5 Recursion Theory

Fix $n \in \omega \setminus \{0\}$. To get a useful enumeration of the recursive functions, we do a uniform version of the construction of the proof of Theorem 4.33. Let $\mathrm{Pr}(a, d)$ hold if and only if d is the $\#$ of a deduction from the axioms of Q of a sentence σ of $\mathcal{L}^{\mathrm{PA}}$ such that $a = \#\sigma$. Define $T_n \subseteq {}^{n+2}\omega$ by letting $T_n(e, a_1, \ldots, a_n, d)$ hold if and only if

(i) for some formula $\varphi(v_1, \ldots, v_{n+1})$ of $\mathcal{L}^{\mathrm{PA}}$, $\#\varphi = e$;

(ii) $\mathrm{Pr}(\#\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), \mathbf{S}^{(d)_0}(\mathbf{0})), (d)_1)$;

(iii) d is the smallest number satisfying (ii).

Define $U : \omega \to \omega$ by setting $U(d) = (d)_0$.

Theorem 5.1. *(a) For each $n \geq 1$, T_n is primitive recursive.*

(b) The function U is primitive recursive.

(c) If $n \geq 1$ and $f : {}^n\omega \to \omega$ is recursive, then there is an $e \in \omega$ such that, for all numbers $a_1, \ldots, a_n$,

$$f(a_1, \ldots, a_n) = U(\mu d\, T_n(e, a_1, \ldots, a_n, d)).$$

(d) Every total (i.e., totally defined) function in this form is recursive.

Proof. For (a), note that clause (ii) is equivalent with

$$\mathrm{Pr}(\mathrm{Sb}(\ldots(\mathrm{Sb}(e, \#v_1, \#\mathbf{S}^{a_1}(\mathbf{0})), \ldots), \#v_{n+1}, \#\mathbf{S}^{(d)_0}(\mathbf{0})), (d)_1).$$

For (c), let φ represent f in Q and let $e = \#\varphi$. (d) follows from (a) and (b). □

For $n \geq 1$, a *partial (number-theoretic) function of n variables* is an $f : A \to \omega$ where $A \subseteq {}^n\omega$.

A partial function of n variables is *partial recursive* if there are recursive g and h such that, for all $a_1, \ldots, a_n$,

$$f(a_1, \ldots, a_n) \simeq h(\mu b\, g(a_1, \ldots, a_n, b) = 0),$$

where "$x \simeq y$" means "x and y are defined and equal or both are undefined."

Lemma 5.2. *For each $n \geq 1$ and each e, the partial function f given by*

$$f(a_1, \ldots, a_n) \simeq U(\mu d\, T_n(e, a_1, \ldots, a_n, d))$$

is partial recursive. Indeed, the partial function g given by

$$g(e, a_1, \ldots, a_n) \simeq U(\mu d\, T_n(e, a_1, \ldots, a_n, d))$$

is partial recursive.

Lemma 5.3. *If f is a partial recursive function of n variables, then there is an e such that, for all $a_1, \ldots, a_n$,*

$$f(a_1, \ldots, a_n) \simeq U(\mu d\, T_n(e, a_1, \ldots, a_n, d))\,.$$

Proof. Let g and h witness that f is partial recursive. Let $\varphi(v_1, \ldots, v_{n+2})$ and $\psi(v_1, v_2)$ represent g and h respectively in Q. Let $\chi(v_1, \ldots, v_{n+1})$ be

$$(\exists z)(\varphi(v_1, \ldots, v_n, z, \mathbf{0}) \wedge (\forall z')(z' {<} z \rightarrow \neg\varphi(v_1, \ldots, v_n, z', \mathbf{0})) \wedge \psi(z, v_{n+1}))\,,$$

for appropriate variables z and z'. It is easy to see that the sentence $\chi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), \mathbf{S}^{c}(\mathbf{0}))$ is provable in Q if and only if $c \simeq f(a_1, \ldots, a_n)$. (The main point is that only sentences true in $\mathfrak{N}$ are provable in Q.) Thus we can let $e = \#\chi$. □

Theorem 5.4. *The partial recursive functions of n variables are exactly the functions $\{e\}_n$, where*

$$\{e\}_n(a_1, \ldots, a_n) \simeq U(\mu d\, T_n(e, a_1, \ldots, a_n, d))\,.$$

Exercise 5.1. Define an operation of composition for partial functions and prove that the partial recursive functions are closed under composition.

A subset A of ω is *recursively enumerable* (*r.e.*) if A is the domain of a partial recursive function.

Theorem 5.5. *If $A \subseteq \omega$, then A is r.e. if and only if A is either empty or the range of a recursive function, where the function can be taken to be of one argument.*

Proof. Suppose the A is r.e. Then there is an e such that $A = \{a \mid (\exists d)\, T_1(e, a, d)\}$. Suppose that $A \neq \emptyset$. Let $a \in A$. Define a recursive g by setting

$$g(b) = \begin{cases} (b)_0 & \text{if } T_1(e, (b)_0, (b)_1)\,; \\ a & \text{otherwise.} \end{cases}$$

Now suppose that $A = \text{range}\,(\tilde{g})$ with $\tilde{g}$ recursive. For $b \in \omega$, let

$$f(b) \simeq \mu c\, \tilde{g}((c)_1, \ldots, (c)_n)) = b\,.$$

Clearly $A = \text{domain}\,(f)$. To see that f is partial recursive, define g and h by:

$$\begin{aligned} g(b, c) &= (\tilde{g}((c)_1, \ldots, (c)_n)) \dot{-} b) + (b \dot{-} \tilde{g}((c)_1, \ldots, (c)_n)\,; \\ h(a) &= a\,. \end{aligned}$$

It is easy to see that there is a partial recursive function with domain $\emptyset$: Note that, e.g., $\{0\}_1 = \emptyset$. □

Theorem 5.6. *A subset A of ω is recursive if and only if both A and $\neg A$ are r.e.*

Proof. Suppose first that A is recursive. Define g and g' by setting

$$\begin{aligned} g(a) &\simeq \mu b\, K_A(a) = 1\,; \\ g'(a) &\simeq \mu b\, K_A(a) = 0\,. \end{aligned}$$

g and g' witness that A and $\neg A$ respectively are r.e.

For the converse, suppose that $A = \{a \mid (\exists d)\, T_1(e, a, d)\}$ and that $\neg A = \{a \mid (\exists d)\, T_1(e', a, d)\}$. Then

$$K_A(a) = K_{T_1}(e, a, \mu d\,(T_1, (e, a, d) \vee T_1(e', a, d)))\,. \qquad \square$$

Let $\mathcal{K} = \{e \mid (\exists d)\, T_1(e, e, d)\}$.

Theorem 5.7. *The set $\mathcal{K}$ is r.e. but not recursive.*

Proof. $\mathcal{K}$ is the domain of the partial recursive function f given by $f(e) \simeq U(\mu d\, T_1(e, e, d))$.

Suppose that $\mathcal{K}$ is recursive. Then $\neg\mathcal{K}$ is r.e., and so there is an e such that $\neg\mathcal{K} = \text{domain}\,(\{e\}_1)$. But then

$$e \in \mathcal{K} \leftrightarrow (\exists d)\, T_1(e, e, d) \leftrightarrow e \notin \mathcal{K}\,. \qquad \square$$

Theorem 5.8 (s-m-n Theorem). *For any positive integers m and n, there is a recursive function S^m_n such that, for all $e, a_1, \ldots, a_m, b_1, \ldots, b_n$,*

$$\{e\}_{m+n}(a_1, \ldots, a_m, b_1, \ldots, b_n) \simeq \{S^m_n(e, a_1, \ldots, a_m)\}_n(b_1, \ldots, b_n)\,.$$

Proof. The idea of the proof is simple. In the case that matters, when e is the number of a formula $\varphi(v_1, \ldots, v_{m+n+1})$, what we would like to do is to set $S^m_n(e, a_1, \ldots, a_m) = \#\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_m}(\mathbf{0}), v_1, \ldots, v_{n+1})$. But, for $1 \leq i \leq n+1$, some occurrences of v_i that replace free occurrences of v_{m+i} may be bound. For this reason, we need to change the bound occurrences of these v_i to occurrences of other variables before we insert the v_i, and even this step requires preparation. We will begin by replacing free occurrences of v_i, $1 \leq i \leq m$, by $\mathbf{S}^{a_i}(\mathbf{0})$ and replacing free occurrences of v_{m+i}, $1 \leq i \leq n+1$, by v_{e+i}. The point of the latter replacements is that no symbol whose # is $\geq e$ can occur in a formula whose # is e.

Let then $f_n^m(e, a_1, \dots, a_m) =$

$$\underbrace{\mathrm{Sb}(\cdots \mathrm{Sb}(}_{n+1}\underbrace{\mathrm{Sb}(\cdots \mathrm{Sb}(}_{m}e\underbrace{, \#v_1, \#\mathbf{S}^{a_1}(\mathbf{0}))\cdots, \#v_m, \#\mathbf{S}^{a_m}(\mathbf{0})),}_{m}$$

$$\underbrace{\#v_{m+1}, \#v_{e+1})\cdots, \#v_{m+n+1}, \#v_{e+n+1})}_{n+1}.$$

Next let

$$g_n(e, c, i) = \begin{cases} (c)_i + 2(e+n+1) & \text{if } (c)_i \text{ is even and } 2 \le (c)_i \le 2(n+1); \\ (c)_i & \text{otherwise.} \end{cases}$$

Then let

$$h_n(e, c) = \prod_{i < \mathrm{lh}(a)} p_i^{g_n(e,c,i)+1}$$

and let

$$k_n^m(e, a_1, \dots, a_m) = h_n(e, f_n^m(e, a_1, \dots, a_m)).$$

Finally let

$$S_n^m(e, a_1, \dots, a_m) = \mathrm{Sb}(k_n^m(e, a_1, \dots, a_m), \#v_{e+1}, \#v_1)\cdots, \#v_{e+n+1}, \#v_{n+1})$$

if e is the # of a formula $\varphi(v_1, \dots, v_{m+n+1})$, and let $S_n^m(e, a_1, \dots, a_m) = 0$ otherwise.

Let us see how the definition works. As we have already indicated, if $e = \#\varphi(v_1, \dots, v_{m+n+1})$ then

$$f_n^m(e, a_1, \dots, a_m) = \#\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \dots, \mathbf{S}^{a_m}(\mathbf{0}), v_{e+1}, \dots, v_{e+n+1}).$$

In this case, $k_n^m(e, a_1, \dots, a_m)$ is the number of a formula we shall call $\psi(\mathbf{S}^{a_1}(\mathbf{0}), \dots, \mathbf{S}^{a_m}(\mathbf{0}), v_{e+1}, \dots, v_{e+n+1})$, the formula that is gotten from $\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \dots, \mathbf{S}^{a_m}(\mathbf{0}), v_{e+1}, \dots, v_{e+n+1})$ by replacing all occurrences of v_i by occurrences of $v_{e+n+1+i}$ for $1 \le i \le n+1$. The replaced occurrences of v_i are bound occurrences, since these are the only occurrences of v_i. Finally,

$$S_n^m(e, a_1, \dots, a_m) = \#\psi(\mathbf{S}^{a_1}(\mathbf{0}), \dots, \mathbf{S}^{a_m}(\mathbf{0}), v_1, \dots, v_{n+1}).$$

□

For subsets A and B of ω, we say that A is *many-one reducible* to B ($A \le_m B$) if there is a recursive f such that

$$(\forall a \in \omega)(a \in A \leftrightarrow f(a) \in B).$$

From now on, we shall usually write $\{e\}$ for $\{e\}_1$. Let $\mathcal{H} = \{b \in \omega \mid \{(b)_0\}((b)_1)$ *is defined*$\}$.

