iterations of such M as special cases of internal iterations of premice. Let us therefore say that $\mathcal{M}$ is a *premouse** if either of the following holds:

(i) $\mathcal{M}$ is a premouse;

(ii) $\mathcal{M} = (M; \in)$ for some transitive class model M of ZFC.

If $\mathcal{M} = (M; \in)$ is a premouse*, then by $\delta^{\mathcal{M}}$ we mean $\mathrm{Ord} \cap M$ (which is not a genuine ordinal number if M is a proper class).

[Mitchell, 1974] proved wellfoundedness results for internal iterations in the special case that all the E_α are given by normal ultrafilters on their critical points. His methods extend to the case of general extenders. We will now present a different and simpler approach to the same theorems. This approach is attributed to R. Jensen in Dodd [1982]. The results we prove with it in this section are essentially from Dodd [19??]. The first lemma is the key to the whole method.

Lemma 7.1.4. *Let $\mathcal{M} = (M; \in, \delta^{\mathcal{M}})$ be a countable premouse*. Suppose that, for some ordinals η and $\delta \leq \eta$, there is a*

$$\tau : \mathcal{M} \prec (V_\eta; \in, \delta).$$

Let E be a (κ, λ)-extender in $\mathcal{M}$ with $\kappa \leq \delta^{\mathcal{M}}$. Then there is a

$$\sigma : \mathrm{Ult}(\mathcal{M}; E) \prec (V_\eta; \in, \delta)$$

such that $\sigma \circ i_E^{\mathcal{M}} = \tau$.

Proof. By the elementarity of τ, we have that $\tau(E)$ is a $(\tau(\kappa), \tau(\lambda))$-extender in the premouse* $(V_\eta; \in, \delta)$ and so in V. For each $a \in [\lambda]^{<\omega}$, let

$$X_a = \bigcap_{Y \in E_a} \tau(Y).$$

Since M is countable, each X_a is a countable intersection of elements of $(\tau(E))_{\tau(a)}$; hence each

$$X_a \in (\tau(E))_{\tau(a)}.$$

By the countability of $[\lambda]^{<\omega}$ and the countable completeness of $\tau(E)$ (clause (4) of Lemma 6.1.2), there is an order preserving $h : \tau''[\lambda]^{<\omega} \to \tau(\kappa)$ such that

$$(\forall a \in [\lambda]^{<\omega})\, h''\tau(a) \in X_a.$$

We define σ by setting

$$\sigma(\pi_E^{\mathcal{M}}([\![a, f]\!]_E^{\mathcal{M}})) = (\tau(f))(h''\tau(a)).$$

To see that σ is well-defined, suppose that $[\![a, f]\!]_E^{\mathcal{M}} = [\![b, g]\!]_E^{\mathcal{M}}$. Then

$$Y = \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) = g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

By the definition of $X_{a\cup b}$,

$$h''\tau(a \cup b) \in X_{a\cup b} \subseteq \tau(Y).$$

Thus

$$(\tau(f))((h''\tau(a\cup b))_{\tau(a),\tau(a\cup b)}) = (\tau(g))((h''\tau(a\cup b))_{\tau(b),\tau(a\cup b)}).$$

Since, e.g., $(h''\tau(a\cup b))_{\tau(a),\tau(a\cup b)} = h''\tau(a)$, we get that

$$(\tau(f))(h''\tau(a)) = (\tau(g))(h''\tau(b)),$$

i.e., that

$$\sigma(\pi_E^{\mathcal{M}}([\![a, f]\!]_E^{\mathcal{M}})) = \sigma(\pi_E^{\mathcal{M}}([\![b, g]\!]_E^{\mathcal{M}})).$$

The proof that σ is elementary is similar to the proof that it is well-defined, and we omit it.

Finally we must prove commutativity. Let $x \in M$.

$$\begin{aligned} \sigma(i_E^{\mathcal{M}}(x)) &= \sigma(\pi_E^{\mathcal{M}}([\![\emptyset, c_x]\!])) \\ &= c_{\tau(x)}(h''\tau(\emptyset)) \\ &= \tau(x), \end{aligned}$$

as required. □

Remark. Note that the proof gives directly an elementary embedding of $\prod_E^{\mathcal{M}} \mathcal{M}$ into $(V_\eta, \in, \delta)$. Since $(V_\eta; \in \delta)$ is wellfounded it follows that $\prod_E^{\mathcal{M}} \mathcal{M}$ is wellfounded. Thus the proof gives a different way of showing e.g. that ultrapowers of V with respect to extenders are wellfounded.

The next lemma extends Lemma 7.1.4 to countable internal iterations.

Lemma 7.1.5. *Let $\mathcal{M} = (M; \in, \delta^{\mathcal{M}})$ be a countable premouse*. Suppose that, for some ordinals η and $\delta \leq \eta$, there is a $\tau : \mathcal{M} \prec (V_\eta; \in, \delta)$. Let $\theta > 0$ be a countable ordinal and let $\langle \mathcal{M}_\alpha \mid \alpha < \theta\rangle$ and $\langle j_{\alpha,\beta} \mid \alpha \leq \beta < \theta\rangle$ witness that $\langle E_\alpha \mid \alpha + 1 < \theta\rangle$ is an internal iteration of $\mathcal{M}$. Let $(\tilde{\mathcal{M}}_\theta, \langle \tilde{j}_{\alpha,\theta} \mid \alpha < \theta\rangle)$ be the direct limit of $(\langle \mathcal{M}_\alpha \mid \alpha < \theta\rangle, \langle j_{\alpha,\beta} \mid \alpha \leq \beta < \theta\rangle)$. Then there is a*

$$\tau^* : \tilde{\mathcal{M}}_\theta \prec (V_\eta; \in, \delta)$$

such that $\tau^* \circ \tilde{j}_{0,\theta} = \tau$.

Proof. Since $\mathcal{M}$ is countable and θ is countable, it follows by induction that all the M_α, $\alpha < \theta$, are countable.

By induction on $\alpha < \theta$, we define embeddings

$$\tau_\alpha : \mathcal{M}_\alpha \prec (V_\eta; \in, \delta).$$

When we define τ_α, we will make sure that

$$(\forall \beta < \alpha)\, \tau_\alpha \circ j_{\beta,\alpha} = \tau_\beta.$$

Let $\tau_0 = \tau$.

Next consider the case that $\alpha = \gamma + 1$ for some γ. We apply Lemma 7.1.4 with $\mathcal{M}_\gamma$ as the $\mathcal{M}$ of that lemma, with τ_γ as the τ, and with E_γ as the E. Let τ_α be the σ given by this application of Lemma 7.1.4. It is easy to see that our induction hypotheses for α are satisfied.

Now consider the case that $\alpha < \theta$ is a limit ordinal. Let $x \in M_\alpha$, where each $\mathcal{M}_\beta = (M_\beta, \in; \delta^{\mathcal{M}_\beta})$. Then $x = j_{\beta,\alpha}(y)$ for some $\beta < \alpha$ and some $y \in M_\beta$. Set

$$\tau_\alpha(x) = \tau_\beta(y).$$

It is easy to see that τ_α is well-defined and that $\tau_\alpha \circ j_{\beta,\alpha} = \tau_\beta$ for all $\beta < \alpha$.

The definition of τ^* is similar to that of limit τ_α. Let $x \in \tilde{M}_\theta$, where $\tilde{\mathcal{M}}_\theta = (\tilde{M}_\theta; \tilde{\in}_\theta, \tilde{\delta}_\theta)$. Then $x = \tilde{j}_{\alpha,\theta}(y)$ for some $\alpha < \theta$ and some $y \in M_\alpha$. Set

$$\tau^*(x) = \tau_\alpha(y).$$

It is easy to see that τ^* has the required properties. □

Corollary 7.1.6. *Assume all the hypotheses of Lemma 7.1.5. Then the direct limit model $\tilde{\mathcal{M}}_\theta$ is wellfounded.*

Proof. If τ^* is given by the lemma, then τ^* embeds $\tilde{\mathcal{M}}_\theta$ into a wellfounded structure. □

Remark. The corollary is trivial in the case θ is a successor ordinal, but the lemma is not.

Theorem 7.1.7. *Every internal iteration of V has a wellfounded direct limit model.*

Proof. Let $\mathcal{J}$ be an internal iteration of V. Assume for a contradiction that the direct limit model of $\mathcal{J}$ is not wellfounded. Let δ be such that $\mathcal{J} \in V_\delta$ and let $\eta > \delta$ be such that $(V_\eta; \in, \delta)$ is a premouse satisfying "the direct limit model of $\mathcal{J}$ is not wellfounded." Let $(X; \in, \delta)$ be a countable elementary submodel of $(V_\eta; \in, \delta)$ such that $\mathcal{J} \in X$. Let $\pi : X \cong M$ with M transitive. Let $\mathcal{I} = \pi(\mathcal{J})$. Then $(M; \in, \pi(\delta))$ is a countable premouse and

$$\pi^{-1} : (M; \in, \pi(\delta)) \prec (V_\eta; \in, \delta).$$

Moreover

$$(M; \in, \pi(\delta)) \models \text{“the direct limit model of } \mathcal{I} \text{ is not wellfounded.”}$$

By the absoluteness of wellfoundedness, it follows that the direct limit model of $\mathcal{I}$ is not wellfounded. But this contradicts Corollary 7.1.6. □

If M is a model of ZFC, then applying Theorem 7.1.7 in M gives that, if $\mathcal{I}$ is an internal iteration of M and $\mathcal{I} \in M$, then the direct limit model of $\mathcal{I}$ is wellfounded. Exercise 7.1.1 eliminates the assumption that $\mathcal{I} \in M$ in the case that $\mathrm{Ord} \cap M$ has uncountable cofinality.

Exercise 7.1.1. Let M be transitive class model of ZFC such that $\mathrm{Ord} \cap M$ is not an ordinal number of cofinality ω. Prove that every internal iteration of M has a wellfounded direct limit model.

Hint. Suppose that $\mathcal{J}$ is a counterexample.

First assume that M is a proper class. Deduce that there is a premouse $(V_\eta^M; \in, \delta) \in M$ such that $\mathcal{J}$ is also an internal iteration of $(V_\eta^M; \in, \delta)$ with illfounded direct limit model. Next use the Löwenheim–Skolem Theorem to show that there are a premouse $\bar{\mathcal{M}}$ with countable universe, an embedding $\tau : \bar{\mathcal{M}} \prec (V_\eta^M; \in, \delta)$, and a countable iteration $\mathcal{I}$ of $\bar{\mathcal{M}}$ whose direct limit model is not wellfounded. Now use an absoluteness argument to show that Corollary 7.1.6 fails in M.

Now assume that M is a set. Use the Löwenheim–Skolem Theorem as above to get a countable $\bar{M}$, an embedding $\tau : \bar{M} \prec M$, and a countable iteration $\mathcal{I}$ of $\bar{M}$ with illfounded direct limit model. Use the hypothesis about $\mathrm{cf}(\mathrm{Ord} \cap M)$ to prove the existence of an ordinal η of M such that $\mathrm{range}\,(\tau) \subseteq V_\eta^M \prec M$. Now contradict Corollary 7.1.6 in M as in the first case.

7.2 General Iteration Trees

The main applications of iterations and iteration trees are in the study of canonical inner models for large cardinal axioms. One defines the property of being a "canonical" model, proves that canonical models exist and satisfy the large cardinal axiom, and one proves (the "Comparison Lemma") that canonical models are indeed *canonical* by proving that any two of them can be elementarily embedded into a third. The elementary embeddings in question are some $j_{0,\alpha}$ of an iteration or an iteration tree. This technique was used in a primitive form in [Kunen, 1970], where the large cardinal axiom is (mainly) the existence of a measurable cardinal, and the iterations were the iterated ultrapowers we discussed in Chapter 3. [Mitchell, 1974] introduced the general Comparison Lemma method, in the context of axioms asserting the existence of measurable cardinals with a rich array of normal ultrafilters. [Mitchell, 1979] extended the method beyond the range of normal ultrafilters. [Dodd,] and [Baldwin, 1986] developed it far enough to get canonical inner models for strong cardinals and more, employing internal iterations in our general sense. At, or a little before, the level of Woodin cardinals, internal iterations are no longer adequate. [Steel,?] used primitive iteration trees in studying inner model theory at about this level. Iteration trees proper were introduced in [Martin and Steel, 1988] and [Martin and Steel, 1989], and they were used as we will use them in Chapter 8, to prove determinacy results. In [Martin and Steel, 1994] their theory was further developed and they were applied to get inner models for Woodin cardinals.

Remark. The historical sketch just given omitted a major part of inner model theory: fine structure and core models. This omitted subject was invented by Ronald Jensen. Dodd, Mitchell, and Steel have, after Jensen, probably made the most important contributions to it. See the introduction to [Martin and Steel, 1994] for a longer historical sketch that does not omit fine structure and core models.

The basic step in generating an iteration tree is, like the basic step in generating an internal iteration, the ultrapower of a given model with respect to an extender. In the case of internal iterations, the extender belongs to the given model and is an extender in it. In the case of iteration trees, the extender may be an extender in a different model and may not belong to the given model at all. Already in the last chapter (page 360), we discussed this kind of ultrapower for the case of transitive class models of ZFC. We now must extend the discussion to the case of premice.

Let us say that premice $\mathcal{M} = (M; \in, \delta^{\mathcal{M}})$ and $\mathcal{N} = (N; \in, \delta^{\mathcal{N}})$ *agree through* α if $\alpha \leq \min\{\delta^{\mathcal{M}}, \delta^{\mathcal{N}}\}$ and M and N agree through α. More generally, let us say that premice* $\mathcal{M}$ and $\mathcal{N}$ *agree through* α if $\alpha \leq \min\{\delta^{\mathcal{M}}, \delta^{\mathcal{N}}\}$ and the universes (first components) of $\mathcal{M}$ and $\mathcal{N}$ agree through α. (The hybrid concept of a premouse* was defined on page 374.)

Suppose that κ is an ordinal and that $\mathcal{M}$ and $\mathcal{N}$ are premice* agreeing through $\kappa + 1$. Suppose that E is a (κ, λ)-extender in M, for some λ. Then we can define $\prod_E^{\mathcal{N}} \mathcal{N}$ just as we defined, on page 360, $\prod_E^N (N; \in)$ for N a transitive class model of ZFC and E a (κ, λ)-extender in another such model M agreeing with N through $\kappa + 1$. Our notation for such ultrapowers will be the obvious combination of that for the $\prod_E^N (N; \in)$ and that for the $\prod_E^{\mathcal{M}} \mathcal{M}$ of the preceding section.

The proof of the following theorem is just like that of the special case $M = N$ (Theorems 7.1.1 and 6.1.3).

Theorem 7.2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be premice**. *Let κ and $\lambda > \kappa$ be ordinals of M with $\kappa < \delta^{\mathcal{N}}$. Assume that $\mathcal{M}$ and $\mathcal{N}$ agree through $\kappa + 1$. Let E be an (κ, λ)-extender in $\mathcal{M}$. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle$ be elements of $\mathcal{D}_E^{\mathcal{N}}$. Let $b = \bigcup_{1 \leq i \leq n} a_i$. Then*

$$\prod_E^{\mathcal{N}} \mathcal{N} \models \varphi[[\![a_1, f_1]\!]_E^{\mathcal{N}}, \ldots, [\![a_n, f_n]\!]_E^{\mathcal{N}}] \leftrightarrow \\ \{z \in [\kappa]^{|b|} \mid \mathcal{N} \models \varphi[f_1(z_{a_1,b}), \ldots, f_n(z_{a_n,b})]\} \in E_b.$$

Theorem 7.2.1 gives us the canonical $(i')_E^{\mathcal{N}} : \mathcal{N} \prec \prod_E^{\mathcal{N}} \mathcal{N}$. The next lemma is proved as were earlier analogous lemmas.

Lemma 7.2.2. *Let $\mathcal{M}$, $\mathcal{N}$, κ, λ, and E be as in the statement of Theorem 7.2.1. Then $\prod_E^{\mathcal{N}} \mathcal{N}$ is set-like.*

$\prod_E^{\mathcal{N}} \mathcal{N}$ may not be wellfounded. If it is, and if $\mathcal{N}$ is a premouse $(N; \in, \delta)$, then we get a unique

$$\pi_E^{\mathcal{N}} : \prod_E^{\mathcal{N}} \mathcal{N} \cong \mathrm{Ult}(\mathcal{N}; E) = (\mathrm{Ult}(N; E); \in, \delta^{\mathrm{Ult}(\mathcal{N}; E)})$$

with $\mathrm{Ult}(N; E)$ transitive. If it is wellfounded and if $\mathcal{N} = (N; \in)$, then we get a unique

$$\pi_E^{\mathcal{N}} = \pi_E^N : \prod_E^{\mathcal{N}} \mathcal{N} \cong \mathrm{Ult}(\mathcal{N}; E) = (\mathrm{Ult}(N; E); \in)$$

with $\mathrm{Ult}(N; E)$ transitive. In either case we define

$$i_E^{\mathcal{N}} = \pi_E^{\mathcal{N}} \circ (i')_E^{\mathcal{N}} : \mathcal{N} \prec \mathrm{Ult}(\mathcal{N}; E).$$

When $\mathcal{N} = (N; \in)$, we may of course also write i_E^N for $i_E^{\mathcal{N}}$. Note that $\mathrm{Ult}(\mathcal{N}; E)$ is a premouse* and is a premouse if and only if $\mathcal{N}$ is a premouse.

The following analogue of Lemma 6.3.17 is proved just as was that lemma.

Lemma 7.2.3. *Let $\mathcal{M}$ be a premouse*. Let E be a (κ, λ)-extender in $\mathcal{M}$. Let $\zeta \geq \kappa$ be an ordinal of $\mathcal{M}$. Let $\mathcal{N}$ be a premouse* such that $\kappa \leq \delta^{\mathcal{N}}$ and such that $\mathcal{M}$ and $\mathcal{N}$ agree through $\zeta + 1$. Assume that $\prod_E^{\mathcal{N}} \mathcal{N}$ is wellfounded. Then*

(a) If $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to (\zeta^+)^{\mathcal{M}} \cup V_{\zeta+1}^{\mathcal{M}}$, then

$$\pi_E^{\mathcal{M}}([\![a, f]\!]_E^{\mathcal{M}}) = \pi_E^{\mathcal{N}}([\![a, f]\!]_E^{\mathcal{N}});$$

(b) $(\forall \alpha)(\alpha \leq (\zeta^+)^{\mathcal{M}} \to i_E^{\mathcal{M}}(\alpha) = i_E^{\mathcal{N}}(\alpha))$; in particular, $i_E^{\mathcal{M}}(\kappa) = i_E^{\mathcal{N}}(\kappa)$;

(c) $\mathrm{Ult}(\mathcal{M}; E)$ and $\mathrm{Ult}(\mathcal{N}; E)$ agree through $i_E^{\mathcal{M}}(\zeta) + 1$; in particular, they agree through $i_E^{\mathcal{M}}(\kappa) + 1$.

If there is a premouse* $\mathcal{M}$ such that E is an (κ, λ)-extender in $\mathcal{M}$, set $\mathrm{crit}\,(E) = \kappa$. Thus, for any premouse* $\mathcal{N}$ such that $i_E^{\mathcal{N}}$ exists, we have

$$\mathrm{crit}\,(E) = \mathrm{crit}\,(i_E^{\mathcal{M}}) = \mathrm{crit}\,(i_E^{\mathcal{N}}).$$

The next lemma is a consequence of Lemma 7.2.3 describing the amount of agreement between $\mathcal{M}$ and $\mathrm{Ult}(\mathcal{N}; E)$ for E an extender in $\mathcal{M}$.

Lemma 7.2.4. *Assume all the hypotheses of Lemma 7.2.3. Let $\rho = \mathrm{strength}^{\mathcal{M}}(E)$. Then*

(1) $\mathrm{Ult}(\mathcal{N}; E)$ and $\mathcal{M}$ agree through ρ;

(2) $V_{\rho+1}^{\mathrm{Ult}(\mathcal{N};E)} \subsetneq V_{\rho+1}^{\mathcal{M}}$.

Proof. By the definition of strength$^{\mathcal{M}}(E)$ and the fact that $\mathrm{Ult}(\mathcal{M};E)$ is a class in $\mathcal{M}$,

$$V_{\rho+1}^{\mathrm{Ult}(\mathcal{M};E)} \subsetneq V_{\rho+1}^{\mathcal{M}}.$$

By (c) of Lemma 7.2.3 and the fact that $\rho \leq i_E^{\mathcal{M}}(\kappa)$,

$$V_{\rho+1}^{\mathrm{Ult}(\mathcal{N};E)} = V_{\rho+1}^{\mathrm{Ult}(\mathcal{M};E)}.$$

The lemma follows. □

Let θ be a non-zero ordinal number. A partial ordering T of θ is a *tree ordering of* θ if

(i) for all $\beta < \theta$, the set of all α such that $\alpha \, T \, \beta$ is wellordered by T;

(ii) T respects the natural order: if $\alpha \, T \, \beta$ then $\alpha < \beta$;

(iii) 0 is the T-least element of θ: if $0 < \alpha < \theta$ then $0 T \alpha$;

(iv) for all $\alpha < \theta$, α is a successor ordinal if and only if α is a T-successor, i.e., if and only if α has an immediate predecessor with respect to T;

(v) for all limit ordinals $\lambda < \theta$, the set of all α such that $\alpha \, T \, \lambda$ is an unbounded subset of λ (with respect to $<$).

For successor ordinals $\alpha < \theta$, we define α_T^- to be the immediate predecessor of α with respect to T, which exists by clause (iv). When there is no ambiguity, we write α^- for α_T^-.

To avoid giving two definitions of "iteration tree," we make use of the concept of premice*. An *iteration tree* is a triple

$$\mathcal{T} = (\mathcal{M}, T, \langle E_\alpha \mid \alpha + 1 < \theta \rangle),$$

such that there are premice* $\mathcal{M}_\alpha$, $\alpha < \theta$, and embeddings $j_{\alpha,\beta}$, $\alpha \, T \, \beta < \theta$, satisfying

(a) T is a tree ordering of θ;

(b) $\mathcal{M}_0 = \mathcal{M}$;

(c) each E_α is an extender in $\mathcal{M}_\alpha$ with $E_\alpha \in V_{\delta^{\mathcal{M}_\alpha}}^{\mathcal{M}_\alpha}$;

(d) for $\alpha \, T \, \beta \, T \, \gamma < \theta$, $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$;

(e) for each α such that $\alpha + 1 < \theta$,

(i) $\mathcal{M}_\alpha$ and $\mathcal{M}_{(\alpha+1)^-_T}$ agree through $\operatorname{crit}(E_\alpha) + 1$;

(ii) $\mathcal{M}_{\alpha+1} = \operatorname{Ult}(\mathcal{M}_{(\alpha+1)^-_T}; E_\alpha)$;

(iii) $j_{(\alpha+1)^-_T, \alpha+1} = i_{E_\alpha}^{\mathcal{M}_{(\alpha+1)^-_T}}$;

(f) for each limit $\lambda < \theta$, $(\mathcal{M}_\lambda, \langle j_{\alpha,\lambda} \mid \alpha \, T \, \lambda \rangle)$ is the canonical limit of $(\langle \mathcal{M}_\alpha \mid \alpha \, T \, \lambda \rangle, \langle j_{\alpha,\beta} \mid \alpha \, T \, \beta \, T \, \lambda \vee \alpha = \beta \, T \, \lambda \rangle)$, where $j_{\alpha,\alpha}$ is the identity embedding of $\mathcal{M}_\alpha$ into itself for each $\alpha < \theta$.

Note that (c) and (e)(i) imply that $\operatorname{crit}(E_\alpha) < \delta^{\mathcal{M}_{(\alpha+1)^-_T}}$. Note also that $\mathcal{T}$ uniquely determines the $\mathcal{M}_\alpha$ and the $j_{\alpha,\beta}$. By $T^{\mathcal{T}}$, $E^{\mathcal{T}}_\alpha$, and $j^{\mathcal{T}}_{\alpha,\beta}$ we mean repectively the tree ordering, the αth extender, and the αth embedding of $\mathcal{T}$. Our notation for the premice* of $\mathcal{T}$ will be in terms of the first component of $\mathcal{T}$: If the first component is $\mathcal{N}$, then $\mathcal{N}^{\mathcal{T}}_\alpha$ is the αth premouse* of $\mathcal{T}$.

If $\mathcal{T} = (\mathcal{M}, T, \langle E_\alpha \mid \alpha + 1 < \theta \rangle)$ is an iteration tree, then $\mathcal{T}$ is an *iteration tree on* $\mathcal{M}$, and θ is the *length* of $\mathcal{T}$. We write $\ell\mathrm{h}(\mathcal{T})$ for the length of $\mathcal{T}$. If M is a transitive class model of ZFC and $\mathcal{T}$ is an iteration tree on $(M; \in)$, then we will also say that $\mathcal{T}$ is an *iteration tree on* M.

If $\mathcal{I}$ is an internal iteration of $\mathcal{M}$, then $(\mathcal{M}, <, \mathcal{I})$ is an iteration tree on $\mathcal{M}$. Thus internal iterations are essentially iteration trees whose tree orderings are linear and so, by property (ii) of tree orderings, are the natural orderings of their lengths.

It would accord better with the concept of internal iterations if we defined iteration trees on $\mathcal{M}$ to be of the form $\langle T, \langle E_\alpha \mid \alpha + 1 < \theta \rangle \rangle$. This would have the additional virtue of making iteration trees sets, where the actual definition makes them proper classes if $\mathcal{M}$ is a proper class. Indeed it was for just this reason that we did not make $\mathcal{M}$ a component of internal iterations on $\mathcal{M}$. Unfortunately, our notation would become too cumbersome if we were to do likewise for iteration trees on $\mathcal{M}$. We will often need notation such as "$j^{\mathcal{T}}_{\alpha,\beta}$," and we do not want to write instead "$j^{\mathcal{T},\mathcal{M}}_{\alpha,\beta}$" or something more complicated when we have a more complex name than "$\mathcal{M}$."

The amount of agreement between two of the models of an iteration tree is related to the strength of the extenders of the tree. If $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ and $\alpha < \beta \leq \ell\mathrm{h}(\mathcal{T})$, then set

$$\rho^{\mathcal{T}}(\alpha, \beta) = \min\{\operatorname{strength}^{\mathcal{M}^{\mathcal{T}}_\gamma}(E^{\mathcal{T}}_\gamma) \mid \alpha \leq \gamma < \beta\}.$$

Lemma 7.2.5. *Let $\mathcal{T}$ be an iteration tree on $\mathcal{M}$ and let $\alpha < \beta < \ell\mathrm{h}(\mathcal{T})$. Then*

(1) $\mathcal{M}^{\mathcal{T}}_\alpha$ and $\mathcal{M}^{\mathcal{T}}_\beta$ agree through $\rho^{\mathcal{T}}(\alpha,\beta)$;

(2) $V^{\mathcal{M}^{\mathcal{T}}_\beta}_{\rho^{\mathcal{T}}(\alpha,\beta)+1} \subsetneq V^{\mathcal{M}^{\mathcal{T}}_\alpha}_{\rho^{\mathcal{T}}(\alpha,\beta)+1}$.

Proof. We suppress the superscript $\mathcal{T}$, and we suppress the subscript T. Fix $\alpha < \ell\mathrm{h}(\mathcal{T})$. We proceed by induction on β for $\alpha < \beta < \ell\mathrm{h}(\mathcal{T})$.

First suppose that $\beta = \gamma + 1$ for some $\gamma \geq \alpha$. Let $\delta = \beta^-$. Now $\mathcal{M}_\beta = \mathrm{Ult}(\mathcal{M}_\delta; E_\gamma)$, and by part (1) of Lemma 7.2.4 $\mathrm{Ult}(\mathcal{M}_\delta; E_\gamma)$ agrees with $\mathcal{M}_\gamma$ through $\mathrm{strength}^{\mathcal{M}_\gamma}(E_\gamma)$. If $\alpha = \gamma$, then $\rho(\alpha,\beta) = \mathrm{strength}^{\mathcal{M}_\gamma}(E_\gamma)$. If $\alpha < \gamma$, then $\rho(\alpha,\beta) = \min\{\mathrm{strength}^{\mathcal{M}_\gamma}(E_\gamma), \rho(\alpha,\gamma)\}$, and induction gives us that $\mathcal{M}_\alpha$ and $\mathcal{M}_\gamma$ agree through $\rho(\alpha,\gamma)$. In either case, we have (1). If $\alpha = \gamma$ or if $\mathrm{strength}^{\mathcal{M}_\gamma}(E_\gamma) \leq \rho(\alpha,\gamma)$, then part (2) of Lemma 7.2.4 and (if $\alpha < \gamma$) induction give that

$$V^{\mathcal{M}_\beta}_{\mathrm{strength}^{\mathcal{M}_\gamma}(E_\gamma)+1} \subsetneq V^{\mathcal{M}_\gamma}_{\mathrm{strength}^{\mathcal{M}_\gamma}(E_\gamma)+1} \subseteq V^{\mathcal{M}_\alpha}_{\mathrm{strength}^{\mathcal{M}_\gamma}(E_\gamma)+1}.$$

If $\alpha < \gamma$ and $\rho(\alpha,\gamma) \leq \mathrm{strength}^{\mathcal{M}_\gamma}(E_\gamma)$, then part (2) of Lemma 7.2.4 and induction give that

$$V^{\mathcal{M}_\beta}_{\rho(\alpha,\gamma)+1} \subseteq V^{\mathcal{M}_\gamma}_{\rho(\alpha,\gamma)+1} \subsetneq V^{\mathcal{M}_\alpha}_{\rho(\alpha,\gamma)+1}.$$

In either case (2) follows.

Now suppose that β is a limit ordinal. We first show that there are only finitely many γ such that

$$(\gamma + 1)\, T\, \beta \ \wedge\ \mathrm{crit}\,(E_\gamma) \leq \rho(\alpha,\beta).$$

Assume that $\gamma_0 < \gamma_1 < \cdots$ witness that this fails. For each $i \in \omega$, $\mathrm{crit}\,(E_{\gamma_i}) \leq \rho(\alpha,\beta)$. Hence we have for each i that

$$\begin{aligned}
j_{(\gamma_i+1)^-,(\gamma_{i+1}+1)^-}(\mathrm{crit}\,(E_{\gamma_i})) &\geq j_{(\gamma_i+1)^-,\gamma_i+1}(\mathrm{crit}\,(E_{\gamma_i}))\\
&= i^{\mathcal{M}_{(\gamma_i+1)^-}}_{E_{\gamma_i}}(\mathrm{crit}\,(E_{\gamma_i}))\\
&= i^{\mathcal{M}_{\gamma_i}}_{E_{\gamma_i}}(\mathrm{crit}\,(E_{\gamma_i}))\\
&\geq \mathrm{strength}^{\mathcal{M}_{\gamma_i}}(E_{\gamma_i})\\
&\geq \rho(\alpha,\beta).
\end{aligned}$$

Since $j_{(\gamma_i+1)^-,(\gamma_{i+1}+1)^-}(\mathrm{crit}\,(E_{\gamma_i})) > \mathrm{crit}\,(E_{\gamma_i})$, it follows that

$$j_{(\gamma_i+1)^-,(\gamma_{i+1}+1)^-}(\rho(\alpha,\beta)) > \rho(\alpha,\beta).$$

This gives us the contradiction that $\langle j_{(\gamma_i+1)^-,\beta}(\rho(\alpha,\beta)) \mid i \in \omega\rangle$ is an infinite descending sequence of ordinals. Next we observe that, since $\{\delta \mid \delta\, T\, \beta\}$ is unbounded in β, there is a δ such that $\alpha < \delta\, T\, \beta$ and

$$(\forall\gamma)((\delta < \gamma \ \wedge\ (\gamma+1)\, T\, \beta) \ \rightarrow\ \mathrm{crit}\,(i_{E_\gamma}) > \rho(\alpha,\beta)).$$

Thus $\mathrm{crit}\,(j_{\delta,\beta}) > \rho(\alpha,\beta)$, and so $\mathcal{M}_\beta$ and $\mathcal{M}_\delta$ agree through $\rho(\alpha,\beta)+1 = \rho(\alpha,\delta)+1$. Hence (1) and (2) for α and β follow from (1) and (2) for α and δ. □

Corollary 7.2.6. *Let $\mathcal{T} = (\mathcal{M}, T, \langle E_\alpha \mid \alpha+1 < \ell\mathrm{h}(\mathcal{T})\rangle)$ be an iteration tree and let $\alpha+1 < \ell\mathrm{h}(\mathcal{T})$. Then*

- *(a)* $\mathrm{crit}\,(E_\alpha)+1 \le \rho^{\mathcal{T}}((\alpha+1)^-_T, \alpha)$;
- *(b)* $(\forall\beta)((\alpha+1)^-_T \le \beta < \alpha \ \rightarrow\ \mathrm{crit}\,(E_\alpha)+1 \le \mathrm{strength}^{\mathcal{M}^{\mathcal{T}}_\beta}(E_\beta))$.

Proof. We omit the superscript $\mathcal{T}$ and the subscript T.

By clause (e) in the definition of an iteration tree, $\mathcal{M}_\alpha$ and $\mathcal{M}_{(\alpha+1)^-}$ agree through $\mathrm{crit}\,(E_\alpha)+1$. But part (2) of Lemma 7.2.5 implies that they do not agree through $\rho((\alpha+1)^-, \alpha)+1$. Thus $\mathrm{crit}\,(E_\alpha)+1 \le \rho((\alpha+1)^-, \alpha)$.

Assume that $(\alpha+1)^- \le \beta < \alpha$. By the definition of the ρ function, $\rho((\alpha+1)^-, \alpha) \le \mathrm{strength}^{\mathcal{M}_\beta}(E_\beta)$. Thus (b) for β follows from (a). □

The main constructions of Chapter 8 will be constructions of iteration trees. There are three ingredients needed to construct an iteration tree on $\mathcal{M}$:

(1) the extenders E_α;

(2) wellfoundedness at successor ordinals $\alpha+1$, i.e., wellfoundedness of $\prod_{E_\alpha}^{\mathcal{M}_{(\alpha+1)^-}} \mathcal{M}_{(\alpha+1)^-}$;

(3) wellfoundedness at limit ordinals λ, i.e., the existence of a T-chain that is unbounded in λ such that the corresponding direct limit model is wellfounded.

For (1) we will use the One-Step Lemma, Lemma 6.3.18. The only successor ordinals that will concern us are the finite ordinals, and in the next section we will prove that, for α finite, (2) holds very generally, e.g. it holds if $\mathcal{M}$ is a model of ZFC. The only limit ordinal that will concern us is ω, and in §4 we will prove that (3) holds for $\lambda = \omega$ under conditions that will be satisfied by our constructions in Chapter 8.

7.3 Finite Trees

In our wellfoundedness proofs, we will be given an iteration tree $\mathcal{T}$ on a premouse $\mathcal{M}$, a tree for which wellfoundedness fails, and we will also be given an elementary embedding $\tau : \mathcal{M} \prec \mathcal{Q} \subsetneq \mathcal{N} = (V; \in, \delta)$. We will construct an iteration tree $\mathcal{U}$ on $\mathcal{N}$ and embeddings $\tau_\alpha : \mathcal{M}_\alpha^{\mathcal{T}} \prec \mathcal{Q}_\alpha \subsetneq \mathcal{N}_\alpha^{\mathcal{U}}$. In the first proof, we will make sure that each τ_α belongs to the universe of $\mathcal{N}_\alpha^{\mathcal{U}}$ and use this fact to derive a contradiction. In the second proof, we will get our contradiction from some further models and embeddings that are constructed at the same time as the $\mathcal{N}_\alpha$ and the τ_α. To keep our inductive constructions going, we will in both cases need a certain amount of agreement among the τ_α. The next definition gives the appropriate notion of "agreement."

Suppose that $\tau : \mathcal{M} \prec \mathcal{N}$ and $\tau' : \mathcal{M}' \prec \mathcal{N}'$, where $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{N}$, and $\mathcal{N}'$ are premice. For ordinals η, say that τ and τ' *agree through* η if

(a) $\mathcal{M}$ and $\mathcal{M}'$ agree through η;

(b) $\tau(V_\eta^{\mathcal{M}}) = \tau'(V_\eta^{\mathcal{M}'})$;

(c) $\tau \restriction V_\eta^{\mathcal{M}} = \tau' \restriction V_\eta^{\mathcal{M}'}$.

Note that (b) can be restated as follows: $\tau(\eta) = \tau'(\eta)$ and $\mathcal{N}$ and $\mathcal{N}'$ agree through $\tau(\eta)$. Note also that (b) follows from (c) if η is a successor ordinal.

The next lemma will be one of the main tools in our construction of the embeddings τ_α. A slight variant of it is called the *Shift Lemma* in [Martin and Steel, 1994].

Lemma 7.3.1. *Let $\tau : \mathcal{M} \prec \mathcal{N}$ and $\tau' : \mathcal{M}' \prec \mathcal{N}'$, with $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{N}$, and $\mathcal{N}'$ premice. Suppose that τ and τ' agree through $\kappa + 1$. Suppose that E is an extender in $\mathcal{M}$ with* $\operatorname{crit}(E) = \kappa$*. Suppose that $\prod_{\tau(E)}^{\mathcal{N}'} \mathcal{N}'$ is wellfounded. Then $\prod_E^{\mathcal{M}'} \mathcal{M}'$ is wellfounded. Moreover, if*

$$\sigma : \mathrm{Ult}(\mathcal{M}'; E) \to \mathrm{Ult}(\mathcal{N}'; \tau(E))$$

is given by

$$\sigma(\pi_E^{\mathcal{M}'}([\![a, f]\!]_E^{\mathcal{M}'})) = \pi_{\tau(E)}^{\mathcal{N}'}([\![\tau(a), \tau'(f)]\!]_{\tau(E)}^{\mathcal{N}'}),$$

then σ *is well-defined and elementary, and* σ *and* τ *agree through* $\mathrm{strength}^{\mathcal{M}}(E)$*. Furthermore, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Ult}(\mathcal{M}'; E) & \xrightarrow{\sigma} & \mathrm{Ult}(\mathcal{N}'; \tau(E)) \\ \uparrow i_E^{\mathcal{M}'} & & \uparrow i_E^{\mathcal{N}'} \\ \mathcal{M}' & \xrightarrow{\tau'} & \mathcal{N}' \end{array}$$

Proof. For $[\![a, f]\!]_E^{\mathcal{M}'} \in \prod_E^{\mathcal{M}'} \mathcal{M}'$, set

$$\tilde{\sigma}([\![a, f]\!]_E^{\mathcal{M}'}) = [\![\tau(a), \tau'(f)]\!]_{\tau(E)}^{\mathcal{N}'}.$$

We will show that $\tilde{\sigma}$ is well-defined and that

$$\tilde{\sigma} : \prod_E^{\mathcal{M}'} \mathcal{M}' \prec \prod_{\tau(E)}^{\mathcal{N}'} \mathcal{N}'.$$

Since $\prod_{\tau(E)}^{\mathcal{N}'} \mathcal{N}'$ is wellfounded by hypothesis, this will show that $\prod_E^{\mathcal{M}'} \mathcal{M}'$ is wellfounded. It will also show that σ is welldefined and elementary.

To show that $\tilde{\sigma}$ is well-defined, suppose that $[\![a, f]\!]_E^{\mathcal{M}'} = [\![b, g]\!]_E^{\mathcal{M}'}$. Then

$$X = \{z \in [\kappa]^{|a \cup b|} \mid f(z_{a, a\cup b}) = g(z_{b, a \cup b})\} \in E_{a \cup b}.$$

Now $X \in V_{\kappa+1}^{\mathcal{M}'}$, so our hypotheses about agreement imply that $X \in V_{\kappa+1}^{\mathcal{M}}$ and that $\tau(X) = \tau'(X)$. The elementarity of τ gives that $\tau(X) \in (\tau(E))_{\tau(a \cup b)}$. Using these facts and the elementarity of τ', we get that

$$\begin{array}{rcl} \{z \in [\tau'(\kappa)]^{|a \cup b|} \mid (\tau'(f))(z_{a, a \cup b}) = (\tau'(g))(z_{b, a\cup b})\} & = & \tau'(X) \\ & = & \tau(X) \\ & \in & (\tau(E))_{\tau(a \cup b)}. \end{array}$$

Hence $[\![\tau(a), \tau'(f)]\!]_{\tau(E)}^{\mathcal{N}'} = [\![\tau(b), \tau'(g)]\!]_{\tau(E)}^{\mathcal{N}'}$.

We omit the proof that $\tilde{\sigma}$ is elementary, as it is similar to the proof that $\tilde{\sigma}$ is well-defined.

By Lemma 7.2.4, $\mathrm{Ult}(\mathcal{M}'; E)$ and $\mathcal{M}$ agree through $\mathrm{strength}^{\mathcal{M}}(E)$. To show that σ and τ agree through $\mathrm{strength}^{\mathcal{M}}(E)$, what we must show is that

$$\sigma \restriction (V_{\mathrm{strength}^{\mathcal{M}}(E)}^{\mathrm{Ult}(\mathcal{M}';\mathrm{E})} \cup \{V_{\mathrm{strength}^{\mathcal{M}}(E)}^{\mathrm{Ult}(\mathcal{M}';\mathrm{E})}\}) = \tau \restriction (V_{\mathrm{strength}^{\mathcal{M}}(E)}^{\mathrm{Ult}(\mathcal{M}';\mathrm{E})} \cup \{V_{\mathrm{strength}^{\mathcal{M}}(E)}^{\mathrm{Ult}(\mathcal{M}';\mathrm{E})}\}).$$

Now strength$^{\mathcal{M}}(E) \leq i_E^{\mathcal{M}}(\kappa)$, and Lemma 7.2.3 gives that $i_E^{\mathcal{M}'}(\kappa) = i_E^{\mathcal{M}}(\kappa)$ and that

$$V_{i_E^{\mathcal{M}'}(\kappa)+1}^{\mathrm{Ult}(\mathcal{M}';\mathrm{E})} = V_{i_E^{\mathcal{M}'}(\kappa)+1}^{\mathrm{Ult}(\mathcal{M};\mathrm{E})},$$

with the latter clearly a subset of the universe of $\mathcal{M}$. Thus it suffices to prove that

$$\sigma \restriction V_{i_E^{\mathcal{M}'}(\kappa)+1}^{\mathrm{Ult}(\mathcal{M}';\mathrm{E})} = \tau \restriction V_{i_E^{\mathcal{M}'}(\kappa)+1}^{\mathrm{Ult}(\mathcal{M}';\mathrm{E})}.$$

Every element of $V_{i_E^{\mathcal{M}'}(\kappa)+1}^{\mathrm{Ult}(\mathcal{M}';\mathrm{E})}$ is of the form $\pi_E^{\mathcal{M}'}([\![a, f]\!]_E^{\mathcal{M}'})$, with $f : [\kappa]^{|a|} \to V_{\kappa+1}^{\mathcal{M}'}$. Consider such a $\pi_E^{\mathcal{M}'}([\![a, f]\!]_E^{\mathcal{M}'})$. By definition,

$$\sigma(\pi_E^{\mathcal{M}'}([\![a, f]\!]_E^{\mathcal{M}'})) = \pi_{\tau(E)}^{\mathcal{N}'}([\![\tau(a), \tau'(f)]\!]_{\tau(E)}^{\mathcal{N}'}).$$

Since f can be coded by an element of $V_{\kappa+1}^{\mathcal{M}'}$, we get that $\tau(f) = \tau'(f)$ and so that

$$\pi_{\tau(E)}^{\mathcal{N}'}([\![\tau(a), \tau'(f)]\!]_{\tau(E)}^{\mathcal{N}'}) = \pi_{\tau(E)}^{\mathcal{N}'}([\![\tau(a), \tau(f)]\!]_{\tau(E)}^{\mathcal{N}'}).$$

The agreement of τ and τ' through $\kappa + 1$ means, in particular, that $\mathcal{N}$ and $\mathcal{N}'$ agree through $\tau(\kappa) + 1$, and so Lemma 7.2.3 gives that

$$\pi_{\tau(E)}^{\mathcal{N}'}([\![\tau(a), \tau(f)]\!]_{\tau(E)}^{\mathcal{N}'}) = \pi_{\tau(E)}^{\mathcal{N}}([\![\tau(a), \tau(f)]\!]_{\tau(E)}^{\mathcal{N}}).$$

By the elementarity of τ,

$$\pi_{\tau(E)}^{\mathcal{N}}([\![\tau(a), \tau(f)]\!]_{\tau(E)}^{\mathcal{N}}) = \tau(\pi_E^{\mathcal{M}}([\![a, f]\!]_E^{\mathcal{M}})).$$

Since $\mathcal{M}$ and $\mathcal{M}'$ agree through $\kappa + 1$, another application of Lemma 7.2.3 gives that

$$\tau(\pi_E^{\mathcal{M}}([\![a, f]\!]_E^{\mathcal{M}})) = \tau(\pi_E^{\mathcal{M}'}([\![a, f]\!]_E^{\mathcal{M}'})).$$

By this chain of equalities, it follows that

$$\sigma(\pi_E^{\mathcal{M}'}([\![a, f]\!]_E^{\mathcal{M}'})) = \tau(\pi_E^{\mathcal{M}'}([\![a, f]\!]_E^{\mathcal{M}'})).$$

It remains only to show that the diagram commutes. Let x belong to the universe of $\mathcal{M}'$. Then

$$\begin{aligned} \sigma(i_E^{\mathcal{M}'}(x)) &= \sigma(\pi_E^{\mathcal{M}'}([\![\emptyset, c_x]\!]_E^{\mathcal{M}'})) \\ &= \pi_{\tau(E)}^{\mathcal{N}'}([\![\emptyset, c_{\tau'(x)}]\!]_{\tau(E)}^{\mathcal{N}'}) \\ &= i_{\tau(E)}^{\mathcal{N}'}(\tau'(x)). \end{aligned}$$

□

We are now ready to prove our wellfoundedness results for finite iteration trees.

Theorem 7.3.2. *Let $n \in \omega$.*

(1) Let $\mathcal{T}$ be an iteration tree of length $n+1$ on a premouse $\mathcal{M}$. Suppose that $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$ for some ordinals ν and δ. Suppose that $n^* < n$ and that E is an extender in $\mathcal{M}_n^{\mathcal{T}}$ with $\mathrm{crit}\,(E) < \rho^{\mathcal{T}}(n^*, n)$. Then $\prod_E^{\mathcal{M}_{n^*}^{\mathcal{T}}} \mathcal{M}_{n^*}^{\mathcal{T}}$ is wellfounded.

(2) Let $\mathcal{U}$ be an iteration tree of length $n+1$ on V. Suppose that $n' < n$ and that F is an extender in $(V;\in)_n^{\mathcal{U}}$ with $\mathrm{crit}\,(F) < \rho^{\mathcal{U}}(n', n)$. Then $\prod_F^{(V;\in)_{n'}^{\mathcal{U}}} (V;\in)_{n'}^{\mathcal{U}}$ is wellfounded.

Proof. We first observe that, for each n, (1) implies (2). To see this, suppose that $\mathcal{U}$, n', and F witness that (2) fails for n. Let δ be large enough that all the $E_m^{\mathcal{U}}$ and F belong to V_δ, and let $\nu > \delta$ be such that $\mathcal{M} = (V_\nu; \in, \delta)$ is a premouse and such that $(i')_F^{(V;\in)_{n'}^{\mathcal{U}}}(j_{0,n}^{\mathcal{U}}(V_\nu))$ is not wellfounded. (This last condition is actually automatic.) Then $(\mathcal{M}, T^{\mathcal{U}}, \langle E_m^{\mathcal{U}} \mid m < n\rangle)$, the identity embedding, n', and F witness that (1) fails for n.

Assume that the theorem is false. We may assume that n is the least number for which the theorem is false.

