MATHEMATICS 61 SPRING 2010

First Midterm Examination

April 19, 2010

Name: ______________________________

Signature: ______________________________

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Instructions:

- No calculators, books, or notes are allowed.
- Answer all 6 questions.
- Use only the scratch paper provided.

1 (18 points). Prove that, for every positive integer n,

$$\frac{2}{3n} \leq \frac{\Pi_{i=1}^{n}(3i-1)}{\Pi_{i=1}^{n} 3i}.$$

Solution. We use induction.

Basis Step:

$$\frac{\Pi_{i=1}^{1}(3i-1)}{\Pi_{i=1}^{1} 3i} = \frac{3 \cdot 1 - 1}{3 \cdot 1} = \frac{2}{3} = \frac{2}{3 \cdot 1}.$$

Inductive Step: Assume that the statement is true for n.

$$\begin{aligned}
\frac{\Pi_{i=1}^{n}(3i-1)}{\Pi_{i=1}^{n} 3i} \quad &= \quad \frac{\Pi_{i=1}^{n}(3i-1)}{\Pi_{i=1}^{n} 3i} \cdot \frac{3n+2}{3(n+1)} \\
&\geq \quad \frac{2}{3n} \cdot \frac{3n+2}{3(n+1)} \\
&=\geq \quad \frac{3n+2}{3n} \cdot \frac{2}{3(n+1)} \\
&\geq \quad \frac{2}{3(n+1)}.
\end{aligned}$$

The first $\geq$ is by the induction hypothesis.

2 (17 points). Find a positive integer k such that:

(a) postage of k cents or more can be achieved by using only 3 and 8 cent stamps;

(b) k is the least number with property (a).

Prove that your k satisfies both (a) and (b).

Solution. To find the least k that works, we look for three consecutive numbers that are of the form $3a + 8b$ for non-negative integers a and b. We find that the first such numbers are 14, 15, and 16. It is easy to see that 13 is not of the form, and

$$\begin{aligned} 14 &= 3 \cdot 2 + 8; \\ 15 &= 3 \cdot 5; \\ 16 &= 8 \cdot 2. \end{aligned}$$

Let $k = 14$. To show that 14 satisfies (a), we use the strong form of induction. Let $n \geq 14$. Assume that every number m with $14 < n$ is of the form $3a+8b$. We have already proved that n has this form if n is 14, 15, or 16. If $n > 16$, then $n-3 \geq 14$, and so the induction hypothesis implies that $n-3 = 3a+8b$ for some a and b. Thu $n = 3(a + 1) + 8b$.

Because 13 is not of the required form, no number < 14 satisfies (a), so 14 satisfies (b).

3 (16 points). Let $\mathbb{Z}^+$ be the set of all positive integers. Let R be the relation on $Z^+ \times Z^+$ defined as follows:

$$(m,n)R(m',n') \Leftrightarrow \begin{cases} (1)\ \max(m,n) < \max(m',n') \ \text{ or} \\ (2)\ \max(m,n) = \max(m',n') \ \&\ m < m' \ \text{ or} \\ (3)\ \max(m,n) = \max(m',n') \ \&\ m = m' \ \&\ n \le n'. \end{cases}$$

Is R reflexive? Is it symmetric? Is it antisymmetric? Is it transitive? Is it a partial order? Is it a total order? Sketch the proofs of your answers.

Solution. We first show:

(†) For any two pairs (distinct or not) one of the pairs is related to the other in R, i.e., any two pairs are comparable.

If $\max(m,n) \neq \max(m',n')$, then one of the pairs is related to the other because of (1). If $\max(m,n) = \max(m',n')$ and $m \neq m'$, then one of the pairs is related to the other because of (2). If $\max(m,n) = \max(m',n')$ and $m = m'$, then one of the pairs is related to the other because of (3).

Note that (†) implies that R is reflexive.

Next we show that no two distinct pairs are both related to the other, i.e., that R is antisymmetric (and so not symmetric). Assume that $(m,n)R(m',n')$ and $(m',n')R(m,n)$. Then $\max(m,n) \le \max(m',n')$ and $\max(m',n') \le \max(m,n)$. This implies that $\max(m,n) = \max(m',n')$. $(m,n)R(m',n')$ and $(m',n')R(m,n)$ thus imply that $m \le m'$ and $m' \le m$. Hence $m = m'$. But then $(m,n)R(m',n')$ and $(m',n')R(m,n)$ imply that $n \le n'$ and $n' \le n$, which means that the two pairs are not distinct.

Next we show that R is transitive. Suppose that $(m_1,n_1)R(m_2,n_2)$ and $(m_2,m_2)R(m_3,n_3)$ but it is not the case that $(m_1,m_1)R(m_3,m_3)$. By (†), $(m_3,n_3)R(m_1,n_1)$. Arguing as in the preceding paragraph, we get that $\max(m_1,n_1) \le \max(m_2,n_2) \le \max(m_3,n_3) \le \max(m_1,n_1)$. This implies that the three maxima are the same. From this we deduce that $m_1 \le m_2 \le m_3 \le m_1$, which implies that $m_1 = m_2 = m_3$. Hence $n_1 \le n_2 \le n_3 \le n_1$, which implies the contradiction that there three pairs are all the same.