Theorem 5.9. *$\mathcal{H}$ is r.e., $\mathcal{K} \leq_m \mathcal{H}$, and $\mathcal{H} \leq_m \mathcal{K}$.*

Proof. $\mathcal{H}$ is obviously r.e.

Let $f(e) = \langle e, e \rangle$. Then, for any $e \in \omega$,

$$e \in \mathcal{K} \leftrightarrow \{e\}(e) \text{ is defined} \leftrightarrow \langle e, e \rangle \in \mathcal{H}\,,$$

so $\mathcal{K} \leq_m \mathcal{H}$.

To show that $\mathcal{H} \leq_m \mathcal{K}$, we use the *s-m-n* Theorem. Define g by

$$g(b, a) \simeq \{(b)_0\}((b)_1)\,.$$

The partial function g is partial recursive, since

$$g(b, a) \simeq U(\mu d\,(T_1((b)_0, (b)_1, d))\,.$$

Hence there is an $e \in \omega$ such that

$$(\forall b)(\forall a)\, g(b, a) \simeq \{e\}_2(b, a)\,.$$

Set $f(b) = S^1_1(e, b)$ for $b \in \omega$. We have that

$$\{f(b)\}(a) \simeq \{S^1_1(e, b)\}(a) \simeq \{e\}_2(b, a) \simeq g(b, a)\,.$$

Suppose that $b \in \mathcal{H}$. Then $\{(b)_0\}((b)_1)$ is defined. Hence $g(b, a)$ is defined for every a, and so $\{f(b)\}(a)$ is defined for every a. In particular, $\{f(b)\}(f(b))$ is defined, and this means that $f(b) \in \mathcal{K}$.

Now suppose that $b \notin \mathcal{H}$. Then $\{(b)_0\}((b)_1)$ is undefined. Thus $\{f(b)\}$ is the completely undefined function, so $f(b) \notin \mathcal{K}$. □

Theorem 5.10. *Let $A \subseteq \omega$ be r.e. Then $A \leq_m \mathcal{H}$ and so $A \leq_m \mathcal{K}$.*

Proof. Let $A = \text{domain}(\{e\})$. Define f by setting $f(n) = \langle e, n \rangle$. Then, for all n,

$$n \in A \;\leftrightarrow\; \{e\}(n) \text{ is defined } \;\leftrightarrow\; \langle e, n \rangle \in \mathcal{H}\,.$$ □

The *s-m-n* Theorem implies that if g is a partial recursive function of $m + n$ variables, then there is a recursive f such that

$$\{f(a_1, \ldots, a_m)\}_n(b_1, \ldots, b_n) \simeq g(a_1, \ldots, a_m, b_1, \ldots, b_n)\,,$$

for all $a_1, \ldots, a_m, b_1, \ldots, b_n$. From now on we shall use this consequence of the *s-m-n* Theorem directly.

Theorem 5.11 (Recursion Theorem). *For all $m \in \omega$ and all recursive $f : \omega \to \omega$, there is an $n \in \omega$ such that $\{n\}_m = \{f(n)\}_m$.*

Proof. Define g by

$$g(u, a_1, \ldots, a_m) \simeq \{\{u\}(u)\}_m(a_1, \ldots, a_m)\,.$$

It is easy to see that g is partial recursive, so the *s-m-n* Theorem gives a recursive h such that, for all $u, a_1, \ldots, a_m$,

$$g(u, a_1, \ldots, a_m) \simeq \{h(u)\}_m(a_1, \ldots, a_m)\,.$$

Let $\{v\} = f \circ h$, the composition of f and h. Let $n = h(v)$. We have that

$$\begin{aligned}
\{n\}_m(a_1, \ldots, a_m) &\simeq \{h(v)\}_m(a_1, \ldots, a_m) \\
&\simeq g(v, a_1, \ldots, a_m) \\
&\simeq \{\{v\}(v)\}_m(a_1, \ldots, a_m) \\
&\simeq \{f(h(v))\}_m(a_1, \ldots, a_m) \\
&\simeq \{f(n)\}_m(a_1, \ldots, a_m)\,.
\end{aligned}$$

□

Theorem 5.12 (Uniform Recursion Theorem). *For each $m \in \omega$, there is a recursive function r_m such that, for all $e \in \omega$,*

$$\{e\} \text{ is total } \to \{r_m(e)\}_m = \{\{e\}(r_m(e))\}_m\,.$$

Proof. Define h as in the proof of Theorem 5.11. By the *s-m-n* Theorem, let v be a recursive function such that

$$(\forall e)(\forall n)\{v(e)\}(n) \simeq (\{e\} \circ h)(n)\,.$$

For each e, set $r_m(e) = h(v(e))$. □

For $e \in \omega$, let $W_e = \text{domain}\,(\{e\})$. Note that $\mathcal{K} = \{e \mid e \in W_e\}$.

An r.e. set C is *creative* if there is a recursive function f such that

$$(\forall e \in \omega)(W_e \cap C = \emptyset \to f(e) \notin W_e \cup C)\,.$$

If C is creative, then C is not recursive, for $f(e)$ witnesses that $\neg C \neq W_e$ whenever $W_e \subseteq \neg C$. (We write $\neg C$ for $\omega \setminus C$.)

The set $\mathcal{K}$ is witnessed creative by the identity function, for

$$W_e \cap \mathcal{K} = \emptyset \;\Rightarrow\; e \notin W_e \Rightarrow e \notin \mathcal{K}\,.$$

Theorem 5.13. *If C is creative and A is r.e., then $A \leq_m C$.*

Proof. Let f witness that C is creative, and let A be r.e. Define h by

$$h(a,b,c) \simeq \begin{cases} 0 & \text{if } a \in A \text{ and } c = f(b)\,; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is easy to show that h is partial recursive. By applications of the s-m-n Theorem, let p and q be recursive and such that

$$\begin{aligned} h(a,b,c) &\simeq \{p(a,b)\}(c)\,; \\ p(a,b) &= \{q(a)\}(b)\,. \end{aligned}$$

Note that, for all a and b,

$$W_{p(a,b)} = \begin{cases} \{f(b)\} & \text{(the singleton) if } a \in A\,; \\ \emptyset & \text{otherwise}\,. \end{cases}$$

Let $r = r_1$. By the Uniform Recursion Theorem, we have for all a that

$$\begin{aligned} \{r(q(a))\} &= \{\{q(a)\}(r(q(a)))\} \\ &= \{p(a, r(q(a)))\}\,. \end{aligned}$$

Hence, for all a, $W_{r(q(a))} = W_{p(a,r(q(a)))}$.

We show that $f \circ r \circ q$ witnesses that $A \leq_m C$. Note first that

$$\begin{aligned} a \in A \quad &\rightarrow \quad W_{p(a,r(q(a)))} = \{f(r(q(a)))\} \\ &\rightarrow \quad W_{r(q(a))} = \{f(r(q(a)))\} \\ &\rightarrow \quad f(r(q(a))) \in C\,. \end{aligned}$$

(Since f witnesses that C is creative, the next-to-last line implies that $W_{r(q(a))} \cap C \neq \emptyset$. This and the next-to-last line imply the last line.) Note finally that

$$\begin{aligned} a \notin A \quad &\rightarrow \quad W_{p(a,r(q(a)))} = \emptyset \\ &\rightarrow \quad W_{r(q(a))} = \emptyset \\ &\rightarrow \quad f(r(q(a))) \notin C\,. \end{aligned}$$

(The last implication holds because f witnesses that C is creative.) □

Exercise 5.2. The *join* of subsets A and B of ω is

$$\{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}\,.$$

Prove that the join of A and B is a $\leq_m$-least upper bound for A and B.

Exercise 5.3. (a) Show that if A is r.e. and $A \leq_m \neg A$ then A is recursive.

(b) Prove that the hypothesis that A is r.e. cannot be dropped from (a). *Hint.* Consider the join of a set and its complement.

Exercise 5.4. A subset A of ω is a *many-one complete r.e. set* if A is r.e. and, for all r.e. B, $B \leq_m A$. Thus all creative sets are many-one complete r.e. sets. Prove that $\{e \in \omega \mid W_e \neq \emptyset\}$ is a many-one complete r.e. set.

Exercise 5.5. Let C be creative. Show that there is a recursive f such that

$$(\forall e \in \omega)(f(e) \in W_e \cap C \ \vee \ f(e) \notin W_e \cup C)\,.$$

Hint. Let $\bar{f}$ witness that C is creative. Use the *s-m-n* Theorem to define a recursive p such that, for all a and b,

$$W_{p(a,b)} = W_a \cap \{\bar{f}(b)\}\,.$$

Now use the *s-m-n* Theorem and the Uniform Recursion Theorem to get a recursive s such that, for all a,

$$W_{s(a)} = W_{p(a,s(a))}\,.$$

Let $f = \bar{f} \circ s$.

Theorem 5.14. *If C is a many-one complete r.e. set, then C is creative.*

Proof. Let g witness that $\mathcal{K} \leq_m C$. By the *s-m-n* theorem, let h be recursive and such that

$$(\forall e)(\forall a)\,\{h(e)\}(a) \simeq \{e\}(g(a))\,.$$

Note that, for all e, $W_{h(e)}$ is the preimage under g of W_e.

Let $f = g \circ h$. To show that f witnesses that C is creative, let e be such that $W_e \cap C = \emptyset$. Taking preimages under g, we get that $W_{h(e)} \cap \mathcal{K} = \emptyset$. By the definition of $\mathcal{K}$, this implies that $h(e) \notin W_{h(e)} \cup \mathcal{K}$. But then $g(h(e)) \notin W_e \cup C$.