We will also assume that the universe M of $\mathcal{M}$ (given by the hypotheses of (1)) is countable. To see that this assumption involves no loss of generality, let η be a limit ordinal such that our given $\mathcal{T}$ belongs to V_η. Let X be a countable elementary submodel of V_η such that $\mathcal{T}$ and E are members of X. Let $\pi : X \cong M'$ with M' transitive. Then $\tau \circ (\pi^{-1} \restriction \pi(M))$ elementarily embeds $\pi(\mathcal{M})$ into $(V_\nu; \in, \delta)$ and, along with $\pi(\mathcal{T})$, n^*, and $\pi(E)$, witnesses that (1) fails for n.

We will get a contradiction by building an iteration tree $\mathcal{U}$ on $\mathcal{N} = (V; \in, \delta)$ of length $n+1$ and proving that for some extender F in $\mathcal{N}_n^{\mathcal{U}}$ the ultrapower $\prod_F^{\mathcal{N}_n^{\mathcal{U}}} \mathcal{N}_n^{\mathcal{U}}$ is not wellfounded.

The tree ordering of $\mathcal{U}$ will be T. We will mostly suppress the subscript T and the superscripts $\mathcal{T}$ and $\mathcal{U}$. In particular, we will write E_m for the extender $E_m^{\mathcal{T}}$ (but never for $E_m^{\mathcal{U}}$).

As we construct $\mathcal{U}$, we will also construct, for $m \le n$, embeddings

$$\tau_m : \mathcal{M}_m \prec (V_{\nu_m}^{\mathcal{N}_m}; \in, \delta_m),$$

where in fact $\nu_m = j^{\mathcal{U}}_{0,m}(\nu)$ and $\delta_m = j^{\mathcal{U}}_{0,m}(\delta)$. We denote $V^{\mathcal{N}_m}_{\nu_m}$ by Q_m and $(V^{\mathcal{N}_m}_{\nu_m}; \in, \delta_m)$ by $\mathcal{Q}_m$. We will make sure that

$$(\forall m \leq n)\, \tau_m \in N_m.$$

This will be the key to obtaining our contradiction.

The iteration tree $\mathcal{U}$ will be

$$(\mathcal{N}, T, \langle \tau_m(E_m) \mid m \leq n \rangle).$$

Let $\mathcal{N}_0 = \mathcal{N} = (V; \in, \delta)$. Let $\delta_0 = \delta$, $\nu_0 = \nu$, and $\tau_0 = \tau$.

Let $m < n$ and assume that we have defined, for each $k \leq m$, $\mathcal{N}_k = (N_k; \in, \delta_k)$ and

$$\tau_k : \mathcal{M}_k \prec \mathcal{Q}_k = (Q_k; \in, \delta_k) = (V^{\mathcal{N}_k}_{\nu_k}; \in, \delta_k),$$

in such a way that $(\mathcal{N}, T \restriction m+1, \langle \tau_k(E_k) \mid k < m \rangle)$ is an iteration tree whose premice* are the $\mathcal{N}_k$ and that $\tau_k \in N_k$ for each $k \leq m$. Assume also that

(a) for all $k < n$ such that $(k+1)^- \leq m \leq k$, $\tau_{(k+1)^-}$ and τ_m agree through crit $(E_k) + 1$;

(b) if $n^* \leq m$ then τ_{n^*} and τ_m agree through crit $(E) + 1$.

The elementarity of τ_m implies that $\tau_m(E_m)$ is an extender in the premouse $\mathcal{Q}_m$ and so is an extender in $\mathcal{N}_m$.

If we can show that

$$(m+1)^- < m \;\rightarrow\; \mathrm{crit}\,(\tau_m(E_m)) < \rho^{\mathcal{U}}((m+1)^-, m),$$

then we can deduce from the minimality of n that $\prod^{\mathcal{N}_{(m+1)^-}}_{\tau_m(E_m)} \mathcal{N}_{(m+1)^-}$ is wellfounded. By Lemma 7.2.5, this is equivalent with showing that $\mathcal{N}_{(m+1)^-}$ and $\mathcal{N}_m$ agree through crit $(\tau_m(E_m)) + 1$. But this is a consequence of the assumption—a special case of (a)—that $\tau_{(m+1)^-}$ and τ_m agree through crit $(E_m) + 1$.

Let

$$\mathcal{N}_{m+1} = \mathrm{Ult}(\mathcal{N}_{(m+1)^-}; \tau_m(E_m)).$$

Since $E_m \in V^{\mathcal{M}_m}_{\delta^{\mathcal{M}_m}}$, it follows that $\tau_m(E_m) \in V^{\mathcal{Q}_m}_{\delta_m} = V^{\mathcal{N}_m}_{\delta_m}$. Hence $(\mathcal{N}_0, T \restriction m+2, \langle \tau_k(E_k) \mid k < m+1 \rangle)$ is an iteration tree on $\mathcal{N}_0$.

Set $\nu_{m+1} = i^{\mathcal{N}_{(m+1)^-}}_{\tau_m(E_m)}(\nu_{(m+1)^-})$ and $\delta_{m+1} = i^{\mathcal{N}_{(m+1)^-}}_{\tau_m(E_m)}(\delta_{(m+1)^-})$. Now

$$\mathrm{Ult}(\mathcal{Q}_{(m+1)^-}; \tau_m(E_m)) = (V^{\mathcal{N}_{m+1}}_{\nu_{m+1}}; \in, \delta_{m+1}).$$

Thus we may apply the Shift Lemma (Lemma 7.3.1) with $\mathcal{M}_m$ for $\mathcal{M}$, $\mathcal{M}_{(m+1)^-}$ for $\mathcal{M}'$, $\mathcal{Q}_m$ for $\mathcal{N}$, $\mathcal{Q}_{(m+1)^-}$ for $\mathcal{N}'$, τ_m for τ, $\tau_{(m+1)^-}$ for τ', $\mathrm{crit}\,(E_m)$ for κ, and E_m for E. This gives us an embedding

$$\sigma : \mathcal{M}_{m+1} \prec \mathrm{Ult}(\mathcal{Q}_{(m+1)^-}; \tau_m(E_m)) = \mathcal{Q}_{m+1},$$

such that σ and τ_m agree through $\mathrm{strength}^{\mathcal{M}_m}(E_m)$.

Suppose that $k < n$ is such that $(k+1)^- < m+1 \leq k$; i.e., suppose that $(k+1)^- \leq m < k < n$. Our induction hypothesis (a) for m guarantees that $\tau_{(k+1)^-}$ and τ_m agree through $\mathrm{crit}\,(E_k) + 1$. By part (b) of Corollary 7.2.6, $\mathrm{crit}\,(E_k)+1 \leq \mathrm{strength}^{\mathcal{M}_m}(E_m)$. Thus $\tau_{(k+1)^-}$ and σ agree through $\mathrm{crit}\,(E_k)+1$.

The argument just given shows that, for all $k < n$,

$$(k+1)^- < m+1 \leq k \;\rightarrow\; \tau_{(k+1)^-} \text{ and } \sigma \text{ agree through } \mathrm{crit}\,(E_k) + 1.$$

Hence induction hypothesis (a) would be true for $m+1$ if we were to set $\tau_{m+1} = \sigma$.

If $n^* < m+1$ then the fact that $\mathrm{crit}\,(E)+1 \leq \rho^{\mathcal{T}}(n^*, n) \leq \mathrm{strength}^{\mathcal{M}_m}(E_m)$ and our induction hypothesis (b) for m imply that that τ_{n^*} and σ agree through $\mathrm{crit}\,(E) + 1$. It follows easily that induction hypothesis (b) would also be true for $m+1$ if we made $\tau_{m+1} = \sigma$.

Nevertheless, we cannot take σ for τ_{m+1} because σ might not belong to N_{m+1}. Let

$$\mu(m) = \sup\,\{\mathrm{crit}\,(E_k) \mid (k+1)^- \leq m < k < n\}.$$

Let

$$\mu'(m) = \begin{cases} \mu(m) & \text{if } m < n^*; \\ \max\{\mu(m); \mathrm{crit}\,(E)\} & \text{if } n^* \leq m. \end{cases}$$

If we can find a $\bar{\tau} : \mathcal{M}_{m+1} \prec \mathcal{Q}_{m+1}$ such that

$$\bar{\tau} \restriction (V^{\mathcal{M}_{m+1}}_{\mu'(m)+1}) = \sigma \restriction (V^{\mathcal{M}_{m+1}}_{\mu'(m)+1}) \quad \wedge \quad \bar{\tau} \in N_{m+1},$$

then, since $\bar{\tau}$ and σ will agree through $\mu'(m) + 1$, we can set $\tau_{m+1} = \bar{\tau}$, and the induction step of our construction will be complete.

Since n is finite, $\mu(m)$ and hence also $\mu'(m)$ are suprema of finite sets of ordinals, each of which is smaller than strength$^{\mathcal{M}_m}(E_m)$. Hence $\mu'(m)+1 \leq$ strength$^{\mathcal{M}_m}(E_m)$. Let $\chi = \sigma \restriction (V^{\mathcal{M}_{m+1}}_{\mu'(m)+1}) = \tau_m \restriction (V^{\mathcal{M}_{m+1}}_{\mu'(m)+1})$. Since $\tau_m \in N_m$, it follows that $\chi \in N_m$. All the universes M_k of the $\mathcal{M}_k$ are countable transitive sets; since crit $(j^{\mathcal{U}}_{0,m+1})$ is a measurable cardinal, they all belong to and are countable in each $\mathcal{N}_k$, $k \leq m+1$. The function χ has a subset of M_{m+1} for its domain, thus χ can be coded by a subset of $V^{\mathcal{N}_m}_{\mu'(m)+1}$ that is countable in $\mathcal{N}_m$. Now $\mu'(m)$ is either 0 or an infinite ordinal. For any infinite ordinal α, a countable subset of $V_{\alpha+1}$ can be coded by an element of $V_{\alpha+1}$. Hence χ can be coded by an element of $V^{\mathcal{N}_m}_{\mu'(m)+1}$. But $\mathcal{N}_m$ and $\mathcal{N}_{m+1}$ agree through $\mu'(m)+1$, and therefore χ is an element of N_{m+1}.

Let $\langle a_i \mid i \in \omega\rangle \in N_{m+1}$ enumerate M_{m+1}. Let U be the tree of all $u \in {}^{<\omega}(Q_{m+1})$ such that

(i) $u(i) = \chi(a_i)$ for all $i < \ell\mathrm{h}(u)$ with $a_i \in$ domain (χ);

(ii) for all formulas $\varphi(v_1, \ldots, v_k)$ of the language of set theory and for all natural numbers $i_1, \ldots, i_k$ with each $i_j < \ell\mathrm{h}(u)$, $\mathcal{M}_{m+1} \models \varphi[a_{i_1}, \ldots, a_{i_k}]$ if and only if $\mathcal{Q}_{m+1} \models \varphi[u(i_1), \ldots, u(i_k)]$.

The function $i \mapsto \sigma(a_i)$ belongs to $[U]$. By absoluteness, there is an $f \in N_{m+1}$ that belongs to $[U]$. Then $a_i \mapsto f(i)$ is our desired $\bar{\tau} : \mathcal{M}_{m+1} \prec \mathcal{Q}_{m+1}$ such that $\bar{\tau} \in N_{m+1}$ and $\bar{\tau}$ extends χ.

Since E is an extender in $\mathcal{M}_n$, we know that $\tau_n(E)$ is an extender in $\mathcal{Q}_n$ and so in $\mathcal{N}_n$. To finish the proof, we will derive a contradiction from the fact that $\mathcal{N} \models$ "$\tau_n(E)$ is countably complete." (By Lemma 6.1.5, this is the same as contradicting the wellfoundedness of $\prod^{\mathcal{N}_n}_{\tau_n(E)} \mathcal{N}_n$.)

Let $\kappa = $ crit (E). By assumption, we have that $\prod^{\mathcal{M}_{n^*}}_{E} \mathcal{M}_{n^*}$ is not wellfounded. Let then

$$\cdots \in^{\mathcal{M}_{n^*}}_{E} [\![a_2, f_2]\!]^{\mathcal{M}_{n^*}}_{E} \in^{\mathcal{M}_{n^*}}_{E} [\![a_1, f_1]\!]^{\mathcal{M}_{n^*}}_{E} \in^{\mathcal{M}_{n^*}}_{E} [\![a_0, f_0]\!]^{\mathcal{M}_{n^*}}_{E}.$$

Without loss of generality, we may assume that

$$(\forall i \in \omega)\, a_i \subseteq a_{i+1}.$$

For each $i \in \omega$, let

$$X_{i+1} = \{z \in [\kappa]^{|a_{i+1}|} \mid f_{i+1}(z) \in f_i(z_{a_i, a_{i+1}})\},$$

and let $X_0 = [\kappa]^{|a_0|}$. By Theorem 7.2.1, $X_{i+1} \in E_{a_{i+1}}$ for all $i \in \omega$; $X_0 \in E_{a_0}$ trivially. Since τ_n is elementary,

$$(\forall i \in \omega)\, \tau_n(X_i) \in (\tau_n(E))_{\tau_n(a_i)}.$$

All subsets of the hereditarily countable M_n belong to N_n, and therefore both $\langle a_i \mid i \in \omega\rangle$ and $\langle X_i \mid i \in \omega\rangle$ belong to N_n. But $\tau_n \in N_n$, and so

$$\langle \tau_n(a_i) \mid i \in \omega\rangle \in N_n \quad \wedge \quad \langle \tau_n(X_i) \mid i \in \omega\rangle \in N_n.$$

Let $b = \bigcup_{i\in\omega} \tau_n(a_i)$. Since $\tau_n(E)$ is countably complete in $\mathcal{N}_n$, there is a function $h : b \to \tau_n(\kappa)$ such that

$$(\forall i \in \omega)\, h''\tau_n(a_i) \in \tau_n(X_i).$$

Since $\mathcal{M}_n$ and $\mathcal{M}_{n^*}$ agree through $\rho^{\mathcal{T}}(n^*, n) \geq \kappa + 1$, all the X_i belong to M_{n^*}. By the elementarity of τ_{n^*},

$$(\forall i \in \omega)(\forall z \in \tau_{n^*}(X_i))\, (\tau_{n^*}(f_{i+1}))(z) \in (\tau_{n^*}(f_i))(z_{a_i,a_{i+1}}).$$

But τ_n and τ_{n^*} agree through $\kappa + 1$, so

$$(\forall i \in \omega)(\forall z \in \tau_n(X_i))\, (\tau_{n^*}(f_{i+1}))(z) \in (\tau_{n^*}(f_i))(z_{a_i,a_{i+1}}).$$

Hence

$$(\forall i \in \omega)\, (\tau_{n^*}(f_{i+1}))(h''\tau_n(a_{i+1})) \in (\tau_{n^*}(f_i))(h''\tau_n(a_i)).$$

But this contradicts the wellfoundedness of N_{n^*}. $\square$

Remark. Theorem 7.3.2 does not cover the case $n^* = n$, but of course $\prod_E^{\mathcal{M}_n^{\mathcal{T}}} \mathcal{M}_n^{\mathcal{T}}$ is always wellfounded for E an extender in $\mathcal{M}_n^{\mathcal{T}}$ with $\operatorname{crit}(E) \leq \delta^{\mathcal{M}_n^{\mathcal{T}}}$. Part (2) of Theorem 7.3.2 thus guarantees that failure of wellfoundedness will never interfere with our construction of iteration trees on V of length $\leq \omega$.

7.4 Trees of length ω

If $\mathcal{T}$ is an iteration tree of length θ, then a *branch* of $\mathcal{T}$ is a nonempty subset b of θ such that

(i) b has no $<$-greatest element;

(ii) b is linearly ordered by $T^{\mathcal{T}}$;

(iii) if $\beta \in b$ and $\alpha \, T^{\mathcal{T}} \beta$, then $\alpha \in b$.

Note that being a branch of $\mathcal{T}$ depends only on $T^{\mathcal{T}}$. If b is a branch of $\mathcal{T}$ then b is a branch of any other iteration tree with the same tree ordering.

If $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ and b is a branch of $\mathcal{T}$, then we denote by

$$(\tilde{\mathcal{M}}_b^{\mathcal{T}}, \langle \tilde{\jmath}_{\alpha,b}^{\mathcal{T}} \mid \alpha \in b \rangle)$$

the direct limit of

$$(\mathcal{M}_\alpha^{\mathcal{T}}, \langle j_{\alpha,\beta}^{\mathcal{T}} \mid \alpha \, T^{\mathcal{T}} \beta \in b \rangle).$$

We say that a branch b of $\mathcal{T}$ is *wellfounded with respect to* $\mathcal{T}$ if the direct limit model $\tilde{\mathcal{M}}_b^{\mathcal{T}}$ is wellfounded. When there is no ambiguity, we will omit the phrase "with respect to $\mathcal{T}$." If b is wellfounded, then we denote by $(\mathcal{M}_b^{\mathcal{T}}, \langle j_{\alpha,b}^{\mathcal{T}} \mid \alpha \in b \rangle)$ the canonical limit of $(\mathcal{M}_\alpha^{\mathcal{T}}, \langle j_{\alpha,\beta}^{\mathcal{T}} \mid \alpha \, T^{\mathcal{T}} \beta \in b \rangle)$.

If $\mathcal{T}$ is an iteration tree of limit length θ, then a *cofinal branch* of $\mathcal{T}$ is a branch b of $\mathcal{T}$ such that b is an unbounded subset of θ. For $\theta = \omega$, all branches are thus cofinal branches.

Two possible problems can arise at limit steps θ in the construction of an iteration tree: (1) There may be no cofinal branch. (2) There may be cofinal branches, but no cofinal wellfounded branches. By clause (f) in the definition of iteration trees, either (1) or (2) would make it impossible to extend the iteration tree to one of length $\theta + 1$. Since the iteration trees we construct in Chapter 8 will all be of length $\leq \omega$, we will not be worried about this problem *per se*. Nevertheless, we will have to rule out (1) and (2) for the trees of length ω that we construct, for it will be crucial for us that each of our trees has a cofinal wellfounded branch. (In fact, problem (1) will not arise: it will be obvious that our trees have cofinal branches.)

It is an open question whether every iteration tree of length ω on V has a wellfounded branch. The trees we construct in Chapter 8 will, fortunately, have two special properties. We next introduce these two properties, one at a time.

An iteration tree $\mathcal{T}$ of length ω on $\mathcal{M}$ is *continuously illfounded* if there are ξ_n, $n \in \omega$, such that each ξ_n is an ordinal of $\mathcal{M}_n^{\mathcal{T}}$ and such that, for m and $n \in \omega$,

$$m \, T \, n \;\rightarrow\; j_{m,n}^{\mathcal{T}}(\xi_m) > \xi_n.$$

A continuously illfounded tree cannot have wellfounded branches, for if b is a branch then

$$\langle \tilde{\jmath}_{n,b}^{\mathcal{T}}(\xi_n) \mid n \in \omega \rangle$$

is an infinite descending sequence of ordinals of $\tilde{\mathcal{M}}_b^{\mathcal{T}}$.

Lemma 7.4.1. *Let $\mathcal{T}$ be an iteration tree of length ω on V with no branches. Then $\mathcal{T}$ is continuously illfounded.*

Proof. Let T be the tree ordering of $\mathcal{T}$. The absence of branches is equivalent with the wellfoundedness of T^*, where where $m\,T^*\,n$ if and only if $n\,T\,m$. For $n \in \omega$ define, by induction on T^*,

$$\xi_m = \sup\{\xi_n + 1 \mid n\,T^*\,m\} = \sup\{\xi_n + 1 \mid m\,T\,n\}.$$

For $m\,T\,n$ we have that

$$j^{\mathcal{T}}_{m,n}(\xi_m) \geq \xi_m > \xi_n.$$

Hence the ξ_n witness that $\mathcal{T}$ is continuously illfounded. □

If $\mathcal{T}$ is an iteration tree of length ω with tree ordering T and if b is a branch of $\mathcal{T}$, then say that $\mathcal{T}$ is *continuously illfounded off* b if there are ξ_n, $n \in \omega$, such that each ξ_n is an ordinal of $\mathcal{M}^{\mathcal{T}}_n$ and such that, for all m and $n \in \omega$,

$$m\,T\,n \;\rightarrow\; \begin{cases} j^{\mathcal{T}}_{m,n}(\xi_m) > \xi_n & \text{if } n \notin b; \\ j^{\mathcal{T}}_{m,n}(\xi_m) = \xi_n & \text{if } n \in b. \end{cases}$$

Each iteration tree $\mathcal{T}$ we construct in Chapter 8 will be continuously illfounded off some branch b of $\mathcal{T}$, and we will want to know that b is wellfounded. Lemma 7.4.3 below shows that this will follow if we know that $\mathcal{T}$ is not continuously illfounded.

First we need to prove an equivalent of illfoundedness for such limit models.

Lemma 7.4.2. *Let $\mathcal{M}_n$, $n \in \omega$, be premice* and let $j_{m,n} : \mathcal{M}_m \prec \mathcal{M}_n$ for $m \leq n \in \omega$. Assume that whenever $m \leq n \leq p \in \omega$ then $j_{m,p} = j_{n,p} \circ j_{m,n}$. Assume also that $j_{0,n}''\mathrm{Ord}^{\mathcal{M}_0}$ is unbounded in the ordinals of $\mathcal{M}_n$ for each $n \in \omega$. Let $(\tilde{\mathcal{M}}, \langle \tilde{\jmath}_n \mid n \in \omega\rangle)$ be the direct limit of $(\mathcal{M}_n, \langle j_{m,n} \mid m \leq n \in \omega\rangle)$. The following are equivalent:*

(a) $\tilde{\mathcal{M}}$ is not wellfounded.

(b) There are ζ_n, $n \in \omega$, such that each ζ_n is an ordinal of $\mathcal{M}_n$ and such that, for all m and $n \in \omega$,

$$m < n \;\rightarrow\; j_{m,n}(\zeta_m) > \zeta_n.$$

Proof. If ζ_n, $n \in \omega$, witness that (b) holds, then $\langle \tilde{\jmath}_n(\zeta_n) \mid n \in \omega\rangle$ is an infinite descending sequence of ordinals of $\tilde{\mathcal{M}}$.

To see that (a) implies (b), assume that $\tilde{\mathcal{M}}$ is illfounded. If $\langle x_i \mid i \in \omega\rangle$ is an infinite descending sequence with repect to the membership relation of $\tilde{\mathcal{M}}$, then $\langle \mathrm{rank}^{\tilde{\mathcal{M}}}(x_i) \mid i \in \omega\rangle$ is an infinite descending sequence of ordinals of $\tilde{\mathcal{M}}$. Since every ordinal of $\tilde{\mathcal{M}}$ is of the form $\tilde{\jmath}_n(\gamma)$ for some $n \in \omega$ and some ordinal γ of $\mathcal{M}_n$, we may assume that there are sequences $\langle n_i \mid i \in \omega\rangle$ and $\langle \gamma_i \mid i \in \omega\rangle$ such that

(i) $(\forall i \in \omega)\, \gamma_i$ is an ordinal of $\mathcal{M}_{n_i}$;

(ii) $\langle \tilde{\jmath}_{n_i}(\gamma_i) \mid i \in \omega\rangle$ is an infinite descending sequence of ordinals of $\tilde{\mathcal{M}}$.

If some number n were n_i for infinitely many i, then the corresponding subsequence of the γ_i would be an infinite descending sequence of ordinals of $\mathcal{M}_n$. Thus we may assume that

$$i < i' \;\rightarrow\; n_i < n_{i'}.$$

Since $j_{0,n_0}{}''\mathrm{Ord}^{\mathcal{M}_0}$ is unbounded in the ordinals of $\mathcal{M}_{n_0}$, we may assume also without loss of generality that $n_0 = 0$. Replacing, if necessary, each γ_i by $\omega\gamma_i$, we may assume that each γ_i is a limit ordinal. For $i \in \omega$ and $n_i \leq n < n_{i+1}$, set

$$\zeta_n = j_{n_i,n}(\gamma_i) + n_{i+1} - n.$$

The ζ_n, $n \in \omega$, witness that (b) holds. □

Lemma 7.4.3. *Let $\mathcal{T}$ be an iteration tree of length ω and let b be a branch of $\mathcal{T}$. Assume that $\mathcal{T}$ is continuously illfounded off b and that b is not well-founded. Then $\mathcal{T}$ is continuously illfounded.*

Proof. Let $\langle \xi_i \mid i \in \omega\rangle$ witness that $\mathcal{T}$ is continuously illfounded off b. Let $\langle \zeta_n \mid n \in \omega\rangle$ be as given by the illfoundedness of b and Lemma 7.4.2. For $i \in \omega$ let

$$\xi^*_i = \begin{cases} \xi_i & \text{if } i \notin b; \\ \xi_i + \zeta_i & \text{if } i \in b. \end{cases}$$

The ξ^*_i, $i \in \omega$, witness that $\mathcal{T}$ is continuously illfounded. □

It is unknown whether there is a continuously illfounded iteration tree of length ω on V. But we can show that there are no such trees that obey a certain technical restriction, which we now describe.

Let $\mathcal{T} = (\mathcal{M}, T, \langle E_\alpha \mid \alpha + 1 < \ell\mathrm{h}(\mathcal{T})\rangle)$ be an iteration tree. For $\beta + 2 < \ell\mathrm{h}(\mathcal{T})$, let

$$\mu^{\mathcal{T}}(\beta) = \sup\{\mathrm{crit}\,(E_\alpha) \mid (\alpha+1)^-_T \leq \beta < \alpha\}.$$

For $\mathcal{T}$ the tree of Theorem 7.3.2 and for $m + 2 < \ell\mathrm{h}(\mathcal{T})$, $\mu^{\mathcal{T}}(m)$ is the same as the ordinal $\mu(m)$ defined in the proof of Theorem 7.3.2. By part (a) of Corollary 7.2.6, $\mu^{\mathcal{T}}(\beta) \leq \mathrm{strength}^{\mathcal{M}_\beta}(E_\beta)$. In the proof of Theorem 7.3.2, we used the fact that, for $\mathcal{T}$ finite, $\mu^{\mathcal{T}}(\beta) < \mathrm{strength}^{\mathcal{M}_\beta}(E_\beta)$. Unfortunately, this may fail for infinite iteration trees.

For $n \in \omega$, we say that an iteration tree $\mathcal{T}$ is a *plus n iteration tree* if, for every β such that $\beta + 2 < \ell\mathrm{h}(\mathcal{T})$,

$$\mu^{\mathcal{T}}(\beta) + n \leq \mathrm{strength}^{\mathcal{M}_\beta}(E_\beta).$$

The technical restriction mentioned above is thus being a plus one iteration tree. Some of the results of [Martin and Steel, 1994] require plus two trees.

Remark. Iteration trees in the sense of [Martin and Steel, 1988] and [Martin and Steel, 1989] are a special kind of plus one trees. See Exercise 7.4.1.

Theorem 7.4.4. *Let $\mathcal{M}$ be a premouse and suppose that $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$ for some ordinals ν and δ. Then there is no continuously illfounded plus one iteration tree of length ω on $\mathcal{M}$.*

Proof. Assume for a contradiction that $\mathcal{T} = (\mathcal{M}, T, \langle E_n \mid n \in \omega\rangle)$ is a plus one iteration tree and that $\langle \xi_n \mid n \in \omega\rangle$ witnesses that $\mathcal{T}$ is continuously illfounded.

As in the proof of Theorem 7.3.2, we may assume that the universe of M is countable.

We will construct an iteration tree $\mathcal{U} = (\mathcal{N}, T, \langle F_m \mid m \in \omega\rangle)$ with $\mathcal{N} = (V; \in, \delta)$. For each $m \in \omega$, we will also construct

(a) an uncountable premouse $\bar{\mathcal{N}}_m$;

(b) $\psi_m : \bar{\mathcal{N}}_m \prec \mathcal{P}_m = (V^{\mathcal{N}_m}_{\eta_m}; \in, \delta_m)$, where $\delta_m = j^{\mathcal{U}}_{0,m}(\delta)$;

(c) $\bar{\tau}_m : \mathcal{M}_m \prec \bar{\mathcal{Q}}_m = (V^{\bar{\mathcal{N}}_m}_{\bar{\nu}_m}; \in, \bar{\delta}_m)$, where $\bar{\delta}_m = \delta^{\bar{\mathcal{N}}_m}$ and hence $\psi_m(\bar{\delta}_m) = \delta_m$, and where $\psi_m(\bar{\nu}_m) = \nu_m = j^{\mathcal{U}}_{0,m}(\nu)$.

We set $\tau_m = \psi_m \circ \bar{\tau}_m$. Thus we will have

$$\tau_m : \mathcal{M}_m \prec \mathcal{Q}_m = (V_{\nu_m}^{\mathcal{P}_m}; \in, \delta_m) = (V_{\nu_m}^{\mathcal{N}_m}; \in, \delta_m).$$

The extender F_m will be $\tau(E_m)$.

For each of our premice whose name is a subscripted calligraphic letter, we denote the universe of the premouse by the corresponding roman letter. We will suppress the subscript T and—as we already have done—suppress the superscripts $\mathcal{T}$ and $\mathcal{U}$ except where there is ambiguity.

For all $m \in \omega$, the following conditions will be satisfied:

(i) for all $k \in \omega$, if $(k+1)^- \leq m \leq k$ then $\bar{\tau}_{(k+1)^-}$ and $\bar{\tau}_m$ agree through $\mathrm{crit}\,(E_k) + 1$;

(ii) for all $k \leq m$, ψ_k and ψ_m agree through

$$\min\,\{\mathrm{strength}^{\bar{\mathcal{N}}_i}(\bar{\tau}_i(E_i)) \mid k \leq i < m\}$$

$(= \min\,\{\bar{\tau}_i(\mathrm{strength}^{\mathcal{M}_i}(E_i)) \mid k \leq i < m\})$;

(iii) $\bar{\tau}_m \in \bar{N}_m$;

(iv) $\{\alpha \mid \bar{\nu}_m < \alpha \in \mathrm{Ord}^{\bar{\mathcal{N}}_m} \ \wedge\ (V_\alpha^{\bar{\mathcal{N}}_m}; \in, \bar{\delta}_m)$ is a premouse$\}$ has order type at least $\bar{\tau}_m(\xi_m)$.

(v) $\bar{N}_{m+1} \in \bar{N}_m$.

Because condition (v) contradicts the Axiom of Foundation, our construction will give the desired *reductio ad absurdum*.

The independent objects we must define are $\bar{N}_m$, η_m, ψ_m, and $\bar{\tau}_m$.

Let η_0 be such that $(V_{\eta_0}; \in, \delta)$ is a premouse and $\{\alpha \mid \nu < \alpha < \eta_0 \wedge (V_\alpha; \in, \delta)$ is a premouse$\}$ has order type $\tau(\xi_0)$. Let $\bar{N}_0 = P_0$ $(= V_{\eta_0})$. Let ψ_0 be the identity. Let $\bar{\tau}_0 = \tau$.

Let $m \in \omega$ and suppose we have defined $\bar{N}_k$, η_k, ψ_k, and $\bar{\tau}_k$ for all $k \leq m$. Suppose that $(\mathcal{N}, T \restriction m+1, \langle F_k \mid k < m \rangle)$ is an iteration tree. Suppose that, for all $m' \leq m$, (a)–(c) hold with "m'" replacing "m." Suppose that (i)–(iv) hold and that, for all $m' < m$, (v) holds with "m'" replacing "m."

The elementarity of τ_m gives that $F_m = \tau_m(E_m)$ is an extender in $\mathcal{Q}_m$ and so in $\mathcal{N}_m$.

We first show that conditions (i) and (ii) imply that $\tau_{(m+1)^-}$ and τ_m agree through $\mathrm{crit}\,(E_m) + 1$. By condition (i), we will have shown this if we prove that conditions (i) and (ii) imply that $\psi_{(m+1)^-}$ and ψ_m agree through

$\operatorname{crit}(\bar{\tau}_m(E_m))+1$. Corollary 7.2.6 gives that $\operatorname{crit}(E_m)+1 \leq \rho^{\mathcal{T}}(m+1)^-, m)$, i.e., that $\operatorname{crit}(E_m)+1 \leq \operatorname{strength}^{\mathcal{M}_i}(E_i)$ for all i such that $(m+1)^- \leq i < m$. Since condition (i) gives that $\bar{\tau}_i(\operatorname{crit}(E_m)) = \bar{\tau}_m(\operatorname{crit}(E_m))$ for all such i, the elementarity of the $\bar{\tau}_i$ implies that $\operatorname{crit}(\bar{\tau}_m(E_m)) + 1 \leq \operatorname{strength}^{\bar{\mathcal{Q}}_i}(\bar{\tau}_i(E_i)) = \operatorname{strength}^{\bar{\mathcal{N}}_i}(\bar{\tau}_i(E_i))$ for all such i. The desired conclusion follows from condition (ii).

The agreement of $\tau_{(m+1)^-}$ and τ_m implies that $\mathcal{Q}_{(m+1)^-}$ and $\mathcal{Q}_m$ agree through $\operatorname{crit}(\tau_m(E_m))+1$, and so $\mathcal{N}_{(m+1)^-}$ and $\mathcal{N}_m$ agree through $\operatorname{crit}(\tau_m(E_m))+1$. Thus either $(m+1)^- = m$ or else $\operatorname{crit}(\tau_m(E_m)) + 1 < \rho^{\mathcal{U}}((m+1)^-, m)$. Thus we can apply part (2) of Lemma 7.3.2 to deduce that $\prod_{\tau_m(E_m)}^{\mathcal{N}_{(m+1)^-}} \mathcal{N}_{(m+1)^-}$ is wellfounded. Since $F_m = \tau_m(E_m)$ belongs to $V_{\delta_m}^{\mathcal{N}_m}$, it follows that $(\mathcal{N}, T \restriction m+2, \langle F_k \mid k < m+1\rangle)$ is an iteration tree.

By the elementarity of $\bar{\tau}_m$, we get that $\bar{F}_m = \bar{\tau}_m(E_m)$ is an extender in $\bar{\mathcal{Q}}_m$ and so in $\bar{\mathcal{N}}_m$ and that $\bar{F}_m$ belongs to $V_{\bar{\delta}_m}^{\bar{\mathcal{N}}_m}$. Moreover $\psi_m(\bar{F}_m) = F_m$. Thus we can apply the Shift Lemma (Lemma 7.3.1) with $\bar{\mathcal{N}}_m$ for $\mathcal{M}$, $\bar{\mathcal{N}}_{(m+1)^-}$ for $\mathcal{M}'$, $\mathcal{P}_m$ for $\mathcal{N}$, $\mathcal{P}_{(m+1)^-}$ for $\mathcal{N}'$, ψ_m for τ, $\psi_{(m+1)^-}$ for τ', $\operatorname{crit}(\bar{F}_m)$ for κ, and $\bar{F}_m$ for E. This gives us that $\prod_{\bar{F}_m}^{\bar{\mathcal{N}}_{(m+1)^-}} \bar{\mathcal{N}}_{(m+1)^-}$ is wellfounded, and it gives us an embedding

$$\hat{\sigma} : \operatorname{Ult}(\bar{\mathcal{N}}_{(m+1)^-}; \bar{F}_m) \prec \operatorname{Ult}(\mathcal{P}_{(m+1)^-}; F_m) = \mathcal{P}_{m+1},$$

such that $\hat{\sigma}$ and ψ_m agree through $\operatorname{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)$.

Let us next see that, for all $k \leq m$, ψ_k and $\hat{\sigma}$ agree through

$$\min\{\operatorname{strength}^{\bar{\mathcal{N}}_i}(\bar{F}_i) \mid k \leq i < m+1\},$$

where each $\bar{F}_i = \bar{\tau}_i(E_i)$. We know that ψ_m and $\hat{\sigma}$ agree through $\operatorname{strength}^{\bar{\mathcal{N}}_m}(\bar{\tau}_m(E_m))$. This gives us the case $k = m$ and, together with condition (ii), gives the case $k < m$ also.

Another application of the Shift Lemma gives us an embedding

$$\bar{\sigma} : \mathcal{M}_{m+1} \prec \operatorname{Ult}(\bar{\mathcal{Q}}_{(m+1)^-}; \bar{F}_m),$$

such that $\bar{\sigma}$ and $\bar{\tau}_m$ agree through $\operatorname{strength}^{\mathcal{M}_m}(E_m)$ and such that $\bar{\sigma} \circ i_{E_m}^{\mathcal{M}_{(m+1)^-}} = i_{\bar{F}_m}^{\bar{\mathcal{Q}}_{(m+1)^-}} \circ \bar{\tau}_{(m+1)^-}$. (We are finally going to make use of the commutativity clause of the Shift Lemma.)

By an argument exactly like that in the analogous step of the proof of Theorem 7.3.2, we get that, for all $k \in \omega$,

$$(k+1)^- \leq m+1 \leq k \;\rightarrow\; \bar{\tau}_{(k+1)^-} \text{ and } \bar{\sigma} \text{ agree through crit}\,(E_k)+1.$$

We next show that there is a $\bar{\tau} \in \text{Ult}(\bar{N}_{(m+1)^-}; \bar{F}_m)$ such that $\bar{\tau} : \mathcal{M}_{m+1} \prec \text{Ult}(\bar{\mathcal{Q}}_{(m+1)^-}; \bar{F}_m)$, such that $\bar{\tau}$ and $\bar{\sigma}$ agree through $\mu^{\mathcal{T}}(m)+1$, and such that $\bar{\tau}(\xi_{m+1}) = \bar{\sigma}(\xi_{m+1})$. The argument is like that in the proof of Theorem 7.3.2 of the existence of the embedding there called $\bar{\tau}$. We will mention only the points of difference. The fact that all the M_k belong to, and are countable in, all the $\bar{N}_{k'}$ follows from the uncountability of the $\bar{N}_k$. In the earlier proof, the finiteness of the length of the iteration tree gave us that $\mu(m)+1 \leq \text{strength}^{\mathcal{M}_m}(E_m)$ and so that $\mu'(m)+1 \leq \text{strength}^{\mathcal{M}_m}(E_m)$. Here the fact that $\mathcal{T}$ is a plus one tree gives us directly that $\mu^{\mathcal{T}}(m)+1 \leq \text{strength}^{\mathcal{M}_m}(E_m)$. To take care of the extra condition that $\bar{\tau}(\xi_{m+1}) = \bar{\sigma}(\xi_{m+1})$, we simply add to our new version of requirement (i) in the definition of the tree U the clause "and $u(i) = \bar{\sigma}(a_i)$ for the i such that $a_i = \xi_{m+1}$."

The only thing preventing us from setting $\bar{N}_{m+1} = \text{Ult}(\bar{N}_{(m+1)^-}; \bar{F}_m)$, $\eta_{m+1} = i_{F_m}^{\mathcal{N}_{(m+1)^-}}(\eta_m)$, $\bar{\tau}_{m+1} = \bar{\tau}$, and $\psi_{m+1} = \hat{\sigma}$ is condition (v).

By hypothesis, $\xi_{m+1} < i_{E_m}^{\mathcal{M}_{(m+1)^-}}(\xi_{(m+1)^-})$. Thus

$$\begin{aligned}
i_{\bar{F}_m}^{\bar{\mathcal{N}}_{(m+1)^-}}(\bar{\tau}_{(m+1)^-}(\xi_{(m+1)^-})) &= i_{\bar{F}_m}^{\bar{\mathcal{Q}}_{(m+1)^-}}(\bar{\tau}_{(m+1)^-}(\xi_{(m+1)^-})) \\
&= \bar{\sigma}(i_{E_m}^{\mathcal{M}_{(m+1)^-}}(\xi_{(m+1)^-}) \\
&> \bar{\sigma}(\xi_{m+1}) \\
&= \bar{\tau}(\xi_{m+1}).
\end{aligned}$$

Let $\bar{\delta} = i_{\bar{F}_m}^{\bar{\mathcal{N}}_{(m+1)^-}}(\bar{\delta}_m)$. It follows from condition (iv) that there are at least $i_{\bar{F}_m}^{\bar{\mathcal{N}}_{(m+1)^-}}(\bar{\tau}_{(m+1)^-}(\xi_{(m+1)^-}))$ ordinals α of $\text{Ult}(\bar{\mathcal{N}}_{(m+1)^-}; \bar{F}_m)$ that are larger than $i_{\bar{F}_m}^{\bar{\mathcal{N}}_{(m+1)^-}}(\bar{\nu}_m)$ and are such that $(V_\alpha^{\text{Ult}(\bar{\mathcal{N}}_{(m+1)^-}; \bar{F}_m)}; \in, \bar{\delta})$ is a premouse. Let $\bar{\eta}$ be the $\bar{\tau}(\xi_{m+1})$st such α. Let $\eta_{m+1} = \hat{\sigma}(\bar{\eta})$.

Let $\bar{N} = \text{Ult}(\bar{N}_{(m+1)^-}; \bar{F}_m)$ and let $\bar{\mathcal{N}} = \text{Ult}(\bar{\mathcal{N}}_{(m+1)^-}; \bar{F}_m)$. Applying the Löwenheim-Skolem Theorem in $\bar{\mathcal{N}}$, we get an $X \subseteq V_{\bar{\eta}}^{\bar{\mathcal{N}}}$ such that

(1) $(X; \in, \bar{\delta}) \prec (V_{\bar{\eta}}^{\bar{\mathcal{N}}}; \in, \bar{\delta})$;

(2) $V^{\bar{\mathcal{N}}}_{\text{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)} \cup \{\bar{\tau}, i_{\bar{F}_m}^{\bar{\mathcal{N}}_{(m+1)^-}}(\bar{\nu}_m)\} \subseteq X$;

(3) in $\bar{\mathcal{N}}$, the cardinal of X is the same as that of $V^{\bar{\mathcal{N}}}_{\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)}$.

Since $(X; \in, \bar{\delta})$ is an elementary submodel of a premouse, the structure to which it is isomorphic via Lemma 3.2.4 is a premouse. Let then

$$\pi : (X; \in, \bar{\delta}) \cong (\bar{N}_{m+1}; \in, \bar{\delta}_{m+1}) = \bar{\mathcal{N}}_{m+1}.$$

Define ψ_{m+1} and $\bar{\tau}_{m+1}$ by

$$\begin{aligned} \psi_{m+1} &= \hat{\sigma} \circ \pi^{-1} : \bar{\mathcal{N}}_{m+1} \prec \mathcal{P}_{m+1}; \\ \bar{\tau}_{m+1} &= \pi(\bar{\tau}) : \mathcal{M}_{m+1} \prec (V^{\bar{\mathcal{N}}_{m+1}}_{\pi(i^{\bar{\mathcal{N}}_{(m+1)^-}}_{\bar{F}_m}(\bar{\nu}_m))}; \in, \bar{\delta}_{m+1}) = \bar{\mathcal{Q}}_{m+1}. \end{aligned}$$

Note that $\bar{\tau}_{m+1} = \pi \circ \bar{\tau}$, since, for $x \in M_{m+1}$, we have that $\bar{\tau}_{m+1}(x) = (\pi(\bar{\tau}))(x) = (\pi(\bar{\tau}))(\pi(x)) = \pi(\bar{\tau}(x))$.

Since $V^{\bar{\mathcal{N}}}_{\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)} \subseteq X$, we have that $\pi \restriction V^{\bar{\mathcal{N}}}_{\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)}$ is the identity and that $\pi(V^{\bar{\mathcal{N}}}_{\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)}) = V^{\bar{\mathcal{N}}}_{\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)}$. This implies that $\hat{\sigma}$ and ψ_{m+1} agree through $\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)$, and it also implies that $\bar{\tau}$ and $\bar{\tau}_{m+1}$ agree through $\mathrm{strength}^{\mathcal{M}_m}(E_m)$.

The agreement of $\hat{\sigma}$ and ψ_{m+1} through $\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)$ and the agreement of ψ_m and $\hat{\sigma}$ through this same ordinal imply that ψ_m and ψ_{m+1} agree through $\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)$. Together with condition (ii), this gives condition (ii) with "$m+1$" replacing m.

Let $k \in \omega$ be such that $(k+1)^- \leq m+1 \leq k$. Since $\bar{\tau}$ and $\bar{\tau}_{m+1}$ agree through $\mathrm{strength}^{\mathcal{M}_m}(E_m) > \mu^{\mathcal{T}} \geq \mathrm{crit} E_k$, they agree through $\mathrm{crit}\,(E_k)+1$. But $\bar{\tau}$ and $\bar{\sigma}$ agree through $\mu^{\mathcal{T}}(m)+1$, so it follows that $\bar{\tau}_{m+1}$ and $\bar{\sigma}$ agree through $\mathrm{crit}\,(E_k)+1$. Because $\bar{\sigma}$ and $\bar{\tau}_{(k+1)^-}$ also agree through $\mathrm{crit}\,(E_k)+1$, we finally deduce that $\bar{\tau}_{(k+1)^-}$ and $\bar{\tau}_{m+1}$ agree through $\mathrm{crit}\,(E_k)+1$. Thus we have verified condition (i) with "$m+1$" replacing "m."

Condition (iii) with "$m+1$" replacing "m" follows from the fact that $\bar{\tau} \in X$.

By the definition of $\bar{\eta}$, there are $\bar{\tau}(\xi_{m+1})$ ordinals α of $\bar{\mathcal{N}}$ that are larger than $i^{\bar{\mathcal{N}}_{(m+1)^-}}_{\bar{F}_m}(\bar{\nu}_m)$ such that $(V^{\bar{\mathcal{N}}}_\alpha; \in, \bar{\delta})$ is a premouse. It follows that there are $\bar{\tau}_{m+1}(\xi_{m+1})$ ordinals of $\bar{\mathcal{N}}$ that are larger than $\bar{\nu}_{m+1}$ such that $(V^{\bar{\mathcal{N}}_{m+1}}_\alpha; \in, \bar{\delta}_{m+1})$ is a premouse. Thus condition (iv) holds with "$m+1$" replacing "m."

By property (3) of X, the cardinal in $\bar{\mathcal{N}}$ of $\bar{\mathcal{N}}_{m+1}$ is the same as that of $V^{\bar{\mathcal{N}}}_{\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)}$. Hence $\bar{\mathcal{N}}_{m+1}$ can be coded as an element of $V^{\bar{\mathcal{N}}}_{\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)+1}$.

But Lemma 7.2.3 implies that $\mathrm{Ult}(\bar{\mathcal{N}}_{(m+1)^-}; \bar{F}_m)$ (i.e., $\bar{\mathcal{N}}$) and $\mathrm{Ult}(\bar{\mathcal{N}}_m; \bar{F}_m)$ agree through $i^{\bar{\mathcal{N}}_m}_{\bar{F}_m}(\mathrm{crit}\,(\bar{F}_m))+1$, which is at least as large as $\mathrm{strength}^{\bar{\mathcal{N}}_m}(\bar{F}_m)+1$. Hence

$$\bar{N}_{m+1} \in \mathrm{Ult}(\bar{N}_m; \bar{F}_m) \subseteq \bar{N}_m.$$

We have verified all the induction hypotheses for $m+1$, and so we have completed our construction and reached our contradiction. □

Theorem 7.4.5. *There is no continuously illfounded plus one iteration tree of length ω on V.*

Proof. By an argument like the proof that part (1) of Theorem 7.3.2 implies part (2) of that theorem, any counterexample to the the present theorem would give rise to a counterexample to Theorem 7.4.4. □

Corollary 7.4.6. *Let $\mathcal{T}$ be a plus one iteration tree of length ω on V and let b be a branch of $\mathcal{T}$. If $\mathcal{T}$ is continuously illfounded off b then b is wellfounded.*

Proof. The corollary follows directly from Lemma 7.4.3 and Theorem 7.4.5. □

Corollary 7.4.6 is the result needed in Chapter 8. Nevertheless, we will now make a few more remarks about further results and questions concerning wellfounded cofinal branches.

