MATHEMATICS 61 SPRING 2010

Second Midterm Examination

May 17, 2010

Name: ____

Signature: ______________________________________

UCLA ID Number: ______________________________

Instructions:

- No calculators, books, or notes are allowed.
- Answer all 6 questions.
- Use only the scratch paper provided.

1 (17 points). Suppose we have a red marker, a blue marker, and a yellow marker, and 15 balls. Suppose we are going to mark each ball with one of the markers.

(a) How many ways are there to do it if the balls are the balls are identical and the red marker has to be used at least once.

(b) How many ways are there to do it if the balls are considered as distinct and we must use each marker five times.

Solution.

(a) It is the number of unordered, 14-element selections from a set with three members: $C(14+3-1, 3-1) = C(16, 2)$.

(b) Let the balls be $B_1, B_2, \ldots, B_n$. The number of ways to mark the balls is the same as the number of ways to form strings using the letters $R\ R\ R\ R\ R\ B\ B\ B\ B\ B\ Y\ Y\ Y\ Y\ Y$:

$$\frac{15}{5!5!5!}.$$

2 (17 points). Charles will work 185 days in 2010, and he will not work on December 31. Prove that there will be a day when he works such that he also works five days later. There are 365 days in the year.

Solution. Let the days when he works be days of the year number a_1, a_2, $\ldots$, and a_{185}. The numbers

$$a_1, \ldots, a_{185}, a_1 + 5, \ldots, a_{185} + 5$$

are all < 370. If they were distinct, then there would have to be $185 + 185 = 370$ of them. Since this is impossible, two of them must be the same. This implies that $a_i = a_j + 5$ for some i and j.

3 (16 points). Solve the recurrence relation $4a_n = 4a_{n-1} - a_{n-2}$ for the initial conditions $a_0 = a_1 = 1$.

Solution. The only root of $4x^2 - 4x + 1 = 0$ is $1/2$. Thus the solution of the recurrence relation is

$$a_n = \frac{b}{2^n} + \frac{dn}{2^n},$$

for some b and d. For $n = 0$, we get

$$1 = b.$$

For $n = 1$, we get

$$1 = \frac{b}{2} + \frac{d}{2}.$$

Hence $d = 1$. Thus the solution is

$$a_n = \frac{1+n}{2^n}.$$

4 (17 points). Let v and w be the two vertices of $K_{3,2}$ that are adjacent to each of the remaining three vertices. Get G from $K_{3,2}$ by adding an edge incident on v and w.

(a) Does G have an Euler cycle?

(b) Does G have a Hamiltonian cycle?

Prove your answers.

Solution:

(a) Yes, because every vertex has even degree and the graph is connected.

(b) No. There are several ways to prove this. Here is one. In a Hamiltonian cycle, each vertex has degree 2. The three vertices other than v and w have degree 2 in G. For any one of these vertices, both edges incident on it must belong to any Hamiltonian cycle in G. This means that v and w must have degree at least three in any Hamiltonian cycle. Hence there can be no Hamiltonian cycle.

5 (16 points). Draw all non-isomorphic connected graphs with 3 edges and no parallel edges (but they can have loops). In other words, draw some connected graphs with 3 edges and no parallel edges and make sure that (i) no two of them are isomorphic and (ii) every connected graph with three edges and no parallel edges is isomorphic to one of them.

Solution. I'll describe the graphs instead of drawing them.

There is one with two vertices. It has an edge incident on the two vertices and a loop on each vertex.

There are three with three vertices. One has a simple cycle of length three. The other two have one vertex v that is adjacent to both of the others, which are not adjacent to one another. One of the two has a loop on v; the other has a loop on one of the other vertices.

There are two with four vertices. One of them has a vertex v that is adjacent to each of the others. The other one is of the form ·—·—·—·.

In all there are 6 non-isomorphic graphs.

6 (17 points). Write the adjacency matrix A of the graph of Problem 4. Put the vertices in the order v, w, x, y, z where v and w are as in Problem 4. What is the entry in row 1, column 2 of A^3? Explain your answer, and not in just a sentence.

Solution. Here is the matrix A:

$$\begin{array}{c|ccccc} & v & w & x & y & z \\ v & 0 & 1 & 1 & 1 & 1 \\ w & 1 & 0 & 1 & 1 & 1 \\ x & 1 & 1 & 0 & 0 & 0 \\ y & 1 & 1 & 0 & 0 & 0 \\ z & 1 & 1 & 0 & 0 & 0 \end{array}$$

The number in row 1, column 2 in A^3, is the number of paths from v to w in G of length 3.

There is one path of length 3 that does not contain x, y, or z.

There are three other paths of length 3 whose second vertex is w. There are three others, with second vertex x, y, or z and third vertex v.

Thus the number asked for is $1 + 3 + 3 = 7$.