□

Theorem 5.15. *For all m and n, there is a one-one function S^m_n that witnesses the truth of the s-m-n Theorem.*

Proof. Fix m and n. Let $\bar{S}^m_n$ have the property required of S^m_n in the statement of the s-m-n Theorem. Define $h : {}^{m+1}\omega \to \omega$ by setting

$$h(a_0, \ldots, a_m) = \#(\mathbf{S}^{a_0}(\mathbf{0}) = \mathbf{S}^{a_0}(\mathbf{0}) \wedge (\cdots \wedge \mathbf{S}^{a_m}(\mathbf{0}) = \mathbf{S}^{a_m}(\mathbf{0})) \cdots).$$

It is easy to see that h is a one-one recursive function and that all the values of h are #'s of valid sentences of $\mathcal{L}^{\mathrm{PA}}$. Define S^m_n by setting

$$S^m_n(e, a_1, \ldots, a_m) = \#\langle(\rangle * h(e, a_1, \ldots, a_m) * \#\langle\wedge\rangle * \bar{S}^m_n(e, a_1, \ldots, a_m) * \#\langle)\rangle.$$

□

Theorem 5.16. *For each $m \in \omega$, there is a one-one function r_m that witnesses the truth of the Uniform Recursion Theorem.*

Proof. Given m, define functions h and v, as in the proof of Theorem 5.12, using one-one functions S^1_m and S^1_1. The h and v so defined are one-one. Hence $r_m = h \circ v$ is also one-one. □

Theorem 5.17. *If C is creative, then there is a one-one function witnessing that C is creative.*

Proof. Define a partial recursive function g by

$$g(e, n, y) \simeq \begin{cases} y & \text{if } (\exists i < \mathrm{lh}(n))\, y = (n)_i\,; \\ \{e\}(y) & \text{otherwise}. \end{cases}$$

Let p be recursive and such that

$$(\forall e)(\forall n)(\forall y)\, \{p(e, n)\}(y) \simeq g(e, n, y).$$

Thus

$$(\forall e)(\forall n)\, W_{p(e,n)} = W_e \cup \{(n)_0, \ldots, (n)_{\mathrm{lh}(n) \dot{-} 1}\}.$$

Let f witness that C is creative. Define a recursive $\tilde{f}$ by

$$\begin{aligned} \tilde{f}(e, 0) &= \prec f(e) \succ; \\ \tilde{f}(e, k+1) &= \tilde{f}(e, k) * \prec f(p(e, \tilde{f}(e, k))) \succ. \end{aligned}$$

By induction, we show that, for all k,

(i) $\tilde{f}(e, k) \in \mathrm{Seq}$;

(ii) $\mathrm{lh}(\tilde{f}(e, k)) = k + 1$;

(iii) $(\forall k' \le k)\, \tilde{f}(e,k') = \tilde{f}(e,k) \restriction k'+1$;

(iv) $W_e \cap C = \emptyset \;\to\; (\forall i \le k)(\forall j < i)\, (\tilde{f}(e,k))_i \ne (\tilde{f}(e,k))_j$;

(v) $W_e \cap C = \emptyset \;\to\; (\forall i \le k)\, (\tilde{f}(e,k))_i \notin W_e \cup C$.

Clauses (i)–(iii) are clear. To verify (iv) and (v), note that

$$W_{p(e,\tilde{f}(e,k))} = W_e \cup \{(\tilde{f}(e,k))_i \mid i \le k\}.$$

Define h by recursion as follows. If the numbers $(\tilde{f}(e,e))_k,\ k \le e$, are distinct, let $h(e)$ be the least of these numbers that is different from all the $h(e'),\ e' < e$. Otherwise let $h(e)$ be the least number that is different from all the $h(e'),\ e' < e$. The recursive function h witnesses that C is creative.

□

For subsets A and B of ω, say that A is *one-one reducible* to B $(A \le_1 B)$ if some one-one f witnesses that $A \le_m B$. Define the notion of a *one-one complete r.e. set* in the obvious way. All our earlier results go through with "one-one" replacing "many-one." Hence we have the following theorem.

Theorem 5.18. *An r.e. set C is creative if and only if C is many-one complete if and only if C is one-one complete.*

A *recursive permutation* is a recursive one-one onto $f : \omega \to \omega$. Two subsets of ω are *recursively isomorphic* if one is the image of the other under a recursive permutation.

Theorem 5.19. *Let A and B be arbitrary subsets of ω. If $A \le_1 B$ and $B \le_1 A$, then A and B are recursively isomorphic.*

Proof. Suppose that g and h witness that $A \le_1 B$ and $B \le_1 A$ respectively.

We define inductively recursive functions $p : \omega \to \omega$, $r : {}^2\omega \to \omega$, and $s : {}^2\omega \to \omega$. There will be numbers m_i $i \in \omega$, and n_i, $i \in \omega$, such that, for each k,

$$p(k) = \langle\langle m_0, n_0\rangle, \ldots, \langle m_{2k-1}, n_{2k-1}\rangle\rangle.$$

The m_i will be distinct, as will the n_i. Moreover we shall have that

$$m_i \in A \leftrightarrow n_i \in B.$$

Given $p(k)$, let m_{2k} be the least number different from all the m_i, $i < 2k$. Set $r(k,0) = g(m_{2k})$ and

$$r(k,i+1) = \begin{cases} r(k,i) & \text{if } r(k,i) \notin \{n_0,\ldots,n_{2k-1}\}; \\ g(m_j), \text{ where } n_j = r(k,i), & \text{otherwise.} \end{cases}$$

Since g is one-one, it follows that, whenever $r(k, i+1)$ is defined by the second clause, the numbers $r(k,0)\ldots,r(k,i+1)$ are distinct. For any i, $r(k,i) \in B$ if and only if $m_{2k} \in A$.

Let $n_{2k} = r(k,i)$ for the least $i \leq 2k$ such that $r(k,i) \notin \{n_0,\ldots,n_{2k-1}\}$.

Now let n_{2k+1} be the least number different from all the n_i, $i \leq 2k$. Define $s(k,i)$ and m_{2k+1} by analogy with the definition of $r(k,i)$ and n_{2k}.

Now define $f : \omega \to \omega$ by setting $f(m_i) = n_i$ for each $i \in \omega$. Clearly f witnesses that A and B are recursively isomorphic. □

Corollary 5.20. *Any two creative sets are recursively isomorphic.*

We now turn to the topic of relative recursion. If $f : \omega \to \omega$, then the functions *recursive in* f form the smallest set $\mathcal{C}$ such that

(I) The function S, all constant functions, all the I_i^m, and f belong to $\mathcal{C}$;

(II) $\mathcal{C}$ is closed under composition;

(III) $\mathcal{C}$ is closed under primitive recursion;

(IV) $\mathcal{C}$ is closed under the μ operator.

For $R \subseteq {}^n\omega$, R is *recursive in* f if K_R is recursive in f. The partial functions *partial recursive in* f and the subsets of ω *recursively enumerable in* f are defined in the obvious way.

Let $\mathcal{L}^{\mathrm{PAF}}$ be the result of adding to $\mathcal{L}^{\mathrm{PA}}$ a new one-place function symbol $\mathbf{F}$. For any $f : \omega \to \omega$, let $\mathrm{Q}(f)$ be the set of all consequences (in $\mathcal{L}^{\mathrm{PAF}}$) of Axioms (1)–(8) plus

$$\{\mathbf{F}(\mathbf{S}^a(\mathbf{0})) = \mathbf{S}^{f(a)}(\mathbf{0}) \mid a \in \omega\}\,.$$

Theorem 5.21. *For all f, the functions recursive in f are the same as the functions representable in* $\mathrm{Q}(f)$*.*

Proof. Our proofs of Theorems 4.17 and 4.33 are easily adapted to give a proof the present theorem, since f is representable in $\mathrm{Q}(f)$ and since the relation Pr for $\mathrm{Q}(f)$ is recursive in f. □

For $n \geq 1$, let T_n^f be defined just as T_n was defined, but using $\mathrm{Q}(f)$ instead of Q.

Theorem 5.22. *For any f, T_n^f is recursive in f. The functions partial recursive in f are exactly the $\{e\}_n^f$, where*

$$\{e\}_n^f(a_1,\ldots,a_n) \simeq U(\mu d\, T_n^f(e,a_1,\ldots,a_n,d))\,.$$

Whenever $T_n^f(e, a_1, \ldots, a_n, d)$ holds, then $(d)_1$ is the # of some deduction. Any axiom of $\mathrm{Q}(f)$ that occurs as a line in this deduction must have # smaller than d. Hence, for any such axiom of the form $\mathbf{F}(\mathbf{S}^a(\mathbf{0})) = \mathbf{S}^{f(a)}(\mathbf{0})$, we must have that $a < d$. In particular, this means that whether $T_n^f(e, a_1, \ldots, a_n, d)$ holds depends only upon $f \restriction d$. This motivates the following definition

For $c \in \mathrm{Seq}$, let $Q[c]$ be the set of all consequences of Axioms (1)-(8) plus

$$\{\mathbf{F}(\mathbf{S}^a(\mathbf{0})) = \mathbf{S}^{(c)_a}(\mathbf{0}) \mid a < \mathrm{lh}(c)\}\,.$$

For $(c, e, a_1, \ldots, a_n, d) \in {}^{n+3}\omega$, let $T_n^1(c, e, a_1, \ldots, a_n, d)$ hold only if $c \in \mathrm{Seq}$ and $\mathrm{lh}(c) = d$. For $c \in \mathrm{Seq}$ and $\mathrm{lh}(c) = d$, define $T_n^1(c, e, a_1, \ldots, a_n, d)$ just as $T_n(e, a_1, \ldots, a_n, d)$ was defined, but using $Q[c]$ instead of Q. The discussion of the preceding paragraph shows why the following lemma holds.

Lemma 5.23. *For* $(c, e, a_1, \ldots, a_n, d) \in {}^{n+3}\omega$, $T_n^1(c, e, a_1, \ldots, a_n, d)$ *holds if and only if*

$$c \in \mathrm{Seq} \;\wedge\; \mathrm{lh}(c) = d \;\wedge\; (\forall f)((\forall i < d)\, f(i) = (c)_i \to T_n^f(e, a_1, \ldots, a_n, d))\,.$$

Theorem 5.24. *For each n, the relation* T_n^1 *is primitive recursive. For any* f*,* n*,* e*, and* $a_1, \ldots, a_n$*,*

$$\{e\}_n^f(a_1, \ldots, a_n) \simeq U(\mu d\, T_n^1(\bar{f}(d), e, a_1, \ldots, a_n, d))\,.$$

Let us extend the definition of recursive enumerability to subsets of ${}^n\omega$ by declaring $A \subseteq {}^n\omega$ to be *recursively enumerable* if A is the domain of a partial recursive function. Similarly define the notion of A's being *recursively enumerable in* f, for $f : \omega \to \omega$ and $A \subseteq {}^n\omega$.

If $n \geq 1$, $A \subseteq {}^n\omega$, and $k \geq 1$, then $A \in \Sigma_k$ (or A *is* Σ_k) if there is a recursive $B \subseteq {}^{n+k}\omega$ such that, for all $a_1, \ldots, a_n$,

$$(a_1, \ldots, a_n) \in A \;\leftrightarrow\; (\exists b_1) \cdots (\mathrm{Q} b_k)\,(a_1, \ldots, a_n, b_1, \ldots, b_k) \in B\,,$$

where the quantifiers alternate between $\exists$ and $\forall$ (so that Q is $\exists$ just in case k is odd). Let $A \in \Pi_k$ if and only if $\neg A \in \Sigma_k$. Let $\Delta_k = \Sigma_k \cap \Pi_k$. Similarly define $\Sigma_k(f)$, $\Pi_k(f)$, and $\Delta_k(f)$, replacing the condition that B is recursive with the condition that it is recursive in f. We shall sometimes say, e.g., that A *is* Σ_k *in* f to mean that $A \in \Sigma_k(f)$.

We omit the easy proof of the following theorem.