Suppose that $\mathcal{M}$ is a premouse and that $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$. It follows from Lemma 7.4.3 and Theorem 7.4.4 that, if $\mathcal{T}$ is a plus one iteration tree of length ω on $\mathcal{M}$ and if $\mathcal{T}$ is continuously illfounded off b, then b is wellfounded. One can also prove this assertion directly, without going through Theorems 7.4.4 and 7.4.5 (and doing so gives an alternate proof of Corollary 7.4.6). Such a direct proof is like the proof of Theorem 7.4.4, except for two modifications. The first is that condition (v) of that proof ($\bar{N}_{m+1} \in \bar{N}_m$) is now restricted to the case $m+1 \notin b$. For $m+1 \in b$, one just sets

$$\begin{aligned} \bar{\mathcal{N}}_{m+1} &= \mathrm{Ult}(\bar{\mathcal{N}}_{(m+1)^-}; \bar{F}_m); \\ \psi_{m+1} &= \hat{\sigma}; \\ \bar{\tau}_{m+1} &= \bar{\tau}. \end{aligned}$$

From the modified condition (v), it follows that

$$(\forall m)(m+1 \in b \to ((m+1)^- = m \vee \bar{N}_m \in \bar{N}_{(m+1)^-}))$$

and hence that

$$(\forall m)(m+1 \in b \rightarrow \bar{F}_m \in \bar{N}_{(m+1)^-}).$$

If $\langle m_i \mid i \in \omega\rangle$ is an enumeration of b in order of magnitude, then one has that $\langle \bar{F}_{m_i} \mid i \in \omega\rangle$ is an internal iteration of $\bar{\mathcal{N}}_0 = (V_{\eta_0}; \in, \delta)$. It is thus a consequence of Theorem 7.1.5 that there is a canonical limit $(\bar{\mathcal{N}}_b, \langle \bar{\jmath}_{n,b} \mid n \in b\rangle)$. To finish the proof, one constructs an elementary embedding $\bar{\tau}_b : \tilde{\mathcal{M}}_b \prec \bar{\mathcal{N}}_b$. The purpose of the second modification of the proof of Theorem 7.4.4 is to make this possible. One arranges, for all $k \in \omega$ and all $x \in M_k$, that

$$(\exists k' \geq k)(\forall m \geq k')\, i^{\bar{\mathcal{N}}_{(m+1)^-}}_{\bar{F}_m}(\bar{\tau}_{(m+1)^-}(j^{\mathcal{T}}_{k,m}(x))) = \bar{\tau}_{m+1}(i^{\mathcal{M}_{(m+1)^-}}_{E_m}(j^{\mathcal{T}}_{k,m}(x))).$$

This can be done, since it involves, for each stage m of the construction, making the function $\bar{\tau}$ agree with $\bar{\sigma}$ on finitely many additional arguments. One then sets

$$\bar{\tau}_b(\tilde{\jmath}_{k,b}(x)) = \lim_{n\in\omega\cap b} \bar{\jmath}_{n,b}(\bar{\tau}_n(j^{\mathcal{T}}_{k,n}(x))).$$

Suppose that $\mathcal{M}$ is a premouse and that $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$. Suppose in addition that the universe of $\mathcal{M}$ is countable. If $\mathcal{T}$ is a plus one iteration tree of length ω on $\mathcal{M}$, then $\mathcal{T}$ has a wellfounded branch, and indeed $\mathcal{T}$ has a branch b such that there is a $\tau^* : \tilde{\mathcal{M}}^{\mathcal{T}}_b \prec (V_\nu; \in, \delta)$ with $\tau^* \circ \tilde{\jmath}^{\mathcal{T}}_{0,b} = \tau$. (See Exercise 7.4.3.)

If $\mathcal{T} = (\mathcal{M}, T, \langle E_\alpha \mid \alpha+1 < \theta\rangle)$ is an iteration tree and if $\theta' \leq \theta$, then by $\mathcal{T} \restriction \theta'$ we mean the iteration tree $(\mathcal{M}, T, \langle E_\alpha \mid \alpha+1 < \theta'\rangle)$.

Let θ be a countable limit ordinal and let $\mathcal{T}$ be an iteration tree of length θ on a premouse* $\mathcal{M}$.

(a) $\mathcal{T}$ is *continuously illfounded* if there is a subset X of θ of order type ω and there are ξ_α, $\alpha < \theta$, such that

  (i) each ξ_α belongs to the universe of $\mathcal{M}^{\mathcal{T}}_\alpha$;
  (ii) $(\forall\alpha)(\forall\beta)((\alpha\, T^{\mathcal{T}}\, \beta \wedge (\exists\gamma \in X)\, \alpha < \gamma \leq \beta) \rightarrow j^{\mathcal{T}}_{\alpha,\beta}(\xi_\alpha) > \xi_\beta)$;
  (iii) $(\forall\alpha)(\forall\beta)((\alpha\, T^{\mathcal{T}}\, \beta \wedge \neg(\exists\gamma \in X)\, \alpha < \gamma \leq \beta) \rightarrow j^{\mathcal{T}}_{\alpha,\beta}(\xi_\alpha) = \xi_\beta)$.

(b) If b is a cofinal branch of $\mathcal{T}$, then $\mathcal{T}$ is *continuously illfounded off* b if there is a subset X of θ of order type ω and there are ξ_α, $\alpha < \theta$, such that

  (i) each ξ_α belongs to the universe of $\mathcal{M}^{\mathcal{T}}_\alpha$;

(ii) $(\forall\alpha)(\forall\beta)((\alpha\, T^{\mathcal{T}}\, \beta \wedge \beta \notin b \wedge (\exists\gamma \in X)\, \alpha < \gamma \leq \beta) \rightarrow j^{\mathcal{T}}_{\alpha,\beta}(\xi_\alpha) > \xi_\beta)$;

(iii) $(\forall\alpha)(\forall\beta)((\alpha\, T^{\mathcal{T}}\, \beta \wedge (\beta \in b \vee \neg(\exists\gamma \in X)\, \alpha < \gamma \leq \beta)) \rightarrow j^{\mathcal{T}}_{\alpha,\beta}(\xi_\alpha) = \xi_\beta)$.

An iteration tree $\mathcal{T}$ of countable length θ is *self-justifying* if, for all limit $\theta' < \theta$, $\mathcal{T}$ is continuously illfounded off the branch $\{\alpha < \theta' \mid \alpha\, T\, \theta'\}$.

Let us consider the problem of extending our results to trees of length greater than ω. (1) Can we show that Theorem 7.3.2 remains true if we replace the natural numbers n and $n^* < n$ by arbitrary countable ordinals α and α^* with $\alpha^* < \alpha$, and if we assume that the given iteration tree is a plus one tree and is self-justifying? (2) Can we similarly generalize Theorems 7.4.4 and 7.4.5, i.e., can we prove, for all countable ordinals θ, that if $\mathcal{M}$ is a premouse and if $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$ then there is no continuously illfounded, self-justifying, plus one iteration tree of length θ on $\mathcal{M}$? For α of the form $\omega + n$, the answer to (1) is yes. We first do the construction outlined on page 402; then do a construction like that of the proof of Theorem 7.3.2. The fact that $(\omega + n) - \omega$ is finite implies, at step ω, that χ (the analogue of the χ of the earlier proof) belongs to some $\bar{N}_k$, $k \in \omega$, that agrees enough with $\bar{N}$ (the analogue of the $\bar{N}$ of the proof of Theorem 7.4.4) to give that $\xi \in \bar{N}$. This allows us to construct the required $\bar{\tau} \in \bar{N}$, and so to get a $\bar{\tau}_\omega \in \bar{\mathcal{N}}_\omega$. We do, however, encounter an obstacle if we try to get a positive answer to (2) for the case $\theta = \omega + \omega$. Now step ω cannot be carried out, for we cannot show that χ belongs to $\bar{N}$.

In [Martin and Steel, 1994], positive answers to questions (1) and (2) are given, except that "plus one" is replaced by "plus two." In other words the following is proved:

Let $\mathcal{T}$ be a self-justifying, plus two iteration tree of countable length θ on a premouse $\mathcal{M}$ that is elementarily embeddable into some $(V_\nu; \in, \delta)$. Then

(a) if θ is a limit ordinal, then $\mathcal{T}$ is not continuously illfounded;

(b) if $\theta = \alpha{+}1$, $\alpha^* < \alpha$, E is an extender in $M^{\mathcal{T}}_\alpha$, and $\mathrm{crit}\,(E) < \rho^{\mathcal{T}}(\alpha^*, \alpha)$, *then $\prod_E^{\mathcal{M}^{\mathcal{T}}_{\alpha^*}} \mathcal{M}^{\mathcal{T}}_{\alpha^*}$ is wellfounded.*

Instead of merely replacing $\bar{\sigma}$ by $\bar{\tau}$, one uses a tree argument to replace the whole construction up to step ω by a new one. This argument requires the assumption that $\mathcal{T}$ is a plus two tree. The necessity of such an argument also means that, for general countable θ, one needs not just a single construction of the kind we have been discussing but a transfinite sequence of such constructions.

The *Cofinal Branches Hypothesis* (the CBH) is the assertion that if $\mathcal{T}$ is an iteration tree on $\mathcal{M} = (V; \in)$ then

(a) if $\mathcal{T}$ has limit length, then $\mathcal{T}$ has a wellfounded cofinal branch;

(b) if $\alpha^* < \alpha < \ell\mathrm{h}(\mathcal{T})$, if $\mathcal{M}^{\mathcal{T}}_{\alpha} \models$ "E is an extender," and if $\mathrm{crit}\,(E) < \rho^{\mathcal{T}}(\alpha^*, \alpha)$, then $\prod_E^{\mathcal{M}^{\mathcal{T}}_{\alpha^*}} \mathcal{M}^{\mathcal{T}}_{\alpha^*}$ is wellfounded.

The *Unique Branches Hypothesis* (the UBH) says that every iteration tree on V has at most one wellfounded cofinal branch.

The CBH, if true, would guarantee that illfoundedness never blocks the construction of iteration trees on V. For sufficiently closed iteration trees, the UBH implies the CBH. Unfortunately, large cardinal hypotheses in the range of Woodin cardinals imply that both the CBH and the UBH are false. See [Neeman and Steel, 2006].

Provable special cases of the UBH are very useful. Knowing that iteration trees have at most one wellfounded cofinal branch is often important in proving the existence of wellfounded cofinal branches of trees. For example, the theorem of [Martin and Steel, 1994] mentioned on page 404 gives wellfounded cofinal branches only for trees whose restrictions have unique wellfounded cofinal branches. Indeed, the application of this result in [Martin and Steel, 1994] is in a situation where the result of Exercise 7.4.6 gives such uniqueness.

Any failure of the UBH gives an inner model with a Woodin cardinal. (See Exercise 7.4.6.) In [Steel, 2002], Steel gets inner models with more Woodin cardinals from the failure of UBH for non-overlapping iteration trees. An iteration tree $\mathcal{T}$ is non-overlapping if whenever $(\alpha + 1)\, T^{\mathcal{T}}\, (\beta + 1)$ then $\mathrm{crit}\,(E^{\mathcal{T}}_{\beta})$ is greater than the λ such that $E^{\mathcal{T}}_{\alpha}$ is a (κ, λ)-extender in $\mathcal{M}^{\mathcal{T}}_{\alpha}$. The trees are used in inner model theory are essentially only non-overlapping trees. Non-overlapping trees are involved also in Exercises 7.4.12 and 7.4.13. [Sargsyan and Trang, 2016] shows that the failure of UBH for *tame* trees yields inner models of strong large cardinal hypotheses.

An important weakening of the CBH is the *Strategic Branches Hypothesis* (the SBH). For each ordinal θ, consider the game in which players I and II attempt to build an iteration tree $\mathcal{T}$ of length θ on V. I must pick the extenders $E^{\mathcal{T}}_{\alpha}$, satisfying the obvious conditions. At limit ordinals γ, player II must choose a cofinal branch of $\mathcal{T} \restriction \gamma$. Any failure of wellfoundedness, either at successor or limit steps, results in a loss for II. If $\mathcal{T}$ is actually built, then II wins. The SBH says that, for every ordinal θ, player II has a winning strategy. Clearly the CBH implies the SBH, since the CBH provides II with a trivial winning quasistrategy. See §5 of [Martin and Steel, 1994] for more on the SBH.

Exercise 7.4.1. In [Martin and Steel, 1988] and [Martin and Steel, 1989] it is required in the definition of an iteration tree $\mathcal{T}$ that there be a nondecreasing sequence $\langle \rho_\alpha \mid \alpha + 1 < \ell\mathrm{h}(\mathcal{T})\rangle$ such that, for each α,

$$\rho_\alpha < \mathrm{strength}^{\mathcal{M}_\alpha}(E_\alpha) \ \wedge \ \mathrm{crit}\,(E_\alpha) \leq \rho_{(\alpha+1)^-}\,.$$

Prove that every such iteration tree is a plus one tree.

Exercise 7.4.2. Let $\mathcal{M}$ be a premouse and suppose that $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$. Let $\mathcal{T}$ be an iteration tree of length ω on $\mathcal{M}$. Construct an iteration tree $\mathcal{U}$ on V and a sequence $\langle \tau_n \mid n \in \omega\rangle$ such that

(a) $T^{\mathcal{U}} = T^{\mathcal{T}}$;

(b) if $k \in \omega$ and $\mathcal{T}$ is a plus k tree, then so is $\mathcal{U}$;

(c) $\tau_0 = \tau$;

(d) for $n \in \omega,\ \tau_n : \mathcal{M}^{\mathcal{T}}_n \prec j^{\mathcal{U}}_{0,n}(V_\nu; \in, \delta)$;

(e) for $m\, T^{\mathcal{T}}\, n \in \omega,\ \tau_n \circ j^{\mathcal{T}}_{m,n} = j^{\mathcal{U}}_{m,n} \circ \tau_m$.

Exercise 7.4.3. Let $\mathcal{M}$ be a premouse whose universe is countable and suppose that $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$. Let $\mathcal{T}$ be a plus one iteration tree of length ω on $\mathcal{M}$. Show that there are a branch b of $\mathcal{T}$ and a $\tau^* : \tilde{\mathcal{M}}^{\mathcal{T}}_b \prec (V_\nu; \in, \delta)$ such that $\tau^* \circ \tilde{j}^{\mathcal{T}}_{0,b} = \tau$.

Hint. Let $T = T^{\mathcal{T}}$. Form a tree W whose members are initial segments of attempts to enumerate b and τ^* with the required properties. To define W, let $\langle y^k_i \mid i \leq k \in \omega\rangle$ be such that each y^k_i belongs to the universe $M^{\mathcal{T}}_k$ of $\mathcal{M}^{\mathcal{T}}_k$, such that $j^{\mathcal{T}}_{k',k}(y^{k'}_i) = y^k_i$ for $i \leq k'\, T\, k \in \omega$, and such that, for any branch b of $\mathcal{T}$,

$$\{\tilde{j}^{\mathcal{T}}_{k,b}(y^k_i) \mid i \leq k \in b\} = \tilde{M}^{\mathcal{T}}_b,$$

where $\tilde{M}^{\mathcal{T}}_b$ is the universe of $\tilde{\mathcal{M}}^{\mathcal{T}}_b$. Let W be the set of all $\langle k_i, a_i \mid i \leq n\rangle$ such that (1) $n \in \omega$, (2) $k_i\, T\, k_{i'}$ for $i < i' \leq n$, (3) $a_i \in V_\nu$, (4) $y^{k_n}_i \mapsto a_i$ is a partial elementary embedding of $\mathcal{M}^{\mathcal{T}}_{k_n}$ into $(V_\nu; \in, \delta)$, and (5) for all x in the universe of $\mathcal{M}$ and for all $i \leq n$, if $j^{\mathcal{T}}_{0,k}(x) = y^k_i$, then $a_i = \tau(x)$. If the desired b and τ^* do not exist, then W is wellfounded.

Assume that W is wellfounded. Let $\mathcal{U}$ and $\langle \tau_n \mid\in \omega\rangle$ be as given by Exercise 7.4.2. For $k \in \omega$, define $w_k = \langle k_i, a_i \mid i \leq n\rangle$ as follows. Let n be the number of T-predecessors of k. Let $k_0\, T \cdots T\, k_n = k$. Let $a_i = \tau_k(y^k_i)$. Show that $w_k \in j^{\mathcal{U}}_{0,k}(W)$. Now set

$$\xi_k = \|w_k\|^{j^{\mathcal{U}}_{0,k}(W)}.$$

Show that the ξ_k witness that $\mathcal{U}$ is continuously illfounded.

Exercise 7.4.4. Assume the result from [Martin and Steel, 1994] stated on page 404, and prove the following (the main wellfoundedness result for iteration trees in that paper).

Let $\mathcal{M}$ be a premouse whose universe is countable and suppose that $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$. Let $\mathcal{T}$ be a plus two iteration tree of countable length θ on $\mathcal{M}$. Assume that there is no maximal (not properly extendable) non-cofinal branch b of $\mathcal{T}$ such that there is a $\tau^* : \tilde{\mathcal{M}}^{\mathcal{T}}_b \prec (V_\nu; \in, \delta)$ such that $\tau^* \circ \tilde{j}^{\mathcal{T}}_{0,b} = \tau$. Then

(a) if θ is a limit ordinal, then there are a cofinal branch b of $\mathcal{T}$ and a $\tau^* : \tilde{\mathcal{M}}^{\mathcal{T}}_b \prec (V_\nu; \in, \delta)$ such that $\tau^* \circ \tilde{j}^{\mathcal{T}}_{0,b} = \tau$;

(b) if $\theta = \alpha + 1$, $\alpha^* < \alpha$, E is an extender in $\mathcal{M}^{\mathcal{T}}_\alpha$, and $\mathrm{crit}\,(E) <$ *$\rho(\alpha^*, \alpha)$, then the ultrapower $\prod_E^{\mathcal{M}^{\mathcal{T}}_{\alpha^*}} \mathcal{M}^{\mathcal{T}}_{\alpha^*}$ is wellfounded and there is a $\tau^* : \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\alpha^*}; E) \prec (V_\nu; \in, \delta)$ such that $\tau^* \circ i_E^{M^{\mathcal{T}}_{\alpha^*}} \circ j^{\mathcal{T}}_{0,\alpha^*} = \tau$.*

Hint. By taking direct limits at limit ordinals, construct $\mathcal{U}$ and $\langle \tau_\gamma \mid \gamma < \theta'\rangle$ having the properties (a)–(e) of Exercise 7.4.2, except that ω is replaced by θ', where θ' is either θ or the least ordinal at which illfoundedness prevents continuing the construction. Use an argument similar to the one in the hint for Exercise 7.4.3 to show that $\mathcal{U}$ is self-justifying. Part (b) of the result on page 404 implies that θ' is not a successor ordinal $< \theta$. Part (a) and a routine generalization of Lemma 7.4.3 imply that θ' is not a limit ordinal $< \theta$. Thus $\theta' = \theta$. Another Exercise 7.4.3 argument then shows that (a) follows from part (a) of the page 404 result. For (b), use part (b) of the page 404 result to establish the wellfoundedness of $\prod_{\tau_\alpha(E)}^{j^{\mathcal{U}}_{0,\alpha^*}(V)} j^{\mathcal{U}}_{0,\alpha^*}(V)$. This gives a $\hat{\tau} : \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\alpha^*}; E) \prec \mathrm{Ult}(j^{\mathcal{U}}_{0,\alpha^*}(V_\nu; \in, \delta); \tau_\alpha(E))$ such that $\hat{\tau} \circ i_E^{M^{\mathcal{T}}_{\alpha^*}} \circ j^{\mathcal{T}}_{0,\alpha^*} =$

$i^{j^{\mathcal{U}}_{\alpha^*}(V)}_{\tau_\alpha(E)} \circ j^{\mathcal{U}}_{0,\alpha^*} \circ \tau = i^{j^{\mathcal{U}}_{\alpha^*}(V)}_{\tau_\alpha(E)} \circ j^{\mathcal{U}}_{0,\alpha^*}(\tau)$. Use the absoluteness of illfoundedness of trees to show that there is such a $\hat{\tau}$ belonging to $\mathrm{Ult}(j^{\mathcal{U}}_{0,\alpha^*}(V); \tau_\alpha(E))$. The existence of τ^* follows from the absoluteness of $i^{j^{\mathcal{U}}_{\alpha^*}(V)}_{\tau_\alpha(E)} \circ j^{\mathcal{U}}_{0,\alpha^*}$.

Remark. The proof of this theorem, and the proof of the theorem on which it depends, go through under weaker assumptions than ZFC. For example (and we mention this example only because it will be used in subsequent exercises), if κ is an ordinal number then the theorem holds in any transitive proper class satisfying ZF + $\mathrm{DC}_{<\kappa}$ + $V = L(V_\kappa)$. ($\mathrm{DC}_{<\kappa}$ is is the assertion that sequences of dependent choices of arbitrary length $\beta < \kappa$ can always be made.)

Exercise 7.4.5. For any class model $\tilde{M}$ for the language of set theory (or an expansion of that language), we let $\mathrm{wford}\,(\tilde{M})$ be the largest ordinal that is order isomorphic to a not necessarily proper initial segment of the ordinals of $\tilde{M}$ if not every ordinal is so isomorphic, and let $\mathrm{wford}\,(\tilde{M}) = \mathrm{Ord}$ otherwise. (This is the same as $\mathrm{wfo}(\mathcal{A})$, where $\mathcal{A}$ is the ordering of the ordinals of $\tilde{M}$. See page 289.)

Assume the result of Exercise 7.4.4, in the version mentioned in the remark above, and prove the following theorem of Hugh Woodin.

Let κ be an ordinal and let M be a transitive proper class satisfying ZF + $\mathrm{DC}_{<\kappa}$ + $V = L(V_\kappa)$. *Note that all extenders of M belong to V^M_κ. Let θ be an ordinal number and let $\mathcal{T}$ be a plus two iteration tree of length θ on M.*

(a) If θ is a limit ordinal, then for every ordinal λ there is a generic maximal branch of $\mathcal{T}$ (i.e., there is in some forcing extension of V a maximal branch of $\mathcal{T}$) such that $\mathrm{wford}\,(\tilde{M}_b)$ *is* $\geq \lambda$.

(b) If $\theta = \alpha+1$, $\alpha^* < \alpha$, E is an extender in $M^{\mathcal{T}}_\alpha$, and $\mathrm{crit}\,(E) < \rho(\alpha^*, \alpha)$, *then either the conclusion of (a) holds or else $\prod_E^{M^{\mathcal{T}}_{\alpha^*}} M^{\mathcal{T}}_{\alpha^*}$ is wellfounded.*

Hint (for part (a); the proof of (b) is similar). Assume that (a) fails. Let $\lambda > \max\{\kappa, \ell\mathrm{h}(\mathcal{T})\}$ be arbitrary. Let $\psi(v_1, v_2, v_3, v_4)$ be a formula saying that v_1 and $v_2 > v_1$ are ordinals, that v_3 is a premouse with $\mathrm{Ord}^{v_3} = v_2$, that v_4 is an countable iteration tree of countable limit length on v_3, and that there is no maximal branch b of v_4 such that $\mathrm{wford}\,((\tilde{v_3})_b) \geq v_1$. Let γ be any ordinal greater than λ such that $(V^M_\gamma; \in, \kappa)$ is a premouse. Let $\mathbf{Coll}(\omega, \gamma)$ be the usual partial ordering for collapsing γ to ω. (See 539 for the definition.)

Show that if $\mathbf{G}$ is $\mathbf{Coll}(\omega,\gamma)$-generic over V, then

$$V[\mathbf{G}] \models \psi[\lambda, \gamma, (V_\gamma^M; \in, \kappa), \mathcal{T}(\gamma)],$$

where $\mathcal{T}(\gamma)$ is the iteration tree on $(V_\gamma^M; \in, \kappa)$ with the same tree ordering and extenders as $\mathcal{T}$. Argue by absoluteness that there is a $\mathcal{T}' \in M[\mathbf{G}]$ such that

$$M[\mathbf{G}] \models \psi[\lambda, \gamma, (V_\gamma^M; \in, \kappa), \mathcal{T}'].$$

Let $\eta > (\gamma^+)^M$ be such that $(V_\eta^M; \in, \kappa)$ is a premouse. Let $X \in M$ be such that X is countable, $\gamma \in X$, and $(X; \in, \kappa) \prec (V_\eta^M; \in, \kappa)$. Let N be transitive with $\pi : X \cong N$. Let $\bar{\mathbf{G}} \in M$ be $\mathbf{Coll}(\omega, \pi(\gamma))$-generic over N. Then

$$N[\bar{\mathbf{G}}] \models \psi[\pi(\lambda), \pi(\gamma), (V_{\pi(\gamma)}^N; \in, \pi(\kappa)), \pi(\mathcal{T}')].$$

Now use the absoluteness of ψ and Exercise 7.4.4, applied in M, to get a contradiction.

Exercise 7.4.6. Let θ be a limit ordinal and let $\mathcal{T} = (\mathcal{M}, T, \langle E_\alpha \mid \alpha + 1 < \theta\rangle)$ be an iteration tree. Suppose that b and c are distinct wellfounded cofinal branches of $\mathcal{T}$. Let

$$\kappa^* = \sup\{\rho^{\mathcal{T}}(\alpha, \theta) \mid \alpha < \theta\}.$$

Assume that κ^* is an ordinal both of $\mathcal{M}_b^{\mathcal{T}}$ and of $\mathcal{M}_c^{\mathcal{T}}$. (This is automatically true if $\mathcal{M}$ is a premouse or if it is a proper class.) Let $\eta = \min\{\mathrm{Ord}_b^{\mathcal{M}}, \mathrm{Ord}_c^{\mathcal{M}}\}$. Show that in $L_\eta(V_{\kappa^*}^{\mathcal{M}_b})$ $(= L_\eta(V_{\kappa^*}^{\mathcal{M}_c}))$ the ordinal κ^* is a Woodin cardinal. This is a result of [Martin and Steel, 1994].

Hint. First show that

$$\kappa^* = \sup\{\mathrm{crit}\,(j_{\alpha,b}^{\mathcal{T}}) \mid \alpha \in b\} = \sup\{\mathrm{crit}\,(j_{\alpha,c}^{\mathcal{T}}) \mid \alpha \in c\}.$$

Next define inductively

$$\begin{aligned}
\kappa_0 &= \min\{\kappa \mid (\exists\alpha)(\alpha+1 \in b \setminus c \wedge \kappa = \mathrm{crit}\,(E_\alpha))\};\\
\alpha_0 &= \max\{\alpha \mid \alpha+1 \in b \wedge \kappa_0 = \mathrm{crit}\,(E_\alpha)\};\\
\nu_n &= \min\{\nu \mid (\exists\beta)(\beta+1 \in c \setminus (\alpha_n+1) \wedge \nu = \mathrm{crit}\,(E_\beta))\};\\
\beta_n &= \max\{\beta \mid \beta+1 \in c \wedge \nu_n = \mathrm{crit}\,(E_\beta)\};\\
\kappa_{n+1} &= \min\{\kappa \mid (\exists\alpha)(\alpha+1 \in b \setminus (\beta_n+1) \wedge \kappa = \mathrm{crit}\,(E_\alpha))\};\\
\alpha_{n+1} &= \max\{\alpha \mid \alpha+1 \in b \wedge \kappa_{n+1} = \mathrm{crit}\,(E_\alpha)\}.
\end{aligned}$$

Clearly all four sequences are strictly increasing, and $\alpha_n < \beta_n < \alpha_{n+1}$ for all $n \in \omega$. Show that the α_n and β_n converge to θ and that the κ_n and ν_n converge to κ^*. Show that, for all $n \in \omega$,

$$\nu_n = \mathrm{crit}\,(E_{\beta_n}) < \mathrm{strength}^{\mathcal{M}_{\alpha_n}}(E_{\alpha_n});$$

$$\kappa_{n+1} = \mathrm{crit}\,(E_{\alpha_{n+1}}) < \mathrm{strength}^{\mathcal{M}_{\beta_n}}(E_{\beta_n}).$$

Now fix $n \in \omega$ and let z belong belong to the ranges of both $j^{\mathcal{T}}_{(\alpha_n+1)^-,b}$ and $j^{\mathcal{T}}_{(\beta_n+1)^-,c}$. Let $\varphi(v_1, v_2, v_3)$ be a Σ_0 formula of the language of set theory. Let $\gamma = \min\{\kappa_n, \nu_n\}$ and let $\gamma' = \min\{\nu_n, \kappa_{n+1}\}$. Prove that

(i) $(x \in V^{\mathcal{M}_b}_{\gamma} \wedge (\exists y \in V^{\mathcal{M}_b}_{\kappa^*})\,\varphi(x, y, z)) \rightarrow (\exists y \in V^{\mathcal{M}_b}_{\gamma})\,\varphi(x, y, z)$;

(ii) $(x \in V^{\mathcal{M}_b}_{\gamma'} \wedge (\exists y \in V^{\mathcal{M}_b}_{\kappa^*})\,\varphi(x, y, z)) \rightarrow (\exists y \in V^{\mathcal{M}_b}_{\gamma'})\,\varphi(x, y, z)$.

Now let $f : \kappa^* \to \kappa^*$ with $f \in L_\eta(V^{\mathcal{M}_b}_{\kappa^*})$. For some n, f belongs to the ranges of both $j^{\mathcal{T}}_{(\alpha_n+1)^-,b}$ and $j^{\mathcal{T}}_{(\beta_n+1)^-,c}$. Assume for definiteness that $\kappa_n < \nu_n$. Let $\xi = \min\{\nu_n, \kappa_{n+1}\}$. Observe that $\kappa_n < \xi \leq \nu_n < \mathrm{strength}^{\mathcal{M}_{\alpha_n}}(E_{\alpha_n})$. By the result just proved, κ_n is closed under f and $f(\kappa_n) < \xi$. Let

$$F = \langle (E_{\alpha_n})_a \mid a \in [\xi]^{<\omega} \rangle.$$

Prove that in $L_\eta(V^{\mathcal{M}_b}_{\kappa^*})$ the cardinal κ_n and the embedding $i_F^{L_\eta(V^{\mathcal{M}_b}_{\kappa^*})}$ witness that κ^* is Woodin for f.

Remarks:

(a) If $\mathcal{M}$ satisfies ZFC, then $L_\eta(V^{\mathcal{M}_b}_{\kappa^*})$ satisfies ZF but may not satisfy the Axiom of Choice. However, Woodin has shown that there is a generic extension of it in which Choice holds and κ^* is still Woodin.

(b) Suppose that $\tilde{\mathcal{M}}^{\mathcal{T}}_b$ and $\tilde{\mathcal{M}}^{\mathcal{T}}_c$ are not necessarily wellfounded, but that η is the minimum of wford$\,(\tilde{\mathcal{M}}^{\mathcal{T}}_b)$ and wford$\,(\tilde{\mathcal{M}}^{\mathcal{T}}_c)$. (See page 408.) Then $L_\eta(V^{\mathcal{M}_b}_{\kappa^*})$ $(L_\eta(V^{\mathcal{M}_c}_{\kappa^*}))$ still makes sense, though it may not satisfy ZF. The argument of the hint shows that κ^* is Woodin in this model.

Exercise 7.4.7. Let κ be an ordinal and let M be a transitive proper class model of ZF + DC$_{<\kappa}$ + "κ is inaccessible" + $V = L(V_\kappa)$. Assume also that there is no $\kappa' < \kappa$ such that $L(V^M_{\kappa'}) \models$ "κ' is Woodin." Show that no iteration tree on M has more than one wellfounded cofinal branch.

Hint. Show that the hypotheses about M imply, for any proper class X of ordinals, that the set of all ordinals $< \kappa$ that are definable in M from κ and elements of X is unbounded in κ.

Now assume that b and c are wellfounded cofinal branches of an iteration tree $\mathcal{T}$ on M. Apply the result of the preceding paragraph to M_b and M_c, with

$$X = \{\alpha \in \mathrm{Ord} \mid \alpha = j^{\mathcal{T}}_{0,b}(\alpha) = j^{\mathcal{T}}_{0,c}(\alpha)\}.$$

Now get a contradiction by generalizing to arbitrary formulas the proposition about Σ_0 formulas in the hint to Exercise 7.4.6.

Exercise 7.4.8. This exercise and the next give corollaries of Woodin's result of Exercise 7.4.5

Let κ and M be as in Exercise 7.4.7. For iteration trees $\mathcal{T}$ of limit length θ on M, let us make the following definitions. Let $\kappa^*(\mathcal{T}) = \sup\{\rho^{\mathcal{T}}(\alpha,\theta) \mid \alpha < \theta\}$. For each $\gamma < \kappa^*(\mathcal{T})$, there is an $\alpha < \theta$ such that $V_\gamma^{M^{\mathcal{T}}_\beta} = V_\gamma^{M^{\mathcal{T}}_{\beta'}}$ for all β and β' such that $\alpha \le \beta \le \beta' < \theta$. Define $M(\mathcal{T})$ by letting $M(\mathcal{T}) = L(V^{M(\mathcal{T})}_{\kappa^*(\mathcal{T})})$, where $V^{M(\mathcal{T})}_{\kappa^*(\mathcal{T})}$ is the limit of the $V^{M^{\mathcal{T}}_\beta}_{\kappa^*(\mathcal{T})}$. If $M \models$ "$\mathcal{E}$ is a set of extenders," then let $\mathcal{E}(\mathcal{T})$ be the limit of the $j^{\mathcal{T}}_{0,\beta}(\mathcal{E}) \cap V_{\kappa^*(\mathcal{T})}$.

Let $\mathcal{T}$ be a plus two iteration tree on M of limit length. Assume that $\mathcal{M}(\mathcal{T}) \not\models$ "$\kappa^*(\mathcal{T})$ is Woodin." Prove that $\mathcal{T}$ has a wellfounded cofinal branch (which must be unique, by Exercise 7.4.7).

Hint. Assume $\mathcal{T}$ has no wellfounded cofinal branch. By Exercise 7.4.6, $\mathcal{T}$ has no wellfounded maximal branch. Use Exercise 7.4.5 and remark (b) following Exercise 7.4.6 to get the contradiction that some $\kappa' \le \kappa^*$ is Woodin in $L(V^{M(\mathcal{T})}_{\kappa'})$.

Exercise 7.4.9. Let κ and M be as in Exercises 7.4.7 and 7.4.8. Let $\mathcal{T}$ be a plus two iteration tree on M of successor length $\alpha + 1$. Let $\alpha^* < \alpha$, and let E be an extender in $M^{\mathcal{T}}_\alpha$ with $\mathrm{crit}\,(E) < \rho(\alpha^*, \alpha)$. Show that $\prod_E^{M^{\mathcal{T}}_{\alpha^*}} M^{\mathcal{T}}_{\alpha^*}$ is wellfounded.

Exercise 7.4.10. This exercise gives an improvement, due to Woodin, of the result of Exercise 7.4.8.

For M as in Exercise 7.4.7 and for positive integers n, say that M is *n-iterable* if, for any plus two iteration tree $\mathcal{T}$ of limit length on M, the following game $\mathcal{G}_\mathcal{T}$ is a win for II. Plays of $\mathcal{G}_\mathcal{T}$ are as follows:

$$\begin{array}{ccccccccc} I & \alpha_0 & & \alpha_1 & & \dots & & \alpha_{n-1} & \\ II & & \beta_0 & & \beta_1 & & \dots & & \beta_{n-1} \end{array}$$

All α_i and all β_i must be ordinal numbers. II wins such a play if and only if there is a generic cofinal branch b of $\mathcal{T}$ such that

(1) for all $i < n$, both α_i and β_i belong to wford$(\tilde{\mathcal{M}}_b^\mathcal{T})$;

(2) $(\forall i < n)\, j_{0,b}^\mathcal{T}(\alpha_i) = \beta_i$, where $j_{0,b}^\mathcal{T}$ is the obvious partial function.

Note that the assertion that $\mathcal{G}_\mathcal{T}$ is a win for II is expressed by a formula of the language of set theory in the parameters $\mathcal{T}$ (i.e., $\mathcal{T}$'s extender sequence) and V_κ^M.

(a) Let $n \in \omega$. Suppose that M (is as in Exercise 7.4.7 and) is $(n+1)$-iterable. Suppose that $\mathcal{T}$ is a plus two iteration tree on M of limit length with no wellfounded cofinal branch. Prove that M and $M(\mathcal{T})$ satisfy the same Σ_n sentences. ($M(\mathcal{T})$ is defined in Exercise 7.4.8.)

(b) Assume that there is a transitive proper class M that satisfies ZFC + "There is a Woodin cardinal." Prove that, for every $n \geq 1$, there is an M as in Exercise 7.4.7 such that M is n-iterable and M satisfies "κ is Woodin."

Hint. To prove (b), first show that there is an M as in Exercise 7.4.7 such that M satisfies "κ is Woodin." Fix such an M and assume for a contradiction that M is not n-iterable.

Show that there exist $\langle \mathcal{T}_i \mid i \in \omega \rangle$ and $\langle M_i \mid M \in \omega \rangle$ such that

(i) $M_0 = M$;

(ii) for each $i \in \omega$, $\mathcal{T}_i$ is a plus two iteration tree of limit length on M_i;

(iii) for each $i \in \omega$, $M_{i+1} = M_i(\mathcal{T}_i)$;

(iv) for each $i \in \omega$, $\mathcal{T}_i$ witnesses that M_i is not n-iterable.

Say that an ordinal γ *has property* P if, for any $\mathbf{G}$ that is $\mathbf{Coll}(\omega, \gamma)$-generic over V, there exist in $V[\mathbf{G}]$ a transitive set N and an ordinal $\delta \leq \kappa$ such that

(a) $V_\delta^M \in N$ and N is a model of ZC plus, say, Σ_{100} Replacement;

(b) $\mathrm{Ord} \cap N = \gamma$;

(c) N satisfies the formula asserting that there are $\langle \mathcal{T}_i \mid i \in \omega\rangle$ and $\langle M_i \mid i \in \omega\rangle$ such that $M_0 = L_\gamma(V_\delta^M)$ and (ii), (iii), and (iv) above hold.

Let γ be the least ordinal that has property P. By the absoluteness for M of property P, M satisfies that γ is the least ordinal with property P. Let $\mathbf{G}$ be $\mathbf{Coll}(\omega, \gamma)$-generic over V. Work in $M[\mathbf{G}]$.

Let $N \in M[\mathbf{G}]$ witness (in $M[\mathbf{G}]$) that γ has property P. Let $\langle \mathcal{T}_i \mid i \in \omega\rangle$ and $\langle M_i \mid i \in \omega\rangle$ be given by (c).

Use Exercises 7.4.8, 7.4.9, and 7.4.5 to show that the tree ordering and extenders of $\mathcal{T}_0$ yield a plus two iteration tree $\mathcal{T}_0^*$ on M and that $\mathcal{T}_0^*$ has a cofinal branch b whose wellfounded part is at least $(\gamma^+)^{M[\mathbf{G}]}$. This branch is of course also a branch of $\mathcal{T}_0$. To simplify notation, let us identify the transitive part of the model $\tilde{\mathcal{M}}_b^{\mathcal{T}_0^*}$ with the transitive set isomorphic to it. With this identification,

$$M(\mathcal{T}_0^*) = L(V_{\kappa^*(\mathcal{T}_0^*)}^{\tilde{\mathcal{M}}_b^{\mathcal{T}_0^*}}).$$

(See Exercise 7.4.8 for the definition of $\kappa^*(\mathcal{T}_0^*)$.)

The model N and the sequences $\langle \mathcal{T}_i \mid 1 \leq i \in \omega\rangle$ and $\langle M_i \mid 1 \leq i \in \omega\rangle$ witness that clauses (a)–(c) above hold with M replaced by $\tilde{\mathcal{M}}_b^{\mathcal{T}_0^*}$ and δ replaced by $\kappa^*(\mathcal{T}_0^*)$. Hence γ has what we might call property $j_{0,b}^{\mathcal{T}_0^*}(\mathrm{P})$. By absoluteness, it is true in $\tilde{\mathcal{M}}_b^{\mathcal{T}_0^*}$ that γ has property $j_{0,b}^{\mathcal{T}_0^*}(\mathrm{P})$.

It follows that $j_{0,b}^{\mathcal{T}_0^*}(\gamma) \leq \gamma$ and so that $j_{0,b}^{\mathcal{T}_0^*}(\gamma) = \gamma$. Hence b is a wellfounded cofinal branch of $\mathcal{T}_0$ and $\mathrm{Ord} \cap (M_0)_b^{\mathcal{T}_0} = \gamma$. But this means that $j_{0,b}^{\mathcal{T}_0^*}$ gives a winning strategy for II for the game $\mathcal{G}_{\mathcal{T}_0}$ (where, of course, moves are restricted to ordinals $< \gamma$). By absoluteness, we get that N satisfies that II wins $\mathcal{G}_{\mathcal{T}_0}$, and that is a contradiction.

Exercise 7.4.11. Let $\mathcal{M}$ be a premouse*. Let $\mathcal{T}$ be an iteration tree of length $|\delta^{\mathcal{M}}|^+$ on $\mathcal{M}$. Let b be a cofinal branch of $\mathcal{T}$. Note that b is closed and unbounded in $|\delta^{\mathcal{M}}|^+$. Prove that there is a stationary subset X of b such that

$$(\forall \alpha \in X)(\forall \beta \in X)(\forall \gamma \in X)(\alpha \leq \beta \leq \gamma \rightarrow j_{\alpha,\beta}^{\mathcal{T}}(\mathrm{crit}\,(j_{\alpha,\beta}^{\mathcal{T}})) = \mathrm{crit}\,(j_{\beta,\gamma}^{\mathcal{T}})).$$

Exercise 7.4.12. This exercise concerns a theorem of Woodin whose proof uses iteration trees in the way they are used in inner model theory. The exercise is a sequel to Exercises 6.3.7 and 6.3.8.

For $x \in {}^{\omega}2$, for N a transitive class model of ZFC, and for $\mathcal{E}$ strongly witnessing in N that some κ is Woodin, one can try to define, as $\mathbf{G}_x$ was defined in Exercise 6.3.8, $\mathbf{G}_x^{N,\mathcal{E}} \subseteq \mathcal{P}_{\mathcal{E}}^N$ by

$$[\![\mathbf{c}]\!]^N_{(\sim_{\mathcal{I}_{\mathcal{E}}^N})^N} \in \mathbf{G}_x^{N,\mathcal{E}} \leftrightarrow x \in B_{\mathbf{c}}.$$

Suppose that M is a transitive class model of ZFC + "there is a Woodin cardinal" and that every plus two iteration tree on M has a wellfounded cofinal branch. (Woodin has shown that the existence of such a proper class M follows from the existence of a transitive proper class model of ZFC + "there is a Woodin cardinal" plus the hypothesis that every set has a #.) Let $\mathcal{E}$ strongly witness in M that κ is Woodin.

Let $x \in {}^{\omega}2$. Show that there is an iteration tree $\mathcal{T}$ of successor length $\theta+1$ on M such that $\mathbf{G}_x^{M_\theta^{\mathcal{T}}, j_{0,\theta}^{\mathcal{T}}(\mathcal{E})}$ is well-defined and is $\mathcal{P}^{M_\theta^{\mathcal{T}}}_{j_{0,\theta}^{\mathcal{T}}(\mathcal{E})}$-generic over $M_\theta^{\mathcal{T}}$ with $M_\theta^{\mathcal{T}}[\mathbf{G}_x^{M_\theta^{\mathcal{T}}, j_{0,\theta}^{\mathcal{T}}(\mathcal{E})}] = M_\theta^{\mathcal{T}}[x]$.

Hint. By the argument of the hint to Exercise 6.3.8, it is enough to construct an iteration tree of successor length $\theta+1$ on M (i.e., on $(M;\in)$), such that $\mathcal{I}^{M_\theta^{\mathcal{T}}}_{j_{0,\theta}^{\mathcal{T}}(\mathcal{E})}$ is x-consistent in V. In doing so, one may assume without loss of generality that

$$\mathcal{E} = \{E \mid E \in V_\kappa^M \text{ and } M \models \text{“}E \text{ is strong”}\}.$$

Construct an iteration tree $\mathcal{T}$ on M and a sequence $\langle \rho_\alpha \mid \alpha+1 < \ell\mathrm{h}(\mathcal{T})\rangle$ of ordinals, with the following properties. (We omit the superscript $\mathcal{T}$ and the subscript T.)

(i) The length of $\mathcal{T}$ is $\theta+1$ for the least $\theta < \kappa^+$ such that $\mathcal{I}^{M_\theta}_{j_{0,\theta}(\mathcal{E})}$ is x-consistent in V, if there is such a θ; otherwise $\ell\mathrm{h}(\mathcal{T}) = \kappa^+$.

(ii) For each $\alpha < \theta$, E_α belongs to $j_{0,\alpha}(\mathcal{E})$, i.e., is in M_α a strong extender belonging to $V^{M_\alpha}_{j_{0,\alpha}(\kappa)}$.

(iii) For all α and β such that $\alpha \le \beta < \theta$, $\mathrm{strength}^{M_\alpha}(E_\alpha) \le \mathrm{strength}^{M_\beta}(E_\beta)$.

(iv) For all α and β such that $\alpha \le \beta < \theta$, $\rho_\alpha \le \rho_\beta$.

(v) For each $\alpha < \theta$, $\rho_\alpha < \mathrm{strength}^{M_\alpha}(E_\alpha)$.

(vi) For each $\alpha < \theta$, $(\alpha+1)^-$ is the least $\beta \leq \alpha$ such that $\mathrm{crit}\,(E_\alpha) \leq \rho_\beta$.

Since the strength of a strong extender must be a limit ordinal, properties (ii) and (v) imply that $\rho_\alpha + \omega \leq \mathrm{strength}^{M_\alpha}(E_\alpha)$ for each α. Properties (iv) and (vi) and the argument for Exercise 7.4.1 then show that $\mathcal{T}$ will be plus n for every n.

Suppose that $\mathcal{T} \restriction \alpha + 1$ and $\langle \rho_\beta \mid \beta < \alpha \rangle$ have been constructed and that $\alpha+1$ does not meet condition (i) above for being the length $\theta+1$ of $\mathcal{T}$. Then there must exist

(1) an $E \in V^{M_\alpha}_{j_{0,\alpha}(\kappa)}$ that is a strong (δ,λ)-extender in M_α, for some δ and λ;

(2) a sequence $\langle \mathbf{c}_\gamma \mid \gamma < \delta \rangle \in M_\alpha$ such that each $\mathbf{c}_\gamma \in \mathbf{C}^{M_\alpha} \cap V^{M_\alpha}_\delta$, such that $\hat{\mathbf{c}}$, the δth term of the sequence $i^M_E(\langle \mathbf{c}_\gamma \mid \gamma < \delta \rangle)$, belongs to $V^{M_\alpha}_\lambda$, and such that $x \in B_{\hat{\mathbf{c}}} \setminus \bigcup_{\gamma<\delta} B_{\mathbf{c}_\gamma}$.

Choose such an E and $\langle \mathbf{c}_\gamma \mid \gamma < \delta \rangle$ with the least possible λ and, subject to this, with the least possible value of $\mathrm{rank}\,(\hat{\mathbf{c}})$. Let $E_\alpha = E$ and let $\rho_\alpha = \mathrm{rank}\,(\hat{\mathbf{c}})$. Also let $\langle \mathbf{c}^\alpha_\gamma \mid \gamma < \delta_\alpha \rangle = \langle \mathbf{c}_\gamma \mid \gamma < \delta \rangle$

At limit steps α, get M_α by choosing any wellfounded cofinal branch.

Show that $\mathcal{T}$ has properties (i)–(vi). The key fact for verifying (iv) is the following: If E and E' are strong extenders with $\mathrm{crit}\,(E) = \mathrm{crit}\,(E')$ and $\mathrm{strength}\,(E) < \mathrm{strength}\,(E')$, then $E' {\restriction} \mathrm{strength}\,(E) = \langle E'_a \mid a \in [\mathrm{strength}\,(E)]^{<\omega} \rangle$ is a strong extender.