Since R is reflexive, antisymmetric, and transitive, it is a partial order. By (†), it is a total order.

Comments on Problem 3:

I have given a detailed proof. The problem only asks for a sketch of a proof. An example of what I had in mind is: the observation that (3) implies that R is reflexive; a counterexample to symmetry; statements that R has comparability, antisymmetry, and transitivity, with small indications about why; a statement that R is a partial order and a total order.

Common misconceptions that showed up were thinking that "antisymmetric" is synonymous with "non-symmetric" and thinking that a total order has to be symmetric.

The relation R is called the *Gödel order*. The *lexicographic* order on $\mathbb{Z}^+ \times \mathbb{Z}^+$ is just the alphabetical order on all words of length 2 on the alphabet $\mathbb{Z}^+$ with the obvious ordering of the alphabet. The Gödel order modifies the lexicographic order by first ordering by max. Its advantage is that each pair has only finitely many predecessors.

4 (16 points). Let R_1 and R_2 be the relations on the power set $\mathcal{P}(\mathbb{Z})$ of the set of all integers defined as follows:

$$\begin{aligned} XR_1Y &\Leftrightarrow |X\Delta Y| \leq 7; \\ XR_2Y &\Leftrightarrow |X\Delta Y| \text{ is finite.} \end{aligned}$$

Recall that the power set of a set A is the set of all the subsets of A and that $X\Delta Y = (X \cup Y) - (X \cap Y)$. For each of R_1 and R_2, either prove that it is an equivalence relation or prove that it is not.

Solution. R_1 is not transitive: $\{1\}R_1\{2,3\}$ and $\{2,3\}R_1\{2,3,4,5,6,7,8,9\}$, but $\{1\}\Delta\{2,3,4,5,6,7,8,9\}$ has 8 elements, and so these two sets are not related by R_1. Hence R_1 is not an equivalence relation. (R_1 is reflexive and symmetric.)

R_2 is an equivalence relation. For each subset X of $\mathbb{Z}$,

$$X\Delta X = X \cup X - X \cap X = X - X = \emptyset,$$

which is finite. Thus R_2 is reflexive. For any X and Y, $X\Delta Y = Y\Delta X$, and so R_2 is symmetric. For any X, Y, and Z,

$$X\Delta Z \subseteq (X\Delta Y) \cup (X\Delta Z).$$

This fact can be seen by, for example, a Venn Diagram. It implies that if $X\Delta Y$ and $Y\Delta Z$ are both finite then so is $X\Delta Z$. Thus R_2 is transitive.

5 (15 points). Let X be the set whose elements are the seven days of the week. Define $f : X \to X$ by letting $f(x)$ be the next day after x whose name begins with the same letter as x's name. For example, $f(\text{Tuesday}) = \text{Thursday}$, $f(\text{Wednesday}) = \text{Wednesday}$, and $f(\text{Thursday}) = \text{Tuesday}$. Recall that functions are defined to be relations (their graphs). Give f's matrix. Is f a symmetric relation? Is f a one-one function? Is it onto X? Explain briefly your answers to these three questions.

Solution. Here is the matrix:

	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Sun	0	0	0	0	0	0	1
Mon	0	1	0	0	0	0	0
Tue	0	0	0	0	1	0	0
Wed	0	0	0	1	0	0	0
Thu	0	0	1	0	0	0	0
Fri	0	0	0	0	0	1	0
Sat	1	0	0	0	0	0	0

f is symmetric. There are only two letters that begin the name of more than one day, and each of these begins exactly two days, which are thus related both ways by f. The other three days are related only to themselves. f is onto X, because because each day is the next day after itself or else another day whose name begins with the same letter. f must be one-one, since its domain is finite and it is onto.

6 (18 points). The executive committee of a certain organization has six members: John, Margaret, George, Sam, Amy, and Sarah. Three officers need to be chosen: a president, a vice president, and a secretary. George is willing to be an officer only if he is the president. Sam is willing to be an officer only if he is not the president. Amy is willing to be an officer only if Margaret is president and John is not an officer.

How many ways are there to choose the officers and satisfy these constraints?

Solution. We first consider selections in which George is president. There is one choice for president, there four choices for vice president (everyone but George and Amy), and there are three remaining choices for secretary. By the multiplication principle, this gives $1 \cdot 4 \cdot 3 = 12$ ways to choose the officers.

We next consider selections in which George is not president and Amy is not an officer. There are three choices for president (John, Margaret, and Sarah), there three choices for vice president (Sam plus the two candidates for president who were not chosen), and there are two remaining choices for secretary. By the multiplication principle, this gives $3 \cdot 3 \cdot 2 = 18$ ways to choose the officers.

Next we consider selections in which Amy is vice president. There is one choice for president (Margaret). There is one choice for vice president. There are two choices for secretary (Sam and Sarah). This gives $1 \cdot 1 \cdot 2 = 2$ ways to choose the officers.

Finally we consider selections in which Amy is secretary. This case is similar to the preceding, and there are again two ways to choose the officers.

By the addition principle, the number of ways to choose the officers and satisfy the constraints is $12 + 18 + 2 + 2 = 34$.

Comments on Problem 6: This is a complicated problem—too complicated—and I regret not having given a less complicated one.

There are other ways than the one chosen above to divide the problem into parts.