Theorem 5.25. *Let $n \geq 1$ and $A \subseteq {}^n\omega$. Then A is Σ_1 if and only if A is r.e., and A is Δ_1 if and only if A is recursive. For $f : \omega \to \omega$, A is Σ_1 in f if and only if A is r.e. in f, and A is Δ_1 in f if and only if A is recursive in f.*

For $f : \omega \to \omega$, let

$$\begin{aligned} \mathcal{K}^f &= \{e \mid \{e\}_1^f(e) \text{ is defined}\} \\ &= \{e \mid (\exists d)\, T_1^1(\bar{f}(d), e, e, d)\} \\ &= \{e \mid e \in W_e^f\}\,, \end{aligned}$$

where $W_e^f = \text{domain}\,(\{e\}_1^f)$.

Theorem 5.26. *For all $f : \omega \to \omega$, we have:*

(1) $\mathcal{K}^f$ is r.e. in f ;

(2) $\mathcal{K}^f$ is not recursive in f ;

(3) if $A \subseteq \omega$ is r.e. in f, then $A \leq_1 \mathcal{K}^f$;

(4) f is recursive in $K_{\mathcal{K}^f}$.

Proof. The proofs of (1) and (2) are like the proofs of the corresponding facts for $\mathcal{K}$.

Note that, for each m and n, a definition like that of the S_n^m function gives a one-one recursive function $\tilde{S}_n^m$ such that, for all f, e, $a_1, \ldots, a_m$, and $b_1, \ldots, b_n$,

$$\{\tilde{S}_n^m(e, a_1, \ldots, a_m)\}_n^f(b_1, \ldots, b_n) \simeq \{e\}_{m+n}^f(a_1, \ldots, a_m, b_1, \ldots, b_n)\,.$$

We leave as an exercise the task of using $\tilde{S}_n^m$ to prove (3) and (4). □

Exercise 5.6. Prove parts (3) and (4) of Theorem 5.26.

For $k \in \omega$, define $0^{(k)} : \omega \to \omega$ as follows:

$$\begin{aligned} 0^{(0)} &= K_\emptyset\,; \\ 0^{(k+1)} &= K_{\mathcal{K}^{0^{(k)}}}\,. \end{aligned}$$

Theorem 5.27. *For any $A \subseteq {}^n\omega$, A is Σ_{k+1} if and only if A is r.e. in $0^{(k)}$.*

Proof. The case $k=0$ follows from Theorem 5.25. so assume that $k \geq 0$ and that the theorem holds for k.

First suppose that A is r.e. in $0^{(k+1)}$. Let e be a number such that $A = \text{domain}\,(\{e\}_n^{0^{(k+1)}})$. Then, for all $a_1, \ldots, a_n$,

$$\begin{aligned} (a_1, \ldots, a_n) \in A \;\; &\leftrightarrow \;\; (\exists d)\, T_n^1(\overline{0^{(k+1)}}(d), e, a_1, \ldots, a_n, d) \\ &\leftrightarrow \;\; (\exists d)(\exists c)(c = \overline{0^{(k+1)}}(d) \,\wedge\, T_n^1(c, e, a_1, \ldots, a_n, d))\,. \end{aligned}$$

Now

$$c = \overline{0^{(k+1)}}(d) \;\leftrightarrow\; (c \in \text{Seq} \wedge \text{lh}(c) = d \wedge (\forall i < d)\,(c)_i = 0^{(k+1)}(i))\,.$$

Moreover

$$(c)_i = 0^{(k+1)}(i) \;\leftrightarrow\; (((c)_i = 1 \,\wedge\, i \in \mathcal{K}^{0^{(k)}}) \,\vee\, ((c)_i = 0 \,\wedge\, i \notin \mathcal{K}^{0^{(k)}}))\,.$$

Since $\mathcal{K}^{0^{(k)}}$ is r.e. in $0^{(k)}$, we have by induction that $\mathcal{K}^{0^{(k)}}$ is Σ_{k+1}. Thus there is a recursive B such that, for each $i \in \omega$,

$$\begin{aligned} i \in \mathcal{K}^{0^{(k)}} \;\; &\leftrightarrow \;\; (\exists b_1) \cdots (Q b_{k+1})\,(i, b_1, \ldots, b_{k+1}) \in B\,; \\ i \notin \mathcal{K}^{0^{(k)}} \;\; &\leftrightarrow \;\; (\forall b_1') \cdots (Q' b_{k+1}')\,(i, b_1', \ldots, b_{k+1}') \notin B\,. \end{aligned}$$

Substituting and bringing all quantifiers to the front, we get that, for all $a_1, \ldots, a_n$, $(a_1, \ldots, a_n) \in A$ if and only if

$$\begin{aligned} &(\exists d)(\exists c)(\forall i < d)(\exists b_1)(\forall b_1') \cdots (Q b_{k+1})(Q' b_{k+1}') \\ &\qquad R(a_1, \ldots, a_n, d, c, i, b_1, b_1', \ldots, b_{k+1}, b_{k+1}')\,, \end{aligned}$$

with R recursive. Now, for any relation P,

$$\begin{aligned} (\forall i < d)(\exists b)\, P(i, b) \;\; &\leftrightarrow \;\; (\exists \hat{b})(\forall i < d)\, P(i, (\hat{b})_i)\,; \\ (\forall i < d)(\forall b)\, P(i, b) \;\; &\leftrightarrow \;\; (\forall b)(\forall i < d)\, P(i, b)\,. \end{aligned}$$

Hence we can move $(\forall i < d)$ to the right past all the other quantifiers. Since

$$\begin{aligned} (\exists b)(\exists b')\, P(b, b') \;\; &\leftrightarrow \;\; (\exists \hat{b})\, P((\hat{b})_0, (\hat{b})_1)\,; \\ (\forall b)(\forall b')\, P(b, b') \;\; &\leftrightarrow \;\; (\forall \hat{b})\, P((\hat{b})_0, (\hat{b})_1)\,; \end{aligned}$$

we can contract adjacent pairs of like quantifiers. The end result is that we show A to be Σ_{k+2}.

Now suppose that A is Σ_{k+2}. There is then a $C \in \Pi_{k+1}$ such that, for all $a_1, \ldots, a_n$,

$$(a_1, \ldots, a_n) \in A \leftrightarrow (\exists b)(a_1, \ldots, a_n, b) \in C\,.$$

By induction, $\neg C$ is r.e. in $0^{(k)}$. Let

$$D = \{\prec a_1, \ldots, a_n, b \succ \mid (a_1, \ldots, a_n, b) \notin C\}\,.$$

Then D is r.e. in $0^{(k)}$, and so Theorem 5.26 implies that $D \leq_1 \mathcal{K}^{0^{(k)}}$. By the definition of $0^{(k+1)}$, this gives that D is recursive in $0^{(k+1)}$. But A is Σ_1 in K_D, hence r.e. in K_D, hence r.e. in $0^{(k+1)}$. □

Theorem 5.28. *For each $k \geq 1$,*

$$\Delta_k \subsetneq \Sigma_k \ \wedge\ \Delta_k \subsetneq \Pi_k \ \wedge\ (\Sigma_k \cup \Pi_k) \subsetneq \Delta_{k+1}\,.$$

Proof. That $\Delta_k \subseteq \Sigma_k$ and $\Delta_k \subseteq \Pi_k$ is by definition. Using vacuous quantifiers, we can see that $\Sigma_k \subseteq \Delta_{k+1}$ and $\Pi_k \subseteq \Delta_{k+1}$.

Since $\mathcal{K}^{0^{(k-1)}}$ is r.e. in $0^{(k-1)}$ but not recursive in $0^{(k-1)}$, we have an example of a set that belongs to $\Sigma_k \setminus \Delta_k$. But then $\neg\mathcal{K}^{0^{(k-1)}}$ belongs to $\Pi_k \setminus \Delta_k$.

The join of $\mathcal{K}^{0^{(k-1)}}$ and $\neg\mathcal{K}^{0^{(k-1)}}$ is recursive in $0^{(k)}$ and so belongs to Δ_{k+1}, but it does not belong to $\Sigma_k \cup \Pi_k$. □

For $n \geq 1$, a subset A of ω is *one-one complete for* Σ_n if $A \in \Sigma_n$ and every Σ_n subset of ω is one-one reducible to A. Similarly define *one-one complete for* Π_n, *many-one complete for* Σ_n, and *many-one complete for* Π_n.

Theorem 5.29. *Let A be the set of all $e \in \omega$ such that W_e is finite. Then A is one-one complete for Σ_2.*

Proof. For each $e \in \omega$,

$$\begin{aligned} e \in A \ &\leftrightarrow\ (\exists m)(\forall n)(n \in W_e \to n \leq m) \\ &\leftrightarrow\ (\exists m)(\forall n)(\forall d)(T_1(e,n,d) \to n \leq m)\,. \end{aligned}$$

Thus $A \in \Sigma_2$.

Let $B \subseteq \omega$ with $B \in \Sigma_2$. There is a recursive C such that

$$(\forall e)(e \in B \leftrightarrow (\exists m)(\forall n)\,(e,m,n) \in C)\,.$$

Define $f : {}^2\omega \to \omega$ by

$$f(e,m) \simeq \mu n\,(\forall m' \leq m)(\exists n' \leq n)\,(e,m',n') \notin C\,.$$

Since f is partial recursive, the *s-m-n* Theorem gives us a one-one recursive g such that

$$(\forall e)(\forall m)\,\{g(e)\}(m) \simeq f(e,m)\,.$$

To see that g witnesses that $B \leq_1 A$, assume first that $e \in B$. Then there is an m such that $(e,m,n) \in C$ for all n. For $m' \geq m$, $f(e,m')$ is undefined. Hence $W_{g(e)} \subseteq m$.

Now assume that $e \notin B$. Then for every m there is an n such that $(e,m,n) \notin C$. Thus $f(e,m)$ is defined for every m, and so $W_{g(e)} = \omega$. □

Exercise 5.4 gives an example of a set many-one complete (indeed, one-one complete) for Σ_1.

Exercise 5.7. Show that $\{e \mid \mathrm{range}(\{e\}) = \omega\}$ is one-one complete for Π_2.

Exercise 5.8. Show that $\{e \mid \neg W_e \text{ is finite}\}$ is one-one complete for Σ_3.

Degrees of unsolvability.

For $f : \omega \to \omega$, define the *degree* $\boldsymbol{d}(f)$ of f by

$$\boldsymbol{d}(f) = \{g \in {}^{\omega}\omega \mid f \leq_T g \wedge g \leq_T f\}\,,$$

where $\leq_T$ means "is recursive in." Let

$$\mathcal{D} = \{\boldsymbol{d}(f) \mid f \in {}^{\omega}\omega\}\,.$$

$\mathcal{D}$ is the set of *degrees of unsolvability*. Partially order $\mathcal{D}$ by

$$\boldsymbol{d}(f) \leq \boldsymbol{d}(g) \leftrightarrow f \leq_T g\,.$$

Theorem 5.30. *The structure $(\mathcal{D}, \leq)$ is an upper semilattice with a least element.*

Proof. The least upper bound of degrees $\boldsymbol{d}(f_1)$ and $\boldsymbol{d}(f_2)$ is $\boldsymbol{f}$, where for each n,

$$\begin{aligned} f(2n) &= f_1(n)\,; \\ f(2n+1) &= f_2(n)\,. \end{aligned}$$

The recursive functions all have the same degree $\mathbf{0}$, and this is the least degree. □

Theorem 5.31. *There exist incomparable degrees, i.e., $\leq$ is not a linear ordering of $\mathcal{D}$.*

Proof. We define inductively $s_0, s_1, \ldots$ and $t_0, t_1, \ldots$ such that

(a) $(\forall i \in \omega)\, s_i \in {}^{<\omega}\omega$;

(b) $(\forall i \in \omega)\, t_i \in {}^{<\omega}\omega$;

(c) $(\forall i \in \omega)(\forall j \in \omega)(i < j \to s_i \subsetneq s_j)$;

(d) $(\forall i \in \omega)(\forall j \in \omega)(i < j \to t_i \subsetneq t_j)$.