Assume that $\ell\mathrm{h}(\mathcal{T}) = \kappa^+$. Let b be a wellfounded cofinal branch of $\mathcal{T}$. Prove that there is a stationary set $X \subseteq b$ such that, for $(\alpha+1)^- \leq (\beta+1)^- \in X$,

$$\begin{aligned} j_{(\alpha+1)^-,(\beta+1)^-}(\delta_\alpha) &= \delta_\beta; \\ j_{(\alpha+1)^-,(\beta+1)^-}(\langle \mathbf{c}^\alpha_\gamma \mid \gamma < \delta_\alpha \rangle) &= \langle \mathbf{c}^\beta_\gamma \mid \gamma < \delta_\beta \rangle. \end{aligned}$$

(The existence of stationary X satisfying the first equation comes directly from Exercise 7.4.11.) Suppose that $(\alpha+1)^-$ and $(\beta+1)^-$ are members of X with $\alpha < \beta$. Now $x \in B_{\hat{\mathbf{c}}_\alpha}$, and $\hat{\mathbf{c}}_\alpha$ is the δ_αth term of the sequence $i^{M_\alpha}_{E_\alpha}(\langle c^\alpha_\gamma \mid \gamma < \delta_\alpha \rangle)$. But $\hat{\mathbf{c}}$ is also the δ_αth term of the seqence $i^{M_{(\alpha+1)^-}}_{E_\alpha}(\langle c^\alpha_\gamma \mid \gamma < \delta_\alpha \rangle)$. Clause (vi) implies that $\mathrm{crit}\,(j_{\alpha+1,(\beta+1)^-}) > \rho_\alpha$. By the definition of ρ_α, this means that $\hat{\mathbf{c}}_\alpha$ is also the δ_αth element of the sequence $j_{(\alpha+1)^-,(\beta+1)^-}(\langle \mathbf{c}^\alpha_\gamma \mid \gamma < \delta_\alpha \rangle)$. This a contradiction.

Exercise 7.4.13. Let κ and M be as in Exercises 7.4.7, 7.4.8, and 7.4.9. Suppose also that κ is Woodin in M. Let $\mathcal{E}$ strongly witness in M that κ is Woodin. Suppose that $x \in {}^{\omega}2$ is such that M is a class in $L[x]$ and $\kappa < \omega_1^{L[x]}$.

Prove that there is an iteration tree $\mathcal{T}$ on M with the following properties:

(1) $\ell\mathrm{h}(\mathcal{T}) = \omega_1^{L[x]}$;

(2) $\mathcal{T} \in L[x]$;

(3) $\omega_1^{L[x]}$ is Woodin in $M(\mathcal{T})$;

(4) $\mathbf{G}_x^{M(\mathcal{T}),\mathcal{E}(\mathcal{T})}$ is well-defined and is $\mathcal{P}^{M(\mathcal{T}),\mathcal{E}(\mathcal{T})}$-generic over $M(\mathcal{T})$ with $M(\mathcal{T})[\mathbf{G}_x^{M(\mathcal{T}),\mathcal{E}(\mathcal{T})}] = M(\mathcal{T})[x] = L[x]$.

(See Exercise 7.4.8 for the definitions of $M(\mathcal{T})$, $\mathcal{E}(\mathcal{T})$, and $\kappa^*(\mathcal{T})$.)

This result is due to Woodin. From it, he gets (a) a proof of Π^1_2 determinacy different from, and using a slightly weaker hypothesis than, the one in Chapter 8 and (b) a proof that the consistency of ZFC + "There is a Woodin cardinal" implies the consistency of ZFC + "Π^1_2 determinacy." (See Exercise 8.3.3.) (b) is half of an equiconsistency result.

Hint. Construct an iteration tree $\mathcal{T} \in L[x]$ as in the hint for Exercise 7.4.12. We are not assuming that every iteration tree on M has a wellfounded cofinal branch, and even if true this might fail in $L[x]$. Therefore we must replace propery (i) of construction of Exercise 7.4.12 by

(i) The length of $\mathcal{T}$ is as small as possible so the one of the following holds:

(a) $\ell\mathrm{h}(\mathcal{T}) = \theta + 1$ and $\theta < \omega_1^{L[x]}$ is the least ordinal such that $\mathcal{I}^{M_\theta}_{j_{0,\theta}(\mathcal{E})}$ is x consistent in $L[x]$.

(b) Wellfoundedness fails: either $\ell\mathrm{h}(\mathcal{T})$ is a limit ordinal and $\mathcal{T}$ has no wellfounded cofinal branch belonging to $L[x]$, or $\ell\mathrm{h}(\mathcal{T}) = \alpha+1$ and, for the chosen candidate for E_α and the $(\alpha+1)^-$ given by (vi), $\prod_{E_\alpha}^{M^{\mathcal{T}}_{(\alpha+1)^-}} M^{\mathcal{T}}_{(\alpha+1)^-}$ is not wellfounded.

(c) $\ell\mathrm{h}(\mathcal{T}) = \omega_1^{L[x]}$.

Since $\mathcal{T}$ is, in particular, a plus two tree, Exercises 7.4.8 and 7.4.9 imply that (b) can hold only if $\ell\mathrm{h}(\mathcal{T})$ is a limit ordinal and $M(\mathcal{T}) \models$ "$\kappa^*(\mathcal{T})$ is Woodin."

If (a) holds, then $j_{0,\theta}(\kappa) < \omega_1^{L[x]}$. Since $\mathcal{P}^{M_\theta}_{j_{0,\theta}(\mathcal{E})}$ has the $j_{0,\theta}(\kappa)$ chain condition in M_θ, this gives the contradiction that

$$\omega_1^{M_\theta[x]} \leq j_{0,\theta}(\kappa) < \omega_1^{L[x]}.$$

If (c) holds, then the last argument of the hint to Exercise 7.4.12 shows that (b) must hold also. Thus (b) holds.

Show that $\mathcal{I}^{M(\mathcal{T})}_{\mathcal{E}(\mathcal{T})}$ is x-consistent in $L[x]$.

If $\kappa^*(\mathcal{T}) < \omega_1^{L[x]}$, then we get a contradiction as in the case of (a)'s holding. Thus $\kappa^* = \ell\mathrm{h}(\mathcal{T}) = \omega_1^{L[x]}$.

Chapter 8

Projective Games

The *projective hierarchy* of subsets of a topological space X is defined (by induction on all X simultaneously) as follows:

(a) $A \in \mathbf{\Sigma}^1_1$ if and only if there is a closed $C \subseteq X \times {}^\omega\omega$ such that $A = \{x \in X \mid (\exists y \in {}^\omega\omega)\, \langle x, y\rangle \in C\}$.

(b) For all positive integers n, $A \in \mathbf{\Pi}^1_n$ if and only if $X \setminus A \in \mathbf{\Sigma}^1_n$.

(c) $A \in \mathbf{\Sigma}^1_{n+1}$ if and only if there is a $B \subseteq X \times {}^\omega\omega$ such that $B \in \mathbf{\Pi}^1_n$ and $A = \{x \in X \mid (\exists y \in {}^\omega\omega)\, \langle x, y\rangle \in B\}$.

(d) $A \in \mathbf{\Delta}^1_n$ if and only if $A \in \mathbf{\Sigma}^1_n$ and $A \in \mathbf{\Pi}^1_n$.

The class of *projective* subsets of X is $\bigcup_n \mathbf{\Sigma}^1_n$.

Remark. Instead of defining $\mathbf{\Sigma}^1_1$ directly by clause (a), one can start with $n = 0$, letting $\mathbf{\Sigma}^1_0$ be the class of all open sets.

In this chapter we will prove, assuming the existence of infinitely many Woodin cardinals greater than $|T|$, the determinacy of all projective games in a game tree T. For the determinacy of all $\mathbf{\Pi}^1_{n+1}$ games in T, we will need n Woodin cardinals greater than $|T|$, plus—say—a measurable cardinal greater than the n Woodin cardinals. These results are from [Martin and Steel, 1988] and [Martin and Steel, 1989].

The proof will proceed via Theorem 4.3.5. By this theorem, the determinacy of all $\mathbf{\Pi}^1_{n+1}$ games in T will follow if we can show that every $\mathbf{\Pi}^1_{n+1}$ subset of $[T]$ is $|T|^+$-homogeneously Souslin. By Theorem 4.3.6, we already have the special case $n = 0$, provided that there is a measurable cardinal greater than $|T|$. What we need is thus a method to propagate homogeneous Souslinness up the projective hierarchy.

In §8.1 we deal with the first half of the problem of propagating homogeneous Souslinness: We give a natural way of transferring homogeneous Souslinness at one level of the projective hierarchy to Souslinness at the next level. We first illustrate the idea by defining an operation

$$\langle U, \langle \mathcal{U}_p \mid p \in U \rangle \rangle \mapsto U^\dagger(\langle \mathcal{U}_p \mid p \in T \rangle).$$

The value $U^\dagger(\langle \mathcal{U}_p \mid p \in T \rangle)$ is defined whenever $\langle \mathcal{U}_p \mid p \in T \rangle$ witnesses that U is homogeneous for T, and this value is a tree witnessing that the complement of the T-projection of U is Ord-Souslin. We next introduce the notions of *weakly homogeneous trees* and *weakly κ-homogeneously Souslin* sets. We study various equivalents of these notions, some of which will be used in Chapter 9. We show that if $A \subseteq [T] \times {}^\omega\omega$ is κ-homogeneously Souslin—in the obvious sense—then pA is weakly κ-homogeneously Souslin. We then define a $U^\ddagger$ operation analogous to the $U^\dagger$ operation, but defined on trees U and witnesses to the weak homogeneity of U. The ultimate origin of the $U^\dagger$ and $U^\ddagger$ operations is [Martin and Solovay, 1969], though the general constuctions were discovered a few years later.

In §8.2 we show that, if κ is a Woodin cardinal and $U^\ddagger$ is the result of applying our operation to a witness that some set is weakly κ^+-homogeneously Souslin, then (any sufficiently large restriction of) $U^\ddagger$ is $(< \kappa)$-homogeneous, i.e., is η-homogeneous for every $\eta < \kappa$. This theorem will enable us to propagate homogeneous Souslinness up the projective hierarchy and so to prove projective determinacy. It will also be the basis for further determinacy theorems in Chapter 9. To make the ideas of the main construction more comprehensible, we first do an analogous construction for the simpler $U^\dagger$ operation. To motivate our constructions, we aim directly at a weaker property than homogeneous Souslinness: the property of having an *embedding normal form*. The homogeneity of $U^\dagger$ and $U^\ddagger$ falls out of our proofs that the T-projections of these trees have embedding normal forms.

The construction given in §8.2 is a modification of that given in [Martin and Steel, 1989]. This modification, based on an idea of Itay Neeman, yields a construction that is slightly more complicated than what could be gotten by a smaller modification of the earlier proof. The slight extra complexity is more than compensated by the new construction's yielding immediately not just an embedding normal form but also homogeneity. The construction of [Martin and Steel, 1989] gave directly gave an embedding normal form, but more work was needed to prove homogeneity.

§8.3 is devoted to variations on the proof given in §8.2. First we give a construction that is not too different from the one in [Martin and Steel, 1989]. We then follow [Martin and Steel, 1989] in proving a lemma asserting roughly that, if $\mathcal{T} \subseteq V_\kappa$ is an iteration tree of length ω on V and $\langle \mathcal{U}_p \mid p \in T\rangle$ witnesses that some tree is κ-homogeneous, then the embeddings of the former and the latter act trivially on one another. Armed with this lemma, we show how the construction of §8.3 can yield the results of §8.2. We next prove a theorem of Katrin Windßus stating that an embedding normal form with $2^{\aleph_0}$-closed models directly implies homogeneous Souslinness. Windßus' theorem provides another way to get the determinacy results of §8.2 from either of our constructions. Finally we mention—and cite references for—machinery due to Neeman for proving the theorem from an optimal hypothesis, machinery that he has used to get a large range of determinacy results.

8.1 Weakly Homogeneous Trees

Let T be a game tree, let Y be a set, and let U be a tree on field$\,(T) \times Y$. Suppose that $\langle \mathcal{U}_p \mid p \in T\rangle$ witnesses that U is homogeneous for T. For $p \in T$, let $\pi_p = \pi_{\mathcal{U}_p} : \prod_{\mathcal{U}_p}(V;\in) \cong (\mathrm{Ult}(V;\mathcal{U}_p);\in)$. For $p \subseteq q \in T$, let

$$i_{p,q} : \mathrm{Ult}(V;\mathcal{U}_p) \prec \mathrm{Ult}(V;\mathcal{U}_q)$$

be defined as on page 200. Similarly define, for $x \in [T]$, the class model $\mathcal{M}_x$ and the embeddings

$$i^x_{x\restriction n} : \mathrm{Ult}(V;\mathcal{U}_p) \prec \mathcal{M}_x$$

as before.

Define a tree $U^\dagger(\langle \mathcal{U}_p \mid p \in T\rangle)$ on field$\,(T) \times \mathrm{Ord}$, as follows. If $p \in T$ and $t \in {}^{\ell\mathrm{h}(p)}\mathrm{Ord}$, then $\langle\!| p, t |\!\rangle \in U^\dagger(\langle \mathcal{U}_p \mid p \in T\rangle)$ if and only if

$$(\forall i_1 < \ell\mathrm{h}(p))(\forall i_2 < \ell\mathrm{h}(p))(i_1 < i_2 \rightarrow t(i_2) < i_{p\restriction i_1, p\restriction i_2}(t(i_1))).$$

Let A be the T-projection of U.

Theorem 8.1.1. *Let T, Y, U, $\langle \mathcal{U}_p \mid p \in T\rangle$, and A be as above. Let $U^\dagger = U^\dagger(\langle \mathcal{U}_p \mid p \in T\rangle)$. Then $[T] \setminus A$ is the T-projection of $U^\dagger$. Moreover $[T] \setminus A$ is also the T-projection of $U^\dagger \restriction \alpha$ for any ordinal $\alpha \geq \max\{\omega, (2^{|Y|})^+\}$, where $U^\dagger \restriction \alpha = U^\dagger \cap \{\langle\!| p, t |\!\rangle \mid \mathrm{range}\,(t) \subseteq \alpha\}$.*

Proof. Let $x \in [T]$ be such that $[U^\dagger(x)] \neq \emptyset$. Let $f \in [U^\dagger(x)]$. For each $n \in \omega$,

$$i_{x\restriction n, x\restriction n+1}(f(n)) < f(n+1).$$

Now

$$i^x_{x\restriction n}(f(n)) = i^x_{x\restriction n+1}(i_{x\restriction n, x\restriction n+1}(f(n))) < i^x_{x\restriction n+1}(f(n+1)).$$

Hence $\langle i^x_{x\restriction n}(f(n)) \mid n \in \omega\rangle$ is an infinite descending sequence of ordinals of $\mathcal{M}_x$. Lemma 4.3.4 implies that $x \notin A$.

Next let $x \in [T] \setminus A$. Thus $U(x)$ is a wellfounded tree. Note that, in the case that Y is finite, it follows by König's Lemma that $U(x)$ is finite. This gives us the function

$$\| \, \|^{U(x)} : U(x) \to |U(x)|^+,$$

defined on page 25. For each $n \in \omega$, let

$$f_n : U[x \restriction n] \to |U(x)|^+$$

be given by setting $f_n(s) = \|s\|^{U(x)}$ for each $s \in U[x \restriction n]$. For $n \in \omega$ let $t_n = \pi_{x\restriction n}([\![f_n]\!]_{\mathcal{U}_{x\restriction n}})$. Since, for each n, $f_{n+1}(s) < f_n(s \restriction n)$ for every $s \in U[x \restriction n+1]$, it follows that $t_{n+1} < i_{x\restriction n, x\restriction n+1}(t_n)$ for each n. This shows that $[U^\dagger(x)] \neq \emptyset$. Moreover, for each $n \in \omega$,

$$\begin{aligned} |t_n| &= |\{[\![g]\!]_{\mathcal{U}_{x\restriction n}} \mid \pi_{x\restriction n}([\![g]\!]_{\mathcal{U}_{x\restriction n}}) < \pi_{x\restriction n}([\![f_n]\!]_{\mathcal{U}_{x\restriction n}})\}| \\ &\leq |\{g : U[x \restriction n] \to \mathrm{Ord} \mid (\forall t \in U[x \restriction n])\, g(t) < f_n(t)\}| \\ &\leq |U(x)|^{|U(x)|}. \end{aligned}$$

For Y infinite, $|U(x)|^{|U(x)|} \leq 2^{|Y|}$. Hence each $t_n < \max\{\omega, (2^{|Y|})^+\}$, and so the t_n witness that $[(U^\dagger \restriction \alpha)(x)] \neq \emptyset$ for all $\alpha \geq \max\{\omega, (2^{|Y|})^+\}$.

□

In order to propagate up the projective hierarchy the property of being homogeneously Souslin, it will be useful to have an operation on homogenous trees that will yield a result like Theorem 8.1.1, but with the complement of pA replacing the complement of A. For this purpose, and for use in Chapter 9, we now introduce the notion of *weak homogeneity.*

For trees T and R, let

$$T \otimes R = \{\langle\!\langle p, r\rangle\!\rangle \mid p \in T \wedge r \in R \wedge \ell\mathrm{h}(p) = \ell\mathrm{h}(r)\}.$$

Let T be a game tree, let Y be a nonempty set, and let U be a tree on field$(T) \times Y$. We say that U is *weakly homogeneous* for T if there is a system

$$\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r \rangle\!\rangle \in T \otimes {}^{<\omega}\omega \rangle$$

satisfying the following conditions:

(1) Each $\mathcal{U}_{p,r}$ is a countably complete ultrafilter on $U[p]$.

(2) The $\mathcal{U}_{p,r}$ are *compatible*: For all $\langle\!\langle p, r \rangle\!\rangle \subseteq \langle\!\langle q, s \rangle\!\rangle \in T \otimes {}^{<\omega}\omega$, $\mathcal{U}_{q,s}$ projects to $\mathcal{U}_{p,r}$ by $\chi_{q,p}$, where $\chi_{q,p} : U[q] \to U[p]$ is given (as on page 200) by $\chi_{q,p}(t) = t \restriction \ell\mathrm{h}(p)$.

(3) Let $x \in [T]$ and $\langle Z_r \mid r \in {}^{<\omega}\omega \rangle$ be such that each Z_r belongs to $\mathcal{U}_{x \restriction \ell\mathrm{h}(r), r}$. Then

$$[U(x)] \neq \emptyset \;\rightarrow\; (\exists y \in {}^{\omega}\omega)(\exists f \in {}^{\omega}Y)(\forall n \in \omega)\, f \restriction n \in Z_{y \restriction n}.$$

As was the case for the corresponding clause in the definition of homogeneous trees, there is an equivalent of condition (3) in terms of wellfoundedness of direct limit models. Suppose that (1) and (2) are satisfied. For $\langle\!\langle p, r \rangle\!\rangle \in T \otimes {}^{<\omega}\omega$, let $\pi_{p,r} = \pi_{\mathcal{U}_{p,r}} : \prod_{\mathcal{U}_{p,r}}(V; \in) \cong (\mathrm{Ult}(V; \mathcal{U}_{p,r}); \in)$. For $\langle\!\langle p, r \rangle\!\rangle \subseteq \langle\!\langle q, s \rangle\!\rangle \in T \otimes {}^{<\omega}\omega \rangle$ let

$$i_{\langle p,r \rangle, \langle q,s \rangle} = \pi_{q,s} \circ i_{\mathcal{U}_{p,r}, \mathcal{U}_{q,s}, \chi_{q,p}} \circ {\pi_{p,r}}^{-1}.$$

(See page 199 for the definition of $i_{\mathcal{U}_{p,r}, \mathcal{U}_{q,s}, \chi_{q,p}}$.) For $x \in [T]$ and $y \in {}^{\omega}\omega$, let

$$(\mathcal{M}_{x,y}; \langle i^{x,y}_{\langle x \restriction n, y \restriction n \rangle} \mid n \in \omega \rangle)$$

be the direct limit of the directed system of elementary embeddings

$$(\langle \mathrm{Ult}(V; \mathcal{U}_{x \restriction n, y \restriction n}) \mid n \in \omega \rangle; \langle i_{\langle x \restriction m, y \restriction m \rangle \langle x \restriction n, y \restriction n \rangle} \mid m \leq n \in \omega \rangle).$$

(3′) $(\forall x \in [T])([U(x)] \neq \emptyset \;\rightarrow\; (\exists y \in {}^{\omega}\omega)\, \mathcal{M}_{x,y}$ is wellfounded).

Lemma 8.1.2. *Let T and $\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r \rangle\!\rangle \in T \otimes {}^{<\omega}\omega \rangle$ be such that (1) and (2) hold. If $x \in [T]$, then x witnesses the falsity of (3) if and only if x witnesses the falsity of (3′). Thus a tree U on* field$(T) \times Y$ *is weakly homogeneous for T if and only if there is a system $\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r \rangle\!\rangle \in T \otimes {}^{<\omega}\omega \rangle$ satisfying (1), (2), and (3′).*

Proof. The proof parallels that of Lemma 4.3.4, with an extra wrinkle in the second part.

Suppose first that x and $\langle Z_r \mid r \in {}^{<\omega}\omega\rangle$ witness the failure of (3). Let $y \in {}^{\omega}\omega$. Let

$$S_y = \{s \in U(x) \mid (\forall n \leq \ell\mathrm{h}(s))\, s \restriction n \in Z_{y\restriction n}\}.$$

Exactly as the tree S was used in the proof of Lemma 4.3.4 to show that $\mathcal{M}_x$ was not wellfounded, the tree S_y can be used to show that $\mathcal{M}_{x,y}$ is not wellfounded.

Now suppose that x witnesses that (3′) fails. For each $y \in {}^{\omega}\omega$, let $\langle z^y_n \mid n \in \omega\rangle$ be an infinite descending sequence with respect to $i^{x,y}_{\langle\emptyset,\emptyset\rangle}(\in)$. For each $y \in {}^{\omega}\omega$ and each $n \in \omega$, let m^y_n and $a^y_n \in \mathrm{Ult}(V;\mathcal{U}_{x\restriction m^y_n, y\restriction m^y_n})$ be such that $z^y_n = i^{x,y}_{\langle x\restriction m^y_n, y\restriction m^y_n\rangle}(a^y_n)$. Without loss of generality, we may assume that

$$(\forall y \in {}^{\omega}\omega)(\forall n' \in \omega)(\forall n \in \omega)(n' < n \rightarrow m^y_{n'} < m^y_n).$$

Let $g^y_n \in {}^{U[x\restriction m^y_n]}V$ be such that

$$\pi_{x\restriction m^y_n, y\restriction m^y_n}([\![g^y_n]\!]_{\mathcal{U}_{x\restriction m_n, y\restriction m_n}}) = a^y_n.$$

For each $y \in {}^{\omega}\omega$ and each $n \in \omega$, let

$$Z^y_{m_{n+1}} = \{s \in U[x \restriction m^y_{n+1}] \mid g^y_{n+1}(s) \in g^y_n(s \restriction m^y_n)\}.$$

For each $m \in \omega$ such that m is not of the form m^y_{n+1}, let $Z^y_m = U[x \restriction m]$. For $m \in \omega$, we have that $Z^y_m \in \mathcal{U}_{x\restriction m, y\restriction m}$. For $r \in {}^{<\omega}\omega$, let

$$Z_r = \bigcap\{Z^y_{\ell\mathrm{h}(r)} \mid y \in {}^{\omega}\omega \wedge r \subseteq y\}.$$

Since any countably complete ultrafilter is $2^{\aleph_0}$-complete, Z_r belongs to $\mathcal{U}_{x\restriction \ell\mathrm{h}(r), r}$ for every $r \in {}^{<\omega}\omega$. To see that $\langle Z_r \mid r \in {}^{<\omega}\omega\rangle$ witnesses the failure of (3) for x, suppose that $y \in {}^{\omega}\omega$ and $f \in {}^{\omega}Y$ are such that $(\forall m \in \omega)\, f \restriction m \in Z_{y\restriction m}$. Then $f(m^y_n) \in Z^y_{m_n}$ for every $n \in \omega$, and so we get the contradiction that $\langle f(m^y_n) \mid n \in \omega\rangle$ is an infinite descending sequence with respect to $\in$. □

Remark. As in the case of homogeneous trees, the "$\rightarrow$" in the last line of condition (3) and that in condition (3′) can be replaced by "$\leftrightarrow$."

For T a game tree, Y a set, and κ a cardinal number, a tree U on field $(T) \times Y$ is *weakly κ-homogeneous* for T if there is a system $\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r\rangle\!\rangle \in T \otimes$

${}^{<\omega}\omega\rangle$ witnessing that U is weakly homogeneous for T and having the further property that each $\mathcal{U}_p$ is κ-complete.

Let T be a game tree. A subset A of $[T]$ is *weakly homogeneously Souslin* if it is the T-projection of a tree weakly homogeneous for T; A is *weakly κ-homogeneously Souslin* if it is the T-projection of a tree weakly κ-homogeneous for T.

We now prove some results giving equivalents of a tree's being weakly κ-homogeneous and of a set's being weakly κ-homogeneously Souslin. The easy half of the first of these results is directly relevant to our goal of propagating homogeneous Souslinness up the projective hierarchy. The other results, particularly Theorem 8.1.7, will be important in Chapter 9.

It will be convenient to extend our notation $\langle\!\lvert p, q\rvert\!\rangle$ to infinite sequences. If x and y are functions with domain ω, let

$$\langle\!\lvert x, y\rvert\!\rangle = \{\langle\!\lvert x \restriction n, y \restriction n\rvert\!\rangle \mid n \in \omega\}.$$

If T and T' are trees, let us say that a subset A of $[T] \times [T']$ is *homogeneously Souslin* if $\{\langle\!\lvert x, x'\rvert\!\rangle \mid \langle x, x'\rangle \in A\}$ is homogeneously Souslin. Similarly define, for subsets of products, the notions of *κ-homogeneously Souslin*, *weakly homogeneously Souslin*, and *weakly κ-homogeneously Souslin*.

Theorem 8.1.3. *Let T be a game tree, let $A \subseteq [T]$, and let κ be a cardinal number greater than $|T|$. Then the following are equivalent:*

(a) A is weakly κ-homogeneously Souslin.

(b) There is a $B \subseteq [T] \times {}^\omega\omega$ such that B is κ-homogeneously Souslin and $A = \mathrm{p}B$

Proof. Suppose first that B is as in (b). Let $\hat{U} \subseteq (T \otimes {}^{<\omega}\omega) \otimes {}^{<\omega}Y$ be a tree witnessing that $B^* = \{\langle\!\lvert x, y\rvert\!\rangle \mid \langle x, y\rangle \in B\}$ is κ-homogeneously Souslin. Let $\langle \hat{\mathcal{U}}_{\langle\!\lvert p,r\rvert\!\rangle} \mid \langle\!\lvert p, r\rvert\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$ witness that $\hat{U}$ is κ-homogeneous for $T \otimes {}^{<\omega}\omega$.

Let

$$U = \{\langle\!\lvert p, \langle\!\lvert r, s\rvert\!\rangle\rvert\!\rangle \mid \langle\!\lvert \langle\!\lvert p, r\rvert\!\rangle, s\rvert\!\rangle \in \hat{U}\}.$$

For $x \in [T]$,

$$\begin{aligned} x \in A \;&\leftrightarrow\; (\exists y \in {}^\omega\omega)\,[\hat{U}(\langle\!\lvert x, y\rvert\!\rangle)] \neq \emptyset \\ &\leftrightarrow\; (\exists y \in {}^\omega\omega)(\exists z \in {}^\omega Y)\,\langle\!\lvert \langle\!\lvert x, y\rvert\!\rangle, z\rvert\!\rangle \in [\hat{U}] \\ &\leftrightarrow\; (\exists y \in {}^\omega\omega)(\exists z \in {}^\omega Y)\,\langle\!\lvert x, \langle\!\lvert y, z\rvert\!\rangle\rvert\!\rangle \in [U] \\ &\leftrightarrow\; [U(x)] \neq \emptyset. \end{aligned}$$

Fix $p \in T$ and $r \in {}^{<\omega}\omega$. For $Z \subseteq U[p]$, define $\hat{Z} \subseteq U[\langle\!|p, r|\!\rangle]$ by

$$\hat{Z} = \{s \in {}^{<\omega}Y \mid \langle\!|r, s|\!\rangle \in Z\}.$$

Set

$$\mathcal{U}_{p,r} = \{Z \subseteq U[p] \mid \hat{Z} \in \hat{\mathcal{U}}_{\langle\!|p,r|\!\rangle}\}.$$

It is easy to check that $\langle \mathcal{U}_{p,r} \mid \langle\!|p, r|\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$ witnesses that U is weakly κ-homogeneous.

To prove the other half of the lemma, suppose U is a tree on field $(T) \times Y$ witnessing that A is weakly κ-homogeneously Souslin and suppose that $\langle \mathcal{U}_{p,r} \mid \langle\!|p, r|\!\rangle \in T \times {}^{<\omega}\omega|\!\rangle$ witnesses that U is weakly κ-homogeneous for T.

Let $x \in [T]$ and $y \in {}^{\omega}\omega$. If there is a system $\langle Z_n \mid n \in \omega\rangle$ such that each $Z_n \in \mathcal{U}_{x\restriction n, y\restriction n}$ and such that

$$(\forall f \in {}^{\omega}Y)(\exists n \in \omega)\, f \restriction n \notin Z_{y\restriction n},$$

then choose such a system and set $Z_n^{x,y} = Z_n$ for each $n \in \omega$. Otherwise set $Z_n^{x,y} = U[x \restriction n]$ for each n.

For $\langle\!|p, r|\!\rangle \in T \otimes {}^{<\omega}\omega$, set

$$Z_{p,r} = \bigcap\{Z^{x,y}_{\ell\mathrm{h}(p)} \mid x \in [T] \,\wedge\, y \in {}^{\omega}\omega \,\wedge\, \langle\!|p, r|\!\rangle \subseteq \langle\!|x, y|\!\rangle\}.$$

Since $\kappa > |T|$, we have that $Z_{p,r} \in \mathcal{U}_{p,r}$ for every $\langle\!|p, r|\!\rangle \in T \otimes {}^{<\omega}\omega$.

Define a tree $\hat{U}$ on (field $(T) \times \omega) \times Y$ by

$$\hat{U} = \{\langle\!|\langle\!|p, r|\!\rangle, s|\!\rangle \mid s \in Z_{p,r}\}.$$

For $\langle\!|p, r|\!\rangle \in T \otimes {}^{<\omega}\omega$, set

$$\hat{\mathcal{U}}_{\langle\!|p,r|\!\rangle} = \mathcal{U}_{p,r} \cap \mathcal{P}(\hat{U}[\langle\!|p, r|\!\rangle]).$$

Thus $\hat{\mathcal{U}}_{\langle\!|p,r|\!\rangle}$ is essentially the same ultrafilter as $\mathcal{U}_{p,r}$.

It is evident that the $\hat{\mathcal{U}}_{\langle\!|p,r|\!\rangle}$ are κ-complete and that $\langle \hat{\mathcal{U}}_{\langle\!|p,r|\!\rangle} \mid \langle\!|p, r|\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$ satisfies conditions (1) and (2) for witnessing that $\hat{U}$ is homogeneous for $T \otimes {}^{<\omega}\omega$.

To verify condition (3), let us suppose that $\langle\!|x, y|\!\rangle \in [\hat{U}]$. Let $f \in [\hat{U}(\langle\!|x, y|\!\rangle)]$. Let $\langle X_n \mid n \in \omega\rangle$ be such that each X_n belongs to $\hat{\mathcal{U}}_{\langle\!|x\restriction n, y\restriction n|\!\rangle}$. Assume for a contradiction that there is no $g : \omega \to Y$ such that $g \restriction n \in X_n$ for every n. By the definition of the $Z_n^{x,y}$, it follows that there is no $g : \omega \to Y$

such that $g \restriction n \in Z_n^{x,y}$ for every n. For each n, $Z_{x \restriction n, y \restriction n} \subseteq Z_n^{x,y}$. Hence there is no $g : \omega \to Y$ such that $g \restriction n \in Z_{x \restriction n, y \restriction n}$ for every n. But

$$(\forall n \in \omega)\, Z_{x \restriction n, y \restriction n} = \hat{U}[\langle\!| x \restriction n, y \restriction n |\!\rangle].$$

Since $f \restriction n \in \hat{U}[|\!\rangle x \restriction n, y \restriction n |\!\rangle]$ for all n, we get our contradiction.

Let $B = \{\langle x, y\rangle \langle\!| x, y |\!\rangle \in [\hat{U}]\}$. It remains to show that $A = \mathrm{p}B$.

Suppose first that $x \in \mathrm{p}B$. Let y be such that $\langle x, y\rangle \in B$. Let f be such that $f \in [\hat{U}(\langle\!| x, y |\!\rangle)]$. We then have, for all $n \in \omega$, that

$$f \restriction n \in \hat{U}[\langle\!| x \restriction n, y \restriction n |\!\rangle] = Z_{x \restriction n, y \restriction n} \subseteq Z_n^{x,y} \subseteq U[x \restriction n].$$

Thus $f \in [U(x)]$, and so $x \in A$.

Now suppose that $x \in A$. Let us apply condition (3) to the system $\langle Z_{x \restriction \ell\mathrm{h}(r), r} \mid r \in {}^{<\omega}\omega\rangle$, which is identical with $\langle \hat{U}[\langle\!| x \restriction n, r] \mid r \in {}^{<\omega}\omega\rangle$. This gives us a $y \in {}^{\omega}\omega$ and an $f : \omega \to Y$ such that each $f \restriction n$ belongs to $\hat{U}[\langle\!| x \restriction n, y \restriction n |\!\rangle]$. But then $f \in [\hat{U}(\langle\!| x, y |\!\rangle)]$, and therefore $\langle x, y\rangle \in B$.

□

Note that our proof that (b) implies (a) made no use of the hypothesis that $\kappa > |T|$. Hence we have the following result.

Corollary 8.1.4. *Let T be a game tree, let κ be a cardinal number, and let B be a κ-homogeneously Souslin subset of $[T] \times {}^{\omega}\omega$. Then* p$B$ *is weakly κ-homogeneously Souslin.*

Remark. The material between here and the end of the proof of Lemma 8.1.7 will not be used until Chapter 9. The reader may thus prefer to skip this material and return to it only when it is about to be used (in §9.6).

Let T be a game tree, let Y be a set, and let U be a tree on field$\,(T) \times Y$. A *T-cover of U by ultrafilters* is a set $\mathbf{V}$ of of countably complete ultrafilters on U such that, for each member x of the T-projection of U, there are $\mathcal{V}_i$, $i \in \omega$, satifying (i)–(iv) below.

(i) each $\mathcal{V}_i$ belongs to $\mathbf{V}$;

(ii) each $\mathcal{V}_i$ is an ultrafilter on $U[x \restriction i]$;

(iii) for $i < j \in \omega$, $\mathcal{V}_j$ projects to $\mathcal{V}_i$ by $\chi_{j,i}$, where $\chi_{j,i} : U[x \restriction j] \to U[x \restriction i]$ is given by $\chi_{j,i}(s) = s \restriction i$;

(iv) if $\langle Z_i \mid i \in \omega\rangle$ is such that each $Z_i \in \mathcal{V}_i$, then there is an $f : \omega \to Y$ such that $f \restriction i \in Z_i$ for all $i \in \omega$.

For cardinal numbers κ, a T-cover $\mathbf{V}$ of U by ultrafilters is *κ-complete* if every member of $\mathbf{V}$ is κ-complete. A T-cover $\mathbf{V}$ of U by ultrafilters is *full* if the following holds. Suppose that $p \in [T]$ and that $\mathcal{V} \in \mathbf{V}$ is an ultrafilter on $U[p]$. Let $p' \in [T]$ with $p' \supseteq p$ and $\ell\mathrm{h}(p') = \ell\mathrm{h}(p) + 1$. Then there is a $\mathcal{W} \in \mathbf{V}$ such that $\mathcal{W}$ is an ultrafilter on $U[p']$ and $\mathcal{W}$ projects to $\mathcal{V}$ by $\chi_{i+1,i}$, where the $\chi_{j,i}$ are as in the statement of (iii).

We need the following technical lemma in order to prove Theorem 8.1.7. It would not have been necessary if we had worked with the definition of homogeneity of Exercise 4.3.5 and with the corresponding definition of weak homogeneity. (See Exercise 8.1.1.)

Lemma 8.1.5. *Let κ be a cardinal number and let ρ be a measurable cardinal. Let T be a game tree, let Y be a set, and let U be a tree on* field $(T) \times Y$*. Suppose that $\mathbf{V}$ is a κ-complete T-cover of U by ultrafilters. Then there exist*

(1) a set $Y' \supseteq Y$ such that $|Y'| \leq \max\{\rho, |Y|\}$;

(2) a tree $U' \supseteq U$ on field $(T) \times Y'$ *such that U and U' have the same T-projection;*

(3) a set $\mathbf{V}' \supseteq \mathbf{V}$ such that $|\mathbf{V}'| \leq \max\{\aleph_0, |\mathbf{V}|\}$ and $\mathbf{V}'$ is a full, $(\min\{\kappa, \rho\})$-complete T-cover of U.

Proof. Let b_β, $\beta < \rho$, be distinct from one another and from all elements of Y. Let $Y' = Y \cup \{b_\beta \mid \beta < \rho\}$.

Let U' be the set of all $\langle\!\langle p, s^\frown\langle b_{\beta_i} \mid i < n\rangle\rangle\!\rangle$ such that

(a) $p \in {}^{<\omega}\mathrm{field}\,(T)$;

(b) $s \in {}^{<\omega}Y$;

(c) $n \in \omega$;

(d) $(\forall i < n)\, \beta_i < \rho$;

(e) $\ell\mathrm{h}(p) = \ell\mathrm{h}(s) + n$;

(f) $\langle\!\langle p \restriction \ell\mathrm{h}(s), s\rangle\!\rangle \in U$;

(g) $(\forall i < n)(\forall j < n)(i < j \to \beta_i > \beta_j)$.

Note that $U \subseteq U'$ and that $[U'(x)] = [U'(x)]$ for all $x \in [T]$.

Let $\mathcal{U}$ be a uniform normal ultrafilter on ρ. Recall the Robottom ultrafilters $\mathcal{U}^{[n]}$ of §3.1. For $n \in \omega$ and $q \in [\rho]^n$ let $g_q : n \to q$ be such that $g_q(i) > g_q(j)$ whenever $i < j < n$. For $n \in \omega$, let $\mathcal{U}^*_n$ be the ultrafilter on ${}^n\rho$ defined by

$$Z \in \mathcal{U}^*_n \leftrightarrow (\exists X \in \mathcal{U}^{[n]})(\forall q \in X)\, g_q \in Z.$$

For each $\mathcal{V} \in \mathbf{V}$ and each $n \in \omega$, let $\mathcal{W}_{\mathcal{V},n}$ be the iterated product of $\mathcal{V}$ and $\mathcal{U}^*_n$, i.e., let

$$X \in \mathcal{W}_{\mathcal{V},n} \leftrightarrow \{s \mid (\exists Z \in \mathcal{U}^*_n)(\forall t \in Z)\, s^\frown t \in X\} \in \mathcal{V}.$$

For each $\langle\!\langle p, \langle b_{\beta_i} \mid i < n\rangle\rangle\!\rangle \in U'$ and each $\mathcal{V} \in \mathbf{V}$ such that $\mathcal{V}$ is an ultrafilter on $U[p \restriction m]$ for some m, $\mathcal{W}_{\mathcal{V},n}$ is a $(\min\{\kappa, \rho\})$-complete ultrafilter on $U'[p]$. Moreover, for each $\mathcal{V} \in \mathbf{V}$ and each $n \in \omega$, $\mathcal{W}_{\mathcal{V},n+1}$ projects to $\mathcal{W}_{\mathcal{V},n}$ by $\chi_{n+1,n}$.

Set

$$\mathbf{V}' = \{\mathcal{W}_{\mathcal{V},n} \mid \mathcal{V} \in \mathbf{V} \wedge n \in \omega\}.$$

It is easy to check that $\mathbf{V}'$ has the required properties. □

Lemma 8.1.6. *Let T be a game tree, let Y be a set, let U be a tree on* field$(T) \times Y$*, and let κ be a cardinal number. Then the following are equivalent:*

(a) U is weakly κ-homogeneous for T.

(b) There is a countable, full, κ-complete T-cover of U by ultrafilters.

Proof. Suppose first that $\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r\rangle\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$ witness that U is weakly κ-homogeneous. Let

$$\mathbf{V} = \{\mathcal{U}_{p,r} \mid \langle\!\langle p, r\rangle\!\rangle \in T \otimes {}^{<\omega}\omega\}.$$

Suppose that $x \in [T]$ and $[U(x)] \neq \emptyset$. By condition (3′), let $y \in {}^\omega\omega$ be such that $\mathcal{M}_{x,y}$ is wellfounded, where $\mathcal{M}_{x,y}$ is defined as on page 423. By the proof of Lemma 4.3.4, clauses (i)–(iv) from the definition of a T-covering of U by ultrafilters hold if we set $\mathcal{V}_i = \mathcal{U}_{x\restriction i, y\restriction i}$ for each i. Thus $\mathbf{V}$ is a T-covering of U by ultrafilters.

Obviously $\mathbf{V}$ is countable, full, and κ-complete.

Now suppose that $\mathbf{V}$ is a countable, full, κ-complete T-cover of U by ultrafilters.

We construct $\mathcal{U}_{p,r}$, $\langle\!\langle p, r\rangle\!\rangle \in T \otimes {}^{<\omega}\omega$, by induction on $\ell\text{h}(p)$. Assume that $U_{p,r}$ has been defined, is an ultrafilter on $U[p]$, and belongs to $\mathbf{V}$. Let $p' \supseteq p$ with $\ell\text{h}(p') = \ell\text{h}(p) + 1$. Since $\mathbf{V}$ is full, there is at least one $\mathcal{V} \in \mathbf{V}$ such that $\mathbf{V}$ is an ultrfilter on $U[p']$ and $\mathcal{V}$ projects to $\mathcal{U}_{p,r}$ by $\chi_{n+1,n}$. Since $\mathbf{V}$ is countable, we can let $\langle \mathcal{U}_{p',r^\frown\langle i\rangle} \mid i \in \omega\rangle$ be an enumeration, possibly with repetitions, of all such $\mathcal{V}$.

To verify clause (3) in the definition of weak homogeneity, let x belong to the T-projection of U and let Z_r, $r \in {}^{<\omega}\omega$, be such that each $Z_r \in \mathcal{U}_{x\restriction \ell\text{h}(r),r}$. Let $\mathcal{V}_i$, $i \in \omega$, be given by (i)–(iv). By construction, there is a $y \in {}^{\omega}\omega$ such that $\mathcal{V}_i = \mathcal{U}_{x\restriction i, y\restriction i}$. By (iv) we get an $f \in {}^{\omega}Y$ such that $f \restriction n \in Z_{y\restriction i}$ for each $i \in \omega$. □

Theorem 8.1.7. *Let κ be a cardinal, and assume that there is a measurable cardinal $\geq \kappa$. Let T be a game tree, and let $A \subseteq [T]$. Then the following are equivalent:*

(a) A is weakly κ-homogeneously Souslin.

(b) A is the T-projection of a tree U such that the exists a countable, κ-complete T-cover of U by ultrafilters.

Proof. That (a) implies (b) follows directly from Lemma 8.1.6.

Let U witness that (b) holds. Applying Lemma 8.1.5 to U with some measurable cardinal $\geq \kappa$ as ρ, we get a tree U' whose T-projection is (a) and which has a countable, full, κ-complete cover by ultrafilters. Lemma 8.1.6 gives (a). □

We now turn to the $U\ddagger$ construction.

Let T be a game tree, let Y be a set, and let U be a tree on field$\,(T) \times Y$. Suppose that $\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r\rangle\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$ witnesses that U is weakly homogeneous for T. Let $\langle \pi_{p,r} \mid \langle\!\langle p, r\rangle\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$, $\langle i_{\langle p,r\rangle,\langle q,s\rangle} \mid \langle\!\langle p, r\rangle\!\rangle \subseteq \langle\!\langle q, s\rangle\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$, and $(\mathcal{M}_{x,y}; \langle i^{x,y}_{\langle x\restriction n, y\restriction n\rangle} \mid n \in \omega\rangle)$ be as on page 423. Let $i \mapsto r_i$ be a one-one correspondence between ω and ${}^{<\omega}\omega$ with the property that

$$(\forall i \in \omega)(\forall i' \in \omega)(r_i \subseteq r_{i'} \to i \leq i').$$

Define a tree $U^\ddagger(\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r\rangle\!\rangle \in T \otimes {}^{<\omega}\omega\rangle)$ on field$\,(T) \times \text{Ord}$, as follows. If $p \in T$ and $t \in {}^{\ell\text{h}(p)}\text{Ord}$, then $\langle\!\langle p, t\rangle\!\rangle \in U^\ddagger(\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r\rangle\!\rangle \in T \otimes {}^{<\omega}\omega\rangle)$ if and only if, for all i_1 and i_2 less than $\ell\text{h}(p)$,

$$r_{i_1} \subsetneq r_{i_2} \;\to\; t(i_2) < i_{\langle p\restriction \ell\text{h}(r_{i_1}), r_{i_1}\rangle, \langle p\restriction \ell\text{h}(r_{i_2}), r_{i_2}\rangle}(t(i_1)).$$

Let A be the T-projection of U.

Theorem 8.1.8. *Let T, Y, U, $\langle \mathcal{U}_p \mid p \in T\rangle$, and A be as above. Let $U^\ddagger = U^\ddagger(\langle \mathcal{U}_{p,r} \mid \langle\!| p, r |\!\rangle \in T \otimes {}^{<\omega}\omega\rangle)$. Then $[T] \setminus A$ is the T-projection of $U^\dagger$. Moreover $[T] \setminus A$ is also the T-projection of $U^\ddagger \restriction \alpha$ for any ordinal $\alpha \geq \max\{\omega, (2^{|Y|})^+\}$, where $U^\dagger \restriction \alpha = U^\ddagger \cap \{\langle\!| p, t |\!\rangle \mid \mathrm{range}\,(t) \subseteq \alpha\}$.*

Proof. Let $x \in [T]$ be such that $[U^\ddagger(x)] \neq \emptyset$. Let $f \in [U^\dagger(x)]$. Let $y \in {}^\omega\omega$. For each $n \in \omega$, let i_n be the number such that $r_{i_n} = y \restriction n$. For each $n \in \omega$,

$$i_{\langle x \restriction n, y \restriction n\rangle, \langle x \restriction n+1, y \restriction n+1\rangle}(f(i_n)) > f(i_{n+1}).$$

It follows that $\langle i^{x,y}_{\langle x \restriction n, y \restriction n\rangle}(f(n)) \mid n \in \omega\rangle$ is an infinite descending sequence of ordinals of $\mathcal{M}_{x,y}$. Since $\mathcal{M}_{x,y}$ is thus illfounded for every $y \in {}^\omega\omega$, Lemma 8.1.2 implies that $x \notin A$.

Next let $x \in [T] \setminus A$. Thus $U(x)$ is a wellfounded tree. As in the proof of Theorem 8.1.1, define, for $n \in \omega$,

$$f_n : U[x \restriction n] \to |U(x)|^+$$

by setting $f_n(s) = \|s\|^{U(x)}$ for each $s \in U[x \restriction n]$. For $i \in \omega$ let

$$t_i = \pi_{x \restriction \ell\mathrm{h}(r_i), r_i}([\![f_{\ell\mathrm{h}(r_i)}]\!]_{\mathcal{U}_{x \restriction \ell\mathrm{h}(r_i), r_i}}).$$

To show that $\langle t_i \mid i \in \omega\rangle \in U^\ddagger(x)$, let i_1 and i_2 be such that $r_{i_1} \subseteq r_{i_2}$. Since $f_{\ell\mathrm{h}(r_{i_2})}(s) < f_{\ell\mathrm{h}(r_{i_1})}(s \restriction \ell\mathrm{h}(r_{i_1}))$ for every $s \in U[x \restriction \ell\mathrm{h}(r_{i_2})]$, it follows that $t_{i_2} < i_{\langle x \restriction \ell\mathrm{h}(r_{i_1}), r_{i_1}\rangle, \langle x \restriction \ell\mathrm{h}(r_{i_2}), r_{i_2}\rangle}(t_{i_1})$, as required.

The proof that $t_n < \max\{\omega, (2^{|Y|})^+\}$ for each n is just like the corresponding part of the proof of Theorem 8.1.1. $\square$

Exercise 8.1.1. Give a modified definition of *weakly homogeneous*, analogous to the modified definition of *homogenous* given in Exercise 4.3.5. Prove that, for any game tree T, every $\boldsymbol{\Sigma}^1_1$ subset of $[T]$ is weakly homogenously Souslin in the modified sense. Prove that if a measurable cardinal exists then the same sets are weakly homogeneously Souslin under the original and the modified definitions.

8.2 Projective Determinacy

From now until the end of the proof of Theorem 8.2.7, let κ be an inaccessible cardinal, let $T \in V_\kappa$ be a game tree, let Y be a set, let U be a tree on field $(T) \times Y$.

In this section we will prove that if κ is Woodin and $\langle \mathcal{U}_{p,r} \mid r \in {}^{<\omega}\omega\rangle$ witnesses that U is weakly κ^+-homogeneous for T, then $U^\ddagger = U^\ddagger(\langle \mathcal{U}_{p,r} \mid \langle\!\langle p, r\rangle\!\rangle \in T \otimes {}^{<\omega}\omega\rangle)$ is γ-homogeneous for every $\gamma < \kappa$.

For the purpose of motivation, it will be helpful to focus on a more modest goal, the goal of showing that the T-projection A of $U^\ddagger$ has an *embedding normal form*, i.e., that there is a system

$$(\langle M_p \mid p \in T\rangle, \langle k_{p_1,p_2} \mid p_1 \subseteq p_2 \in T\rangle)$$

such that

(a) $M_0 = V$ and each M_p is a transitive proper class model of ZFC;

(b) for each $p \in T$, $k_{p_1,p_2} : M_{p_1} \prec M_{p_2}$;

(c) for $p_1 \subseteq p_2 \subseteq p_3 \in T$, $k_{p_1,p_3} = k_{p_2,p_3} \circ k_{p_1,p_2}$;

(d) for each $x \in [T]$, $x \in A$ if and only if the direct limit model of the directed system $(\langle M_{x\restriction n} \mid n \in \omega\rangle, \langle k_{x\restriction m, x\restriction n} \mid m < n \in \omega\rangle)$ is wellfounded.

Note that if $U^\ddagger$ is homogeneous for T then A has an embedding normal form. Actually our construction of an embedding normal form for A will give the desired homogeneity of $U^\ddagger$.

The construction of the tree for $U^\ddagger$ is notationally somewhat complex. To exhibit the main ideas of the construction, we will first do a somewhat simpler construction. We will assume that we have $\langle \mathcal{U}_p \mid p \in T\rangle$ witnessing the κ^+-homogeneity of U, and we will show that the T-projection of $U^\dagger = U^\dagger(\langle \mathcal{U}_p \mid p \in T\rangle)$ has an embedding normal form, indeed that $U^\dagger$ is γ-homogeneous for every $\gamma < \kappa$..

To get an embedding normal form for the T-projection of $U^\dagger$, we will construct, for each $x \in [T]$, a special kind of iteration tree, an *alternating chain*. An *alternating chain* is an iteration tree of length $\leq \omega$ whose tree ordering S is the restriction of the tree ordering C of ω given by

$$m\, C\, n \leftrightarrow (0 = m < n \vee (\exists k \geq 1)\, m + 2k = n).$$

Thus the models and embeddings of an alternating chain (of length at least 7) begin as follows:

$$\begin{tikzcd}
M_0 \arrow[r, "i^{M_0}_{E_0}"] \arrow[dr, "i^{M_0}_{E_1}"] & M_1 \arrow[r, "i^{M_1}_{E_2}"] & M_3 \arrow[r, "i^{M_3}_{E_4}"] & \cdots \\
& M_2 \arrow[r, "i^{M_2}_{E_3}"] & M_4 \arrow[r, "i^{M_4}_{E_5}"] & \cdots
\end{tikzcd}$$

If $\mathcal{C}$ is an alternating chain of length ω, then $\mathcal{C}$ has exactly two branches:

$$\begin{aligned} \text{Even} &= \{2n \mid n \in \omega\}; \\ \text{Odd} &= \{0\} \cup \{2n+1 \mid n \in \omega\}. \end{aligned}$$

With each $x \in [T]$ we will associate an alternating chain $\mathcal{C}_x$ of length ω on V. For each $n \in \omega$, the restrictions $\mathcal{C}_x \restriction 2n+1$ will depend only on $x \restriction n$. To get our embedding normal form for the projection of $U^\dagger$, we will use the branches Even of the $\mathcal{C}_x$. We will set $M_{x\restriction n} = M^{\mathcal{C}_x}_{2n}$; for $m \leq n \in \omega$, we will set $k_{x\restriction m, x\restriction n} = j^{\mathcal{C}_x}_{2m,2n}$. Our task will then be to build the $\mathcal{C}_x$ so that, for each $x \in [T]$,

$$[U^\dagger(x)] \neq \emptyset \leftrightarrow \tilde{\mathcal{M}}^{\mathcal{C}}_{\text{Even}} \text{ is wellfounded.}$$

If we are to carry out our plan, then we must find a method of constructing alternating chains, and we must be able to control the wellfoundedness or illfoundedness of the branch Even for these alternating chains. Before giving the full construction we will discuss, one at a time, how we intend to solve these two problems.

Our tool for building alternating chains will be the One-Step Lemma, Lemma 6.3.18. To illustrate the method, we give a result whose proof will show how to build finite alternating chains.

For the tree ordering C of infinite alternating chains, note that $(k+1)^-_C = k \dot{-} 1$, where

$$m \dot{-} n = \begin{cases} m-n & \text{if } m \geq n; \\ 0 & \text{if } m < n. \end{cases}$$

Lemma 8.2.1. *Assume that κ is Woodin. Let $n \in \omega$. Then there is an alternating chain $\mathcal{C}$ of length $n+1$ on V such that $\mathcal{C} \in V_\kappa$.*

Proof. If $n = 0$, there is nothing to do, so assume that $n > 0$. Let $\gamma < \kappa$ be such that $T \in V_\gamma$. Let $\delta_0 > \gamma$ be $(n-1)$-reflecting in in $\emptyset$ relative to κ. Let

$k < n$ and assume inductively that we have an alternating chain $\mathcal{C}_k$ of length $k+1$ on $\mathcal{M} = (V; \in)$. We denote the extenders $E_m^{\mathcal{C}}$ by E_m, the models $\mathcal{M}_m^{\mathcal{C}}$ by $(M_m; \in)$, and the embeddings $j_{m,m'}^{\mathcal{C}}$ by $j_{m,m'}$. Assume also that there is an ordinal δ_k such that $\gamma < \delta_k < \kappa$ and such that

(i) M_k and $M_{k \dot- 1}$ agree through $\delta_k + 1$;

(ii) $(\mathrm{tp}^{\delta_k}_{\kappa,n-k-1})^{M_k}(\emptyset) = (\mathrm{tp}^{\delta_k}_{\kappa,n-k-1})^{M_{k \dot- 1}}(\emptyset)$;

(iii) δ_k is $(n-k-1)$-reflecting in $\emptyset$ relative to κ in M_k.

Assume first that $n - k > 1$. Since all the $j_{m,m'}$ fix κ, κ is Woodin in M_k. Thus the hypotheses of the One-Step Lemma hold for κ with

$$\begin{aligned} M &= M_k; \\ N &= M_{k \dot- 1}; \\ \delta &= \delta_k; \\ \eta &= \delta_k; \\ \beta &= n-k-1; \\ \xi &= n-k-2; \\ \beta' &= n-k-1; \\ x &= \emptyset; \\ y &= \emptyset; \\ x' &= \emptyset; \\ \chi(v) &= \text{“}\kappa + v \text{ is the greatest ordinal.”} \end{aligned}$$

Let λ and E be given by the One-Step Lemma. If $k = 0$ then the model $\prod_E^{M_{k \dot- 1}}(M_{k \dot- 1}; \in) = \prod_E^V(V; \in)$ and so is wellfounded. If $k > 0$, then $\prod_E^{M_{k \dot- 1}}(M_{k \dot- 1}; \in) = \prod_E^{M_{k-1}}(M_{k-1}; \in)$. By part (2) of Lemma 7.2.5 and the (i) above, we have that

$$\rho^{\mathcal{C}_k}(k-1, k) > \delta_k = \mathrm{crit}\,(E).$$

Thus part (2) of Theorem 7.3.2 implies that $\prod_E^{M_{k-1}}(M_{k-1}; \in)$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. By clause (2^*) of the One-Step Lemma, $y^* = \emptyset$. By clause (4^*), $\xi^* = n-k-2$. Extend $\mathcal{C}_k$ to an alternating chain $\mathcal{C}_{k+1}$ of length $k+2$ by setting $E_k = E$. Let $\delta_{k+1} = \delta^*$. Our inductive assumptions hold for $k+1$.

Now assume that $n = k+1$. Using Lemmas 7.2.5 and 7.3.2 as in the preceding paragraph, we may extend $\mathcal{C}_k$ to an alternating chain $\mathcal{C}_{k+1}$ of length $k+1$ by letting E_k witness that κ is 0-reflecting in $\emptyset$ relative to κ in M_k. $\square$

Remark. Clause (4*) of the One-Step Lemma was not really needed for the proof of Lemma 8.2.1. The fact that that $\xi = n - k - 2$ could have been deduced from clause (2*) instead of from clause (4*). In the proofs of subsequent lemmas, it will be necessary to use clause (4*), and we will use it in just the way we used it in the proof of Lemma 8.2.1.

We next show how to use the One-Step Lemma to construct infinite alternating chains. At first this appears impossible, because of that Lemma's requirement that $\xi < \beta$.

Lemma 8.2.2. *There exist ordinals ν, ζ_0, ζ_1, and ρ such that*

(1) $\nu < \zeta_0 < \zeta_1 < \rho$;

(2) ν, ζ_0, ζ_1, *and* ρ_2 *are strong limit cardinals of cofinality greater than* κ;

(3) $\mathrm{tp}^{\nu}_{\rho,0}(\langle\zeta_0\rangle) = \mathrm{tp}^{\nu}_{\rho,0}(\langle\zeta_1\rangle)$;

(4) $U \in V_\nu$.

Proof. Let Z be the class of all strong limit cardinals of cofinality greater than κ. Let ν be the least element of Z such that $U \in V_\nu$. Let ρ be the $|V_{\nu+1}|^+$th element of Z. There are at most $|V_{\nu+1}|$ distinct values of $\mathrm{tp}^{\nu}_{\rho,0}(\zeta)$. Hence there must exist ζ_0 and ζ_1 belonging to Z and satisfying (1) and (3). $\square$

From now until the end of the proof of Theorem 8.2.7, let ν, ζ_0, ζ_1, and ρ be as in the statement of Lemma 8.2.2.

The next lemma gives the key facts about these ordinals.

Lemma 8.2.3. *(a) If $\mathcal{T} \in V_\kappa$ is an iteration tree, then each of the ordinals ν, ζ_0, ζ_1, and ρ is fixed by each of the embeddings $j^{\mathcal{T}}_{\beta,\gamma}$.*

(b) If $z \in {}^{<\omega}(V_\nu)$ and $\alpha < \kappa$, then $\mathrm{tp}^{\alpha}_{\kappa,\zeta_0}(z) = \mathrm{tp}^{\alpha}_{\kappa,\zeta_1}(z)$.

(c) If $z \in {}^{<\omega}(V_\nu)$, then $\delta < \kappa$ is ζ_0-reflecting in z relative to κ if and only if δ is ζ_1-reflecting in z relative to κ.

Proof. (a) This follows from property (2) of $\langle\nu, \zeta_0, \zeta_1, \rho\rangle$.

(b) Let $n \in \omega$ and let $z = \langle z_1, \ldots, z_n\rangle \in {}^n(V_\nu)$. Let $\alpha < \kappa$ and let $a = \mathrm{tp}^{\alpha}_{\kappa,\zeta_0}$. By Lemma 6.3.12,

$$\mathrm{TYPE}_n(v_1, \ldots, v_{n+1}, c_a, c_\alpha, d) \in \mathrm{tp}^{\kappa}_{\kappa,\rho}(z^\frown\langle\zeta_0\rangle).$$

Since $\kappa + \rho = \rho$, it follows from the definition of $\mathrm{tp}^{\eta}_{\beta,\gamma}$ that

$$\mathrm{TYPE}_n(c_{z_1}, \ldots, c_{z_n}, v_1, c_a, c_\alpha, c_\kappa) \in \mathrm{tp}^{\nu}_{\rho,0}(\langle\zeta_0\rangle).$$

By property (3) of $\langle\zeta_0, \zeta_1, \rho\rangle$, we get that

$$\mathrm{TYPE}_n(c_{z_1}, \ldots, c_{z_n}, v_1, c_a, c_\alpha, c_\kappa) \in \mathrm{tp}^{\nu}_{\rho,0}(\langle\zeta_1\rangle).$$

By definition this gives that

$$\mathrm{TYPE}_n(v_1, \ldots, v_{n+1}, c_a, c_\alpha, d) \in \mathrm{tp}^{\kappa}_{\kappa,\rho}(z^\frown\langle\zeta_0\rangle).$$

Another application of Lemma 6.3.12 then gives that $a = \mathrm{tp}^{\alpha}_{\kappa,\zeta_0}$.

(c) Let z be a in the proof of (b). Let $\delta < \kappa$. Property (3) of $\langle\zeta_0, \zeta_1, \rho\rangle$, Lemma 6.3.13, and the fact that $\kappa + \zeta_0 = \zeta_0$, yield the following chain of equivalences:

$$\begin{array}{ll} & \delta \text{ is } \zeta_0\text{-reflecting in } z \text{ relative to } \kappa \\ \leftrightarrow & \mathrm{REFL}_n(v_1, \ldots, v_{n+1}, c_\delta, d) \in \mathrm{tp}^{\delta+1}_{\kappa,\rho}(z^\frown\langle\zeta_0\rangle) \\ \leftrightarrow & \mathrm{REFL}_n(c_{z_1}, \ldots, c_{v_n}, v_1, c_\delta, c_\kappa) \in \mathrm{tp}^{\nu}_{\rho,0}(\langle\zeta_0\rangle) \\ \leftrightarrow & \mathrm{REFL}_n(c_{z_1}, \ldots, c_{v_n}, v_1, c_\delta, c_\kappa) \in \mathrm{tp}^{\nu}_{\rho,0}(\langle\zeta_1\rangle) \\ \leftrightarrow & \mathrm{REFL}_n(v_1, \ldots, v_{n+1}, c_\delta, d) \in \mathrm{tp}^{\delta+1}_{\kappa,\rho}(z^\frown\langle\zeta_1\rangle) \\ \leftrightarrow & \delta \text{ is } \zeta_1\text{-reflecting in } z \text{ relative to } \kappa \end{array}$$

□

The proof of the following lemma will show how we can construct infinite alternating chains and how we can make the branch Even illfounded.

Lemma 8.2.4. *Assume that κ is Woodin. Then there is a an infinite alternating chain $\mathcal{C}$ on V such that $\mathcal{C} \in V_\kappa$ and such that the branch* Even *of $\mathcal{C}$ is not wellfounded.*

Proof. Let $\gamma < \kappa$ be such that $T \in V_\gamma$. Let $\delta_0 > \gamma$ be $(\zeta_0 + 1)$-reflecting in $\emptyset$ relative to κ. Let $\beta_0 = \zeta_0$. Let $k < n$ and assume inductively that we have an alternating chain $\mathcal{C}_{2k}$ of length $2k + 1$ on V. As in the proof of Lemma 8.2.1 let $\mathcal{C}_{2k}$ have extenders $E_m \in V_\kappa$, $m < 2k$, models M_m, $m \leq 2k$, and embeddings $j_{m,m'}$, $m \, C \, m' \leq 2k$. Assume also that there is an ordinal δ_{2k} such that $\gamma < \delta_{2k} < \kappa$ and that there are ordinals β_m, $m \leq k$, satifying the following conditions:

(i) M_{2k} and $M_{2k \dot{-} 1}$ agree through $\delta_{2k}+1$;

(ii) $(\mathrm{tp}^{\delta_{2k}}_{\kappa,\beta_k+1})^{M_{2k}}(\emptyset) = (\mathrm{tp}^{\delta_{2k}}_{\kappa,\zeta_0+1})^{M_{2k \dot{-} 1}}(\emptyset)$;

(iii) δ_{2k} is (β_k+1)-reflecting in $\emptyset$ relative to κ in M_{2k};

(iv) $(\forall m < k)(\forall m' \leq k)(m < m' \to \beta_{m'} < j_{2m,2m'}(\beta_m))$.

Since all the $j_{m,m'}$ fix κ, κ is Woodin in M_{2k}. Thus the hypotheses of the One-Step Lemma hold for κ with

$$\begin{aligned}
M &= M_{2k};\\
N &= M_{2k \dot{-} 1};\\
\delta &= \delta_{2k};\\
\eta &= \delta_{2k};\\
\beta &= \beta_k + 1;\\
\xi &= \beta_k;\\
\beta' &= \zeta_0 + 1;\\
x &= \emptyset;\\
y &= \emptyset;\\
x' &= \emptyset;\\
\chi(v) &= \text{“}\kappa + v \text{ is the greatest ordinal.”}
\end{aligned}$$

Let λ and E be given by the One-Step Lemma. As in the analogous step in the proof of Lemma 8.2.1, $\prod_E^{M_{2k \dot{-} 1}}(M_{2k \dot{-} 1}; \in)$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. By clause (2^*) of the One-Step Lemma, $y^* = \emptyset$. By clause (4^*), $\xi^* = \zeta_0$. Extend $\mathcal{C}_k$ to an alternating chain $\mathcal{C}_{2k+1}$ of length $2k+2$ by setting $E_{2k} = E$. Let $\delta_{2k+1} = \delta^*$. We have then that

(a) M_{2k+1} and M_{2k} agree through $\delta_{2k+1}+1$;

(b) $(\mathrm{tp}^{\delta_{2k+1}}_{\kappa,\zeta_0})^{M_{2k+1}}(\emptyset) = (\mathrm{tp}^{\delta_{2k+1}}_{\kappa,\beta_k})^{M_{2k}}(\emptyset)$;

(c) δ_{2k+1} is ζ_0-reflecting in $\emptyset$ relative to κ in M_{2k+1}.

By (b) and (c) together with parts (b) and (c) of Lemma 8.2.3, we have that

(b′) $(\mathrm{tp}^{\delta_{2k+1}}_{\kappa,\zeta_1})^{M_{2k+1}}(\emptyset) = (\mathrm{tp}^{\delta_{2k+1}}_{\kappa,\beta_k})^{M_{2k}}(\emptyset)$;

(c′) δ_{2k+1} is ζ_1-reflecting in $\emptyset$ relative to κ in M_{2k+1}.

Since κ is Woodin in M_{2k+1}, the hypotheses of the One-Step Lemma hold for κ with

$$\begin{array}{rcl} M & = & M_{2k+1}; \\ N & = & M_{2k}; \\ \delta & = & \delta_{2k+1}; \\ \eta & = & \delta_{2k+1}; \\ \beta & = & \zeta_1; \\ \xi & = & \zeta_0 + 1; \\ \beta' & = & \beta_k; \\ x & = & \emptyset; \\ y & = & \emptyset; \\ x' & = & \emptyset; \\ \chi(v) & = & \text{“}v = v.\text{”} \end{array}$$

Let λ and E be given by the One-Step Lemma. By Theorem 7.3.2, the model $\prod_E^{M_{2k}}(M_{2k}; \in)$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. Once more, $y^* = \emptyset$. Extend $\mathcal{C}_{2k+1}$ to an alternating chain $\mathcal{C}_{2k+2}$ of length $2k+3$ by setting $E_{2k+1} = E$. Let $\delta_{2k+2} = \delta^*$. Let $\beta_{k+1} = \xi^*$. The inequality $\xi^* < i_E^N(\beta')$of the One-Step Lemma gives us that

$$\beta_{k+1} = \xi^* < i_E^{M_{2k}}(\beta_k) = j_{2k,2k+2}(\beta_k).$$

From this and induction hypothesis (iv) for k we get hypothesis (iv) for $k+1$. Thus all our induction hypotheses hold for $k+1$.

Let $\mathcal{C}$ be the alternating chain of length ω whose restrictions are the $\mathcal{C}_k$.

We must verify that the model $\tilde{\mathcal{M}}^{\mathcal{C}}_{\text{Even}}$ is not wellfounded. But condition (iv) implies that

$$(\forall m \in \omega)(\forall n \in \omega)(m < n \to \tilde{\jmath}^{\mathcal{C}}_{2n,\text{Even}}(\beta_n) < \tilde{\jmath}^{\mathcal{C}}_{2m,\text{Even}}(\beta_m)).$$

Thus the $\tilde{\jmath}^{\mathcal{C}}_{2n,\text{Even}}(\beta_n)$ are an infinite descending sequence of ordinals of $\tilde{\mathcal{M}}^{\mathcal{C}}_{\text{Even}}$. $\square$

We now know how to build infinite alternating chains. Moreover we know a way to arrange that the Even branch of our chains is illfounded. But how are we are going to build chains $\mathcal{C}_x$ whose Even branches are illfounded if and only if $[U(x)] \neq \emptyset$? How are we going to make our illfoundedness construction work only if $[U(x)] \neq \emptyset$? And how are we to guarantee that the branch Even is wellfounded in the case $[U(x)] = \emptyset$?

To solve this last problem, we will make sure that $\mathcal{C}_x$ is continuously illfounded off Even if $[U(x)] = \emptyset$. Lemma 8.2.5 below asserts that every alternating chain is plus one. Therefore Corollary 7.4.6 will imply that Even is wellfounded.

Making an alternating chain continuously illfounded off Even is equivalent with making Odd illfounded. (See Exercise 8.2.1.) To make Odd illfounded when $[U(x)] = \emptyset$, we will arrange that $\tilde{\jmath}^{\mathcal{C}_x}_{0,\mathrm{Odd}}(U(x))$ is illfounded for every x. At the beginning of kth stage of the construction of $\mathcal{C}_x$ we will have chosen an element s_k of $j^{\mathcal{C}_x}_{0,2k\dot{-}1}(U[x \restriction k])$. During the kth stage, we will choose an element s_{k+1} of $j^{\mathcal{C}_x}_{0,2k+1}(U[x \restriction k+1])$ such that $s_{k+1} \supseteq j^{\mathcal{C}_x}_{2k\dot{-}1,2k+1}(s_k)$. The $\tilde{\jmath}^{\mathcal{C}_x}_{2k\dot{-}1,\mathrm{Odd}}(s_k)$ will thus witness the illfoundedness of $\tilde{\jmath}^{\mathcal{C}_x}_{0,\mathrm{Odd}}(U(x))$. Moreover the s_k will directly provide us with ordinals witnessing that $\mathcal{C}$ is continuously illfounded off Even.

We will get the s_k via the One-Step Lemma. Assume that $\langle \mathcal{U}_p \mid p \in T\rangle$ witnesses that U is κ^+-homogeneous. Given k, consider $\mathrm{Ult}(V;\mathcal{U}_{x\restriction k})$ This is the same as $i_{\emptyset,x\restriction k}(V)$, where the $i_{p,q}$, $p \subseteq q \in T$, are defined as on page 200. In this model there is a canonical element of $i_{\emptyset,x\restriction k}(U[x \restriction k])$, namely

$$\mathbf{s}_{x\restriction k} = \pi_{\mathcal{U}_{x\restriction k}}([\![\mathrm{id}]\!]_{\mathcal{U}_{x\restriction k}}).$$

From $\mathbf{s}_{x\restriction k}$, we get

$$j^{\mathcal{C}_x}_{0,2k}(\mathbf{s}_{x\restriction k}),$$

an element of $j^{\mathcal{C}_x}_{0,2k}(i_{\emptyset,x\restriction k}(U[x \restriction k]))$. With the aid of the One-Step-Lemma, we will arrange inductively that, for some ordinal β_k, the type

$$(\mathrm{tp}^{\delta_{2k}}_{\kappa,\beta_k+1})^{j^{\mathcal{C}_x}_{0,2k}(i_{\emptyset,x\restriction k}(V))}(\langle j^{\mathcal{C}_x}_{0,2k}(i_{\emptyset,x\restriction k}(U))\rangle^\frown j^{\mathcal{C}_x}_{0,2k}(\mathbf{s}_{x\restriction k}))$$

is the same as

$$(\mathrm{tp}^{\delta_{2k}}_{\kappa,\zeta_0+1})^{M_{2k\dot{-}1}}(\langle j^{\mathcal{C}_x}_{0,2k\dot{-}1}(U)\rangle^\frown s_k).$$

We will also arrange inductively that δ_{2k} is (β_k+1)-reflecting in the finite sequence $\langle j^{\mathcal{C}_x}_{0,2k}(i_{\emptyset,x\restriction k}(U))\rangle^\frown j^{\mathcal{C}_x}_{0,2k}(\mathbf{s}_{x\restriction k})$ relative to κ in $j^{\mathcal{C}_x}_{0,2k}(i_{\emptyset,x\restriction k}(V))$.

The ordinals β_k will play a role analogous to the role of the β_k appearing in the proof of Lemma 8.2.4. But, whereas the latter gave an infinite descending sequence of ordinals of $\tilde{\mathcal{M}}^{\mathcal{C}_x}_{\mathrm{Even}}$, the β_k of our new construction will give instead an infinite descending sequence of ordinals of

$(\tilde{\jmath}^{\mathcal{C}_x}_{0,\mathrm{Even}}(i^x_\emptyset))(\tilde{\mathcal{M}}^{\mathcal{C}_x}_{\mathrm{Even}}) = \tilde{\jmath}^{\mathcal{C}_x}_{0,\mathrm{Even}}(i^x_\emptyset(V))$. When $\tilde{\mathcal{M}}^{\mathcal{C}_x}_{\mathrm{Even}}$ is wellfounded, the sequence will show that $j^{\mathcal{C}_x}_{0,\mathrm{Even}}(i^x_\emptyset(V))$ is illfounded. By absoluteness and the elementarity of $j^{\mathcal{C}_x}_{0,\mathrm{Even}}$, $\mathcal{M}_x = i^x_\emptyset(V)$ will be illfounded and so we will have $[U(x)] \neq \emptyset$

The β_m will also give us ultrafilters that will witness the homogeneity of $U^\dagger$. For each $k \in \omega$, the finite sequence, $\langle j^{\mathcal{C}_x}_{2m,2k}(\beta_m) \mid m < k\rangle$ will be an element of $j^{\mathcal{C}_x}_{0,2k}(U^\dagger[x \restriction k])$. This finite sequence and Lemma 6.1.1 will yield an ultrafilter on $U^\dagger[x \restriction k]$.

In the preceding discussion, we have been dealing with a fixed $x \in [T]$. The following theorem and its proof will involve all such x simultaneously, so we will have to modify some of our notation. Thus s_k will become $s_{x\restriction k}$, δ_{2k} will become $\delta_{x\restriction k}$, β_k will become $\beta_{x\restriction k}$, M_{2k} will now become $M^{x\restriction k}_{2k}$, etc.

Before proceeding to the theorem, let us first verify that alternating chains are all plus one.

Lemma 8.2.5. *Every alternating chain is plus one.*

Proof. By the definition (given on page 397, an iteration tree $\mathcal{T} = (\langle \mathcal{M}, T, \langle E_\alpha \mid \alpha + 2 < \ell\mathrm{h}(\mathcal{T})\rangle\rangle$ is plus one if and only if, for all β such that $\beta + 2 < \ell\mathrm{h}(\mathcal{T})$,

$$\mu^{\mathcal{T}}(\beta) < \mathrm{strength}^{\mathcal{M}_\beta}(E_\beta),$$

where

$$\mu^{\mathcal{T}}(\beta) = \sup\{\mathrm{crit}\,(E_\alpha) \mid (\alpha+1)^-_T \leq \beta < \alpha\}.$$

Part (b) of Corollary 7.2.6 implies that, for $\alpha + 1 < \ell\mathrm{h}(\mathcal{T})$ and $(\alpha+1)^-_T \leq \beta < \alpha$,

$$\mathrm{crit}\,(E_\alpha) < \mathrm{strength}^{\mathcal{M}_\beta}(E_\beta).$$

It follows that $\mu^{\mathcal{T}}(\beta) < \mathrm{strength}^{\mathcal{M}_\beta}(E_\beta)$ for every β such that $\{\alpha \mid (\alpha+1)^-_T \leq \beta < \alpha\}$ is finite. But, for m and $n \in \omega$,

$$(m+1)^-_C \leq n < m \leftrightarrow m = n+1,$$

where C is the tree ordering of alternating chains. □

Recall that, for κ a cardinal number, a set is $(<\kappa)$-homogeneously Souslin if it is γ-homogeneously Souslin for every $\gamma < \kappa$. If T is a game tree, Y is a set, and U is a tree on field $(T) \times Y$, then let us say that U is $(<\kappa)$-*homogeneous for* T if U is γ-homogeneous for T for every $\gamma < \kappa$.

Theorem 8.2.6. *Assume that κ is Woodin and that $\langle \mathcal{U}_p \mid p \in T\rangle$ witnesses that U is κ^+-homogeneous for T. Let $U^\dagger = U^\dagger(\langle \mathcal{U}_p \mid p \in T\rangle)$. Then, for every sufficiently large ordinal α, $U^\dagger \restriction \alpha$ is $(<\kappa)$-homogeneous for T.*

Proof. Let $\gamma < \kappa$ be such that $T \in V_\gamma$.

Define the embeddings $i_{p,q}$, $p \subseteq q \in T$, the models $\mathcal{M}_x$, $x \in [T]$, and the embeddings i^x_p, $p \in T$ and $p \subseteq x \in [T]$, as on page 200. For $p \in T$ let $\mathbf{s}_p = \pi_{\mathcal{U}_p}([\![\mathrm{id}]\!]_{\mathcal{U}_p})$.

We will define, by induction on $p \in T$, objects δ_p, β_p, $\mathcal{C}_p$, and s_p. Both δ_p and β_p will be ordinals, with $\delta_p < \kappa$. s_p will be a sequence with $\ell\mathrm{h}(s_p) = \ell\mathrm{h}(p)$, $\mathcal{C}_p$ will be an alternating chain of length $2\ell\mathrm{h}(p)+1$ on V. Its extenders will be E^p_m, $m < 2\ell\mathrm{h}(p)$, its models will be M^p_m, $m \leq 2\ell\mathrm{h}(p)$, and its embeddings will be $j^p_{m,n}$, $m \, C \, n \leq 2\ell\mathrm{h}(p)$. Whenever $p \subseteq q \in T$ then we will have $\mathcal{C}_p = \mathcal{C}_q \restriction 2\ell\mathrm{h}(p) + 1$.

To avoid having excessively cumbersome notation, let us make the following definitions. Let p, q, and q' be elements of T with $q \subseteq q'$. Let $m \leq \ell\mathrm{h}(p)$. Set

$$\begin{aligned}
\breve{\imath}^p_{q,q'} &= j^p_{0,2\ell\mathrm{h}(p)}(i_{q,q'});\\
\breve{N}^p_q &= \breve{\imath}^p_{\emptyset,q}(M^p_{2\ell\mathrm{h}(p)});\\
\breve{U}^p_q &= j^p_{0,2\ell\mathrm{h}(p)}(i_{\emptyset,q}(U));\\
\bar{\beta}^p_m &= j^p_{2m,2\ell\mathrm{h}(p)}(\beta_{p\restriction m}).
\end{aligned}$$

The embedding $\breve{\imath}^p_{q,q'}$ is the image of $i_{q,q'}$ in $M^p_{2\ell\mathrm{h}(p)}$. The class model $\breve{N}^p_q$ is the image of $\mathrm{Ult}(V;\mathcal{U}_q)$ in $M^p_{2\ell\mathrm{h}(p)}$. In other words,

$$\breve{N}^p_q = j^p_{0,2\ell\mathrm{h}(p)}(i_{\emptyset,q}(V)).$$

The tree $\breve{U}^p_q$ is the image of $i_{\emptyset,q}(U)$ in $M^p_{2\ell\mathrm{h}(p)}$; i.e., it is the image of U in $\breve{N}^p_q$.

For $p = \emptyset$ we have only to define $\delta_\emptyset$ and $\beta_\emptyset$. Choose $\delta_\emptyset > \gamma$ to be $\zeta_0 + 1$-reflecting in $\langle U\rangle$ relative to κ. Let $\beta_\emptyset = \zeta_0$.

For the induction step of our definition, let $p \in T$. Let $k = \ell\mathrm{h}(p)$. Assume that $\delta_{p'}$, $\beta_{p'}$, $\mathcal{C}_{p'}$, and $s_{p'}$ are defined for all $p' \subseteq p$ so as to satisfy the conditions stated above and also the following conditions:

(i) M^p_{2k} and $M^p_{2k\dot{-}1}$ agree through $\delta_p + 1$;

(ii) $(\mathrm{tp}^{\delta_p}_{\kappa,\beta_p+1})^{\breve{N}^p_p}(\langle \breve{U}^p_p\rangle^\frown j^p_{0,2k}(\mathbf{s}_p)) = (\mathrm{tp}^{\delta_p}_{\kappa,\zeta_0+1})^{M^p_{2k\dot{-}1}}(\langle j^p_{0,2k\dot{-}1}(U)\rangle^\frown s_p)$;

(iii) δ_p is $(\beta_p + 1)$-reflecting in $\langle \breve{U}^p_p \rangle^\frown j^p_{0,2k}(\mathbf{s}_p)$ relative to κ in $\breve{N}^p_p$;

(iv) for all m and m' with $m < m' \leq k$, $\bar{\beta}^p_{m'} < \breve{\imath}^p_{p\restriction m, p\restriction m'}(\bar{\beta}^p_m)$;

(v) s_p belongs to $j^p_{0,2k\dot{-}1}(U[p])$, and, for all $m \leq k$,

$$j^p_{2m\dot{-}1,2k\dot{-}1}(s_{p\restriction m}) \subseteq s_p;$$

(vi) for all $m < k$, $\gamma < \delta_{p\restriction m} = \text{crit}\,(E^p_{2m}) < \text{crit}\,(E^p_{2m+1}) < \delta_{p\restriction m+1}$;

Note that these conditions all hold for for $p = \emptyset$.

Remarks:

(a) Condition (i) and the fact that $\text{crit}\,(\breve{\imath}^p_{\emptyset,p}) \geq \text{crit}\,(i_{\emptyset,p}) > \kappa$ guarantee that $\breve{N}^p_p$ and $M^p_{2k\dot{-}1}$ agree through $\delta_p + 1$. (That they agree through δ_p is also implied by condition (ii).)

(b) Conditions (i), (ii), and (iii) are to make possible our applications of the One-Step Lemma. Condition (iv) will yield that $\tilde{\mathcal{M}}^{\mathcal{C}_x}_{\text{Even}}$ is illfounded whenever $[U(x)] \neq \emptyset$, and it will allow us to use the β_p to define ultrafilters witnessing homogeneity. Condition (v) will yield that $\tilde{\mathcal{M}}^{\mathcal{C}_x}_{\text{Odd}}$ is illfounded whenever $[U(x)] = \emptyset$. Condition (vi) guarantees that T, all members of T, and all members of $[T]$, are fixed by the embeddings of our alternating chains.

Let q be any element of T such that $p \subseteq q$ and $\ell\text{h}(q) = k + 1$.

By (i) and the fact that $\text{crit}\,(\breve{\imath}^p_{\emptyset,q}) > \kappa$, it follows that $\breve{N}^p_q$ and $M^p_{2k\dot{-}1}$ agree through $\delta_p + 1$.

Note that

$$\begin{aligned}\breve{N}^p_q &= \breve{\imath}^p_{\emptyset,q}(M^p_{2k}) \\ &= \breve{\imath}^p_{p,q}(\breve{\imath}^p_{\emptyset,p}(M^p_{2k})) \\ &= \breve{\imath}^p_{p,q}(\breve{N}^p_p),\end{aligned}$$

that

$$\begin{aligned}\breve{U}^p_q &= j^p_{0,2k}(i_{\emptyset,q}(U)) \\ &= j^p_{0,2k}(i_{p,q}(i_{\emptyset,p}(U))) \\ &= (j^p_{0,2k}(i_{p,q}))(j^p_{0,2k}(i_{\emptyset,p}(U))) \\ &= \breve{\imath}^p_{p,q}(\breve{U}^p_p),\end{aligned}$$

and that

$$\begin{aligned} j^p_{0,2k}(i_{p,q}(\mathbf{s}_p)) &= (j^p_{0,2k}(i_{p,q}))(j^p_{0,2k}(\mathbf{s}_p)) \\ &= \breve{\imath}^p_{p,q}(j^p_{0,2k}(\mathbf{s}_p)). \end{aligned}$$

The fact that $\mathrm{crit}\,(\breve{\imath}^p_{p,q}) > \kappa$, together with the facts just mentioned, gives that

$$\begin{aligned} &(\mathrm{tp}^{\delta_p}_{\kappa,\beta_p+1})^{\breve{N}^p_p}(\langle \breve{U}^p_p\rangle ^\frown j^p_{0,2k}(\mathbf{s}_p)) \\ &= \breve{\imath}^p_{p,q}((\mathrm{tp}^{\delta_p}_{\kappa,\beta_p+1})^{\breve{N}^p_p}(\langle \breve{U}^p_p\rangle ^\frown j^p_{0,2k}(\mathbf{s}_p))) \\ &= (\mathrm{tp}^{\delta_p}_{\kappa,\breve{\imath}^p_{p,q}(\beta_p+1)}))^{\breve{N}^p_q}(\langle \breve{U}^p_q\rangle ^\frown j^p_{0,2k}(i_{p,q}(\mathbf{s}_p))). \end{aligned}$$

and so by (ii) this last is the same as $(\mathrm{tp}^{\delta_p}_{\kappa,\zeta_0+1})^{M^p_{2k\dot{-}1}}(\langle j^p_{0,2k\dot{-}1}(U)\rangle ^\frown s_p)$.

From (iii) it similarly follows that δ_p is $(\breve{\imath}^p_{p,q}(\beta_p+1))$-reflecting in the finite sequence $\langle \breve{U}^p_q\rangle ^\frown j^p_{0,2k}(i_{p,q}(\mathbf{s}_p))$ relative to κ in $\breve{N}^p_q$.

Since $j^p_{0,2k}$ and $\breve{\imath}^p_{\emptyset,q}$ fix κ, we have that κ is Woodin in $\breve{N}^p_q$.

Thus the hypotheses of the One-Step Lemma hold for κ with

$$\begin{aligned} M &= \breve{N}^p_q; \\ N &= M^p_{2k\dot{-}1}; \\ \delta &= \delta_p; \\ \eta &= \delta_p; \\ \beta &= \breve{\imath}^p_{p,q}(\beta_p)+1; \\ \xi &= \breve{\imath}^p_{p,q}(\beta_p); \\ \beta' &= \zeta_0+1; \\ x &= \langle \breve{U}^p_q\rangle ^\frown j^p_{0,2k}(i_{p,q}(\mathbf{s}_p)); \\ y &= \langle (j^p_{0,2k}(\mathbf{s}_q))(k)\rangle; \\ x' &= \langle j^p_{0,2k\dot{-}1}(U)\rangle ^\frown s_p; \\ \chi(v) &= \text{“}\kappa+v \text{ is the greatest ordinal.”} \end{aligned}$$

Let λ and E be given by the One-Step Lemma. Since $\breve{\imath}^p_{\emptyset,q}$ fixes λ, E, and δ_p, it follows that E is a (δ_p,λ)-extender in M^p_{2k}. Thus Theorem 7.3.2 guarantees that $\prod_E^{M^p_{2k\dot{-}1}}(M^p_{2k\dot{-}1};\in)$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. By clause (4*) of the One-Step Lemma, $\xi^*=\zeta_0$. Extend $\mathcal{C}_p$ to an alternating chain that will be $\mathcal{C}_q\restriction 2k+2$ by setting $E^q_{2k}=E$. The ordinal δ^* we will call δ'_q. Set $s_q=(j^q_{2k\dot{-}1,2k+1}(s_p))^\frown y^*$.

We will use without comment in the sequel the facts $M^q_n = M^p_n$, $E^q_n = E^p_n$, and $j^q_{m,n} = j^p_{m,n}$ whenever these equations make sense.

By the elementarity of $j^q_{0,2k}$ and the definition of the $\mathbf{s}_{p'}$,

$$\begin{aligned} x^\frown y &= \langle \breve{U}^p_q\rangle^\frown j^p_{0,2k}(i_{p,q}(\mathbf{s}_p))^\frown\langle (j^p_{0,2k}(\mathbf{s}_q))(k)\rangle \\ &= \langle \breve{U}^p_q\rangle^\frown j^p_{0,2k}(i_{p,q}(\mathbf{s}_p)^\frown\langle \mathbf{s}_q(k)\rangle) \\ &= \langle \breve{U}^p_q\rangle^\frown j^p_{0,2k}(\mathbf{s}_q\restriction k^\frown\langle \mathbf{s}_q(k)\rangle) \\ &= \langle \breve{U}^p_q\rangle^\frown j^p_{0,2k}(\mathbf{s}_q). \end{aligned}$$

Since $j^p_{0,2k}(\mathbf{s}_q)$ is an element of $\breve{\imath}^p_{\emptyset,q}(U) = \breve{U}^p_q$, clause (2*) of the One-Step Lemma implies that $s_q \in j^q_{0,2k+1}(U[q])$. Thus the first clause of condition (v) holds for q. Since $j^q_{2k\dot{-}1,2k+1}(s_p) \subseteq s_q$, the second clause of condition (v) holds for q in the case $m = k$.

We have that

(a) M^q_{2k+1} and $\breve{N}^p_q$ agree through $\delta'_q + 1$;

(b) $(\mathrm{tp}^{\delta'_q}_{\kappa,\zeta_0})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q)$
$= (\mathrm{tp}^{\delta'_q}_{\kappa,\breve{\imath}^p_{p,q}(\beta_p)})^{\breve{N}^p_q}(\langle \breve{U}^p_q\rangle^\frown j^p_{0,2k}(\mathbf{s}_q))$;

(c) δ'_q is ζ_0-reflecting in $\langle j^q_{0,2k+1}(U)\rangle^\frown s_q$ relative to κ in M^q_{2k+1}.

By (a) and the fact that $\mathrm{crit}\,(\breve{\imath}^p_{\emptyset,q}) > \delta'_q$, it follows that M^q_{2k+1} and M^q_{2k} agree through $\delta'_q + 1$.

By (b) and (c) together with parts (b) and (c) of Lemma 8.2.3, we have that

(b′) $(\mathrm{tp}^{\delta'_q}_{\kappa,\zeta_1})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q)$
$= (\mathrm{tp}^{\delta'_q}_{\kappa,\breve{\imath}^p_{p,q}(\beta_p)})^{\breve{N}^p_q}(\langle \breve{U}^p_q\rangle^\frown j^p_{0,2k}(\mathbf{s}_q))$;

(c′) δ'_q is ζ_1-reflecting in $\langle j^q_{0,2k+1}(U)\rangle^\frown s_q$ relative to κ in M^q_{2k+1}.

Let $f : j^p_{0,2k}(U[p]) \to \mathrm{Ord}$ belong to M^p_{2k} and be such that

$$\beta_p = \pi^{M^p_{2k}}_{j^p_{0,2k}(\mathcal{U}_p)}([\![f]\!]^{M^p_{2k}}_{j^p_{0,2k}(\mathcal{U}_p)}).$$

In other words, let f be such that $\beta_p = (\breve{\imath}^p_{\emptyset,p}(f))(j^p_{0,2k}(\mathbf{s}_p))$. By (b′) and the fact that $\mathrm{crit}\,(\breve{\imath}^p_{\emptyset,q}) > \delta'_q$, there is a set $X \in j^p_{0,2k}(\mathcal{U}_q)$ such that, for all $t \in X$, (b″) $(\mathrm{tp}^{\delta'_q}_{\kappa,\zeta_1})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q) = (\mathrm{tp}^{\delta'_q}_{\kappa,f(t\restriction k)})^{M^p_{2k}}(\langle j^p_{0,2k}(U)\rangle^\frown t)$.

Choose any element t of X. Since κ is Woodin in M^q_{2k+1}, the hypotheses of the One-Step Lemma hold for κ with

$$\begin{aligned}
M &= M^q_{2k+1};\\
N &= M^p_{2k};\\
\delta &= \delta'_q;\\
\eta &= \delta'_q;\\
\beta &= \zeta_1;\\
\xi &= \zeta_0+1;\\
\beta' &= f(t\restriction k);\\
x &= \langle j^q_{0,2k+1}(U)\rangle^\frown s_q;\\
y &= \emptyset;\\
x' &= (\langle j^p_{0,2k}(U)\rangle^\frown t;\\
\chi(v) &= \text{“}v=v.\text{”}
\end{aligned}$$

Let λ and E be given by the One-Step Lemma. By Theorem 7.3.2, the model $\prod_E^{M^p_{2k}}(M^p_{2k};\in)$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. (We will make no use of ξ^* and y^*.)

For all elements u of X,

$$(\mathrm{tp}^{\delta'_q}_{\kappa,f(u\restriction k)})^{M^p_{2k}}(\langle j^p_{0,2k}(U)\rangle^\frown u) = (\mathrm{tp}^{\delta'_q}_{\kappa,f(t\restriction k)})^{M^p_{2k}}(\langle j^p_{0,2k}(U)\rangle^\frown t).$$

Thus the elementarity of $i_E^{M^p_{2k}}$ gives that, for all $u\in i_E^{M^p_{2k}}(X)$,

$$\begin{aligned}
&(\mathrm{tp}^{i_E^{M^p_{2k}}(\delta'_q)}_{\kappa,(i_E^{M^p_{2k}}(f))(u\restriction k)})^{\mathrm{Ult}(M^p_{2k};E)}(\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown u)\\
&\quad= (\mathrm{tp}^{i_E^{M^p_{2k}}(\delta'_q)}_{\kappa,(i_E^{M^p_{2k}}(f))(i_E^{M^p_{2k}}(t)\restriction k)})^{\mathrm{Ult}(M^p_{2k};E)}(\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown i_E^{M^p_{2k}}(t)).
\end{aligned}$$

Since $\delta^* < i_E^{M^p_{2k}}(\delta'_q)$, for all $u\in i_E^{M^p_{2k}}(X)$ we have in particular that

$$\begin{aligned}
&(\mathrm{tp}^{\delta^*+1}_{\kappa,(i_E^{M^p_{2k}}(f))(u\restriction k)})^{\mathrm{Ult}(M^p_{2k};E)}(\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown u)\\
&\quad= (\mathrm{tp}^{\delta^*+1}_{\kappa,(i_E^{M^p_{2k}}(f))(i_E^{M^p_{2k}}(t)\restriction k)})^{\mathrm{Ult}(M^p_{2k};E)}(\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown i_E^{M^p_{2k}}(t)).
\end{aligned}$$

For each $u \in i_E^{M_{2k}^p}(X)$, we make an application of the last part of the One-Step Lemma, with $z = \langle i_E^{M_{2k}^p}(j_{0,2k}^p(U))\rangle^\frown u$ and with $\alpha = (i_E^{M_{2k}^p}(f))(u \restriction k)$. If $\hat{\xi}$ and $\hat{y}$ are as given given by this application, then clause $(\hat{2})$ of the One-Step Lemma implies that $\hat{y} = \emptyset$ and $\hat{\xi}$ is a successor ordinal. Let $g(u)$ be the least ordinal μ such that clauses $(\hat{2})$ and $(\hat{3})$ of the One-Step Lemma hold with $\hat{\xi} = \mu + 1$ (and $\hat{y} = \emptyset$).