Let $s_0 = t_0 = \langle\, \rangle$.
Assume that s_e and t_e are defined.

For $s \in {}^{<\omega}\omega$ and $n \leq \ell\mathrm{h}(s)$ let

$$\bar{s}(n) = \langle s(0), \ldots, s(n-1) \rangle .$$

I.e., let $\bar{s}(n)$ be the common value of $\bar{f}(n)$ for $f \in {}^{\omega}\omega$ and $s \subseteq f$.
If there is an $s \in \in^{<\omega}\omega$ such that

(i) $s_e \subseteq s$;

(ii) $(\exists d \leq \ell\mathrm{h}(s))\, T^1_1(s \restriction d, e, \ell\mathrm{h}(t_e), d)$;

then let s'_e be the such an s and let

$$t'_e = t_e{}^\frown\langle U(\mu d\, T^1_1(s'_e \restriction d, e, \ell\mathrm{h}(t_e), d)) + 1\rangle .$$

Otherwise let $s'_e = s_e$ and $t'_e = t_e$.
If there is a $t \in \in^{<\omega}\omega$ such that

(i) $t'_e \subseteq t$;

(ii) $(\exists d \leq \ell\mathrm{h}(t))\, T^1_1(t \restriction d, e, \ell\mathrm{h}(s'_e), d)$;

then let t_{e+1} be the such a t and let

$$s_{e+1} = s'_e{}^\frown\langle U(\mu d\, T^1_1(t_{e+1} \restriction d, e, \ell\mathrm{h}(s'_e), d)) + 1\rangle .$$

Otherwise let $s_{e+1} = s'_e{}^\frown\langle 0\rangle$ and $t_{e+1} = t'_e{}^\frown\langle 0\rangle$.
Let $f : \omega \to \omega$ be such that $\bar{f}(\ell\mathrm{h}(s_i)) = s_i$ for all $i \in \omega$ and let $g : \omega \to \omega$ be such that $\bar{g}(\ell\mathrm{h}(t_i)) = t_i$ for all $i \in \omega$.
To show that $g \not\leq_T f$, let $e \in \omega$. We show that $\{e\}^f \neq g$. To see this, note that $\{e\}^f(\ell\mathrm{h}(t_e)) \neq g(\ell\mathrm{h}(t_e))$; for, if $\{e\}^f(\ell\mathrm{h}(t_e))$ is defined, then

$$g(\ell\mathrm{h}(t_e)) = t'_e(\ell\mathrm{h}(t_e)) = \{e\}^f(\ell\mathrm{h}(t_e)) + 1 .$$

Similarly, for each $e \in \omega$,

$$f(\ell\mathrm{h}(s'_e)) = s_{e+1}(\ell\mathrm{h}(s'_e)) = \{e\}^g(\ell\mathrm{h}(s'_e)) + 1\,.$$

Hence $f \not\leq_T g$. □

For subsets A of ω, let $\boldsymbol{d}(A) = \boldsymbol{d}(K_A)$. A degree is *recursively enumerable* if it is $\boldsymbol{d}(A)$ for some r.e. A. There is a least r.e. degree, $\mathbf{0}$, and there is a greatest r.e. degree, $\mathbf{0'} = \boldsymbol{d}(0^{(1)}) = \boldsymbol{d}(\mathcal{K})$.

Theorem 5.32. *There is an r.e. degree* $\boldsymbol{d}$ *such that*

$$\mathbf{0} < \boldsymbol{d} < \mathbf{0'}\,.$$

Proof. We shall construct a recursive function $f : {}^2\omega \to \omega$ satisfying

$$(\forall s)(\forall e)(f(s,e) > 0 \to (e < s \wedge f(s+1,e) = f(s,e)))\,.$$

For $s \in \omega$, we let

$$A^s = \{f(s,e) \mid f(s,e) > 0\}\,.$$

The stated properties of f imply the recursiveness of $\{(s,m) \mid m \in A^s\}$. For $e \in \omega$, we let

$$m_e \simeq f(\mu s\, f(s,e) > 0, e)\,.$$

Finally we let

$$A = \{m_e \mid m_e \text{ is defined}\} = \bigcup_s A^s\,.$$

Thus A will be r.e.

We shall make $\boldsymbol{d}(A) > \mathbf{0}$ by arranging that $\neg A$ is infinite and, for all e,

(1) $$W_e \text{ is infinite } \to\ W_e \cap A \neq 0\,.$$

To achieve this, we shall arrange that $m_e \in W_e$ if W_e is infinite.

We shall make $\boldsymbol{d}(A) < \mathbf{0'}$ by arranging that

(2) $$\mathcal{K}^{K_A} \in \Delta_2\,.$$

By Theorem 5.27, (2) implies that $\mathcal{K}^{K_A}$ is recursive in $0^{(1)}$, and so that

$$\boldsymbol{d}(A) < \boldsymbol{d}(\mathcal{K}^{K_A}) \leq \mathbf{0'}\,.$$

As we define f, we shall simultaneously define another recursive function $g : {}^2\omega \to \omega$.

Set $f(0,e) = 0$ for all e.

Let $s \in \omega$. Suppose $f(s,e)$ is defined for all e. Suppose inductively that

$$(\forall e)(f(s,e) > 0 \to f(s,e) \in W_e^s)\,,$$

where

$$W_e^s = \{n \mid (\exists d \le s)\, T_1(e,n,d)\}\,.$$

For each e, let

$$g(s,e) = \begin{cases} \mu d\,(d \le s \,\wedge\, T_1^1(\overline{K_{A^s}}(d), e, e, d)) & \text{if } (\exists d \le s)\, T_1^1(\overline{K_{A^s}}(d), e, e, d)\,; \\ 0 & \text{otherwise.} \end{cases}$$

For each $e < s$, if $f(s,e) = 0$ and

$$(\exists m \le s)(m \in W_e^s \,\wedge m > 2e \,\wedge\, (\forall e' < e)\, m \ge g(s,e'))$$

then, for the least such m, let $f(s+1,e) = m$. Otherwise let $f(s+1,e) = f(s,e)$.

Lemma 5.33. *$\neg A$ is infinite.*

Proof. Each $m_e > 2e$, and therefore

$$\{n \mid n \in A \,\wedge\, n \le 2e\} \subseteq \{m_{e'} \mid e' < e\}\,,$$

a set of size $\le e$. □

Lemma 5.34. *For each e, $\lim_s g(s,e)$ exists.*

Proof. Fix e. Let s_0 be such that

$$(\forall e' \le e)\,(m_e \text{ defined} \to f(s_0, e') > 0)\,.$$

Suppose that $s \ge s_0$ and $g(s,e) > 0$. Any e' such that $f(s,e') = 0$ and $f(s+1,e') > 0$ must be greater than e, and so, by the definition above of $f(s+1,e')$, must satisfy $m_{e'} \ge g(s,e)$. Thus $A^{s+1} \cap g(s,e) = A^s \cap g(s,e)$. This implies that $g(s+1,e) = g(s,e)$. We have then shown that if $g(s,e) > 0$ for some $s \ge s_0$ then $g(s',e) = g(s,e)$ for every $s' \ge s$. □

Lemma 5.35. *For each e, $\lim_s g(s,e) > 0$ if and only if $e \in \mathcal{K}^{K_A}$.*

Proof. Let $\hat{g}(e) = \lim_s g(s,e)$ and assume that $\hat{g}(e) > 0$. For all sufficiently large s, $A^s \cap \hat{g}(e) = A \cap \hat{g}(e)$. By the definition of $g(s,e)$, $e \in \mathcal{K}^{K_A}$.

Now assume that $e \in \mathcal{K}^{K_A}$. Then $(\exists d)\, T_1^1(\overline{K_A}(d), e, e, d)$. Hence, for every large enough s, $(\exists d \le s)\, T_1^1(\overline{K_{A^s}}(d), e, e, d)$, and so $g(s,e) > 0$. □

Lemma 5.36. *(1) holds.*

Proof. Let $e \in \omega$ and suppose that W_e is infinite. Let $m \in W_e$ with $m > 2e$ and $m \geq \hat{g}(e')$ for all $e' < e$. Let s be such that $e < s,\ m \leq s,\ m \in W_e^s$, and $g(s, e') = \hat{g}(e')$ for all $e' < e$. If $f(s, e) = 0$, then $0 < f(s+1, e) \leq m$, by the definition above. Thus m_e is defined and so $m_e \in W_e \cap A$. □

Lemma 5.37. *(2) holds.*

Proof. For each e,

$$\begin{aligned} e \in \mathcal{K}^{K_A} &\leftrightarrow \lim_s g(s, e) > 0 \\ &\leftrightarrow (\exists s)(\forall s')(s' \geq s \rightarrow g(s', e) > 0) \\ &\leftrightarrow (\forall s)(\exists s')(s' \geq s \wedge g(s', e) > 0)\,. \end{aligned}$$

□

Exercise 5.9. Prove that there is set of size $2^{\aleph_0}$ of pairwise incomparable degrees of unsolvability.

Hint. Modify the proof of Theorem 5.31 by defining $\langle s_u \mid u \in {}^{<\omega}2\rangle$.

Exercise 5.10. Show that there is no partial recursive function f such that, for all $e \in \omega$, if $\neg W_e$ is finite then $f(e)$ is defined and every number $\geq f(e)$ belongs to W_e.

Exercise 5.11. Show that there are recursive functions $f : {}^2\omega \rightarrow \omega$ and $g : {}^2\omega \rightarrow \omega$ such that

(a) for all e_1 and e_2, $W_{f(e_1,e_2)}$ and $W_{g(e_1,e_2)}$ are disjoint and recursive;

(b) for all e_1 and e_2, if $W_{e_1} = \neg W_{e_2}$ then $W_{f(e_1,e_2)} = W_{e_1}$ and $W_{g(e_1,e_2)} = W_{e_2}$.

Hint. All finite sets are recursive.

Exercise 5.12. Let A be a recursively enumerable set such that $\neg A$ is infinite. Let $f : \omega \rightarrow \neg A$ be one-one onto and order preserving. Assume that f eventually dominates every partial recursive function, i.e., that, for every partial recursive g,

$$(\exists m)(\forall n \geq m)(g(n) \text{ is defined} \rightarrow g(n) \leq f(n))\,.$$

Prove that $\boldsymbol{d}(A) = \boldsymbol{0'}$.

6 Constructible Sets

In this section, as in §1, our notation and terminology is pretty much the same as that of Kenneth Kunen's *Set Theory: an Introduction to Independence Proofs*. In addition, our treatment of constructible sets is derived from Kunen's.