Observe that the function $g : i_E^{M_{2k}^p}(X) \to \mathrm{Ord}$ belongs to $\mathrm{Ult}(M_{2k}^p; E)$.

We finish the definition of $\mathcal{C}_q$ by setting $E_{2k+1}^q = E$. Thus $E_{2k+2}^q = \mathrm{Ult}(M_{2k}^p; E)$ and $j_{2k,2k+2}^q = i_E^{M_{2k}^p}$. Clause (1^*) of the One-Step Lemma gives inductive condition (i) for q .

Let $\delta_q = \delta^*$. Clauses $(\hat{2})$ and $(\hat{3})$ of the One-Step Lemma give that, for all $u \in j_{2k,2k+2}^q(X)$,

$(\hat{2})$ $(\mathrm{tp}_{\kappa,g(u)+1}^{\delta_q})^{M_{2k+2}^q}(\langle j_{0,2k+2}^q(U)\rangle^\frown u) = (\mathrm{tp}_{\kappa,\zeta_0+1}^{\delta_q})^{M_{2k+1}^q}(\langle j_{0,2k+1}^q(U)\rangle^\frown s_q)$;

$(\hat{3})$ δ_q is $(g(u)+1)$-reflecting in $\langle j_{0,2k+2}^q(U)\rangle^\frown u$ relative to κ in M_{2k+2}^q.

The set $j_{2k,2k+2}^q(X)$ belongs to $j_{0,2k+2}^q(\mathcal{U}_q)$. This fact allows us to complete our definitions by setting

$$\beta_q = \pi_{j_{0,2k+2}^q(\mathcal{U}_q)}^{M_{2k+2}^q}([\![g]\!]_{j_{0,2k+2}^q(\mathcal{U}_q)}^{M_{2k+2}^q}).$$

Using Łoś's Theorem in M_{2k+2}^q and using the fact that $j_{0,2k+2}^q(\mathbf{s}_q) = \pi_{j_{0,2k+2}^q(\mathcal{U}_q)}^{M_{2k+2}^q}([\![\mathrm{id}]\!]_{j_{0,2k+2}^q(\mathcal{U}_q)}^{M_{2k+2}^q})$, we see that $(\hat{2})$ and $(\hat{3})$ imply that

(ii′) $(\mathrm{tp}_{\kappa,\beta_q+1}^{\delta_q})^{\breve{N}_q^q}(\langle j_{0,2k+2}^q(i_{\emptyset,q}(U))\rangle^\frown j_{0,2k+2}^q(\mathbf{s}_q))$
$= (\mathrm{tp}_{\kappa,\zeta_1}^{\delta_q})^{M_{2k+1}^q}(\langle j_{0,2k+1}^q(U)\rangle^\frown s_q)$;

(iii′) δ_q is (β_q+1)-reflecting in $\langle j_{0,2k+2}^q(i_{\emptyset,q}(U))\rangle^\frown j_{0,2k+2}^q(\mathbf{s}_q)$ relative to κ in $\breve{N}_q^q$.

(ii′) and (iii′) are just our inductive conditions (ii) and (iii) for q.

The inequality $\hat{\xi} < \alpha$ of the One-Step Lemma gives us that

$$(\forall u \in j_{0,2k+2}^q(X))\, g(u) + 1 < (j_{2k,2k+2}^q(f))(u \restriction k).$$

This in turn implies that

$$\beta_q + 1 < \breve{i}_{p,q}^q(j_{2k,2k+2}^q(\beta_p)).$$

Since $\bar{\beta}^q_{k+1} = \beta_q$ and $\bar{\beta}^q_k = j^q_{2k,2k+2}(\beta_p)$, condition (iv) for q holds in the case $m = k$ and $m' = k+1$. To verify condition (iv) for q for $m < m' \leq k$, note that $j^p_{2k,2k+2}(\bar{\beta}^p_m) = \bar{\beta}^q_m$, $j^p_{2k,2k+2}(\bar{\beta}^p_{m'}) = \bar{\beta}^q_{m'}$, and $j^q_{2k,2k+2}(\breve{\imath}^p_{p\restriction m,p\restriction m'}) = \breve{\imath}^q_{q\restriction m,q\restriction m'}$. By these facts, by condition (iv) for p, and by the elementarity of $j^q_{2k,2k+2}$,

$$\bar{\beta}^q_{m'} = j^q_{2k,2k+2}(\bar{\beta}^p_{m'}) < j^q_{2k,2k+2}(\breve{\imath}^p_{p\restriction m,p\restriction m'}(\bar{\beta}^p_m)) = \breve{\imath}^q_{p\restriction m,p\restriction m'}(\bar{\beta}^q_m).$$

Condition (iv) for q holds for $m < k$ and $m' = k+1$ because

$$\bar{\beta}^q_{k+1} < \breve{\imath}^q_{p\restriction k,p\restriction k+1}(\bar{\beta}^q_k) < \breve{\imath}^q_{p\restriction k,p\restriction k+1}(\breve{\imath}^q_{p\restriction m,p\restriction k}(\bar{\beta}^q_m)) = \breve{\imath}^q_{p\restriction m,p\restriction k+1}(\bar{\beta}^q_m).$$

We have already verified for q the first clause of condition (v) and the case $m = k$ of the second clause of that condition. The other cases of the second clause follow easily from the corresponding cases for for p and the case $m = k$ for q.

Because the One-Step Lemma gives that $\eta < \delta^*$, we have that

$$\delta_p < \delta'_q < \delta_q.$$

Now $\mathrm{crit}\,(E^q_{2k}) = \delta_p$ and $\mathrm{crit}\,(E^q_{2k+1}) = \delta'_q$. Since condition (vi) holds for p and (in the case $p = \emptyset$) since $\gamma < \delta_\emptyset$, it follows that condition (vi) holds for q.

This completes our construction and the verification that it has the desired properties.

We will show that the system

$$(\langle M^p_{2\ell\mathrm{h}(p)} \mid p \in T\rangle, \langle j^p_{m,2\ell\mathrm{h}(p)} \mid p \in T \,\wedge\, m < \ell\mathrm{h}(p) \in T\rangle)$$

gives an embedding normal form for the T-projection of $U^\dagger$.

Fix $x \in [T]$. Let $\mathcal{C}_x$ be the alternating chain of length ω whose restrictions are the $\mathcal{C}_{x\restriction n}$.

We must show that $[U^\dagger(x)] \neq \emptyset$ if and only if $\tilde{\mathcal{M}}^{\mathcal{C}_x}_{\mathrm{Even}}$ is wellfounded.

Assume first that $[U^\dagger(x)] \neq \emptyset$. By Theorem 8.1.1, $[U(x)] = \emptyset$, i.e., $U(x)$ is a wellfounded tree. For each $k \in \omega$, let

$$\begin{aligned} \xi_{2k} &= j^{\mathcal{C}_x}_{0,2k}(\|s_\emptyset\|^{U(x)}); \\ \xi_{2k+1} &= \|s_{x\restriction k+1}\|^{j^{\mathcal{C}_x}_{0,2k+1}(U(x))}. \end{aligned}$$

Condition (v) of our construction implies that, for all m and $n \in \omega$ with $m < n$,

$$\begin{aligned} j^{\mathcal{C}_x}_{2m\dot{-}1,2n\dot{-}1}(\xi_{2m\dot{-}1}) &= \|j^{\mathcal{C}_x}_{2m\dot{-}1,2n\dot{-}1}(s_{x\restriction m})\|^{j^{\mathcal{C}_x}_{0,2n\dot{-}1}(U(x))} \\ &> \|s_{x\restriction n}\|^{j^{\mathcal{C}_x}_{0,2n\dot{-}1}(U(x))} \\ &= \xi_{2n\dot{-}1}. \end{aligned}$$

Thus the sequence $\langle \xi_n \mid n \in \omega\rangle$ witnesses that $\mathcal{C}_x$ is continuously illfounded off Even. Since Lemma 8.2.5 implies that $\mathcal{C}_x$ is plus one, Corollary 7.4.6, gives that $\tilde{M}^{\mathcal{C}_x}_{\mathrm{Even}}$ is wellfounded.

Now assume that $\tilde{M}^{\mathcal{C}_x}_{\mathrm{Even}}$ is wellfounded. For $m \in \omega$, $p \in T$ with $p \subseteq x$, and $q \subseteq q' \in T$, let

$$\begin{aligned} \bar{\beta}^x_m &= = j^{\mathcal{C}_x}_{0,\mathrm{Even}}(\beta_{x\restriction m}); \\ \breve{\imath}^x_{q,q'} &= j^{\mathcal{C}_x}_{0,\mathrm{Even}}(i_{q,q'}); \\ \breve{\imath}^x_p &= j^{\mathcal{C}_x}_{0,\mathrm{Even}}(i^x_p). \end{aligned}$$

Condition (iv) gives that, for all $m < m' \leq k \in \omega$,

$$\bar{\beta}^{x\restriction k}_{m'} < \breve{\imath}^{x\restriction k}_{x\restriction m, x\restriction m'}(\bar{\beta}^{x\restriction k}_m).$$

Applying $j^{\mathcal{C}_x}_{2k,\mathrm{Even}}$ to both sides of this inequality, we find that

$$\bar{\beta}^x_{m'} < \breve{\imath}^x_{x\restriction m, x\restriction m'}(\bar{\beta}^x_m).$$

Applying $\breve{\imath}^x_{x\restriction m'}$ to both sides, we get that

$$\breve{\imath}^x_{x\restriction m'}(\bar{\beta}^x_{m'}) < \breve{\imath}^x_{x\restriction m}(\bar{\beta}^x_m).$$

Thus the $\breve{\imath}^x_{x\restriction m}(\bar{\beta}^x_m)$, $m \in \omega$, form an infinite descending chain in the ordinals of the model $j^{\mathcal{C}_x}_{0,\mathrm{Even}}(\tilde{\mathcal{M}}_x)$, and so that model is illfounded. It follows by absoluteness that $\tilde{\mathcal{M}}_x$ is illfounded. This means that $[U(x)] = \emptyset$. Therefore $[U^\dagger(x)] \neq \emptyset$.

Remark. We did not really have to give the argument of the preceding paragraph, for our proof of the theorem will not use the fact that $[U^\dagger(x)] \neq \emptyset$ when $\mathcal{M}^{\mathcal{C}_\S}_{\mathrm{Even}}$ is wellfounded. The converse—proved two paragraphs ago—will, however, be an essential ingredient in our proof of the theorem.

To complete the proof of the theorem, let α be any ordinal such that $\beta_p < \alpha$ for all $p \in T$.

For $p \in T$, set

$$\mathcal{V}_p = \{X \subseteq (U^\dagger \restriction \alpha)[p] \mid \langle \bar{\beta}^p_m \mid m < \ell\mathrm{h}(p)\rangle \in j^p_{0,2\ell\mathrm{h}(p)}(X)\}.$$

We will show that $\langle \mathcal{V}_p \mid p \in T\rangle$ witnesses that $U^\dagger \restriction \alpha$ is γ-homogenous.

We first prove that clause (1) of the definition of homogeneity holds—that each $\mathcal{V}_p$ is a γ-complete ultrafilter on $(U^\dagger \restriction \alpha)[p]$. Let $p \in T$. We begin by proving that $(U^\dagger \restriction \alpha)[p] \in \mathcal{V}_p$, i.e., that

(i) $(\forall m < \ell\mathrm{h}(p))\, \bar{\beta}^p_m < j^p_{0,2\ell\mathrm{h}(p)}(\alpha)$;

(ii) $\langle \bar{\beta}^p_m \mid m < \ell\mathrm{h}(p)\rangle \in j^p_{0,2\ell\mathrm{h}(p)}(U^\dagger[p])$.

For (i), let $m < \ell\mathrm{h}(p)$. Then

$$\bar{\beta}^p_m = j^p_{2m,2\ell\mathrm{h}(p)}(\beta_{p\restriction m}) < j^p_{2m,2\ell\mathrm{h}(p)}(\alpha) \leq j^p_{0,2\ell\mathrm{h}(p)}(\alpha).$$

By the definition of $U^\dagger$, what we must show to prove (ii) is that, for all m and m' such that $m < m' < \ell\mathrm{h}(p)$,

$$\bar{\beta}^p_{m'} < (j^p_{0,2\ell\mathrm{h}(p)}(i_{p\restriction m, p\restriction m'}))(\bar{\beta}^p_m).$$

But this follows directly from condition (iv) of our construction.

Since $(U^\dagger \restriction \alpha)[p] \in \mathcal{V}_p$, it follows by Lemma 6.1.1 that $\mathcal{V}_p$ is an ultrafilter on $(U^\dagger \restriction \alpha)[p]$.

By condition (vi), $\gamma \leq \operatorname{crit}(j^p_{0,2\ell\mathrm{h}(p)})$. Hence Lemma 6.1.1 yields that $\mathcal{V}_p$ is γ-complete.

To check clause (2) in the definition of homogeneity, let $p \subseteq q \in T$. We must verify that

$$(\forall X \in \mathcal{V}_p)\, \{t \in U^\dagger[q] \mid t \restriction \ell\mathrm{h}(p) \in X\} \in \mathcal{V}_q.$$

Let $X \subseteq U^\dagger[p]$. Then

$$\begin{aligned}
&\{t \in U^\dagger[q] \mid t \restriction \ell\mathrm{h}(p) \in X\} \in \mathcal{V}_q\\
&\quad\leftrightarrow\ \langle \bar{\beta}^q_m \mid m < \ell\mathrm{h}(q)\rangle \restriction \ell\mathrm{h}(p) \in j^q_{0,2\ell\mathrm{h}(q)}(X)\\
&\quad\leftrightarrow\ \langle j^p_{2\ell\mathrm{h}(p),2\ell\mathrm{h}(q)}(\bar{\beta}^p_m) \mid m < \ell\mathrm{h}(p)\rangle \in j^q_{2\ell\mathrm{h}(p),2\ell\mathrm{h}(q)}(j^p_{0,2\ell\mathrm{h}(p)}(X))\\
&\quad\leftrightarrow\ \langle \bar{\beta}^p_m \mid m < \ell\mathrm{h}(p)\rangle \in j^p_{0,2\ell\mathrm{h}(p)}(X))\\
&\quad\leftrightarrow\ X \in \mathcal{V}_p.
\end{aligned}$$

To complete the proof, we verify clause (3′) in the definition of homogeneity. For $p \subseteq q \in T$, let $i^\dagger{}_{p,q} : \mathrm{Ult}(V;\mathcal{V}_p) \prec \mathrm{Ult}(V;\mathcal{V}_q)$ be the canonical elementary embedding. Fix $x \in [T]$. Let $(\mathcal{M}^\dagger{}_x, \langle i^{\dagger x}_{x\restriction n} \mid n \in \omega\rangle)$ be the direct limit of the system $(\langle \mathrm{Ult}(V;\mathcal{V}_{x\restriction n}) \mid n \in \omega\rangle, \langle i^\dagger{}_{x\restriction m, x\restriction n} \mid m \leq n \in \omega\rangle)$.

Assume that x belongs to the T-projection of $U^\dagger$. We have already shown that $\tilde{\mathcal{M}}^{\mathcal{C}_x}_{\mathrm{Even}}$ is wellfounded for any such x. We must show that $\mathcal{M}^\dagger{}_x$ is wellfounded. It will be sufficient for us to prove that $\mathcal{M}^\dagger{}_x$ can be elementarily embedded into $\tilde{\mathcal{M}}^{\mathcal{C}_x}_{\mathrm{Even}}$.

For $m \in \omega$ and $\pi_{\mathcal{V}_{x\restriction n}}([\![f]\!]_{\mathcal{V}_{x\restriction n}}) \in \mathrm{Ult}(V;\mathcal{V}_{x\restriction n})$, set

$$k_n(\pi_{\mathcal{V}_{x\restriction n}}([\![f]\!]_{\mathcal{V}_{x\restriction n}})) = (j^{\mathcal{C}_x}_{0,2n}(f))(\langle \bar{\beta}^{x\restriction n}_m \mid m < n\rangle).$$

To see that k_n is well-defined, assume that $[\![f]\!]_{\mathcal{V}_{x\restriction n}} = [\![g]\!]_{\mathcal{V}_{x\restriction n}}$. Then

$$\{t \in U^\dagger[x \restriction n] \mid f(t) = g(t)\} \in \mathcal{V}_{x\restriction n},$$

and the definition of $\mathcal{V}_{x\restriction n}$ gives that

$$(j^{\mathcal{C}_x}_{0,2n}(f))(\langle \bar{\beta}^{x\restriction n}_m) \mid m < n\rangle) = (j^{\mathcal{C}_x}_{0,2n}(g))(\langle \bar{\beta}^{x\restriction n}_m) \mid m < n\rangle),$$

and so that $k_n(\pi_{\mathcal{V}_{x\restriction n}}([\![f]\!]_{\mathcal{V}_{x\restriction n}}) = \pi_{\mathcal{V}_{x\restriction n}}([\![g]\!]_{\mathcal{V}_{x\restriction n}})$. A similar argument shows that $k_n : \mathrm{Ult}(V;\mathcal{V}_{x\restriction n}) \prec M^{\mathcal{C}_x}_{2n}$. Furthermore, if $m \leq n \in \omega$ and $\pi_{\mathcal{V}_{x\restriction m}}([\![f]\!]_{\mathcal{V}_{x\restriction m}}) \in \mathrm{Ult}(V;\mathcal{V}_{x\restriction m})$, then

$$\begin{aligned}
&j^{\mathcal{C}_x}_{2m,2n}(k_m(\pi_{\mathcal{V}_{x\restriction m}}([\![f]\!]_{\mathcal{V}_{x\restriction m}})))\\
&\quad= (j^{\mathcal{C}_x}_{2m,2n}((j^{\mathcal{C}_x}_{0,2m}(f))(\langle \bar{\beta}^{x\restriction m}_{m'}) \mid m' < m\rangle))\\
&\quad= (j^{\mathcal{C}_x}_{0,2n}(f))(\langle \bar{\beta}^{x\restriction n}_{m'}) \mid m' < m\rangle))\\
&\quad= (j^{\mathcal{C}_x}_{0,2n}(f))(\langle \bar{\beta}^{x\restriction n}_{m'})) \mid m' < n\rangle \restriction m))\\
&\quad= k_n(i^\dagger{}_{x\restriction m, x\restriction n}(\pi_{\mathcal{V}_{x\restriction m}}([\![f]\!]_{\mathcal{V}_{x\restriction m}}))).
\end{aligned}$$

This argument shows that

$$j^{\mathcal{C}_x}_{2m,2n} \circ k_m = k_n \circ i^\dagger{}_{x\restriction m, x\restriction n}.$$

Thus we can define an elementary embedding $k : \mathcal{M}^\dagger{}_x \prec \tilde{\mathcal{M}}^{\mathcal{C}_x}_{\mathrm{Even}}$ by setting

$$k(i^{\dagger x}_{x\restriction n}(z)) = \tilde{\jmath}^{\mathcal{C}_x}_{n,\mathrm{Even}}(k_n(z))$$

for $z \in \mathrm{Ult}(V;\mathcal{V}_{x\restriction m})$. □

We now turn to the $U^\ddagger$ construction.

Assume that $\langle \mathcal{U}_{p,r} \mid \langle\!| p, r |\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$ witnesses that U is weakly κ^+-homogeneous for T. Let $\langle \pi_{p,r} \mid \langle\!| p, r |\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$, $\langle i_{\langle p,r\rangle,\langle q,s\rangle} \mid \langle\!| p, r |\!\rangle \subseteq \langle\!| q, s |\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$, and $(\mathcal{M}_{x,y}; \langle i^{x,y}_{\langle x \restriction n, y \restriction n\rangle} \mid n \in \omega\rangle)$ be as on page 423. Let $i \mapsto r_i$ be the function introduced on page 430. Let $U^\ddagger = U^\ddagger(\langle U_{p,r} \mid \langle\!| p, r |\!\rangle \in T \otimes {}^{<\omega}\omega\rangle)$.

To show that the T-projection of $U^\ddagger$ has an embedding normal form (indeed that $U^\ddagger$ is homogeneous), we will build, for each $x \in [T]$, a plus one iteration tree $\mathcal{S}_x$ of length ω. The set of all branches of $\mathcal{S}_x$ will be $\{\text{Even}\} \cup \{b_y \mid y \in {}^{<\omega}\omega\}$, where $\text{Even} = \{2m \mid m \in \omega\}$ as before and

$$b_y = \{2k \dot{-} 1 \mid k \in \omega \wedge r_k \subseteq y\}.$$

We will arrange that $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\text{Even}}$ is wellfounded if and only if $[U(x)] = \emptyset$.

To guarantee that $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\text{Even}}$ is wellfounded when $[U(x)] = \emptyset$, we will make sure that $\mathcal{S}_x$ is continuously illfounded off Even if $[U(x)] = \emptyset$. To do this we will make sure, in a sufficiently continuous fashion, that the trees $\tilde{\jmath}^{\mathcal{S}_x}_{0,b_y}(U(x))$ are illfounded for every $x \in [T]$ and every $y \in {}^{\omega}\omega$. This will be done with the aid of objects s_p, $p \in T$, that will play a role similar to the role played by the s_p of the proof of Theorem 8.2.6. Each $s_{x\restriction k}$ will belong to $j^{\mathcal{S}_x}_{0,2k\dot{-}1}(U[x \restriction \ell\text{h}(r_k)])$, and whenever $r_k \subseteq r_{k'}$ then we will have $j^{\mathcal{S}_x}_{2k\dot{-}1,2k'\dot{-}1}(s_{x\restriction k}) \subseteq s_{x\restriction k'}$. To get the s_p, we will use elements $\mathbf{s}_{q,r}$ of of $i_{\langle\emptyset,\emptyset\rangle,\langle q,r\rangle}(U[q])$ that are analogous to the $\mathbf{s}_p$ of the proof of Theorem 8.2.6. From the $\mathbf{s}_{q,r}$ we will get elements $j^{\mathcal{S}_x}_{0,2k}(\mathbf{s}_{p\restriction \ell\text{h}(r_k),r_k})$ of $(j^{\mathcal{S}_x}_{0,2k}(i_{\langle\emptyset,\emptyset\rangle,\langle p\restriction \ell\text{h}(r_k),r_k\rangle}(U[x \restriction \ell\text{h}(r_k)]))$, and from these elements we will get the s_p with the aid of the One-Step Lemma.

To arrange that when $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\text{Even}}$ is wellfounded then $[U(x)] = \emptyset$, we will use ordinals β_p, $p \in T$. For each $y \in {}^{\omega}\omega$, the $\beta_{x\restriction k}$, $r_k \subseteq y$, will give rise to an infinite descending chain of ordinals of $\tilde{\jmath}^{\mathcal{S}_x}_{0,\text{Even}}(i^{x,y}_{\langle\emptyset,\emptyset\rangle}(V))$. When $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\text{Even}}$ is wellfounded, these chains will show that $\mathcal{M}_{x,y}$ is illfounded for every $y \in {}^{\omega}\omega$ and hence that $[U(x)] = \emptyset$.

The β_n will generate ultrafilters witnessing the homogeneity of $U^\ddagger$ in pretty much the same way that the corresponding ordinals performed the analoguous task in the proof of Theorem 8.2.7.

Theorem 8.2.7. *Assume that κ is Woodin and that $\langle \mathcal{U}_{p,r} \mid \langle\!| p, r |\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$ witnesses that U is κ^+-homogeneous for T. Let $U^\ddagger = U^\ddagger(\langle \mathcal{U}_{p,r} \mid \langle\!| p, r |\!\rangle \in T \otimes {}^{<\omega}\omega\rangle)$ Then, for every sufficiently large ordinal α, $U^\ddagger \restriction \alpha$ is $(< \kappa)$-homogeneous for T.*

Proof. Let $\gamma < \kappa$ be such that $T \in V_\gamma$.

Let $\langle \pi_{p,r} \mid \langle\!\langle p, r \rangle\!\rangle \in T \otimes {}^{<\omega}\omega \rangle$, $\langle i_{\langle p,r \rangle, \langle q,s \rangle} \mid \langle\!\langle p, r \rangle\!\rangle \subseteq \langle\!\langle q, s \rangle\!\rangle \in T \otimes {}^{<\omega}\omega \rangle$, $(\mathcal{M}_{x,y}; \langle i^{x,y}_{\langle x \restriction n, y \restriction n \rangle} \mid n \in \omega \rangle)$, and $i \mapsto r_i$ be as in the discussion preceding the statement of the theorem. For $\langle\!\langle p, r \rangle\!\rangle \in T \otimes {}^{<\omega}\omega$, let $\mathbf{s}_{p,r} = \pi_{\mathcal{U}_{p,r}}([\![\mathrm{id}]\!]_{\mathcal{U}_{p,r}})$.

Let S be the tree ordering of ω defined as follows:

(i) $0 \, S \, n$ for every $n > 0$;

(ii) $2m \, S \, 2n$ if $m < n$;

(iii) $2m + 1 \, S \, 2n + 1$ if $r_{m+1} \subsetneq r_{n+1}$;

(iv) $m \, S \, n$ only if (i), (ii), or (iii) requires that $m \, S \, n$.

Note that the branches of an iteration tree whose tree ordering is $\mathcal{S}$ are just Even and the b_y, $y \in {}^\omega\omega$, defined above.

We will define, by induction on $p \in T$, objects δ_p, β_p, $\mathcal{S}_p$, and s_p. Both δ_p and β_p will be ordinals, with $\delta_p < \kappa$. s_p will be a sequence such that $\ell\mathrm{h}(s_p) = \ell\mathrm{h}(r_{\ell\mathrm{h}(p)})$. $\mathcal{S}_p$ will be an iteration tree of length $2\ell\mathrm{h}(p) + 1$ on V. Its tree ordering will be the restriction of S. Its extenders will be E^p_m, $m < 2\ell\mathrm{h}(p)$, its models will be M^p_m $m \le 2\ell\mathrm{h}(p)$, and its embeddings will be $j^p_{m,n}$, $m \, S \, n \le 2\ell\mathrm{h}(p)$. Whenever $p \subseteq q \in T$ then we will have $\mathcal{S}_p = \mathcal{S}_q \restriction 2\ell\mathrm{h}(p) + 1$.

To simplify notation, we make some definitions. Let p and q belong to T and let $m \le n \le \ell\mathrm{h}(q)$ such that $r_m \subseteq r_n$. Let

$$
\begin{aligned}
i_{q;m,n} &= i_{\langle q \restriction \ell\mathrm{h}(r_m), r_m \rangle, \langle q \restriction \ell\mathrm{h}(r_n), r_n \rangle};\\
\breve{\imath}^p_{q;m,n} &= j^p_{0,2\ell\mathrm{h}(p)}(i_{q;m,n});\\
\breve{N}^p_{q;m} &= \breve{\imath}^p_{q;0,m}(M^p_{2\ell\mathrm{h}(p)});\\
\breve{U}^p_{q;m} &= j^p_{0,2\ell\mathrm{h}(p)}(i_{q;0,m}(U));\\
\bar{\beta}^q_m &= j^q_{2m,2\ell\mathrm{h}(q)}(\beta_{q \restriction m}));\\
\mathbf{s}^q_m &= \mathbf{s}_{q \restriction \ell\mathrm{h}(r_m), r_m}.
\end{aligned}
$$

The embedding $\breve{\imath}^p_{q;m,n}$ is the image of $i_{q;m,n}$ in $M^p_{2\ell\mathrm{h}p}$. The class model $\breve{N}^p_{q;m}$ is the image of $\mathrm{Ult}(V; \mathcal{U}_{q \restriction \ell\mathrm{h}(r_m), r_m})$ in $M^p_{2\ell\mathrm{h}(p)}$. In other words,

$$\breve{N}^p_{q;m} = j^p_{0,2\ell\mathrm{h}(p)}(i_{q;0,m}(V)).$$

The tree $\breve{U}^p_{q;m}$ is the image of $i_{q;0,m}(U)$ in $M^p_{2\ell\mathrm{h}(p)}$; i.e., it is the image of U in $\breve{N}^p_{q;m}$

For $p = \emptyset$ we have only to define $\delta_\emptyset$ and $\beta_\emptyset$. (Note that $s_\emptyset = \emptyset$; for $\ell\mathrm{h}(s_\emptyset) = \ell\mathrm{h}(r_0) = \ell\mathrm{h}(\emptyset)$, because of the property of $i \mapsto r_i$ stated on page 430.) Choose $\delta_\emptyset > \gamma$ to be $\zeta_0 + 1$-reflecting in $\langle U \rangle$ relative to κ. Let $\beta_\emptyset = \zeta_0$.

Let $p \in T$. Let $k = \ell\mathrm{h}(p)$. Assume that $\delta_{p'}$, $\beta_{p'}$, $\mathcal{S}_{p'}$, and $s_{p'}$ are defined for all $p' \subseteq p$ so as to satisfy the conditions stated above and also the following conditions:

(i) for all $m \leq k$, M^p_{2k} and $M^p_{2m \dot- 1}$ agree through $\delta_{p \restriction m} + 1$;

(ii) for all $m \leq k$, the type $(\mathrm{tp}^{\delta_{p\restriction m}}_{\kappa, \bar\beta^p_m + 1})^{\breve N^p_{p;m}}(\langle \breve U^p_{p;m} \rangle {}^\frown j^p_{0,2k}(\mathbf{s}^p_m))$ is the same as $(\mathrm{tp}^{\delta_{p\restriction m}}_{\kappa, \zeta_0 + 1})^{M^p_{2m \dot- 1}}(\langle j^p_{0,2m \dot- 1}(U) \rangle {}^\frown s_{p \restriction m})$;

(iii) for all $m \leq k$, $\delta_{p\restriction m}$ is $(\bar\beta^p_m + 1)$-reflecting in $\langle \breve U^p_{p;m} \rangle {}^\frown j^p_{0,2k}(\tilde{\mathbf{s}}^p_m)$ relative to κ in $\breve N^p_{p;m}$;

(iv) for all m and m with $m < m' \leq k$, if $r_m \subseteq r_{m'}$ then

$$\bar\beta_{p\restriction m'} < \breve\imath^p_{p;m,m'}(\bar\beta^p_m);$$

(v) for all $m \leq k$, $s_{p\restriction m}$ belongs to $(j^p_{0,2m \dot- 1})(U[p \restriction \ell\mathrm{h}(r_m)])$, and

$$r_m \subseteq r_k \to j^p_{2m \dot- 1, 2k \dot- 1}(s_{p\restriction m}) \subseteq s_p);$$

(vi) for all $m < k$,

$$\gamma < \delta_{p\restriction m} < \mathrm{crit}\,(E^p_{2m+1}) < \delta_{p\restriction m+1},$$

and $\mathrm{crit}\,(E^p_{2m}) = \delta_{p\restriction \bar m}$, where $2\bar m \dot- 1 = (2m+1)^-_S$.

Note that these conditions all hold for for $p = \emptyset$.

Remark. The first seven conditions have pretty much the same roles as in the proof of Theorem 8.2.6. Indeed, the present construction can be thought of as a whole tree of constructions, each one like the construction of the proof of that lemma. Condition (vi), besides doing the work of the old condition (vi), will guarantee that these constructions do not conflict with one another. Condition (vi) will also be used in proving that our iteration trees $\mathcal{S}_x$ are plus one.

Let q be any element of T such that $p \subseteq q$ and $\ell\mathrm{h}(q) = k + 1$.

Let n be the largest number $\leq k$ such that $r_n \subseteq r_{k+1}$. Let $e = \ell\mathrm{h}(r_n)$. By the property of $i \mapsto r_i$ stated on page 430, $\ell\mathrm{h}(r_{k+1}) = e + 1$.

By (i) and the fact that $\mathrm{crit}\,(\breve{\imath}^p_{q;0,k+1}) > \kappa$, it follows that $\breve{N}^p_{q;k+1}$ and $M^p_{2n\dot{-}1}$ agree through $\delta_{p\restriction n} + 1$.

Note that

$$\begin{aligned}\breve{N}^p_{q;k+1} &= \breve{\imath}^p_{q;0,k+1}(M^p_{2k})\\ &= \breve{\imath}^p_{q;n,k+1}(\breve{\imath}^p_{q;0,n}(M^p_{2k}))\\ &= \breve{\imath}^p_{q;n,k+1}(\breve{N}^p_{p;n}),\end{aligned}$$

that

$$\begin{aligned}\breve{U}^p_{q;k+1} &= j^p_{0,2k}(i^p_{q;0,k+1}(U))\\ &= j^p_{0,2k}(i^p_{q;n,k+1}(i^p_{q;0,n}(U)))\\ &= (j^p_{0,2k}(i^p_{q;n,k+1}))(j^p_{0,2k}(i^p_{q;0,n}(U)))\\ &= \breve{\imath}^p_{q;n,k+1}(\breve{U}^p_{p;n}),\end{aligned}$$

and that

$$\begin{aligned}j^p_{0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n)) &= (j^p_{0,2k}(i_{q;n,k+1}))(j^p_{0,2k}(\mathbf{s}^p_n))\\ &= \breve{\imath}^p_{q;n,k+1}(j^p_{0,2k}(\mathbf{s}^p_n)).\end{aligned}$$

Since $\mathrm{crit}\,(\breve{\imath}^p_{q;n,k+1}) > \kappa$, it follows that

$$\begin{aligned}&(\mathrm{tp}^{\delta_{p\restriction n}}_{\kappa,\bar{\beta}^p_n+1})^{\breve{N}^p_{p;n}}(\langle \breve{U}^p_{p;n}\rangle ^\frown j^p_{0,2k}(\mathbf{s}^p_n))\\ &= \breve{\imath}^p_{q;n,k+1}((\mathrm{tp}^{\delta_{p\restriction n}}_{\kappa,\bar{\beta}^p_n+1})^{\breve{N}^p_{p;n}}(\langle \breve{U}^p_{p;n}\rangle ^\frown j^p_{0,2k}(\mathbf{s}^p_n)))\\ &= (\mathrm{tp}^{\delta_{p\restriction n}}_{\kappa,\breve{\imath}^p_{q;n,k+1}(\bar{\beta}^p_n)+1})^{\breve{N}^p_{q;k+1}}(\langle \breve{U}^p_{q;k+1}\rangle ^\frown j^p_{0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n)))\end{aligned}$$

and so by (ii) this last is the same as $(\mathrm{tp}^{\delta_{p\restriction n}}_{\kappa,\zeta_0+1})^{M^p_{2n\dot{-}1}}(\langle j^p_{0,2n\dot{-}1}(U)\rangle ^\frown s_{p\restriction n})$.

From (iii) it similarly follows that $\delta_{p\restriction n}$ is $(\breve{\imath}^p_{q;n,k+1}(\bar{\beta}^p_n+1))$-reflecting in the sequence $\langle \breve{U}^p_{q;k+1}\rangle ^\frown j^p_{0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n))$ relative to κ in $\breve{N}^p_{q;k+1}$.

Since $j^p_{0,2k}$ and $\breve{\imath}^p_{q;0,k+1}$ fix κ, we have that κ is Woodin in $\breve{N}^p_{q;k+1}$.

Thus the hypotheses of the One-Step Lemma hold for κ with

$$\begin{aligned}M &= \breve{N}^p_{q;k+1};\\ N &= M^p_{2n\dot{-}1};\end{aligned}$$

$$\begin{array}{rcl}
\delta & = & \delta_{p\restriction n};\\
\eta & = & \delta_p;\\
\beta & = & \breve{\imath}^p_{q;n,k+1}(\bar{\beta}^p_n)+1;\\
\xi & = & \breve{\imath}^p_{q;n,k+1}(\bar{\beta}^p_n);\\
\beta' & = & \zeta_0+1;\\
x & = & \langle \breve{U}^p_{q;k+1}\rangle^\frown j^p_{0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n));\\
y & = & \langle (j^p_{0,2k}(\mathbf{s}^q_{k+1}))(e)\rangle;\\
x' & = & \langle j^p_{0,2n\dot{-}1}(U)\rangle^\frown s_{p\restriction n};\\
\chi(v) & = & \text{“}\kappa+v \text{ is the greatest ordinal.”}
\end{array}$$

Let λ and E be given by the One-Step Lemma. Since $\breve{\imath}^p_{q;0,k+1}$ fixes λ, E, and δ_p, it follows that E is a $(\delta_{p\restriction n},\lambda)$-extender in M^p_{2k}. Thus Theorem 7.3.2 gives that $\prod_E^{M^p_{2n\dot{-}1}}(M^p_{2n\dot{-}1};\in)$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. By clause (4*) of the One-Step Lemma, $\xi^*=\zeta_0$. Extend $\mathcal{S}_p$ to an alternating chain that will be $\mathcal{S}_q\restriction 2k+2$ by setting $E^q_{2k}=E$. The ordinal δ^* we will call δ'_q. Set $s_q=(j^q_{2n\dot{-}1,2k+1}(s_p))^\frown y^*$.

Note that $M^q_m=M^p_m$, $E^q_m=E^p_m$, $j^q_{m,m'}=j^p_{m,m'}$, and $i^q_{m,m'}=i^p_{m,m'}$ whenever these equations make sense. We will use these identities without comment in the sequel.

We have that

$$x^\frown y=\langle \breve{U}^p_{q;k+1}\rangle^\frown j^p_{0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n))^\frown\langle (j^p_{0,2k}(\mathbf{s}^q_{k+1}))(e)\rangle.$$

Observe that

$$\begin{array}{rcl}
i_{q;n,k+1}(\mathbf{s}^p_n)) & = & i_{q;n,k+1}(\mathbf{s}_{p\restriction e,r_n})\\
& = & \mathbf{s}_{q\restriction e+1,r_{k+1}}\restriction e\\
& = & \mathbf{s}^q_{k+1}\restriction e.
\end{array}$$

Thus $x^\frown y$ is the concatenation of $\langle \breve{U}^p_{q;k+1}\rangle$ and

$$j^p_{0,2k}(\mathbf{s}^q_{k+1}\restriction e)^\frown(j^p_{0,2k}(\mathbf{s}^q_{k+1}))(e)\rangle.$$

By the elementarity of $j^p_{0,2k}$, we finally get that

$$x^\frown y=\langle \breve{U}^p_{q;k+1}\rangle^\frown j^p_{0,2k}(\mathbf{s}^q_{k+1}).$$

Now $\mathbf{s}^q_{k+1} = \mathbf{s}_{q\restriction e+1, r_{k+1}}$ belongs to $i^q_{0,k+1}(U[q\restriction e+1])$. Therefore the sequence $j^p_{0,2k}(\mathbf{s}^q_{k+1})$ belongs to $j^p_{0,2k}(i^q_{0,k+1}(U[q\restriction e+1])) = \breve{U}^p_{q;k+1}$. It follows by clause (2*) of the One-Step Lemma that $s_q \in j^q_{0,2k+1}(U[q\restriction e+1])$, and so the first clause of condition (v) holds for q. Since $j^q_{2n\dot{-}1,2k+1}(s_{q\restriction n}) \subseteq s_q$, the second clause of condition (v) holds for q the case $m = n$.

We have that

(a) M^q_{2k+1} and $\breve{N}^p_{q;k+1}$ agree through $\delta'_q + 1$;

(b) $(\mathrm{tp}^{\delta'_q}_{\kappa,\zeta_0})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q) =$
$(\mathrm{tp}^{\delta'_q}_{\kappa,\breve{\imath}^p_{q;n,k+1}(\bar{\beta}^p_n)})^{\breve{N}^p_{q;k+1}}(\langle \breve{U}^p_{q;k+1}\rangle^\frown j^p_{0,2k}(\mathbf{s}^q_{k+1}))$;

(c) δ'_q is ζ_0-reflecting in $\langle j^q_{0,2k+1}(U)\rangle^\frown s_q$ relative to κ in M^q_{2k+1}.

By (b) and (c) together with parts (b) and (c) of Lemma 8.2.3, we have that

(b′) $(\mathrm{tp}^{\delta'_q}_{\kappa,\zeta_1})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q) =$
$(\mathrm{tp}^{\delta'_q}_{\kappa,\breve{\imath}^p_{q;n,k+1}(\bar{\beta}^p_n)})^{\breve{N}^p_{q;k+1}}(\langle \breve{U}^p_{q;k+1}\rangle^\frown j^p_{0,2k}(\mathbf{s}^q_{k+1}))$;

(c′) δ'_q is ζ_1-reflecting in $\langle j^q_{0,2k+1}(U)\rangle^\frown s_q$ relative to κ in M^q_{2k+1}.

Let $f : j^p_{0,2k}(U[p\restriction e]) \to \mathrm{Ord}$ belong to M^p_{2k} and be such that

$$\bar{\beta}^p_n = \pi^{M^p_{2k}}_{j^p_{0,2k}(\mathcal{U}_{p\restriction e,r_n})}([\![f]\!]^{M^p_{2k}}_{j^p_{0,2k}(\mathcal{U}_{p\restriction e,r_n})}).$$

In other words, let f be such that $\bar{\beta}^p_n = (\breve{\imath}^p_{p;0,n}(f))(j^p_{0,2k}(\mathbf{s}^p_n))$. By (b′) and the fact that $\mathrm{crit}\,(\breve{\imath}^p_{q;0,k+1}) > \delta'_q$, there is a set $X \in j^p_{0,2k}(\mathcal{U}_{q\restriction e+1,r_{k+1}})$ such that, for all $t \in X$,

(b″) $(\mathrm{tp}^{\delta'_q}_{\kappa,\zeta_1})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q) = (\mathrm{tp}^{\delta'_q}_{\kappa,f(t\restriction e)})^{M^p_{2k}}(\langle j^p_{0,2k}(U)\rangle^\frown t$.

Choose any element t of X. Since κ is Woodin in M^q_{2k+1}, the hypotheses of the One-Step Lemma hold for κ with

$$\begin{aligned} M &= M^q_{2k+1}; \\ N &= M^p_{2k}; \\ \delta &= \delta'_q; \\ \eta &= \delta'_q; \end{aligned}$$

$$\begin{aligned}
\beta &= \zeta_1;\\
\xi &= \zeta_0+1;\\
\beta' &= f(t\restriction e);\\
x &= \langle j^q_{0,2k+1}(U)\rangle^\frown s_q;\\
y &= \emptyset;\\
x' &= \langle j^p_{0,2k}(U)\rangle^\frown t;\\
\chi(v) &= \text{“}v=v.\text{”}
\end{aligned}$$

Let λ and E be given by the One-Step Lemma. By Theorem 7.3.2, the model $\prod_E^{M^p_{2k}}(M^p_{2k};\in)$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. (We will make no use of ξ^* and y^*.)

For all elements u of X,

$$(\mathrm{tp}^{\delta'_q}_{\kappa,f(u\restriction e)})^{M^p_{2k}}(\langle j^p_{0,2k}(U)\rangle^\frown u=(\mathrm{tp}^{\delta'_q}_{\kappa,f(t\restriction e)})^{M^p_{2k}}(\langle j^p_{0,2k}(U)\rangle^\frown t.$$

Thus the elementarity of $i_E^{M^p_{2k}}$ gives that, for all $u\in i_E^{M^p_{2k}}(X)$,

$$\begin{aligned}
&(\mathrm{tp}^{i_E^{M^p_{2k}}(\delta'_q)}_{\kappa,(i_E^{M^p_{2k}}(f))(u\restriction e)})^{\mathrm{Ult}(M^p_{2k};E)}(\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown u)\\
&\quad=(\mathrm{tp}^{i_E^{M^p_{2k}}(\delta'_q)}_{\kappa,(i_E^{M^p_{2k}}(f))(i_E^{M^p_{2k}}(t)\restriction e)})^{\mathrm{Ult}(M^p_{2k};E)}(\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown i_E^{M^p_{2k}}(t)).
\end{aligned}$$

Since $\delta^*_q<i_E^{M^p_{2k}}(\delta'_q)$, for all $u\in i_E^{M^p_{2k}}(X)$ we have that

$$\begin{aligned}
&(\mathrm{tp}^{\delta^*+1}_{\kappa,(i_E^{M^p_{2k}}(f))(u\restriction e)})^{\mathrm{Ult}(M^p_{2k};E)}(\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown u)\\
&\quad=(\mathrm{tp}^{\delta^*+1}_{\kappa,(i_E^{M^p_{2k}}(f))(i_E^{M^p_{2k}}(t)\restriction e)})^{\mathrm{Ult}(M^p_{2k};E)}(\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown i_E^{M^p_{2k}}(t)).
\end{aligned}$$

Let $u\in i_E^{M^p_{2k}}(X)$. We make an application of the last part of the One-Step Lemma, with $z=\langle i_E^{M^p_{2k}}(j^p_{0,2k}(U))\rangle^\frown u$ and with $\alpha=(i_E^{M^p_{2k}}(f))(u\restriction e)$. If $\hat\xi$ and $\hat y$ are as given given by this application, then clause $(\hat 2)$ of the One-Step Lemma implies that $\hat y=\emptyset$ and $\hat\xi$ is a successor ordinal. Let $g(u)$ be the least ordinal ν such that clauses $(\hat 2)$ and $(\hat 3)$ of the One-Step Lemma hold with $\hat\xi=\nu+1$ and $\hat y=\emptyset$.

Observe that the function $g: i_E^{M^p_{2k}}(X)\to\mathrm{Ord}$ belongs to $\mathrm{Ult}(M^p_{2k};E)$.

We finish the definition of $\mathcal{S}_q$ by setting $E^q_{2k+1} = E$. Thus $E^q_{2k+2} = \mathrm{Ult}(M^p_{2k}; E)$ and $j^q_{2k,2k+2} = i^{M^p_{2k}}_E$. Clause (1*) of the One-Step Lemma gives the case $k+1$ of inductive condition (i) for q .

Let $\delta_q = \delta^*$. Clauses $(\hat{2})$ and $(\hat{3})$ of the One-Step Lemma give that, for all $u \in j^q_{2k,2k+2}(X)$,

$(\hat{2})$ $(\mathrm{tp}^{\delta_q}_{\kappa,g(u)+1})^{M^q_{2k+2}}(\langle j^q_{0,2k+2}(U)\rangle^\frown u) = (\mathrm{tp}^{\delta_q}_{\kappa,\zeta_0+1})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q)$;

$(\hat{3})$ δ_q is $(g(u)+1)$-reflecting in $\langle j^q_{0,2k+2}(U)\rangle^\frown u$ relative to κ in M^q_{2k+2}.

The set $j^q_{2k,2k+2}(X)$ belongs to $j^q_{0,2k+2}(\mathcal{U}_{q\restriction e+1,r_{k+1}})$. This fact allows us to complete our definitions by setting

$$\beta_q = \pi^{M^q_{2k+2}}_{j^q_{0,2k+2}(\mathcal{U}_{q\restriction e+1,r_{k+1}})}([\![g]\!]^{M^q_{2k+2}}_{j^q_{0,2k+2}(\mathcal{U}_{q\restriction e+1,r_{k+1}})}).$$

Using Łoś's Theorem in M^q_{2k+2} and using the fact that

$$\mathbf{s}^q_{k+1} = \pi^{M^q_{2k+2}}_{j^q_{0,2k+2}(\mathcal{U}_{q\restriction e+1,r_{k+1}})}([\![\mathrm{id}]\!]^{M^q_{2k+2}}_{j^q_{0,2k+2}(\mathcal{U}_{q\restriction e+1,r_{k+1}})}),$$

we see that $(\hat{2})$ and $(\hat{3})$ imply that

(ii′) $(\mathrm{tp}^{\delta_q}_{\kappa,\beta_q+1})^{\breve{N}^q_{q;k+1}}(\langle j^q_{0,2k+2}(i^q_{0,k+1}(U))\rangle^\frown j^q_{0,2k+2}(\mathbf{s}_q))$
$= (\mathrm{tp}^{\delta_q}_{\kappa,\zeta_0+1})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q)$;

(iii′) δ_q is (β_q+1)-reflecting in $\langle j^q_{0,2k+2}(i_{\emptyset,q}(U))\rangle^\frown j^q_{0,2k+2}(\mathbf{s}_q)$ relative to κ in $\breve{N}^q_{q;k+1}$.