In ZFC without the axiom of Foundation, we proved (Theorem 1.9) the existence of the class function $\alpha \mapsto V_\alpha$. Still working in ZFC − Foundation, we can define the proper class WF by

$$\mathrm{WF} = \bigcup \{V_\alpha \mid \alpha \in \mathrm{ON}\}\,.$$

Moreover it is easy to convince oneself that all the axioms of ZFC, including Foundation, hold in $(\mathrm{WF}; \in \restriction \mathrm{WF})$. Can one not show in this way the consistency of the Axiom of Foundation? The answer is yes, but we have to be careful about several things.

We can't hope to show that the consistency of ZFC is a theorem of ZFC − Foundation, for the second incompleteness theorem implies that the consistency of ZFC cannot be proved even in ZFC (unless ZFC is inconsistent). Of course, the argument outlined above doesn't actually establish the consistency of ZFC, since $(\mathrm{WF}; \in \restriction \mathrm{WF})$ isn't actually a (set) model. And we can't really be "working in ZFC − Foundation" if we show that all the axioms of ZFC hold in WF, for this assertion isn't even expressible in the formal language of set theory.

Let M be a class. For formulas φ (of the language of set theory), we define φ^M, the *relativization* of φ to M, inductively as follows:

(a) $(x = y)^M$ is $x = y$;

(b) $(x \in y)^M$ is $x \in y$;

(c) $(\neg\varphi)^M$ is $\neg\varphi^M$;

(d) $(\varphi \wedge \psi)^M$ is $(\varphi^M \wedge \psi^M)$;

(e) $((\exists x)\varphi)^M$ is $(\exists x)(x \in M \wedge \varphi^M)$.

This definition requires some explanation.

Classes are the (sometimes nonexistent, from the point of view of ZFC) extensions of formulas. So we should think of M as being $\{x \mid \chi(x)\}$ for some formula χ. Thus clause (e) should really read

(e) $((\exists x)\,\varphi)^M$ is $(\exists x)(\chi(x) \wedge \varphi^M)$.

Hence the operation $\varphi \mapsto \varphi^M$ depends not just on M but also on a formula χ defining M.

Even this amended account of the definition is not really accurate. A class need not be *definable*. It may be given by a formula $\chi(x, y_1, \ldots, y_n)$. (If we are *using* the language, then the formula is, in effect, specifying for us a class; if we are *talking about* the language, then the formula isn't specifying a class unless we assign sets to the variables y_i.) For classes M given by such formulas, the definition of φ^M must be modified so that the quantifiers of φ^M do not bind any of the variables $y_1, \ldots, y_n$ occurring free in the the defining formula.

For any class M and formula φ, *φ is true in M*, *φ holds in M*, and *M is a model of φ* all mean the same as the formula φ^M.

Lemma 6.1. *Let S and T be sets of sentences in the language of set theory and let M be a definable class. Suppose that (for some formula defining M)*

(1) $T \models M \neq \emptyset$ *;*

(2) $(\forall \sigma \in S)\, T \models \sigma^M$.

Then S is consistent if T is consistent.

Proof. Let $\chi(x)$ be the given formula defining M for which (1) and (2) hold. (Note that the Lemma is really about the defining formula χ and has nothing to do with M *qua* class.)

Assume that T is consistent. Let $\mathfrak{A}$ be a model of T. Let $\mathfrak{B}$ be given by

$$\begin{aligned} B &= \{a \in A \mid \mathfrak{A} \models \chi[a]\}\,; \\ \in_{\mathfrak{B}} &= \in_{\mathfrak{A}} \restriction B\,. \end{aligned}$$

(1) implies that $B \neq \emptyset$ and so that $\mathfrak{B}$ is a model. It is routine to show that, for any sentence σ,

$$\mathfrak{B} \models \sigma \;\leftrightarrow\; \mathfrak{A} \models \sigma^M\,.$$

Thus (2) implies that $\mathfrak{B} \models S$. □

Remarks:

(a) It is easy to give a direct proof-theoretic argument for the (equivalent) version of Lemma 6.1 formulated in terms of deductive consistency.

(b) Suppose that S and T are, say, recursively axiomatizable theories. Then the deductive consistency version of Lemma 6.1 for S and T can be formulated in, for example, Peano Arithmetic. Moreover it can be proved in PA. The applications we make of Lemma 6.1 will all involve recursively axiomatizable theories, and the arithmetic versions of (1) and (2) will be provable in PA. Thus our relative consistency results are all essentially theorems of PA.

Lemma 6.2. *If M is a transitive class, then the Axiom of Extensionality holds in M.*

Proof. Let M be transitive. The relativization of Extensionality to M is equivalent to

$$(\forall x \in M)(\forall y \in M)((\forall z \in M)(z \in x \leftrightarrow z \in y) \rightarrow x = y)\,.$$

Fix elements x and y of M and assume that $(\forall z \in M)(z \in x \leftrightarrow z \in y)$. Since M is transitive, this implies that $(\forall z)(z \in x \leftrightarrow z \in y)$. By Extensionality, $x = y$. □

Lemma 6.3. *The Axiom of Foundation holds in every subclass of* WF*.*

Proof. Let $M \subseteq \mathrm{WF}$. The relativization of Foundation to M is

$$(\forall x \in M)((\exists y \in M)\, y \in x \;\rightarrow\; (\exists y \in x \cap M)(\forall z \in x \cap M)\, z \notin y)\,.$$

Let $x \in M$. Assume that $x \cap M \neq \emptyset$. Since $M \subseteq \mathrm{WF}$, there is a least ordinal α such that $x \cap M \cap V_\alpha \neq \emptyset$. For this least α, let $y \in x \cap M \cap V_\alpha$. Since all members of y belong to V_β for some $\beta < \alpha$, y is disjoint from $x \cap M$. □

Lemma 6.4. *Let M be a class with the following property: For each formula $\varphi(x, z, w_1, \ldots, w_n)$ and for any elements z, w_1, …, w_n of M,*

$$\{x \in z \mid \varphi^M(x, z, w_1, \ldots, w_n)\} \in M\,.$$

Then every instance of the Axiom Schema of Comprehension holds in M.

Proof. Any relativization to M of an instance of Comprehension is of the form

$$(\forall w_1 \in M) \cdots (\forall w_n \in M)(\forall z \in M)(\exists y \in M)(\forall x \in M)(x \in y \leftrightarrow (x \in z \wedge \varphi^M))\,,$$

for φ as in the statement of the lemma. Fix such a φ and fix elements z, $w_1, \ldots w_n$ of M. Let $y = \{x \in z \mid \varphi^M(x, z, w_1, \ldots, w_n)\}$. By hypothesis, $y \in M$. Since $(\forall x)(x \in y \leftrightarrow (x \in z \wedge \varphi^M))$, we have in particular that $(\forall x \in M)(x \in y \leftrightarrow (x \in z \wedge \varphi^M))$. □

In our applications, M will be transitive, so that the set y will not have elements $x \notin M$ satisfying $\varphi^M(x)$. Note that a class M (transitive or not) satisfies the hypothesis of Lemma 6.4 if M if all subsets of elements of M belong to M.

The following two lemmas are easy to prove.

Lemma 6.5. *If M is a class such that, for all x and y belonging to M, there is a $z \in M$ with $\{x, y\} \subseteq z$, then the Axiom of Pairing holds in M.*

Lemma 6.6. *If M is a class such that for all $x \in M$ there is a $y \in M$ such that $\mathcal{U}(x) \subseteq y$, then the Axiom of Union holds in M.*

Lemma 6.7. *Let M be a class with the following property: For each formula $\varphi(x, z, w_1, \ldots, w_n)$ and for any elements z, w_1, ..., w_n of M, if*

$$(\forall x \in z \cap M)(\exists! y \in M)\varphi^M(x, y, z, w_1, \ldots, w_n),$$

then there is a $u \in M$ such that

$$\{y \in M \mid (\exists x \in z \cap M)\varphi^M(x, y, z, w_1, \ldots, w_n)\} \subseteq u.$$

Then every instance of the Axiom Schema of Replacement holds in M.

Proof. The proof is similar to that of Lemma 6.4. □

We postpone discussing the Axioms of Infinity, Power Set, and Choice until we have proved some results about *absoluteness*.

Let $\varphi(x_1, \ldots, x_n)$ be a formula. If M and N are classes such that $M \subseteq N$, then φ is *absolute* for (M, N) if, for any elements $x_1, \ldots x_n$ of M,

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi^N(x_1, \ldots, x_n).$$

We say that φ is *absolute* for a class M if φ is absolute for (M, V), i.e., if, for any elements $x_1, \ldots, x_n$ of M,

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n).$$

Lemma 6.8. *If $M \subseteq N$, then the set of formulas absolute for (M, N) is closed under negation and conjunction.*

Proof. The lemma follows directly from the facts that the relativization of $\neg\varphi$ is the negation of the relativization of φ and that the relativization of $\varphi \wedge \psi$ is the conjunction of the relativizations of φ and ψ. □

Lemma 6.9. *Let M and N be transitive classes such that $M \subseteq N$. Then the set of formulas absolute for (M, N) is closed under* bounded quantification*; that is to say, if φ is absolute for (M, N) then*

$$(\exists x)(x \in y \wedge \varphi)$$

is absolute for (M, N).

Proof. Given $\varphi(x, y, z_1, \ldots, z_n)$ absolute for (M, N) and given elements y, $z_1, \ldots, z_n$ of M, we have

$$\begin{aligned} &((\exists x)(x \in y \wedge \varphi(x, y, z_1, \ldots, z_n)))^M \\ &\quad \leftrightarrow (\exists x)(x \in y \wedge \varphi^M(x, y, z_1, \ldots, z_n)) \\ &\quad \leftrightarrow (\exists x)(x \in y \wedge \varphi^N(x, y, z_1, \ldots, z_n)) \\ &\quad \leftrightarrow ((\exists x)(x \in y \wedge \varphi(x, y, z_1, \ldots, z_n)))^N . \end{aligned}$$

The first biconditional follows from the transitivity of M, the second from the absoluteness of φ for (M, N), and the third from the transitivity of N. □

The Δ_0 formulas form the smallest set of formulas satisfying the following conditions:

(1) All atomic formulas are Δ_0.

(2) If φ is Δ_0 then so is $\neg\varphi$.

(3) If φ and ψ are Δ_0 then so is $(\varphi \wedge \psi)$.

(4) If φ is Δ_0 then so is $(\exists x)(x \in y \wedge \varphi)$.

Lemma 6.10. *If M and N are transitive classes and $M \subseteq N$, then all Δ_0 formulas are absolute for (M, N).*

The following useful lemma is easy to prove.

Lemma 6.11. *Let T be a theory and let $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots x_n)$ be formulas such that*

$$T \models (\forall x_1) \cdots (\forall x_n)(\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)) .$$

Let M and N be models of T such that $M \subseteq N$. Then φ is absolute for (M, N) if ψ is.

Exercise 6.1. A formula of the language of set theory is Σ_1 if it is of the form $(\exists x_1)\ldots(\exists x_n)\,\varphi$ with φ a Δ_0 formula. A formula is Π_1 if it is of the form $(\forall x_1)\ldots(\forall x_n)\,\varphi$ with φ a Δ_0 formula. If T is a theory, a formula φ is *provably* Δ_1 *in* T if there are formulas ψ and χ such that ψ is Σ_1, χ is Π_1, and $T \models$ both $(\varphi \leftrightarrow \psi)$ and $(\varphi \leftrightarrow \chi)$.

Let φ be provably Δ_1 in T and let M and N be transitive models of T such that $M \subseteq N$. Prove that φ is absolute for (M, N).

If $(\forall v_1)\cdots(\forall v_n)(\exists! v_{n+1})\varphi(v_1, \ldots, v_{n+1})$, then the formula φ defines, in an obvious sense, an n-argument operation. If T is a theory in the language of set theory and $\varphi(v_1, \ldots, v_{n+1})$ is a formula of that language, then let us say that φ *defines an an n-argument operation in T* if

$$T \models (\forall v_1)\cdots(\forall v_n)(\exists! v_{n+1})\varphi(v_1, \ldots, v_{n+1})\,.$$

Every formula $\varphi(v_1, \ldots, v_n)$ defines an n-ary relation. To have a uniform terminology, lets also say, for every $\varphi(v_1, \ldots, v_n)$ and every T, that φ *defines an n-ary relation in T*.

When we speak of a defined operation or relation as being *absolute*, we mean that the defining formula is absolute.

Let ZF be ZFC − Choice.

Lemma 6.12. *The following relations and operations are defined in ZF − Foundation − Power Set − Infinity by formulas provably equivalent in ZF − Foundation − Power Set − Infinity to Δ_0 formulas. Hence they are absolute for any transitive class M that is a model of ZF − Foundation − Power Set − Infinity.*

(a) $x \in y$;	(h) $x \cup y$;
(b) $x = y$;	(i) $x \cap y$;
(c) $x \subseteq y$;	(j) $x \setminus y$;
(d) $\{x, y\}$;	(k) $\mathcal{S}(x)$;
(e) $\{x\}$;	(l) x is transitive ;
(f) $\langle x, y\rangle$;	(m) $\mathcal{U}(x)$;
(g) $\emptyset$;	(n) $\bigcap x$.

In (n), we construe $\bigcap\emptyset$ to be $\emptyset$ in order to make $\bigcap$ into an operation.

Proof. That we defined these relations and functions in ZF − Foundation − Power Set − Infinity, we leave to the reader to check. We content ourselves with making it clear that the defining formulas are equivalent in that theory to Δ_0 formulas.

(a) and (b) are obvious.

For (c), note that $x \subseteq y$ if and only if $(\forall z \in x)\, z \in y$.

For (d), observe that

$$z = \{x, y\} \;\leftrightarrow\; (x \in z \,\wedge\, y \in z \,\wedge\, (\forall w \in z)(w = x \,\vee\, w = y))\,.$$

(e) is similar.

For (f), note that $z = \langle x, y\rangle$ if and only if

$$(\exists w \in z)\, w = \{x\} \wedge (\exists w \in z)\, w = \{x, y\} \wedge (\forall w \in z)(w = \{x\} \vee w = \{x, y\})\,.$$

Since $w = \{x\}$ and $w = \{x, y\}$ are equivalent to Δ_0 formulas, so is $z = \langle x, y\rangle$.

For (g)–(k), note that

$$\begin{aligned}
z = \emptyset \quad &\leftrightarrow \quad (\forall w \in z)\, w \neq w\,;\\
z = x \cup y \quad &\leftrightarrow \quad x \subseteq z \,\wedge\, y \subseteq z \,\wedge\, (\forall w \in z)(w \in x \,\vee\, w \in y)\,;\\
z = x \cap y \quad &\leftrightarrow \quad z \subseteq x \,\wedge\, z \subseteq y \,\wedge\, (\forall w \in x)(w \in y \to w \in z)\,;\\
z = x \setminus y \quad &\leftrightarrow \quad z \subseteq x \,\wedge\, z \cap y = \emptyset \,\wedge\, (\forall w \in x)(w \notin y \to w \in z)\,;\\
z = \mathcal{S}(x) \quad &\leftrightarrow \quad x \in z \,\wedge\, x \subseteq z \wedge (\forall w \in z)(w = x \,\vee\, w \in x)\,.
\end{aligned}$$

For (l), observe that x is transitive if and only if $(\forall z \in x)(\forall w \in z)\, w \in x$.

For (m) and (n), note that

$$y = \mathcal{U}(x) \;\leftrightarrow\; (\forall z \in x)\, z \subseteq y \,\wedge\, (\forall z \in y)(\exists w \in x)\, z \in w$$

and that

$$\begin{aligned}
y = \bigcap x \;\leftrightarrow\; & (\forall z \in x)\, y \subseteq z \,\wedge\, (\forall z \in x)(\forall w \in z)((\forall u \in x)\, w \in u \to w \in y)\\
& \wedge (x = \emptyset \to y = \emptyset)\,.
\end{aligned}$$

□

Lemma 6.13. *Suppose that M is a transitive model of ZF − Foundation − Power Set − Infinity such that $(\forall x \in M)(\exists y \in M)\, \mathcal{P}(x) \cap M \subseteq y$. Then the Axiom of Power Set holds in M.*

Proof. The relativization to M of Power Set is

$$(\forall x \in M)(\exists y \in M)(\forall z \in M)((z \subseteq x)^M \to z \in y)\,.$$

By Lemma 6.12, $\subseteq$ is absolute for M, so the relativization of Power Set to M is equivalent to

$$(\forall x \in M)(\exists y \in M)(\forall z \in M)(z \subseteq x \to z \in y)\,.$$

But this is just what the second part of the hypothesis of the lemma says.

□

Remark. Since $\subseteq$ is literally defined by a Δ_0 formula, the lemma holds without the assumption that M is a model of ZF − Foundation − Power Set − Infinity.

Lemma 6.14. *Let M be a transitive model of ZF − Foundation − Power Set − Infinity. If $\omega \in M$, then the Axiom of Infinity holds in M.*

Proof. The relativization to M of Infinity is

$$(\exists x \in M)(\emptyset^M \in x \wedge (\forall y \in x \cap M)\, \mathcal{S}^M(y) \in x)\,.$$

By the transitivity of M and the absoluteness of $\emptyset$ and $\mathcal{S}$, this is equivalent to

$$(\exists x \in M)(\emptyset \in x \wedge (\forall y \in x)\, \mathcal{S}(y) \in x)\,.$$

But ω witnesses that this is true. □

Lemma 6.15. (Uses Choice) *Let M be a transitive model of ZF − Foundation − Power Set − Infinity such that every subset of an element of M belongs to M. Then the Axiom of Choice holds in M.*

Proof. Using the transitivity of M and the absoluteness of $\emptyset$ and $\cap$, we get that the relativization to M of Choice is

$$\begin{aligned}&(\forall x \in M)\\ &\quad((\forall y_1)(\forall y_2)\,((y_1 \in x \wedge y_2 \in x) \to (y_1 \neq \emptyset \wedge (y_1 = y_2 \vee y_1 \cap y_2 = \emptyset)))\\ &\qquad\quad \to\; (\exists z \in M)(\forall y)(y \in x \to (\exists! w \in M)\, w \in y \cap z)))\end{aligned}$$

Let $x \in M$ satisfy the antecedent of the conditional. Let z be given by Choice. Then

$$(\forall y)(y \in x \to (\exists! w)\, w \in y \cap z)\,.$$

The transitivity of M implies that

$$(\forall y)(y \in x \to (\exists! w \in M)\, w \in y \cap z)\,.$$

This in turn implies that

$$(\forall y)(y \in x \to (\exists! w \in M)\, w \in y \cap (z \cap \mathcal{U}(x))\,.$$

Since the $\mathcal{U}$ operation is defined in M and is absolute for M, the set $\mathcal{U}(x)$ belongs to M. Since $z \cap \mathcal{U}(x) \subseteq \mathcal{U}(x)$, the hypotheses of the lemma give that $z \cap \mathcal{U}(x) \in M$. □

Theorem 6.16. *(a) The class* WF *is a model of ZF.*

(b) (Uses Choice) *The class* WF *is a model of ZFC.*

Proof. Since WF is transitive, Lemma 6.2 implies that Extensionality holds in WF. Since WF $\subseteq$ WF, Lemma 6.3 gives that Foundation holds in WF. All subsets of WF belong to WF, so, by the remark after the proof of Lemma 6.4, Comprehension holds in WF. It is easy to see that WF is closed under the operations of pairing and $\mathcal{U}$; hence Pairing and Union hold in WF by Lemmas 6.5 and 6.6. We leave as an exercise to prove that the hypothesis of Lemma 6.7 holds for WF. By that lemma we then get that Replacement holds in WF. We now have that WF is a model of ZF − Foundation − Power Set − Infinity. For $x \in \mathrm{WF}$,

$$\mathcal{P}(x) \cap \mathrm{WF} = \mathcal{P}(x) \in \mathrm{WF}\,.$$

Hence, by Lemma 6.13, Power Set holds in WF. By Lemma 6.14 and the fact that $\omega \in \mathrm{WF}$, we have that Infinity holds in WF. Since the hypotheses of Lemma 6.15 hold in WF, Choice holds in WF if it holds in V. □

Theorem 6.17. *(a) If ZF − Foundation is consistent, then so is ZF.*

(b) If ZFC − Foundation is consistent, then so is ZFC.

Proof. (a) follows from Lemma 6.1 and and part (a) Theorem 6.16, and (b) follows from Lemma 6.1 and and part (b) Theorem 6.16. □

Exercise 6.2. Prove the the Schema of Replacement holds in WF.

Announcement. We shall no longer note uses of Foundation.

Lemma 6.18. *The composition of absolute operations and relations is absolute: Suppose that T is a theory, that $M \subseteq N$, that M and N are models of T, and that that $G_1, \dots, G_m$ are n-argument operations defined in T that are absolute for (M, N).*

(a) Let R be an m-ary relation defined in T that is absolute for (M, N). Then the n-ary relation R' given by

$$R'(x_1, \dots, x_n) \leftrightarrow R(G_1(x_1, \dots, x_n), \dots, G_m(x_1, \dots, x_n))$$

is defined in T and is absolute for (M, N).

(b) Let F be an m-argument operation defined in T that is absolute for (M, N). Then the n-argument operation H given by

$$H(x_1, \dots, x_n) = F(G_1(x_1, \dots, x_n), \dots, G_m(x_1, \dots, x_n))$$

is defined in T and is absolute for (M, N).