Observe that (ii′) and (iii′) are just the case $k+1$ of our inductive conditions (ii) and (iii) for q.

Let us now verify our all our inductive conditions for q. To verify (i), (ii), and (iii) for q, let $m \le k+1$ be arbitrary.

We have already noted that condition (i) for the case $m = k+1$ follows from clause (1*) of the One-Step Lemma. Suppose that $m \le k$. By condition (vi) for p, we have that $\delta_{q\restriction m} \le \delta_p$. Because the One-Step Lemma gives that $\eta < \delta^*$, we also have that

$$\delta_p < \delta'_q < \delta_q.$$

By condition (i) for p, M^q_{2k} and $M^q_{2m\dot{-}1}$ agree through $\delta_{p\restriction m}+1$. Since M^q_{2k+1} and M^q_{2k} agree through δ'_q+1, it follows that M^q_{2k+1} and $M^q_{2m\dot{-}1}$ agree through

$\delta_{p\restriction m}+1$. Since M^q_{2k+2} and M^q_{2k+1} agree through δ_q+1, we get that M^q_{2k+2} and $M^q_{2m\dot{-}1}$ agree through $\delta_{p\restriction m}+1=\delta_{q\restriction m}+1$.

We have already checked that (ii) and (iii) hold for the case $m=k+1$. Now let $m\leq k$. By the fact that $\operatorname{crit}(j^q_{2k,2(k+1)})=\delta'_q>\delta_{q\restriction m}$ and by our definitions, we have that

(1) $j^q_{2k,2(k+1)}(\delta_{q\restriction m})=\delta_{q\restriction m}$;

(2) $j^q_{2k,2(k+1)}(\bar\beta^p_m)=\bar\beta^q_m$;

(3) $j^q_{2k,2(k+1)}(\breve N^p_{p;m})=\breve N^q_{q;m}$;

(4) $j^q_{2k,2(k+1)}(\breve U^p_{p;m})=\breve U^q_{q;m}$.

The fact that $\operatorname{crit}(j^q_{2k,2(k+1)})>\delta_{q\restriction m}$ also implies that

$$(\mathrm{tp}^{\delta_{p\restriction m}}_{\kappa,\bar\beta^p_m+1})^{\breve N^p_{p;m}}(\langle\breve U^p_{p;m}\rangle^\frown j^p_{0,2k}(\mathbf{s}^q_m))$$

is fixed by $j^q_{2k,2(k+1)}$. By (1)–(4),

$$\begin{aligned}j^q_{2k,2k+2}&(\mathrm{tp}^{\delta_{p\restriction m}}_{\kappa,\bar\beta^p_m+1})^{\breve N^p_{p;m}}(\langle\breve U^p_{p;m}\rangle^\frown j^p_{0,2k}(\mathbf{s}^q_m))\\ &=(\mathrm{tp}^{\delta_{q\restriction m}}_{\kappa,\bar\beta^q_m+1})^{\breve N^q_{q;m}}(\langle\breve U^q_{q;m}\rangle^\frown j^q_{0,2(k+1)}(\mathbf{s}^q_m)).\end{aligned}$$

Hence (ii) for q follows from (ii) for p. Similarly case m of (iii) for q follows from (iii) for p.

The inequality $\hat\xi<\alpha$ of the One-Step Lemma gives us that

$$(\forall u\in j^q_{2k,2(k+1)}(X))\,g(u)+1<(j^q_{2k,2(k+1)}(f))(u\restriction k).$$

This in turn implies that

$$\beta_q+1<\breve\imath^q_{q;n,k+1}(j^q_{2k,2k+2}(\beta_p)).$$

Since $\beta_q=\bar\beta^q_{k+1}$ and $j^q_{2k,2(k+1)}(\beta_p)=\bar\beta^q_n$, we have condition (iv) for q in the case $m=n$. Assume that $m<m'\leq k$ and that $r_m\subseteq r_{m'}$. Recall that $j^p_{2k,2k+2}(\bar\beta^p_m)=\bar\beta^q_m$ and $j^p_{2k,2k+2}(\bar\beta^p_{m'})=\bar\beta^q_{m'}$ and observe that $j^q_{2k,2k+2}(\breve\imath^p_{p;m,m'})=\breve\imath^q_{q;m,m'}$. By these facts, condition (iv) for p, and the elementarity of $j^q_{2k,2k+2}$,

$$\bar\beta^q_{m'}=j^q_{2k,2k+2}(\bar\beta^p_{m'})<j^q_{2k,2k+2}(\breve\imath^p_{p;m,m'}(\bar\beta^p_m))=\breve\imath^q_{q;m,m'}(\bar\beta^q_m).$$

The remaining case of condition (iv) for q is $m \neq n$ and $m' = k+1$. Assume that $m \leq k$, $m \neq n$, and $r_m \subseteq r_{k+1}$. By the definition of n, we must have $m < n$ and $r_m \subseteq r_n$. But then

$$\bar{\beta}^q_{k+1} < \breve{\imath}^q_{q;n,k+1}(\bar{\beta}^q_k) < \breve{\imath}^q_{q;n,k+1}(\breve{\imath}^q_{q;m,n}(\bar{\beta}^q_m)) = \breve{\imath}^q_{q;m,k+1}(\bar{\beta}^q_m).$$

We have already verified for q the first clause of condition (v) and the case $m = k$ of the second clause of that condition. The other cases of the second clause follow easily from the corresponding cases for for p and the case $m = k$ for q.

We have noted that $\delta_p < \delta'_q < \delta_q$. Note also that $2n \dot{-} 1 = (2k+1)^-_S$. Now $\operatorname{crit}(E^q_{2k}) = \delta_{q\restriction n}$ and $\operatorname{crit}(E^q_{2k+1}) = \delta'_q$. Since condition (vi) holds for p and since $\gamma < \delta_\emptyset$, it follows that condition (vi) holds for q.

This completes our construction and the verification that it has the desired properties.

We will show that the system

$$(\langle M^p_{2\ell\mathrm{h}(p)} \mid p \in T\rangle, \langle j^p_{m,2\ell\mathrm{h}(p)} \mid p \in T \wedge m < \ell\mathrm{h}(p) \in T\rangle)$$

gives an embedding normal form for the T-projection of $U^\ddagger$.

Fix $x \in [T]$. Let $\mathcal{S}_x$ be the iteration tree of length ω whose restrictions are the $\mathcal{S}_{x\restriction n}$.

We will need to know that $\mathcal{S}_x$ is a plus one tree. To prove this, let $n \in \omega$. It is not true for S, as it was for the alternating chain ordering C, that the set

$$\{m \mid (m+1)^-_S \leq n < m\}$$

is finite. But the first part of the proof of Lemma 8.2.5 shows that it suffices to prove prove the weaker fact that, for every $k \in \omega$, the set

$$\{\operatorname{crit}(E^{\mathcal{S}_x}_m) \mid (m+1)^-_S \leq n < m\}$$

is finite. Condition (vi) of our construction implies that, for all m and $m' \in \omega$,

$$(m+1)^-_S = (m'+1)^-_S \to \operatorname{crit}(E^{\mathcal{S}_x}_m) = \operatorname{crit}(E^{\mathcal{S}_x}_{m'}),$$

and there are only finitely many numbers $\leq n$ of the form $(m+1)^-_S$.

We must show that $[U^\ddagger(x)] \neq \emptyset$ if and only if $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\mathrm{Even}}$ is wellfounded.

Assume first that $[U^\ddagger(x)] \neq \emptyset$. By Theorem 8.1.8, $[U(x)] = \emptyset$, i.e., $U(x)$ is a wellfounded tree. For each $k \in \omega$, let

$$\begin{aligned} \xi_{2k} &= j^{\mathcal{S}_x}_{0,2k}(\|s_\emptyset\|^{U(x)}); \\ \xi_{2k+1} &= \|s_{x\restriction k+1}\|^{j^{\mathcal{S}_x}_{0,2k+1}(U(x))}. \end{aligned}$$

Using condition (v), we get that, for all m and $n \in \omega$ with $r_m \subsetneq r_n$,

$$\begin{aligned} j^{\mathcal{S}_x}_{2m\dot{-}1,2n\dot{-}1}(\xi_{2m\dot{-}1}) &= \|j^{\mathcal{S}_x}_{2m\dot{-}1,2n\dot{-}1}(s_{x\restriction m})\|^{j^{\mathcal{S}_x}_{0,2n\dot{-}1}(U(x))} \\ &> \|s_{x\restriction n}\|^{j^{\mathcal{S}_x}_{0,2n\dot{-}1}(U(x))} \\ &= \xi_{2n\dot{-}1}. \end{aligned}$$

Thus the sequence $\langle \xi_n \mid n \in \omega\rangle$ witnesses that $\mathcal{S}_x$ is continuously illfounded off Even. Since $\mathcal{S}_x$ is plus one, Corollary 7.4.6 gives that $\mathcal{M}^{\mathcal{S}_x}_{\mathrm{Even}}$ is wellfounded.

Now assume that $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\mathrm{Even}}$ is wellfounded. For elements m and m' of ω with $m \leq m'$, let

$$\begin{aligned} \bar{\beta}^x_m &= j^{\mathcal{S}_x}_{2m,\mathrm{Even}}(\beta_{x\restriction m}); \\ \breve{\imath}^x_{m,m'} &= j^{\mathcal{S}_x}_{0,\mathrm{Even}}(i_{x\restriction n;m,n}) \\ \breve{\imath}^x_m &= j^{\mathcal{S}_x}_{0,\mathrm{Even}}(i^x_{\langle x\restriction \ell\mathrm{h}(r_m),r_m\rangle}). \end{aligned}$$

Let $y \in {}^\omega\omega$ be arbitrary. For each $n \in \omega$, let m_n be such that $y \restriction n = r_{m_n}$. Let $n < n' \in \omega$ and let $k \in \omega$ be such that $m_{n'} \leq k$. By condition (iv), we have that

$$\bar{\beta}^{x\restriction k}_{m_{n'}} < \breve{\imath}^{x\restriction k}_{x\restriction k;m_n,m_{n'}}(\bar{\beta}^{x\restriction k}_{m_n}).$$

Applying $j^{\mathcal{S}_x}_{2k,\mathrm{Even}}$ to both sides of this inequality, we find that

$$\bar{\beta}^x_{m_{n'}} < \breve{\imath}^x_{m_n,m_{n'}}(\bar{\beta}^x_{m_n}).$$

Applying $\breve{\imath}^x_{m_{n'}}$ to both sides, we get that

$$\breve{\imath}^x_{m_{n'}}(\bar{\beta}^x_{m_{n'}}) < \breve{\imath}^x_{m_n}(\bar{\beta}^x_{m_n}).$$

This argument shows that the $\breve{\imath}^x_{m_n}(\bar{\beta}^x_{m_n})$, $n \in \omega$, form an infinite descending chain in the ordinals of the model $j^{\mathcal{C}_x}_{0,\mathrm{Even}}(\mathcal{M}_{x,y})$, and so that model is illfounded. It follows by absoluteness that $\mathcal{M}_{x,y}$ is illfounded. Since y was an arbitrary element of ${}^\omega\omega$, this means that $[U(x)] = \emptyset$. Therefore $[U^\ddagger(x)] \neq \emptyset$.

Remark. As in the proof of Theorem 8.2.6, the second half of the proof of embedding normal form is not needed for the proof of the theorem.

To complete the proof of the theorem, let α be any ordinal such that $\beta_p < \alpha$ for all $p \in T$.

For $p \in T$, set

$$\mathcal{V}_p = \{X \subseteq (U^\ddagger \restriction \alpha)[p] \mid \langle \bar{\beta}^p_m \mid m < \ell\mathrm{h}(p)\rangle \in j^p_{0,2\ell\mathrm{h}(p)}(X)\}.$$

We will show that $\langle \mathcal{V}_p \mid p \in T\rangle$ witnesses that $U^\dagger \restriction \alpha$ is γ-homogenous.

We first prove that clause (1) of the definition of homogeneity holds—that each $\mathcal{V}_p$ is a γ-complete ultrafilter on $(U^\dagger \restriction \alpha)[p]$. Let $p \in T$. We begin by proving that $(U^\ddagger \restriction \alpha)[p] \in \mathcal{V}_p$, i.e., that

(i) $(\forall m < \ell\mathrm{h}(p))\, \bar{\beta}^p_m < j^p_{0,2\ell\mathrm{h}(p)}(\alpha)$;

(ii) $\langle \bar{\beta}^p_m \mid m < \ell\mathrm{h}(p)\rangle \in j^p_{0,2\ell\mathrm{h}(p)}(U^\ddagger[p])$.

For (i), let $m < \ell\mathrm{h}(p)$. Then

$$\bar{\beta}^p_m = j^p_{2m,2\ell\mathrm{h}(p)}(\beta_{p\restriction m}) < j^p_{2m,2\ell\mathrm{h}(p)}(\alpha) \leq j^p_{0,2\ell\mathrm{h}(p)}(\alpha).$$

By the definition of $U^\ddagger$, what we must show to prove (ii) is that, for all m and m' such that $m < m' < \ell\mathrm{h}(p)$,

$$\bar{\beta}^p_{m'} < (j^p_{0,2\ell\mathrm{h}(p)}(i_{\langle p\restriction \ell\mathrm{h}(r_m), r_m\rangle, \langle p \restriction \ell\mathrm{h}(r_{m'}), r_{m'}\rangle}))(\bar{\beta}^p_m),$$

i.e., that

$$\bar{\beta}^p_{m'} < (j^p_{0,2\ell\mathrm{h}(p)}(i_{p;m,m'}))(\bar{\beta}^p_m).$$

But this follows directly from condition (iv) of our construction.

Since $(U^\ddagger \restriction \alpha)[p] \in \mathcal{V}_p$, it follows by Lemma 6.1.1 that $\mathcal{V}_p$ is an ultrafilter on $(U^\dagger \restriction \alpha)[p]$.

By condition (vi), $\gamma \leq \mathrm{crit}\,(j^p_{0,2\ell\mathrm{h}(p)})$. Hence Lemma 6.1.1 yields that $\mathcal{V}_p$ is γ-complete.

The verifications of clauses (2) and (3′) in the definition of homogeneity is exactly like the corresponding verifications in the proof of Theorem 8.2.6, and we omit them. □

Corollary 8.2.8. *Let κ be a Woodin cardinal, let T be a game tree such that $|T| < \kappa$, and let $A \subseteq [T]$ be such that $[T] \setminus A$ is weakly κ^+-homogeneously Souslin. Then A is $(<\kappa)$-homogeneously Souslin.*

Proof. Let Y be a set and U a tree on field$(T) \otimes Y$ witnessing that T is weakly κ^+-homogeneously Souslin. Let $\langle \mathcal{U}_{p,r} \mid p \in T \wedge r \in {}^{<\omega}\omega \rangle$ witness that U is weakly κ^+-homogeous for T. Let $U^\ddagger = U^\ddagger(\langle \mathcal{U}_{p,r} \mid p \in T \wedge r \in {}^{<\omega}\omega \rangle)$. By Theorem 8.2.7, let $\alpha \geq \max\{\omega, (2^{|Y|})^+\}$ be such that $U^\ddagger \restriction \alpha$ is $(< \kappa)$-homogeneous for T. By Theorem 8.1.8, A is the T-projection of $U^\ddagger \restriction \alpha$, and so A is $(< \kappa)$-homogeneously Souslin. □

Theorem 8.2.9. *Let T be a game tree and let $n \in \omega$. Let $\langle \kappa_i \mid i \leq n$ be a strictly increasing sequence of cardinals such that $|T| < \kappa_0$, κ_n is measurable and, for $i < n$, κ_i is Woodin. Then every $\mathbf{\Pi}^1_{n+1}$ subset of $[T]$ is $(< \kappa_0)$-homogeneously Souslin.*

Proof. We prove by induction on $m \leq n$ that, for every game tree T' such that $|T'| \leq \max\{\aleph_0, |T|\}$, every $\mathbf{\Pi}^1_{m+1}$ subset of $[T']$ is $(< \kappa_{n-m})$-homogeneously Souslin.

By Theorem 4.3.6, every $\mathbf{\Pi}^1_1$ subset of such a $[T']$ is κ_n-homogeneously Souslin and so $(< \kappa_n)$-homogeneously Souslin.

Let $m < n$ and assume that what we want to prove holds of m. Let T' be a game tree such that $|T'| \leq \max\{\aleph_0, |T|\}$. Let $A \subset [T]$ with $A \in \mathbf{\Pi}^1_{m+2}$. Let $B \subseteq [T] \times {}^\omega\omega$ be such that $B \in \mathbf{\Pi}^1_n$ and $A = [T] \setminus \mathrm{p}B$. Let $B^* = \{\langle\!\langle x, y \rangle\!\rangle \mid \langle x, y \rangle \in B\}$. Then $B^* \subseteq [T \otimes {}^{<\omega}\omega]$ and $B^* \in \mathbf{\Pi}^1_n$. Since $\kappa_{n-(m+1)} < \kappa_{n-m}$, we have that B^* is $(\kappa_{n-(m+1)})^+$-homogeneously Souslin. But this means that B is $(\kappa_{n-(m+1)})^+$-homogeneously Souslin. (See page 425.) By Theorem 8.1.3, $[T] \setminus A$ is weakly κ-homogeneously Souslin. Since $\kappa_{n-(m+1)}$ is Woodin, Corollary 8.2.8 implies that A is $(< \kappa)$-homogeneously Souslin. □

Theorem 8.2.10. *Let T be a game tree and let $n \in \omega$. Assume that there are n distinct Woodin cardinals all greater than $|T|$ and that there is a measurable cardinal greater than all of them. Then every $\mathbf{\Pi}^1_{n+1}$ game in T is determined.*

Proof. The theorem follows directly from Theorems 8.2.9 and 4.3.5. □

Exercise 8.2.1. Let b be one of the two branches of an alternating chain $\mathcal{C}$ of length ω. Assume that the other branch of $\mathcal{S}$ is illfounded. Prove that $\mathcal{C}$ is continuously illfounded off b.

Hint. Use Lemma 7.4.2.

Exercise 8.2.2. Let $n \in \omega$. Assume that there is a Woodin cardinal. Prove that there is an infinite plus n alternating chain on V.

Hint. The most natural way to proceed is to prove a variant of the One-Step Lemma which will let one build a plus n alternating chain by a construction just like that of Lemma 8.2.4. It is possible, nevertheless, to get by with the One-Step Lemma itself.

8.3 Variations

In this section we discuss variants of the construction of the proof of Theorem 8.2.7.

The first variant is a cleaned-up version of the construction of [Martin and Steel, 1989]. It has the advantage of being a little simpler than the construction of the proof of Theorem 8.2.7. Its disadvantage lies in its not immediately yielding homogeneity ultrafilters for $U^{\ddagger}$. After giving the construction, we prove the lemma needed to get the existence of these ultrafilters. We then prove a result of K. Windßus that allows one to sidestep the problem, propagating homogeneous Souslinness without proving the homogeneity of $U^{\ddagger}$.

The second variant construction is due to Itay Neeman. Rather than propagate homogeneous Souslinness, Neeman propagates what he calls the *auxiliary game property.* His method has various advantages, one of which is that it seems to yield sharper results.

In the proof in §8.2 of Theorem 8.2.7, we maintained inductively relations between the models

$$\breve{N}^p_{q;m} = j^p_{0,2k}(i^p_{p;0,m}(V))$$

and the models $M^p_{2m \dot{-} 1} = j^p_{0,2m \dot{-} 1}(V)$. The induction step of the construction involved successive applications of the One-Step Lemma. The first of these applications was to the models $\breve{N}^p_{q;k+1}$ and $M^p_{2n \dot{-} 1}$, for some $n \leq k$. But the second application was not, as one might have anticipated, to M^p_{2n+1} and $\breve{N}^p_{q;k+1}$. Instead we undid the embedding $\breve{\imath}^p_{q;k+1}$, made a whole set of applications of the second half of the One-Step Lemma to the models M^p_{2n+1} and M^p_{2k}, and then applied $\breve{\imath}^q_{q;0,k+1}$ to the results of these applications. The reason for this round-about process was that the direct method would have yielded, for example, the model $i^{\breve{N}^p_{q;k+1}}_{E^q_{2k}}(\breve{N}^p_{q;k+1})$ in place of $\tilde{N}^q_{q;k+1}$.

The analogous construction in [Martin and Steel, 1989] could—at least, after a mostly cosmetic rearrangement—be regarded as maintaining inductively certain relations between the models

$$i^p_{p;0,m}(j^p_{0,2k}(V))$$

and the models $M^p_{2m\dot{-}1}$ The induction step could be seen as involving two straightforward applications of the One-Step Lemma. In [Martin and Steel, 1989] embedding normal form was not hard to demonstrate, but homogeneity of $U^\ddagger$ needed a substantial lemma.

Let us see how would work after the "cosmetic rearrangement."

Let T, κ, Y, U, ν, ζ_0, ζ_1, ρ, $\langle U_{p,r} \mid \langle\!|p,r|\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$, $U^\ddagger$, γ, $\langle \pi_{p,r} \mid \langle\!|p,r|\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$, $\langle i_{\langle p,r\rangle,\langle q,s\rangle} \mid \langle\!|p,r|\!\rangle \subseteq \langle\!|q,s|\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$, $(\mathcal{M}_{x,y}; \langle i^{x,y}_{\langle x\restriction n, y\restriction n\rangle} \mid n \in \omega\rangle)$, $i \mapsto r_i$, $\langle \mathbf{s}_{p,r} \mid \langle\!|p,r|\!\rangle \in T \otimes {}^{<\omega}\omega\rangle$, and S be as in the proof of Theorem 8.2.7.

As in the proof of Theorem 8.2.7, we will define, by induction on $p \in T$, objects δ_p, β_p, $\mathcal{S}_p$, and s_p. Both δ_p and β_p will be ordinals, with $\delta_p < \kappa$. s_p will be a sequence such that $\ell\mathrm{h}(s_p) = \ell\mathrm{h}(r_{\ell}\mathrm{h}(p))$. $\mathcal{S}_p$ will be an iteration tree of length $2\ell\mathrm{h}(p) + 1$ on V. Its tree ordering will be the restriction of S. Its extenders will be E^p_m, $m < 2\ell\mathrm{h}(p)$, its models will be M^p_m $m \leq 2\ell\mathrm{h}(p)$, and its embeddings will be $j^p_{m,n}$, $m\, S\, n \leq 2\ell\mathrm{h}(p)$. Whenever $p \subseteq q \in T$ then we will have $\mathcal{S}_p = \mathcal{S}_q \restriction 2\ell\mathrm{h}(p) + 1$.

We introduce some notation. Let p and q belong to T. Let $k \leq k' \leq 2\ell\mathrm{h}(p)$. Let $m \leq n \leq \ell\mathrm{h}(q)$ such that $r_m \subseteq r_n$. Let

$$\begin{aligned}
i_{q;m,n} &= i_{\langle q\restriction \ell\mathrm{h}(r_m), r_m\rangle, \langle q\restriction \ell\mathrm{h}(r_n), r_n\rangle};\\
\hat{\jmath}^{q;m}_{p;k,k'} &= i_{q;0,m}(j^p_{k,k'});\\
\hat{N}^{q;m}_p &= i_{q;0,m}(M^p_{2\ell\mathrm{h}(p)});\\
\hat{U}^{q;m}_p &= i_{q;0,m}(j^p_{0,2\ell\mathrm{h}(p)}(U));\\
\tilde{\beta}^p_m &= \hat{\jmath}^{q;m}_{p;2m,2\ell\mathrm{h}(p)}(\beta_{p\restriction m}));\\
\mathbf{s}^q_m &= \mathbf{s}_{q\restriction \ell\mathrm{h}(r_m), r_m}.
\end{aligned}$$

The embedding $\hat{\jmath}^{q;m}_{p;k,k'}$ is the image of $j^p_{k,k'}$ in $\mathrm{Ult}(V; \mathcal{U}_{q\restriction \ell\mathrm{h}(r_m), r_m}) = i_{q;0,m}(V)$. The class model $\hat{N}^{q;m}_p$ is the image of $M^p_{2\ell\mathrm{h}(p)}$ in $i_{q;0,m}(V)$. In other words,

$$\hat{N}^{q;m}_p = i_{q;0,m}(j^p_{0,2\ell\mathrm{h}(p)}(V)).$$

The tree $\hat{U}^{q;m}_p$ is the image of $j^p_{0,2\ell\mathrm{h}(p)}(U)$ in $i_{q;0,m}(V)$; i.e., it is the image of U in $\hat{N}^{q;m}_p$

For $p = \emptyset$ we proceed exactly as in the proof of Theorem 8.2.7. We choose $\delta_\emptyset > \gamma$ to be $\zeta_0 + 1$-reflecting in $\langle U \rangle$ relative to κ, and we let $\beta_\emptyset = \zeta_0$.

Let $p \in T$. Let $k = \ell\mathrm{h}(p)$. Assume that $\delta_{p'}$, $\beta_{p'}$, $\mathcal{S}_{p'}$, and $s_{p'}$ are defined for all $p' \subseteq p$ so as to satisfy the conditions stated above and also the following conditions:

(i) for all $m \leq k$, M^p_{2k} and $M^p_{2m\dot{-}1}$ agree through $\delta_{p\restriction m} + 1$;

(ii) for all $m \leq k$, the type $(\mathrm{tp}^{\delta_{p\restriction m}}_{\kappa,\tilde{\beta}^p_m+1})^{\hat{N}^{p;m}_p}(\langle \hat{U}^{p;m}_p \rangle \frown \hat{\jmath}^{p;m}_{p;0,2k}(\mathbf{s}^p_m))$ is the same as $(\mathrm{tp}^{\delta_{p\restriction m}}_{\kappa,\zeta_0+1})^{M^p_{2m\dot{-}1}}(\langle j^p_{0,2m\dot{-}1}(U)\rangle \frown s_{p\restriction m})$;

(iii) for all $m \leq k$, $\delta_{p\restriction m}$ is $(\tilde{\beta}^p_m + 1)$-reflecting in $\langle \hat{U}^{p;m}_p \rangle \frown \hat{\jmath}^{p;m}_{p;0,2k}(\tilde{\mathbf{s}}^p_m)$ relative to κ in $\hat{N}^{p;m}_p$;

(iv) for all m and m with $m < m' \leq k$, if $r_m \subseteq r_{m'}$ then

$$\tilde{\beta}^p_{m'} < i_{p;m,m'}(\tilde{\beta}^p_m);$$

(v) for all $m \leq k$, $s_{p\restriction m}$ belongs to $(j^p_{0,2m\dot{-}1})(U[p \restriction \ell\mathrm{h}(r_m)])$, and

$$r_m \subseteq r_k \to j^p_{2m\dot{-}1,2k\dot{-}1}(s_{p\restriction m}) \subseteq s_p);$$

(vi) for all $m < k$,

$$\gamma < \delta_{p\restriction m} < \mathrm{crit}\,(E^p_{2m+1}) < \delta_{p\restriction m+1},$$

and $\mathrm{crit}\,(E^p_{2m}) = \delta_{p\restriction \bar{m}}$, where $2\bar{m} \dot{-} 1 = (2m+1)^-_S$.

Note that these conditions all hold for for $p = \emptyset$.

Let q be any element of T such that $p \subseteq q$ and $\ell\mathrm{h}(q) = k + 1$.

Let n be the largest number $\leq k$ such that $r_n \subseteq r_{k+1}$. Let $e = \ell\mathrm{h}(r_n)$. Thus $\ell\mathrm{h}(r_{k+1}) = e + 1$.

By (i) and the fact that $\mathrm{crit}\,(i_{q;0,k+1}) > \kappa$, it follows that $\hat{N}^{q;k+1}_p$ and $M^p_{2n\dot{-}1}$ agree through $\delta_{p\restriction n} + 1$.

One readily computes that

$$\begin{aligned} \hat{N}^{q;k+1}_p &= i_{q;n,k+1}(\hat{N}^{p;n}_p); \\ \hat{U}^{q;k+1}_p &= i^{q;n,k+1}(\breve{U}^{p;n}_p); \\ \hat{\jmath}^{q;k+1}_{p;0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n)) &= i_{q;n,k+1}(\hat{\jmath}^{p;n}_{p;0,2k}(\mathbf{s}^p_n)). \end{aligned}$$

Since crit $(i_{q;n,k+1}) > \kappa$, it follows that

$$\begin{aligned}&(\mathrm{tp}^{\delta_{p\restriction n}}_{\kappa,\tilde\beta^p_n+1})^{\hat N^{p;n}_p}(\langle \hat U^{p;n}_p\rangle^\frown \hat\jmath^{p;n}_{p;0,2k}(\mathbf{s}^p_n))\\ &= i_{q;n,k+1}((\mathrm{tp}^{\delta_{p\restriction n}}_{\kappa,\tilde\beta^p_n+1})^{\hat N^{p;n}_p}(\langle \hat U^{p;n}_p\rangle^\frown \hat\jmath^{p;n}_{p;0,2k}(\mathbf{s}^p_n)))\\ &= (\mathrm{tp}^{\delta_{p\restriction n}}_{\kappa,i_{q;n,k+1}(\tilde\beta^p_n)+1})^{\hat N^{q;k+1}_p}(\langle \hat U^{q;k+1}_p\rangle^\frown \hat\jmath^{q;k+1}_{p;0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n)))\end{aligned}$$

and so by (ii) this last is the same as $(\mathrm{tp}^{\delta_{p\restriction n}}_{\kappa,\zeta_0+1})^{M^p_{2n\dot-1}}(\langle j^p_{0,2n\dot-1}(U)\rangle^\frown s_{p\restriction n})$.

From (iii) it similarly follows that $\delta_{p\restriction n}$ is $(i_{q;n,k+1}(\tilde\beta^p_n+1))$-reflecting in the sequence $\langle \hat U^{q;k+1}_p\rangle^\frown \hat\jmath^{q;k+1}_{p;0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n))$ relative to κ in $\hat N^{q;k+1}_p$.

Since $j^p_{0,2k}$ and $i_{q;0,k+1}$ fix κ, we have that κ is Woodin in $\hat N^{q;k+1}_p$.

Thus the hypotheses of the One-Step Lemma hold for κ with

$$\begin{aligned}M &= \hat N^{q;k+1}_p;\\ N &= M^p_{2n\dot-1};\\ \delta &= \delta_{p\restriction n};\\ \eta &= \delta_p;\\ \beta &= i_{q;n,k+1}(\tilde\beta^p_n)+1;\\ \xi &= i_{q;n,k+1}(\tilde\beta^p_n);\\ \beta' &= \zeta_0+1;\\ x &= \langle \hat U^{q;k+1}_p\rangle^\frown \hat\jmath^{q;k+1}_{p;0,2k}(i_{q;n,k+1}(\mathbf{s}^p_n));\\ y &= \langle(\hat\jmath^{q;k+1}_{p;0,2k}(\mathbf{s}^q_{k+1}))(e)\rangle;\\ x' &= \langle j^p_{0,2n\dot-1}(U)\rangle^\frown s_{p\restriction n};\\ \chi(v) &= \text{“}\kappa+v \text{ is the greatest ordinal.”}\end{aligned}$$

Let λ and E be given by the One-Step Lemma. Since $i_{q;0,k+1}$ fixes λ, E, and δ_p, it follows that E is a $(\delta_{p\restriction n},\lambda)$-extender in M^p_{2k}. Thus Theorem 7.3.2 gives that $\prod_E^{M^p_{2n\dot-1}}(M^p_{2n\dot-1};\in)$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. By clause (4*) of the One-Step Lemma, $\xi^*=\zeta_0$. Extend $\mathcal{S}_p$ to an alternating chain that will be $\mathcal{S}_q\restriction 2k+2$ by setting $E^q_{2k}=E$. The ordinal δ^* we will call δ'_q. Set $s_q=(j^q_{2n\dot-1,2k+1}(s_p))^\frown y^*$.

Note that $M^q_m = M^p_m$, $E^q_m = E^p_m$, $j^q_{m,m'} = j^p_{m,m'}$, and $i^q_{m,m'} = i^p_{m,m'}$ whenever these equations make sense. We will use these identities without comment in the sequel.

A computation like the analogous one in the proof of Theorem 8.2.7 gives that

$$x^\frown y = \langle \hat U^{q;k+1}_p\rangle^\frown \hat\jmath^{q;k+1}_{p;0,2k}(\mathbf{s}^q_{k+1}).$$

Since $\mathbf{s}^q_{k+1}$ belongs to $i_{q;0,k+1}(U[q\restriction e+1])$, the sequence $\hat{\jmath}^{q;k+1}_{p;0,2k}(\mathbf{s}^q_{k+1})$ belongs to $\hat{\jmath}^{q;k+1}_{p;0,2k}(i^q_{0,k+1}(U[q\restriction e+1])) = (i_{q;0,k+1}(j^p_{0,2k}))(i^q_{0,k+1}(U[q\restriction e+1])) = i_{q;0,k+1}(j^p_{0,2k}(U[q\restriction e+1])) = \hat{U}^{q;k+1}_p$. It follows by clause (2*) of the One-Step Lemma that $s_q \in j^q_{0,2k+1}(U[q\restriction e+1])$, and so the first clause of condition (v) holds for q. Since $j^q_{2n\dot{-}1,2k+1}(s_{q\restriction n}) \subseteq s_q$, the second clause of condition (v) holds for q the case $m=n$.

We have that

(a) M^q_{2k+1} and $\hat{N}^{q;k+1}_p$ agree through δ'_q+1;

(b) $(\mathrm{tp}^{\delta'_q}_{\kappa,\zeta_0})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q) =$
$(\mathrm{tp}^{\delta'_q}_{\kappa,i_{q;n,k+1}(\tilde{\beta}^p_n)})^{\hat{N}^{q;k+1}_p}(\langle \hat{U}^{q;k+1}_p\rangle^\frown \hat{\jmath}^{q;k+1}_{p;0,2k}(\mathbf{s}^q_{k+1}))$;

(c) δ'_q is ζ_0-reflecting in $\langle j^q_{0,2k+1}(U)\rangle^\frown s_q$ relative to κ in M^q_{2k+1}.

By (b) and (c) together with parts (b) and (c) of Lemma 8.2.3, we have that

(b′) $(\mathrm{tp}^{\delta'_q}_{\kappa,\zeta_1})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q) =$
$(\mathrm{tp}^{\delta'_q}_{\kappa,i_{q;n,k+1}(\tilde{\beta}^p_n)})^{\hat{N}^{q;k+1}_p}(\langle \hat{U}^{q;k+1}_p\rangle^\frown \hat{\jmath}^{q;k+1}_{p;0,2k}(\mathbf{s}^q_{k+1}))$;

(c′) δ'_q is ζ_1-reflecting in $\langle j^q_{0,2k+1}(U)\rangle^\frown s_q$ relative to κ in M^q_{2k+1}.

Since κ is Woodin in M^q_{2k+1}, the hypotheses of the One-Step Lemma hold for κ with

$$\begin{aligned}
M &= M^q_{2k+1};\\
N &= \hat{N}^{q;k+1}_p\\
\delta &= \delta'_q;\\
\eta &= \delta'_q;\\
\beta &= \zeta_1;\\
\xi &= \zeta_0+1;\\
\beta' &= i_{q;n,k+1}(\tilde{\beta}^p_m));\\
x &= \langle j^q_{0,2k+1}(U)\rangle^\frown s_q;\\
y &= \emptyset;\\
x' &= \langle \hat{U}^{q;k+1}_p\rangle^\frown \hat{\jmath}^{q;k+1}_{p;0,2k}(\mathbf{s}^q_{k+1});\\
\chi(v) &= \text{“}v=v.\text{”}
\end{aligned}$$

Let λ and E be given by the One-Step Lemma. From (a) above and the fact that that crit$\,(i_{q;0,k+1}) > \kappa > \delta'_q + 1$, it follows that that M^q_{2k+1} and M^q_{2k} agree through $\delta'_q + 1$. Thus the model $\prod_E^{M^q_{2k}}(M^q_{2k};\in)$ is wellfounded. By the elementarity of $i_{q;0,k+1}$, $\prod_E^{i_{q;0,k+1}(M^q_{2k})}(i^q_{0,k+1}(M^q_{2k};\in))$ is wellfounded. Let then δ^*, ξ^*, and y^* be given by the One-Step Lemma. Clause (2^*) of the One-Step Lemma implies that $y^* = \emptyset$ and that ξ^* is a successor ordinal. We finish the definition of $\mathcal{S}_q$ by setting $E^q_{2k+1} = E$. Let $\delta_q = \delta^*$. Let β_q be such that $\beta_q + 1 = \xi^*$.

Observe that

$$\begin{aligned}
\mathrm{Ult}(\hat{N}_p^{q;k+1}; E) &= \mathrm{Ult}(i_{q;0,k+1}(M^p_{2k}); E)\\
&= i_{q;0,k+1}(\mathrm{Ult}(M^p_{2k}; E))\\
&= i_{q;0,k+1}(M^q_{2k+2})\\
&= \hat{N}_q^{q;k+1}
\end{aligned}$$

and that

$$\begin{aligned}
i_E^{\hat{N}_p^{q;k+1}} &= i_{q;0,k+1}(i_E^{M^p_{2k}})\\
&= i_{q;0,k+1}(j^q_{2k,2k+2})\\
&= \hat{\jmath}^{q;k+1}_{q;2k,2k+2}.
\end{aligned}$$

Thus

$$\begin{aligned}
i_E^{\hat{N}_p^{q;k+1}}(\hat{U}_p^{q;k+1}) &= ((i_{q;0,k+1}(j^q_{2k,2k+2}))(i_{q;0,k+1}(j^p_{0,2\ell\mathrm{h}(p)}(U))\\
&= i_{q;0,k+1}(j^q_{0,2k+2}(U))\\
&= \hat{U}_q^{q;k+1}
\end{aligned}$$

and

$$\begin{aligned}
i_E^{\hat{N}_p^{q;k+1}}(\hat{\jmath}^{q;k+1}_{p;0,2k}(\mathbf{s}^q_{k+1})) &= \hat{\jmath}^{q;k+1}_{q;2k,2k+2}(\hat{\jmath}^{q;k+1}_{p;0,2k}(\mathbf{s}^q_{k+1}))\\
&= \hat{\jmath}^{q;k+1}_{q;0,2k+2}(\mathbf{s}^q_{k+1}).
\end{aligned}$$

To verify inductive conditions (i)–(iii) for q, let $m \le k+1$ be arbitrary.

Since $\mathrm{Ult}(\hat{N}_p^{q;k+1}; E) = i_{q;0,k+1}(M^q_{2k+2})$, condition (i) for the case $m = k+1$ follows from clause (1^*) of the One-Step Lemma and the fact that crit$\,(i_{q;0,k+1}) > \kappa$. The verification of (i) for $m \le k$ is exactly like the corresponding step in the proof of Theorem 8.2.7.

Clauses (2^*) and (3^*) of the One-Step Lemma and the identities above give that

(ii′) $(\mathrm{tp}^{\delta_q}_{\kappa,\beta_q+1})^{\hat{N}^{q;k+1}_q}(\langle\hat{U}^{q;k+1}_q\rangle^\frown\hat{\jmath}^{q;k+1}_{q;0,2k+2}(\mathbf{s}_q))$
$= (\mathrm{tp}^{\delta_q}_{\kappa,\zeta_0+1})^{M^q_{2k+1}}(\langle j^q_{0,2k+1}(U)\rangle^\frown s_q)$;

(iii′) δ_q is (β_q+1)-reflecting in $\langle\hat{U}^{q:k+1}_q\rangle^\frown\hat{\jmath}^{q;k+1}_{q;0,2k+2}(\mathbf{s}_q)$ relative to κ in $\hat{N}^{q;k+1}_q$.

(ii′) and (iii′) are just our inductive conditions (ii) and (iii) for q.

Now let $m \leq k$. By the fact that crit $(\hat{\jmath}^{q;m}_{q;2k,2(k+1)}) = \delta'_q > \delta_{q\restriction m}$ and by our definitions, we have that

(1) $\hat{\jmath}^{q;m}_{p;2k,2(k+1)}(\delta_{q\restriction m}) = \delta_{q\restriction m}$;

(2) $\hat{\jmath}^{q;m}_{p;2k,2(k+1)}(\tilde{\beta}^p_m) = \tilde{\beta}^q_m$;

(3) $\hat{\jmath}^{q;m}_{p;2k,2(k+1)}(\hat{N}^{p;m}_p) = \hat{N}^{q;m}_q$;

(4) $\hat{\jmath}^{q;m}_{p;2k,2(k+1)}(\hat{U}^{p;m}_p) = \hat{U}^{q;m}_q$.

The fact that crit $(\hat{\jmath}^{q;m}_{p;2k,2(k+1)}) > \delta_{q\restriction m}$ also implies that

$$(\mathrm{tp}^{\delta_{p\restriction m}}_{\kappa,\tilde{\beta}^p_m+1})^{\hat{N}^{p;m}_p}(\langle\hat{U}^{p;m}_p\rangle^\frown\hat{\jmath}^{p;m}_{p;0,2k}(\mathbf{s}^q_m))$$

is fixed by $\hat{\jmath}^{q;m}_{p;2k,2(k+1)}$. Thus we get condition (ii) for q just as we got the corresponding fact in the proof of Theorem 8.2.7. Condition (iii) for q follows similarly.

The inequality $\xi^* < i^N_E$ of the One-Step Lemma gives us that

$$\begin{array}{rcl}
\beta_q & < & \xi^* \\
 & < & i^{\hat{N}^{q;k+1}_p}_E(i_{q;n,k+1}(\tilde{\beta}^p_n)) \\
 & = & (i_{q;0,k+1}(j^q_{2k,2(k+1)}))(i_{q;n,k+1}(\tilde{\beta}^p_n)) \\
 & = & i_{q;n,k+1}((i_{q;0,n}(j^q_{2k,2(k+1)}))(\tilde{\beta}^p_n)) \\
 & = & i_{q;n,k+1}(\hat{\jmath}^{q;n}_{q;2k,2(k+1)}(\tilde{\beta}^p_n)) \\
 & = & i_{q;n,k+1}(\tilde{\beta}^p_n).
\end{array}$$

Since $\beta_q = \tilde{\beta}^q_{k+1}$, this gives us condition (iv) for q in the case $m = n$. The other cases are handled as were the corresponding cases in the proof of Theorem 8.2.7.

Assume that $m < m' \leq k$ and that $r_m \subseteq r_{m'}$. Note that $\hat{j}^{q;m'}_{q;2k,2k+2} = i_{q;m,m'}(\hat{j}^{q;m}_{q;2k,2k+2})$. Using condition (iv) for p, we get that

$$\begin{aligned} \tilde{\beta}^q_{m'} &= \hat{j}^{q;m'}_{q;2k,2k+2}(\tilde{\beta}^p_{m'}) \\ &< \hat{j}^{q;m'}_{q;2k,2k+2}(i_{q;m,m'}(\tilde{\beta}^p_m)) \\ &= i_{q;m,m'}(\hat{j}^{q;m}_{q;2k,2k+2}(\tilde{\beta}^p_m)) \\ &= i_{q;m,m'}(\tilde{\beta}^q_m). \end{aligned}$$

The remaining case of condition (iv) for q is $m \neq n$ and $m' = k+1$. The proof for this case is analogous to the corresponding step in the proof of Theorem 8.2.7.

The proofs of conditions (v) and (vi) for q are like the corresponding steps in the proof of Theorem 8.2.7.

We will show that the system

$$(\langle M^p_{2\ell\mathrm{h}(p)} \mid p \in T\rangle, \langle j^p_{m,2\ell\mathrm{h}(p)} \mid p \in T \wedge m < \ell\mathrm{h}(p) \in T\rangle)$$

gives an embedding normal form for the T-projection of $U^\ddagger$.

Fix $x \in [T]$. Let $\mathcal{S}_x$ be the iteration tree of length ω whose restrictions are the $\mathcal{S}_{x\restriction n}$.

The proof that $\mathcal{M}^{\mathcal{S}_x}_{\mathrm{Even}}$ is wellfounded if $[U^\ddagger(x)] \neq \emptyset$, is exactly like the corresponding step in the proof of Theorem 8.2.7.

Now assume that $[U^\ddagger(x)] = \emptyset$. Thus $[U(x)] \neq \emptyset$. By Lemma 8.1.2, let $y \in {}^\omega\omega$ be such that $\mathcal{M}_{x,y}$ is wellfounded. For each $n \in \omega$, let $y \restriction n = r_{m_n}$. Let $n < n' \in \omega$. Applying condition (iv) with $p = x \restriction m'_n$, we get that

$$\tilde{\beta}^{x\restriction m_{n'}}_{m_{n'}} < i_{x\restriction m_{n'},m_n,m_{n'}}(\tilde{\beta}^{x\restriction m_{n'}}_{m_n}).$$

Unpacking our definitions, we find that this means that

$$\beta_{x\restriction m_{n'}} < i_{\langle x\restriction n, y\restriction n\rangle,\langle x\restriction n', y\restriction n'\rangle}((i_{\langle\emptyset,\emptyset\rangle,\langle x\restriction n,y\restriction n\rangle}(j^{\mathcal{S}_x}_{2m_n,2m_{n'}}))(\beta_{x\restriction m_n})).$$

This in turn gives that

$$\beta_{x\restriction m_{n'}} < (i_{\langle\emptyset,\emptyset,\rangle,\langle x\restriction n',y\restriction n'\rangle}(j^{\mathcal{S}_x}_{2m_n,2m_{n'}}))(i_{\langle x\restriction n,y\restriction n\rangle,x\restriction n',y\restriction n'\rangle}(\beta_{x\restriction m_n})).$$

Applying $i^{x,y}_{\langle x\restriction n',y\restriction n'\rangle}$ to both sides of this inequality, we get that

$$i^{x,y}_{\langle x\restriction n',y\restriction n'\rangle}(\beta_{x\restriction m_{n'}}) < (i^{x,y}_{\langle\emptyset,\emptyset\rangle}(j^{\mathcal{S}_x}_{2m_n,2m_{n'}}))(i^{x,y}_{\langle x\restriction n,y\restriction n\rangle}(\beta_{x\restriction m_n})).$$

Applying $i^{x,y}_{\langle\emptyset,\emptyset\rangle}(\tilde{\jmath}^{\mathcal{S}_x}_{2m_{n'},\mathrm{Even}})$ to both sides, we get that

$$(i^{x,y}_{\langle\emptyset,\emptyset\rangle}(\tilde{\jmath}^{\mathcal{S}_x}_{2m_{n'},\mathrm{Even}}))(i^{x,y}_{\langle x\restriction n',y\restriction n'\rangle}(\beta_{x\restriction m_{n'}})) < (i^{x,y}_{\langle\emptyset,\emptyset\rangle}(\tilde{\jmath}^{\mathcal{S}_x}_{2m_n,\mathrm{Even}}))(i^{x,y}_{\langle x\restriction n,y\restriction n\rangle}(\beta_{x\restriction m_n})).$$

Thus the $(i^{x,y}_{\emptyset,\emptyset}(\tilde{\jmath}^{\mathcal{S}_x}_{2m_n,\mathrm{Even}}))(i^{x,y}_{\langle x\restriction n,y\restriction n\rangle}(\beta_{x\restriction m_n}))$, $n \in \omega$, form an infinite descending chain in the ordinals of the model $i^{x,y}_{\langle\emptyset,\emptyset\rangle}(\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\mathrm{Even}})$. This implies that $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\mathrm{Even}}$ is illfounded.

There are two ways to use our construction and the result just proved about it to get the homogeneous Souslinness of the T-projection of $U^\ddagger$. One way is to use it to get the homogeneity of $U^\ddagger$. We now show how this can be done.

For $p \in T$ and $m \leq \ell\mathrm{h}(p)$, define $\bar{\beta}^p_m$ as in the earlier construction: $\bar{\beta}^p_m = j^p_{2m,2\ell\mathrm{h}(p)}(\beta_{p\restriction m})$. Let $\alpha > \beta_p$ for all p. We can use the ordinals $\bar{\beta}^p_m$ to define ultrafilters $\mathcal{V}_p$ on ${}^{\ell\mathrm{h}(p)}\alpha$. For $X \subseteq {}^{\ell\mathrm{h}(p)}\alpha$, let

$$X \in \mathcal{V}_p \leftrightarrow \langle \bar{\beta}^p_m \mid m < \ell\mathrm{h}(p)\rangle \in j^p_{0,2\ell\mathrm{h}(p)}(X).$$

The arguments of the proofs of Theorems 8.2.6 and 8.2.7 show that the $\mathcal{V}_p$ are compatible, that each $\mathcal{V}_p$ is a γ complete ultrafilter on ${}^{\ell\mathrm{h}(p)}\alpha$, and that whenever $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\mathrm{Even}}$ is wellfounded then the direct limit model for the system given by the $\mathcal{V}_{x\restriction n}$, $n \in \omega$, is wellfounded. Since $\tilde{\mathcal{M}}^{\mathcal{S}_x}_{\mathrm{Even}}$ is wellfounded whenever x belongs to the T-projection of $U^\ddagger$, we can verify homogeneity condition (3′).

There is only one problem in showing that $\langle \mathcal{V}_p \mid p \in T\rangle$ witnesses that $U^\ddagger$ is homogeneous for T: we must prove $(U^\ddagger \restriction \alpha)[p] \in \mathcal{V}_p$ for each p. If we can do this, then we will have shown that each $\mathcal{V}_p$ induces an ultrafilter on $(U^\ddagger \restriction \alpha)[p]$ and that the corresponding system of ultrafilters witnesses the γ-homogeneity of $U^\ddagger \restriction \alpha$.

To prove that $(U^\ddagger \restriction \alpha)[p] \in \mathcal{V}_p$, we need to show that, for all m and m' such that $m < m' < \ell\mathrm{h}(p)$ and $r_m \subseteq r_{m'}$,

$$\bar{\beta}^p_m < (j^p_{0,2\ell\mathrm{h}(p)}(i_{p;m,m'}))(\bar{\beta}^p_{m'}).$$

To see in simpler terms what we need to show, fix m and m' with $m < m' < \ell\mathrm{h}(p)$ and $r_m \subseteq r_{m'}$. By the definitions of the ordinals $\bar{\beta}^p_n$, what we must show is that

$$j^p_{2m',2\ell\mathrm{h}(p)}(\beta_{p\restriction m'}) < (j^p_{0,2\ell\mathrm{h}(p)}(i_{p;m,m'}))(j^p_{2m,2\ell\mathrm{h}(p)}(\beta_{p\restriction m})).$$

By the elementarity of $j^p_{2m',2\ell\mathrm{h}(p)}$, this is equivalent with the assertion that

$$\beta_{p\restriction m'} < (j^p_{0,2m'}(i_{p;m,m'}))(j^p_{2m,2m'}(\beta_{p\restriction m})).$$

Condition (iv) of our construction gives that

$$(i_{p;0,m'}(j^p_{2m',2\ell\mathrm{h}(p)}))(\beta_{p\restriction m'}) < i_{p;m,m'}((i_{p;0,m}(j^p_{2m,2\ell\mathrm{h}(p)}))(\beta_{p\restriction m})).$$

By the elementarity of $i_{p;m,m'}$ and of $i_{p;0,m'}(j^p_{2m',2\ell\mathrm{h}(p)})$, this is equivalent with the assertion that

$$\beta_{p\restriction m'} < (i_{p;0,m'}(j^p_{2m,2m'}))(i_{p;m,m'}(\beta_{p\restriction m})).$$

We will prove a theorem showing that that, on the ordinals,

a) $j^p_{0,2m'}(i_{p\restriction m,p\restriction m'})$ agrees with $i_{p\restriction m,p\restriction m'}$;
b) $i_{\emptyset,p\restriction m'}(j^p_{2m,2m'})$ agrees with $j^p_{2m,2m'}$;
c) $i_{p\restriction m,p\restriction m'} \circ j^p_{2m,2m'}$ agrees with $j^p_{2m,2m'} \circ i_{p\restriction m,p\restriction m'}$.

It will be easy to see that these facts give the equivalence of the two inequalities above.

Instead of proceeding directly to the proof of this result, we first illustrate the ideas in a simplified form.

Lemma 8.3.1. *Let κ be a strong limit cardinal and let Let $\mathcal{U}$ be a κ-complete ultrafilter on some set X. Let $\mathcal{V} \in V_\kappa$ be an ultrafilter a some set $\bar{X}$ (which must also belong to V_κ). Then*

(a) $i_\mathcal{V}(i_\mathcal{U}) = i_\mathcal{U} \restriction \mathrm{Ult}(V;\mathcal{V})$;
(b) $i_\mathcal{U}(i_\mathcal{V}) = i_\mathcal{V} \restriction \mathrm{Ult}(V;\mathcal{U})$;
(c) $i_\mathcal{U} \circ i_\mathcal{V} \restriction \mathrm{ON} = i_\mathcal{V} \circ i_\mathcal{U} \restriction \mathrm{ON}$.

Proof. Let $j = i_\mathcal{V}$ and $i = i_\mathcal{U}$.

(a) We will prove two technical facts and deduce (a) from them.

(i) *If $Y \in \mathrm{Ult}(V;\mathcal{V})$ and $Y \subseteq j(X)$, then*

$$Y \in j(\mathcal{U}) \leftrightarrow \{z \in X \mid j(z) \in Y\} \in \mathcal{U}.$$

To prove (i), first let Y be an element of $j(X)$. Let $f : \bar{X} \to \mathcal{P}(X)$ be such that $Y = \pi_{\mathcal{V}}([\![f]\!]_{\mathcal{V}})$. Let

$$K = \{x \in \bar{X} \mid f(x) \in \mathcal{U}\}.$$

By Theorem 3.2.5, $K \in \mathcal{V}$. Let

$$Z = \bigcap_{x \in K} f(x).$$

The κ-completeness of $\mathcal{U}$ implies that $Z \in \mathcal{V}$. The following chain of implications show that $j''Z \subseteq Y$.

$$\begin{aligned} z \in Z &\to (\forall x \in K)\, z \in f(x) \\ &\to \{x \mid z \in f(x)\} \in \mathcal{V} \\ &\to j(z) \in \pi_{\mathcal{V}}([\![f]\!]_{\mathcal{V}}) \\ &\to j(z) \in Y. \end{aligned}$$

Now let $Y \in \mathrm{Ult}(V;\mathcal{V})$, $Y \subseteq j(X)$, and $Y \notin j(\mathcal{U})$. Then $j(X) \setminus Y \in j(\mathcal{U})$. By what we have already proved, $\{z \in X \mid j(z) \notin Y\} \in j(\mathcal{U})$. Thus $\{z \in X \mid j(z) \in Y\} \notin \mathcal{U}$.

(ii) *For each* $F \in \mathrm{Ult}(V;\mathcal{V}) \cap {}^{j(X)}\mathrm{Ult}(V;\mathcal{V})$, *define* $\mathbf{\Phi}(F) : X \to \mathrm{Ult}(V;\mathcal{V})$ *by*

$$(\mathbf{\Phi}(F))(z) = F(j(z)).$$

Then for each $G \in {}^{X}\mathrm{Ult}(V;\mathcal{V})$ *there is an* $F \in \mathrm{Ult}(V;\mathcal{V}) \cap {}^{j(X)}\mathrm{Ult}(V;\mathcal{V})$ *such that* $F : j(X) \to \mathrm{Ult}(V;\mathcal{V})$ *and*

$$[\![\mathbf{\Phi}(F)]\!]_{\mathcal{U}} = [\![G]\!]_{\mathcal{U}}.$$

Let $G \in {}^{X}\mathrm{Ult}(V;\mathcal{V})$. Choose for each $z \in X$ an f_z such that $G(z) = \pi_{\mathcal{V}}([\![f_z]\!]_{\mathcal{V}}) = G(z)$. Define $h : \bar{X} \to {}^{X}V$ by

$$(h(x))(z) = f_z(x).$$

Let

$$F = \pi_{\mathcal{V}}([\![h]\!]_{\mathcal{V}}).$$

Clearly $F \in \mathrm{Ult}(V;\mathcal{V}) \cap {}^{j(X)}\mathrm{Ult}(V;\mathcal{V})$. For any $z \in X$,

$$\begin{aligned} F(j(z)) &= (\pi_{\mathcal{V}}([\![h]\!]_{\mathcal{V}}))(j(z)) \\ &= \pi_{\mathcal{V}}([\![f_z]\!]_{\mathcal{V}}) \\ &= G(z). \end{aligned}$$

To prove (a), let $\mathbf{\Phi}$ be as in (ii). For $F \in \mathrm{Ult}(V;\mathcal{V}) \cap {}^{j(X)}\mathrm{Ult}(V;\mathcal{V})$, set

$$\mathbf{\Phi}^*([\![F]\!]_{j(\mathcal{U})}) = [\![\mathbf{\Phi}(F)]\!]_{\mathcal{U}}.$$

To see that $\mathbf{\Phi}^*$ is well-defined, note that

$$\begin{aligned} [\![F_1]\!]_{j(\mathcal{U})} = [\![F_2]\!]_{j(\mathcal{U})} &\leftrightarrow \{y \in j(X) \mid F_1(y) = F_2(y)\} \in j(\mathcal{U}) \\ &\leftrightarrow \{z \in X \mid F_1(j(z)) = F_2(j(z))\} \in \mathcal{U} \\ &\leftrightarrow \{z \in X \mid (\mathbf{\Phi}(F_1))(z) = (\mathbf{\Phi}(F_2))(z)\} \in \mathcal{U} \\ &\leftrightarrow [\![\mathbf{\Phi}(F_1)]\!]_{\mathcal{U}} = [\![\mathbf{\Phi}(F_2)]\!]_{\mathcal{U}}. \end{aligned}$$

The second of the biconditionals is a consequence of (i). A similar chain of equivalences shows that

$$\mathbf{\Phi}^* : \prod_{j(\mathcal{U})}^{\mathrm{Ult}(V;\mathcal{V})} \mathrm{Ult}(V;\mathcal{V}) \prec \prod_{\mathcal{U}} \mathrm{Ult}(V;\mathcal{V}).$$

(See the proof of Lemma 8.3.3 below.) By (i), the elementary embedding $\mathbf{\Phi}^*$ is a surjection, and so it is an isomorphism. This in turn gives that $\pi_{\mathcal{U}} \circ \mathbf{\Phi}^* \circ (\pi_{j(\mathcal{U})}^{\mathrm{Ult(V;\mathcal{V})}})^{-1}$ is an isomorphism between $(j(i))(\mathrm{Ult}(V;\mathcal{V}))$ and $i(\mathrm{Ult}(V;\mathcal{V}))$. Since these two classes are transitive, they must be identical, and the isomorphism must be the identity. If $w \in \mathrm{Ult}(V;\mathcal{V})$, we have

$$\begin{aligned} (j(i))(w) &= \pi_{j(\mathcal{U})}^{\mathrm{Ult(V;\mathcal{V})}}([\![c_w]\!]_{j(\mathcal{U})}^{\mathrm{Ult}(V;\mathcal{V})}) \\ &= \pi_{\mathcal{U}}(\mathbf{\Phi}^*([\![c_w]\!]_{j(\mathcal{U})}^{\mathrm{Ult}(V;\mathcal{V})})) \\ &= \pi_{\mathcal{U}}([\![c_w]\!]_{\mathcal{U}}) \\ &= i(w). \end{aligned}$$

Here we have ambiguously used "c_w" for two distinct functions with constant value w.

The proof of (a) is now complete.

(b) The proof of (b) is simpler. Since $\mathcal{U}$ is κ-complete and since $\bar{X} \in V_\kappa$, Lemma 3.2.11 implies that V and $\mathrm{Ult}(V;\mathcal{U})$ have the same functions $f : \bar{X} \to \mathrm{Ult}(V;\mathcal{U})$. Since $i(\mathcal{V}) = \mathcal{V}$, we have that $\prod_{i(\mathcal{V})}^{\mathrm{Ult}(V;\mathcal{U})} \mathrm{Ult}(V;\mathcal{U})$ and $\prod_{\mathcal{V}} \mathrm{Ult}(V;\mathcal{U})$ are identical and that $i(j) \restriction \mathrm{Ult}(V;\mathcal{U}) = j \restriction \mathrm{Ult}(V;\mathcal{U})$.

(c) Let $\alpha \in \mathrm{ON}$. By (a) and by the elementarity of j, we have that

$$i(j(\alpha)) = (j(i))(j(\alpha)) = j(i(\alpha)).$$

(We could just as well have deduced (c) from (b) and the elementarity of i.) $\square$

From now through Theorem 8.3.6, let κ be a strong limit cardinal and let $\mathcal{T} \in V_\kappa$ be an iteration tree of length $\leq \omega$ on V with tree ordering T, extenders E_n, $n+1 < \ell\mathrm{h}(\mathcal{T})$, models M_n, $n < \ell\mathrm{h}(\mathcal{T})$, and embeddings $j_{m,n}$, $m\,T\,n < \ell\mathrm{h}(\mathcal{T})$. For $n+1 < \ell\mathrm{h}(\mathcal{T})$, let δ_n and λ_n be such that E_n is a (δ_n, λ)-extender in M_n.

The following lemma is a generalization of assertion (i) of the proof of Lemma 8.3.1, with the embedding $i_{\mathcal{V}}$ replaced by the embeddings $j_{0,n}$. (See Exercise 8.3.1, however.) Because the $j_{0,n}$ are the embeddings of an iteration as opposed to a single ultrapower, our proof is by induction. The individual steps are similar to the proof of Lemma 8.3.1. No extra problems are caused by (1) the fact that the $j_{(m+1)^-_T,m+1}$ come from extenders rather than ultrafilters or (2) the fact that the ultrapowers are of models to which the extenders to not belong.

Lemma 8.3.2. *Let $\mathcal{U}$ be a κ-complete ultrafilter on a set X. Let $n < \ell\mathrm{h}(\mathcal{T})$. If $Y \in M_n$ and $Y \subseteq j_{0,n}(X)$, then*

$$Y \in j_{0,n}(\mathcal{U}) \leftrightarrow \{z \in X \mid j_{0,n}(z) \in Y\} \in \mathcal{U}.$$

Proof. We prove the lemma (for fixed $\mathcal{U}$) by induction. It trivially holds for $n = 0$. Suppose that it holds for all $n' \leq n$. Let $m = (n+1)^-_T$.

First let Y be any element of $j_{0,n+1}(\mathcal{U})$. Let a and f be such that $Y = \pi^{\mathcal{M}_m}_{E_n}([\![a, f]\!]^{\mathcal{M}_m}_{E_n})$. Let

$$K = \{x \in [\delta_n]^{|a|} \mid f(x) \in j_{0,m}(\mathcal{U})\}.$$

By Theorem 6.3.15, we have that $K \in (E_n)_a$. By our induction hypothesis for m, choose for each $x \in K$ a set $Z_x \subseteq X$ such that

$$Z_x \in \mathcal{U} \ \wedge\ (\forall z \in Z_x)\, j_{0,m}(z) \in f(x).$$

Let $Z = \bigcap_{x \in K}(Z_x)$. The κ-completeness of $\mathcal{U}$ implies that $Z \in \mathcal{U}$. The following chain of implications shows that $j''_{0,n+1} Z \subseteq Y$.

$$\begin{aligned} z \in Z &\to (\forall x \in K)\, z \in Z_x \\ &\to (\forall x \in K)\, j_{0,m}(z) \in f(x) \\ &\to \{x \mid j_{0,m}(z) \in f(x)\} \in (E_n)_a \\ &\to j_{m,n+1}(j_{0,m}(z)) \in \pi^{\mathcal{M}_m}_{E_n}([\![a,f]\!]^{\mathcal{M}_m}_{E_n}) \\ &\to j_{0,n+1}(z) \in Y. \end{aligned}$$

Now let $Y \in M_n, Y \subseteq j_{0,n+1}(x)$, and $Y \notin j_{0,n+1}(\mathcal{U})$. Then $j_{0,n+1}(X) \setminus Y \in j_{0,n+1}(\mathcal{U})$. By what we have already proved, $\{z \in X \mid j_{0,n+1}(z) \notin Y\}$ belongs to $\mathcal{U}$. Thus $\{z \in x \mid j_{0,n+1}(z) \in Y\} \notin \mathcal{U}$. $\square$

The next lemma generalizes assertion (ii) of the proof of Lemma 8.3.1 in the way that Lemma 8.3.2 generalizes (i). The proof is in two ways more complicated than the proof of Lemma 8.3.1: Like the proof of Lemma 8.3.2, it proceeds by induction. (But see Exercise 8.3.1.) The fact that the E_n are extenders rather than ultrafilters occasions an additional use of the κ-completeness of $\mathcal{U}$.

Lemma 8.3.3. *Let $\mathcal{U}$ be a κ-complete ultrafilter on a set X. Let $n < \ell\mathrm{h}(\mathcal{T})$. For each $F \in M_n \cap {}^{j_{0,n}(X)}M_n$, define $\mathbf{\Phi}_n(F) : X \to M_n$ by*

$$(\mathbf{\Phi}_n(F))(z) = F(j_{0,n}(z)).$$

Then for each $G \in {}^X M_n$ there is an $F \in M_n \cap {}^{j_{0,n}(X)}M_n$ such that $F : j_{0,n}(X) \to M_n$ and

$$[\![\mathbf{\Phi}_\mathrm{n}(\mathrm{F})]\!]_{\mathcal{U}} = [\![\mathrm{G}]\!]_{\mathcal{U}}.$$

Proof. We prove the lemma by induction. The case $n = 0$ is trivial. Assume that the lemma holds for $\mathcal{U}$ for all $n' \leq n$. Let $m = (n+1)^-_T$.

Let $G \in {}^X M_{n+1}$. Choose for each $z \in X$ a pair $\langle a_z, f_z \rangle$ such that $G(z) = \pi^{\mathcal{M}_m}_{E_n}([\![a_z, f_z]\!]^{\mathcal{M}_m}_{E_n})$. By the κ-completeness of $\mathcal{U}$, let $a \in [\lambda_n]^{<\omega}$ be such that $\{z \mid a_z = a\} \in \mathcal{U}$. Thus

$$\{z \mid G(z) = \pi^{\mathcal{M}_m}_{E_n}([\![a, f_z]\!]^{\mathcal{M}_m}_{E_n})\} \in \mathcal{U}.$$

Define $G^* : X \to M_m$ by

$$G^*(z) = f_z.$$

By our induction hypothesis for M, let $F^* \in M_m \cap {}^{j}_{0,m}(X)$ be such that

$$[\![\mathbf{\Phi}_m(F^*)]\!]_{\mathcal{U}} = [\![G^*]\!]_{\mathcal{U}}.$$

Define $h : [\delta_n]^{|a|} \to M_m$ by letting $h(x) \in {}^{j_{0,m}(X)}M_m$ be given by

$$(h(x))(y) = (F^*(y))(x).$$

Let $F = \pi^{\mathcal{M}_m}_{E_n}([\![a, h]\!]^{\mathcal{M}_m}_{E_n})$. Clearly F belongs to $M_{n+1} \cap {}^{j_{0,n}(X)}M_{n+1}$. We must show that

$$\{z \in X \mid F(j_{0,m}(z)) = G(z)\} \in \mathcal{U}.$$

We know that

$$\{z \in X \mid \{z \mid G(z) = \pi^{\mathcal{M}_m}_{E_n}([\![a, f_z]\!]^{\mathcal{M}_m}_{E_n})\} \in \mathcal{U} \ \wedge \ F^*(j_{0,m}(z)) = G^*(z)\} \in \mathcal{U}.$$

Let z be any member of this set. For any $x \in j_{0,m}(X)$, we have that

$$\begin{array}{rcl} (h(x))(j_{0,m}(z)) & = & (F^*(j_{0,m}(z)))(x) \\ & = & (G^*(z))(x) \\ & = & f_z(x). \end{array}$$

Hence

$$\begin{array}{rcl} F(j_{0,n+1}(z)) & = & F(j_{m,n+1}(j_{0,m}(z))) \\ & = & \pi^{\mathcal{M}_m}_{E_n}([\![a, f_z]\!]^{\mathcal{M}_m}_{E_n}) \\ & = & G(z). \end{array}$$

$\square$

The next lemma and its corollary generalize part (a) of Lemma 8.3.1. Their proof is essentially the same as the proof of Lemma 8.3.1 from (i) and (ii).

Lemma 8.3.4. *Let $\mathcal{U}$ be a κ-complete ultrafilter on a set X. Let $i = i_{\mathcal{U}}$. Let $n < \ell\mathrm{h}(\mathcal{T})$. Define $\mathbf{\Phi}_n$ as in the statment of Lemma 8.3.3. For $F \in {}^{j_{0,n}(X)}M_n$, set*

$$\mathbf{\Phi}^*_n([\![F]\!]_{j_{0,n}(\mathcal{U})}) = [\![\mathbf{\Phi}_n(F)]\!]_{\mathcal{U}}.$$

Then $\mathbf{\Phi}_n^*$ *is well-defined and*

$$\mathbf{\Phi}_n^* : \prod_{j_{0,n}(\mathcal{U})}^{M_n} M_n \cong \prod_{\mathcal{U}} M_n.$$

Proof. Let $\varphi(v_1, \ldots, v_k)$ be a formula of the language of set theory and let $F_1, \ldots F_k$ be elements of $M_n \cap {}^{j_{0,n}(X)}M_n$. We have the following chain of equivalences, where Lemma 8.3.2 is used to get the third line from the second:

$$\begin{aligned}
&\textstyle\prod_{j_{0,n}(\mathcal{U})}^{M_n} M_n \models \varphi[[\![F_1]\!]_{j_{0,n}(\mathcal{U})}^{M_n}, \ldots, [\![F_k]\!]_{j_{0,n}(\mathcal{U})}^{M_n}] \\
&\quad \leftrightarrow \{y \in j_{0,n}(X) \mid M_n \models \varphi[F_1(y), \ldots, F_k(y)]\} \in j_{0,n}(\mathcal{U}) \\
&\quad \leftrightarrow \{z \in X \mid M_n \models \varphi[F_1(j_{0,n}(z)), \ldots, F_k(j_{0,n}(z))]\} \in \mathcal{U} \\
&\quad \leftrightarrow \{z \in X \mid M_n \models \varphi[(\mathbf{\Phi}_n(F_1))(z), \ldots, (\mathbf{\Phi}_n(F_k))(z)]\} \in \mathcal{U} \\
&\quad \leftrightarrow \textstyle\prod_{\mathcal{U}} M_n \models \varphi[[\![\mathbf{\Phi}_n(F_1)]\!]_{\mathcal{U}}, \ldots, [\![\mathbf{\Phi}_n(F_k)]\!]_{\mathcal{U}}].
\end{aligned}$$

Taking $v_1 = v_2$ for φ, we see that $\mathbf{\Phi}_n^*$ is well-defined. Taking φ as arbitrary, we then see that $\mathbf{\Phi}_n^* : \prod_{j_{0,n}(\mathcal{U})}^{M_n} M_n \prec \prod_{\mathcal{U}} M_n$.

By Lemma 8.3.3, the elementary embedding $\mathbf{\Phi}_n^*$ is a surjection, and so it is an isomorphism. □

Corollary 8.3.5. *Let* $\mathcal{U}$ *be a* κ*-complete ultrafilter on a set* X*. Let* $i = i_{\mathcal{U}}$*. Let* $n < \ell\text{h}(\mathcal{T})$*. Then*

(i) $(j_{0,n}(i))(M_n) = i(M_n)$*;*

(ii) $j_{0,n}(i) = i \restriction M_n$*.*

Proof. (i) follows immediately from Lemma 8.3.4. For (ii), note that, for $w \in M_n$,

$$\begin{aligned}
((j_{0,n}(i))(w) &= \pi_{j_{0,n}(\mathcal{U})}^{M_n}([\![c_w]\!]_{j_{0,n}(\mathcal{U})}^{M_n}) \\
&= \pi_{\mathcal{U}}(\mathbf{\Phi}_n^*([\![c_w]\!]_{j_{0,n}(\mathcal{U})}^{M_n})) \\
&= \pi_{\mathcal{U}}([\![c_w]\!]_{\mathcal{U}}) \\
&= i(w),
\end{aligned}$$

where we have once more ambiguously used "c_w" for two distinct constant functions. □

The theorem that follows is analogous to Lemma 8.3.1. Besides the extra complications due to the replacement of the $\mathcal{V}$ of Lemma 8.3.1 by the $j_{0,n}$, there are other complications. Part (b) of the theorem asserts that i acts trivially on the $j_{m,n}$, not just on the $j_{0,n}$. Part (a) involves not just $\mathcal{U}$ but also another ultrafilter $\mathcal{U}'$ that projects to $\mathcal{U}$. It says that the $j_{0,n}$ act trivially on the canonical embedding $i^* : \mathrm{Ult}(V;\mathcal{U}) \prec \mathrm{Ult}(V;\mathcal{U}')$.

Theorem 8.3.6. *Let $\mathcal{U}$ and $\mathcal{U}'$ be κ-complete ultrafilters on sets X and X' respectively. Let $i = i_{\mathcal{U}}$ and let $i' = i_{\mathcal{U}'}$. Assume that $\mathcal{U}'$ projects to $\mathcal{U}$ by $\chi : X' \to X$. (See page 199.) Let $i^* = i_{\mathcal{U},\mathcal{U}'} : \mathrm{Ult}(V;\mathcal{U}) \prec \mathrm{Ult}(V;\mathcal{U}')$. Let $m\,T\,n < \ell\mathrm{h}(\mathcal{T})$. Then*

(a) $j_{0,n}(i^*) = i^* \restriction j_{0,m}(\mathrm{Ult}(V;\mathcal{U})) = i^* \restriction i(M_n)$;

(b) $i(j_{m,n}) = j_{m,n} \restriction i(M_m) = j_{m,n} \restriction j_{0,m}(\mathrm{Ult}(V;\mathcal{U}))$;

(c) $i^* \circ j_{m,n} \restriction \mathrm{ON} = j_{m,n} \circ i^* \restriction \mathrm{ON}$.

Proof. Note first that

$$\begin{aligned} j_{0,n}(\mathrm{Ult}(V;\mathcal{U})) &= j_{0,n}(i(V)) \\ &= (j_{0,n}(i))(j_{0,n}(V) \\ &= (j_{0,n}(i))(M_n) \\ &= i(M_n), \end{aligned}$$

where the last line follows from its predecessor by Corollary 8.3.5.

(a) Define $\mathbf{\Phi}_n$ as in the statment of Lemma 8.3.3. Similarly define $\mathbf{\Phi}'_n$ from $\mathcal{U}'$. Let $x \in j_{0,n}(\mathrm{Ult}(V;\mathcal{U})) = i(M_n)$. By Let $F \in {}^{j_{0,n}(X)}M_n$ be such that $x = \pi^{M_n}_{j_{0,n}(\mathcal{U})}([\![F]\!]^{M_n}_{j_{0,n}(\mathcal{U})})$. By Lemma 8.3.4,

$$\pi^{M_n}_{j_{0,n}(\mathcal{U})}([\![F]\!]^{M_n}_{j_{0,n}(\mathcal{U})}) = \pi_{\mathcal{U}}([\![\mathbf{\Phi}(F)]\!]_{\mathcal{U}}).$$

We have that

$$\begin{aligned} (j_{0,n}(i^*))(\pi^{M_n}_{j_{0,n}(\mathcal{U})}(x)) &= (j_{0,n}(i^*))(\pi^{M_n}_{j_{0,n}(\mathcal{U})}([\![F]\!]^{M_n}_{j_{0,n}(\mathcal{U})}) \\ &= \pi^{M_n}_{j_{0,n}(\mathcal{U}')}([\![F \circ j_{0,n}(\chi)]\!]^{M_n}_{j_{0,n}(\mathcal{U}')}). \end{aligned}$$

But we also have that

$$i^*(x) \;=\; i^*(\pi_{\mathcal{U}}([\![\mathbf{\Phi}(F)]\!]_{\mathcal{U}}))$$

$$\begin{aligned}
&= \pi_{\mathcal{U}}([\![\mathbf{\Phi}(F) \circ \chi]\!]_{\mathcal{U}'}) \\
&= \pi_{\mathcal{U}}([\![F \circ j_{0,n} \circ \chi]\!]_{\mathcal{U}'}) \\
&= \pi_{\mathcal{U}}([\![F \circ j_{0,n}(\chi) \circ j_{0,n}]\!]_{\mathcal{U}'}) \\
&= \pi_{\mathcal{U}}([\![\mathbf{\Phi}'(F \circ j_{0,n}(\chi))]\!]_{\mathcal{U}'}) \\
&= \pi^{M_n}_{j_{0,n}(\mathcal{U}')}([\![F \circ j_{0,n}(\chi)]\!]^{M_n}_{j_{0,n}(\mathcal{U}')}).
\end{aligned}$$

(b) We proceed by induction on n. Suppose (b) holds for all m' and n' such that $m' \, T \, n' \leq n$. Let $\bar{m} = (n+1)^-_T$. Since $\mathcal{U}$ is κ-complete, it follows from Lemma 3.2.11 that V and $\mathrm{Ult}(V;\mathcal{U})$ have the same functions from ${}^{<\omega}\kappa$ to $\mathrm{Ult}(V;\mathcal{U})$. By the elementarity of $j_{0,\bar{m}}$, this implies that $M_{\bar{m}}$ and $j_{0,\bar{m}}(\mathrm{Ult}(V;\mathcal{U})$ have the same functions from ${}^{<\omega}j_{0,\bar{m}}(\kappa)$ to $j_{0,m}(\mathrm{Ult}(V;\mathcal{U}))$, i.e., to $i(M_{\bar{m}})$. It follows that $M_{\bar{m}}$ and $i(M_{\bar{m}})$ have the same functions from ${}^{<\omega}\delta_n$ to $i(M_{\bar{m}})$. Since $i(E_n) = E_n$ and since $i(M_{\bar{m}})$ is the domain of $i(j_{\bar{m},n+1}$, we get that

$$i(j_{\bar{m},n+1}) = j_{\bar{m},n+1} \restriction i(M_{\bar{m}}).$$

To finish the proof of (b) for $n+1$, let $m \, T \, n+1$. By our induction hypothesis, we have that

$$i(j_{m,\bar{m}}) = j_{m,\bar{m}} \restriction i(M_m).$$

Thus

$$\begin{aligned}
i(j_{m,n+1}) &= i(j_{\bar{m},n+1} \circ j_{m,\bar{m}}) \\
&= i(j_{\bar{m},n+1}) \circ i(j_{m,\bar{m}}) \\
&= (j_{\bar{m},n+1} \restriction i(M_{\bar{m}})) \circ i(j_{m,\bar{m}}) \\
&= (j_{\bar{m},n+1} \restriction i(M_{\bar{m}})) \circ (j_{m,\bar{m}} \restriction i(M_m)) \\
&= (j_{\bar{m},n+1} \circ j_{m,\bar{m}}) \restriction i(M_{\bar{m}}) \\
&= j_{m,n+1} \restriction i(M_m).
\end{aligned}$$

(c) Let $\alpha \in \mathrm{ON}$. By two instances of (a) and the elementarity of $j_{m,n}$, we have that

$$\begin{aligned}
i^*(j_{m,n}(\alpha)) &= (j_{0,n}(i^*))(j_{m,n}(\alpha)) \\
&= j_{m,n}((j_{0,m}(i^*))(\alpha)) \\
&= j_{m,n}(i^*(\alpha)).
\end{aligned}$$

$\square$

Let us now apply Theorem 8.3.6 to show that the construction of the beginning of this section gives the homogeneity of $U^\ddagger$. In the notation of that construction and the subsequent discussion, let $p \in T$ and let $m < m' < \ell\mathrm{h}(p)$ with $r_m \subseteq r_{m'}$. By the discussion on page 473, it the γ-homogeneity of $U \restriction \alpha$ will will follow if we can prove that

$$\beta_{p\restriction m'} < (j^p_{0,2m'}(i_{p;m,m'}))(j^p_{2m,2m'}(\beta_{p\restriction m})).$$

By condition (iv) of our construction and by the discussion on page 473, we have that

$$\beta_{p\restriction m'} < (i_{p;0,m'}(j^p_{2m,2m'}))(i_{p;m,m'}(\beta_{p\restriction m})).$$

By part (b) of Theorem 8.3.6, we get that

$$\beta_{p\restriction m'} < j^p_{2m,2m'}(i_{p;m,m'}(\beta_{p\restriction m})).$$

By part (c) of Theorem 8.3.6, we get that

$$\beta_{p\restriction m'} < i_{p;m,m'}(j^p_{2m,2m'}(\beta_{p\restriction m}).$$

Finally we apply part (a) of Theorem 8.3.6 to get the desired inequality.

The second way to use our construction to get the homogeneous Souslinness of $U^\ddagger$ is due to Katrin Windßus.

Let T be any game tree and let $A \subseteq [T]$. Let λ be an infinite cardinal number. Say that A of $U^\ddagger$ has a *λ-closed embedding normal form*, if there is a system

$$(\langle M_p \mid p \in T\rangle, \langle k_{p_1,p_2} \mid p_1 \subseteq p_2 \in T\rangle)$$

witnessing that A has an embedding normal form and such that

$$(\forall p \in T)\, {}^\lambda M_p \subseteq M_p.$$

Here is Windßus' theorem:

Theorem 8.3.7. *Let T be a game tree, let $A \subseteq [T]$, and let*

$$(\langle M_p \mid p \in T\rangle, \langle k_{p_1,p_2} \mid p_1 \subseteq p_2 \in T\rangle)$$

witness that A has a $2^{\aleph_0}$-closed embedding normal form. Let γ be a cardinal number such that $\gamma \geq \mathrm{crit}\, k_{p_1,p_2}$ for all p_1 and $p_2 \in T$ with $p_1 \subsetneq p_2$.

Then A is γ-homogeneously Souslin.

Proof. For $p \in T$, let A_p be the T-projection of $T[p]$. Let $\alpha \in \mathrm{Ord}$. Let U_α be the set of all $\langle\!| p, u |\!\rangle$ such that $p \in T$, $u \in {}^{\ell\mathrm{h}(p)}V$, and, for all m and n smaller than $\ell\mathrm{h}(p)$,

(a) $u(m) : [T[x \restriction m]] \setminus A_{p \restriction m} \to \alpha$;

(b) $m < n \to (\forall x \in [T[x \restriction n]] \setminus A_{p \restriction n})\, (u(n))(x) < (u(m))(x)$.

We first show how to define a tree whose T-projection is A. This construction will make no use of the given embedding normal form.

Let α be infinite. Let $\langle\!| x, f |\!\rangle \in [U_\alpha]$. We show that A is the T-projection of U_α. Let $x \in [T]$. First assume $\langle\!| x, f |\!\rangle \in [U]$ and that $x \in [T] \setminus A$. Then $x \in \operatorname{domain} f(n)$ for every $n \in \omega$ and $(f(n))(x) < (f(m))(x)$ whenever $m < n \in \omega$. This is a contradiction. Now assume that $x \in A$. For $n \in \omega$ and $y \in [T[x \restriction m]] \setminus A_{p \restriction m})$, let $(f(m))(y)$ be $n - m$, where n is the least number such that $x(n) \neq y(n)$. Then $\langle\!| x, f |\!\rangle \in [U]$.

We now use our embedding normal form to prove that U_α is homogeneous.

For each $x \in [T]$, let $(\mathcal{M}_x, \langle k^x_{x \restriction m} \mid m \in \omega \rangle)$ be the direct limit of $(\langle M_{x \restriction m} \mid M \in \omega \rangle, k_{x \restriction m, x \restriction n} \mid m < n \in \omega \rangle)$.

Let $X \in [T] \setminus A$. Let $\langle z_n \mid n \in \omega \rangle$ be an infinite descending sequence of ordinals of the illfounded model $\mathcal{M}_x$. For each $n \in \omega$, let β^x_n and k_n be such that $z_n = k^x_{x \restriction k_n}(\beta^x_n)$. We may assume that the sequence $\langle k_n \mid n \in \omega \rangle$ is strictly increasing. Multiplying the given β^x_n by ω if necessary, we may assume that each β^x_n is a limit ordinal. Filling in if necessary, we may assume that $k_n = n$ for all n. Thus the sequence β^x_n has the property that

$$(\forall m \in \omega)(\forall n \in \omega)(m < n \to \beta^x_n < k_{x \restriction m, x \restriction n}(\beta^x_m)).$$

Let α be an ordinal larger than all the ordinals β^x_n.

For each $p \in T$, let $g_p : [T[p]] \setminus A \to \mathrm{Ord}$ be defined by setting

$$g_p(x) = \beta^x_{\ell\mathrm{h}(p)}$$

for all $x \in [T[p]] \setminus A$. Since ${}^{2^{\aleph_0}}M_p \subseteq M_p$, the function g_p belongs to M_p.

For $p \in T$, let

$$\mathcal{U}_p = \{X \subseteq (U_\alpha)[p] \mid \langle k_{p \restriction m, p}(g_{p \restriction m}) \mid m < \ell\mathrm{h}(p) \rangle \in k_{\emptyset, \ell\mathrm{h}(p)}(X)\}.$$

To see that $U[p] \in \mathcal{U}_p$, observe that, for m and $n < \ell\mathrm{h}(p)$ and $x \in [T[p]] \setminus A$,

$$\begin{aligned} m < n \quad &\to \quad \beta^x_n < k_{x \restriction m, x \restriction n}(\beta^x_m) \\ &\to \quad k_{p \restriction n, p}(\beta^x_n) < k_{p \restriction m, p}(\beta^x_n) \\ &\to \quad k_{p \restriction n, p}(g_{p \restriction n}(x)) < k_{p \restriction m, p}(g_{p \restriction m}(x)). \end{aligned}$$

By arguments like those in the last parts of the proofs of Theorems 8.2.6 and 8.2.7, one can show that $\langle \mathcal{U}_p \mid p \in T \rangle$ witnesses that U_α is γ-homogeneous. $\square$

Using Theorem 6.3.7, it is easy to prove a strengthened One-Step Lemma in which it is demanded that if the models $\gamma \leq \delta$, ${}^{2^{\aleph_0}}M \subseteq M$, and ${}^{2^{\aleph_0}}N \subseteq N$, then ${}^{2^{\aleph_0}}(\mathrm{Ult}(N; E)) \subseteq \mathrm{Ult}(N; E)$. (In [Martin and Steel, 1989], this is done, but with the unimportant difference that $\aleph_0$ replaces $2^{\aleph_0}$.) If we do the construction of the beginning of this section using this version of the One-Step Lemma, then all the $\mathcal{M}_p^{\mathcal{S}}$ will satisfy ${}^{2^{\aleph}_0}\mathcal{M}_p^{\mathcal{S}} \subseteq \mathcal{M}_p^{\mathcal{S}}$. Thus the construction will yield a $2^{\aleph_0}$-closed embedding normal form for the T-projection of $U^{\ddagger}$. Thus Theorem 8.3.7 will yield Theorems 8.2.8 and 8.2.9.

In [Koepke, 1998], the Windßus' construction is used to prove the results of this chapter without any use of homogeneous trees. This is done by propagating directly the property of having an embedding normal form.

The hypothesis of $2^{\aleph_0}$-closure cannot be omitted from Theorem 8.3.7. This seems to have been noted by several people, including Menachem Magidor and Koepke. See Exercise 8.3.2.

In [Neeman, 1995], [Neeman, 2002], [Neeman, 2004], [Neeman, 2010], and [Neeman, 2007], Neeman develops machinery that starts with something like the Martin-Steel construction but goes far beyond it. With this machinery, he is able to prove determinacy for large classes of games. These include not just larger classes of ordinary ω-length games, but also classes of games with transfinitely many moves, even certain games of length ω_1. Furthermore his theorems have large-cardinal hypotheses that are provably optimal.

Exercise 8.3.1. Let $\mathcal{T}$ be an iteration tree on M and let $n \in \omega$. Prove that there is an extender E in the sense of Exercise 6.1.2 such that $E \in V_\kappa$ and $j_{0,n}^{\mathcal{T}} = i_E$. Using this result one can eliminate the use of induction from Lemmas 8.3.2 and 8.3.3.

Hint. Let λ be such that each $E_m^{\mathcal{T}}$, $m < n$, is in $M_m^{\mathcal{T}}$ a (δ, λ') extender for some $\lambda' \leq \lambda$. Prove inductively that every element of M_n is of the form $(j_{0,n}^{\mathcal{T}}(f))(a)$ for some $a \in [\lambda]^{<\omega}$. (The basic fact is that

$$[\![b, g]\!]_{E_m^{\mathcal{T}}}^{M_{(m+1)_T^-}} = (j_{(m+1)_T^-, m+1}^{\mathcal{T}}(g))(b).$$

Use this fact to deduce that $j_{0,n} = i_E$, where E the (generalized) $(\mathrm{crit}\,(j_{0,n}), \lambda)$-extender derived from $j_{0,n}$.

Exercise 8.3.2. Assume that there is a measurable cardinal and prove that every $A \subseteq {}^\omega\omega$ has an embedding normal form.

Hint. (The following example is from [Koepke, 1998]. The result was probably first proved by Magidor.) Let $\alpha \mapsto x_\alpha$ be a one-one correspondence between $2^{\aleph_0}$ and ${}^\omega\omega$. Let $A \subseteq {}^\omega\omega$. Let κ be measurable and let $\mathcal{U}$ be a uniform normal ultrafilter on κ. Let $i = i_{\mathcal{U}}$. Let $N = \mathrm{Ult}_{2^{\aleph_0}}(V;\mathcal{U})$. For $s \in {}^{<\omega}\omega$, and $\alpha \leq 2^{\aleph}_0$, define $j^s_{\alpha,\beta} : N \prec N$ inductively as follows. Let j^s_0 be the identity. Let

$$j^s_{\alpha,\alpha+1} = \begin{cases} i_{\omega\alpha} & \text{if } s \subseteq x_\alpha \text{ and } x_\alpha \notin A; \\ \text{the identity} & \text{otherwise.} \end{cases}$$

(See §3.3 for definitions.) For $\gamma < \alpha$ let

$$j^s_{\gamma,\alpha+1} = j^s_{\alpha,\alpha+1} \circ j^s_{\gamma,\alpha}.$$

For limit ordinals $\lambda \leq 2^{\aleph_0}$, let

$$(N, \langle j^s_{\alpha,\lambda} \mid \alpha < \lambda \rangle)$$

be the direct limit of the system

$$(\langle N \mid \alpha < \lambda \rangle, \langle j^s_{\alpha,\beta} \mid \alpha \leq \beta < \lambda \rangle),$$

which one can verify inductively to be directed. Now let $M_\emptyset = V$ and $M_s = N$ for $s \in {}^{<\omega}\omega \setminus \emptyset$. For $s \subseteq t \in {}^{<\omega}\omega$ with $\ell\mathrm{h}(t) = \ell\mathrm{h}(s) + 1$, let

$$k_{s,t} = \begin{cases} i_{0,2^{\aleph_0}} & \text{if } s = \emptyset; \\ j^s_{0,2^{\aleph_0}} & \text{if } s \neq \emptyset. \end{cases}$$

Define $k_{s,t}$ for other $s \subseteq t \in {}^{<\omega}\omega$ by commutativity. Show that

$$(\langle M_s \mid s \in {}^{<\omega}\omega \rangle, \langle j_{s,t} \mid s \subseteq t \in {}^{<\omega}\omega \rangle)$$

gives an embedding normal form for A.

Exercise 8.3.3. Recall that a *cone of degrees of unsolvability* is the set of all degrees of unsolvability above some particular degree. For any degree of unsolvability $\mathbf{d}$, let $\mathrm{T}(\mathbf{d})$ be the set of sentences true in class models $(L[x])[\mathbf{H}]$ where x has degee $\mathbf{d}$ and $\mathbf{H}$ is $\mathbf{Coll}(\omega, \omega_1^{L[x]})$-generic over $L[x]$. (See page 539.)

Assume that, for each positive integer n, there is a transitive proper class M and a countable ordinal κ such that

(1) M is a model of ZF + $V = L(V_\kappa)$ + DC_κ + "There is no $\kappa' < \kappa$ such that κ' is Woodin" + "κ is Woodin"; i.e., M is as in Exercises 7.4.7, 7.4.8, 7.4.9, 7.4.10, and 7.4.13 and κ is Woodin in M;

(2) M is n-iterable.

(See Exercise 7.4.10.) Use the result of Exercise 7.4.13 to show that $\mathrm{T}(\mathbf{d})$ is constant on a cone of degrees.

This result is due to Woodin, who has also proved that if $\mathrm{T}(\mathbf{d})$ is constant on a cone of degrees, then all Π^1_2 games in ${}^{<\omega}\omega$ are determined. Using the fact that Exercise 7.4.10 and a simple forcing argument show that the consistency of ZFC + "There is a Woodin cardinal" gives the consistency of the hypothesis of the present exercise, Woodin also deduced the following theorem:

> If ZFC + "There is a Woodin cardinal" is consistent, then so is ZFC + "All Π^1_2 games in ${}^{<\omega}\omega$ are determined."

Woodin has proved the converse of this theorem, so it is an equiconsistency result. (See part (4) of the hint to Exercise 9.6.4.)

Hint. Fix $n \in \omega$, and let M be as given by the assumption for $n+1$. Let

$$A_n = \{x \in {}^\omega 2 \mid \kappa < \omega_1^{L[x]} \wedge V_\kappa^M \in L[x]\}.$$

It suffices to show that, for $\mathbf{H}$ as above, the set of Σ_n sentences true in $L[x]$ is constant on A_n. The set $\mathcal{A}_n$ of degrees of members of A_n contains a cone. Hence $\bigcap_n \mathcal{A}_n$ contains a cone, and we will have that $\mathrm{T}(\mathbf{d})$ is constant on $\bigcap_n \mathcal{A}_n$.

Argue as follows. For $x \in A$, let $\mathcal{T}_x$ be the iteration tree given by Exercise 7.4.13. Let $\mathbf{H}$ be as above. Since $|\mathcal{P}^{M(\mathcal{T}_x),\mathcal{E}(\mathcal{T}_x)}|^{M(\mathcal{T}_x)} = \omega_1^{L[x]}$, it follows by Lemma 25.11 of Jech [1978] that there is an $\mathbf{H}'$ such that $\mathbf{H}'$ is $\mathbf{Coll}(\omega, \omega_1^{L[x]})$-generic over $M(\mathcal{T}_x)$ and

$$(L[x])[\mathbf{H}] = (M(\mathcal{T}_x))[\mathbf{G}_x^{M(\mathcal{T}_x),\mathcal{E}(\mathcal{T}_x)}])[\mathbf{H}] = (M(\mathcal{T}_x))[\mathbf{H}'].$$

Hence for $x \in A$ the set of Σ_n sentences true in $(L[x])[\mathbf{H}]$ depends only on the set of Σ_n sentences true in $M(\mathcal{T}_x)$. Since M is $(n+1)$-iterable, this is the same as the set of Σ_n sentences true in M.

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