We prove (b). The argument for (a) is similar. It is easy to see that H is defined in T. To prove its absoluteness, let $x_1, \ldots, x_n$ be elements of M. Then

$$\begin{aligned} H^M(x_1,\ldots,x_n) &= F^M(G_1^M(x_1,\ldots,x_n),\ldots,G_m^M(x_1,\ldots,x_n)) \\ &= F^N(G_1^M(x_1,\ldots,x_n),\ldots,G_m^M(x_1,\ldots,x_n)) \\ &= F^N(G_1^N(x_1,\ldots,x_n),\ldots,G_m^N(x_1,\ldots,x_n)) \\ &= H^N(x_1,\ldots,x_n)\,. \end{aligned}$$

□

Lemma 6.19. *The following relations and operations are defined in ZF − Power Set and are absolute for transitive models of ZF − Power Set.*

(a) z *is an ordered pair;*

(b) $u \times v$ *;*

(c) z *is a relation;*

(d) $\mathrm{domain}\,(z)$ $(= \{x \mid (\exists y)\langle x,y\rangle \in z\})$ *;*

(e) $\mathrm{range}\,(z)$ $(= \{y \mid (\exists x)\langle x,y\rangle \in z\})$ *;*

(f) z *is a function;*

(g) $z(x)$ $\left(= \begin{cases} z(x) & \textit{if } (\exists! y)\, \langle x,y\rangle \in z; \\ \emptyset & \textit{otherwise}; \end{cases} \right)$

(h) z *is a one-one function.*

Proof. (a) z is an ordered pair if and only if $(\exists x \in \mathcal{U}(z))(\exists y \in \mathcal{U}(z))\, z = \langle x,y\rangle$.

(b) The first of our two proofs of the existence of $u \times v$ was in ZF − Power Set, so $\times$ is defined in ZF − Power Set. For absoluteness, note that $z = u \times v$ if and only if

$$(\forall x \in u)(\forall y \in v)\, \langle x,y\rangle \in z \ \wedge \ (\forall w \in z)(\exists x \in u)(\exists y \in v)\, w = \langle x,y\rangle\,.$$

(c) z is a relation if and only if every element of z is an ordered pair.

(d) $u = \mathrm{domain}\,(z)$ if and only if

$$\begin{aligned} &(\forall x \in u)(\exists y \in \mathcal{U}(\mathcal{U}(z)))\langle x,y\rangle \in z \\ &\quad \wedge\ (\forall x \in \mathcal{U}(\mathcal{U}(z)))(\forall y \in \mathcal{U}(\mathcal{U}(z)))(\langle x,y\rangle \in z \rightarrow x \in u)\,. \end{aligned}$$

(e) $v = \mathrm{range}\,(z)$ if and only if

$$\begin{aligned} &(\forall y \in v)(\exists x \in \mathcal{U}(\mathcal{U}(z)))\langle x,y\rangle \in z \\ &\quad \wedge\ (\forall x \in \mathcal{U}(\mathcal{U}(z)))(\forall y \in \mathcal{U}(\mathcal{U}(z)))(\langle x,y\rangle \in z \rightarrow y \in v)\,. \end{aligned}$$

(f) z is a function if and only if z is a relation and

$$(\forall x \in \mathcal{U}(\mathcal{U}(z)))(\forall y \in \mathcal{U}(\mathcal{U}(z)))(\forall y' \in \mathcal{U}(\mathcal{U}(z))) \\ ((\langle x, y\rangle \in z \wedge \langle x, y'\rangle \in z) \to y = y')\,.$$

(g) $y = z(x)$ if and only if

$$(\langle x, y\rangle \in z \wedge (\exists! v \in \mathcal{U}(\mathcal{U}(z)))\,\langle x, v\rangle \in z) \\ \vee\ (y = \emptyset \wedge \neg(\exists! v \in \mathcal{U}(\mathcal{U}(z)))(\langle x, v\rangle \in z))\,.$$

(h) z is a one-one function if and only if z is a function and

$$(\forall x \in \mathcal{U}(\mathcal{U}(z)))(\forall x' \in \mathcal{U}(\mathcal{U}(z)))(z(x) = z(x') \to x = x')\,. \qquad \square$$

From now on, when we state that an operation or relation is absolute for transitive models of a theory T, we mean that the operation or relation is defined in T and is absolute for transitive models of T.

Lemma 6.20. *The following operations and relations are absolute for transitive models of ZF − Power Set.*

(a) x *is an ordinal;*

(b) x *is a limit ordinal;*

(c) x *is a successor ordinal;*

(d) x *is a finite ordinal;*

(e) ω*;*

(f) $0, 1, 2 \ldots.$

Proof. (a) x is an ordinal if and only if x is transitive and $\in \restriction x$ is a linear ordering of x. The first clause is absolute by Lemma 6.12 and the second is given by a Δ_0 formula (all the quantifiers are bounded to x).

(b) x is a limit ordinal if and only if x is an ordinal and $x \neq \emptyset$ and $(\forall y \in x)\, \mathcal{S}(y) \in x$.

(c) x is a successor ordinal if and only if x is an ordinal and x is neither $\emptyset$ nor a limit ordinal.

(d) x is a natural number if and only if x is an ordinal number and neither x nor any member of x is a limit ordinal.

(e) $x = \omega$ if and only if x is a limit ordinal and no member of x is a limit ordinal.

(f) $z = 0 \leftrightarrow z = \emptyset$; $z = a + 1 \leftrightarrow (\exists x \in z)(x = a \wedge z = \mathcal{S}(x))$. $\square$

Exercise 6.3. Explain briefly which axioms of ZFC are true in the following transitive classes. (The classes are all sets, so "true in M" can be taken in either of our two senses.)

(1) V_ω ;

(2) $V_{\omega+1}$;

(3) $V_{\omega+\omega}$;

(4) V_{ω_1} ;

(5) V_κ for κ inaccessible.

A cardinal κ is *inaccessible* if κ is uncountable and *regular* (if $\lambda < \kappa$ and $f : \lambda \to \kappa$ then range(f) is bounded in κ) and if, for all $\kappa' < \kappa$, $2^{\kappa'} < \kappa$.

Lemma 6.21. *Let M be a transitive model of ZF − Power Set. Then every finite subset of M belongs to M.*

Proof. There is only one subset x of M with card $(x) = 0$, namely $\emptyset$, and this set belongs to M. Assume inductively that every size n subset of M belongs to M. Let $x \subseteq M$ with card $(x) = n + 1$. Then there is a $y \subseteq M$ and there is a $z \in M$ such that card $(y) = n$ and $x = y \cup \{z\}$. By induction $y \in M$, and so Lemma 6.12 gives that $x \in M$. □

Lemma 6.22. *The following are absolute for transitive models of ZF − Power Set.*

(a) x is finite;

(b) ${}^{<\omega}x$.

Proof. (a) x is finite if and only if there is a one-one function f with domain $(f) \in \omega$ and range $(f) = x$. If $x \in M$, then Lemmas 6.12 and 6.20 imply that any such f is a subset of M and so, by Lemma 6.21, an element of M.

(b) We must show that ${}^{<\omega}x$ is defined in ZF − Power Set. To do this we first use induction to prove in ZF − Power Set that ${}^{n}x$ exists for every set x and every $n \in \omega$. This is true for $n = 0$, because ${}^{0}x = \{\emptyset\}$. It is easy to define a one-one correspondence between ${}^{n}x \times x$ and ${}^{n+1}x$, so our assertion for $n + 1$ follows from the assertion for n using Lemma 6.19 and Replacement. Next we use Replacement to get the existence of $\{{}^{n}x \mid n \in \omega\}$. Since ${}^{<\omega}x = \mathcal{U}\{{}^{n}x \mid n \in \omega\}$, we finally get the existence of ${}^{<\omega}x$. By Lemma 6.21, ${}^{<\omega}x \subseteq M$. Absoluteness holds because $z \in {}^{<\omega}x$ if and only if z is a function and domain $(z) \in \omega$ and range $(z) \subseteq x$. □

Lemma 6.23. *The following are absolute for transitive models of ZF − Power Set.*

(a) r *wellorders* x*;*

(b) $\mathrm{ot}(x, r)$*, that is, the unique ordinal* α *such that* $\langle x, r\rangle$ *is isomorphic to* $\langle \alpha, \in\restriction\alpha\rangle$ *if* r *wellorders* x *and* 0 *otherwise.*

Proof. That r linearly orders x is expressible by a Δ_0 formula.

Suppose that r wellorders x. Then every non-empty subset of x has an r-least element. Let $y \in M$ be such that $(y \subseteq x)^M$ and $(y \neq \emptyset)^M$. Then $y \subseteq x$ and $y \neq \emptyset$. Let z be an r-least element of y. Then $z \in M$ and it is true in M that z is the r-least element of y.

Now suppose that "r wellorders x" is true in M. Since the proof of Theorem 1.14 goes through in ZF − Power Set, it is true in M that there is an ordinal number α such that $\langle x, r\rangle$ is isomorphic to $\langle \alpha, \in\restriction\alpha\rangle$. Let f be such that in M it is true that f is an isomorphism between $\langle \alpha, \in\restriction\alpha\rangle$ and $\langle x, r\rangle$. By the absoluteness of the relevant notions, this is also true in V. Hence r wellorders x.

The argument just given proves (a), but it also shows that if r wellorders x then $\mathrm{ot}^M(x, r) = \mathrm{ot}(x, r)$. By (a) and the absoluteness of 0, we have (b).

□

We can extend our notion of absolute definable relations to relations defined from set parameters. For simplicity, we make this extension only for unary relations, i.e., for classes. Fix a class M. If A is the class $\{x \mid \varphi(x, a_1, \ldots, a_n)\}$, let us say that A is *defined in* M if $a_1, \ldots, a_n$ are elements of M. If A is defined in M, then

$$A^M = \{x \in M \mid \varphi^M(x, a_1, \ldots, a_n)\}\,.$$

We say that A is *absolute for* M if A is defined in M and $A^M = A \cap M$.

In an analogous fashion we now introduce the notion of absolute class functions. For a class function $F = \{\langle x, y\rangle \mid \varphi(\langle x, y\rangle, a_1, \ldots, a_n)\}$, let us say that F is *defined in* M (*as a function*) if $a_1, \ldots, a_n$ are elements of M and "F is a function" is true in M. If F is defined in M, then

$$F^M = \{\langle x, y\rangle \in M \mid \varphi^M(\langle x, y\rangle, a_1, \ldots, a_n)\}\,.$$

We say that F is *absolute for* M *(as a function)* if F is defined in M and $F^M = F \restriction M$ (so that, in particular, $\mathrm{domain}\,(F^M) = \mathrm{domain}\,(F) \cap M$).

Remarks:

(a) Being defined in M and being absolute for M depend upon the defining formula and parameters and not just on the class or function.

(b) *Definability in M* could be defined in a natural way for defined operations, although we have not done so.

(c) We have required that defined operations of n arguments be defined on any $x_1, \ldots, x_n$, but we allow absolute class functions to have domains that are not all of V.

Lemma 6.24. *Let $F : V \to V$. Let $G : \mathrm{ON} \to V$ be defined as in the proof of Theorem 1.8. Thus*

$$(\forall \alpha \in \mathrm{ON})\, G(\alpha) = F(G \restriction \alpha).$$

Let M be a transitive model of ZF $-$ Power Set. Assume that F is absolute for M. Then G is absolute for M.

Proof. Since the proof of Theorem 1.8 goes through in ZF $-$ Power Set and since $(F : V \to V)^M$, we have by earlier absoluteness results that G is defined in M, that $G^M : \mathrm{ON} \cap M \to M$, and that

$$(\forall \alpha \in \mathrm{ON} \cap M)\, G^M(\alpha) = F^M(G^M \restriction \alpha)\,.$$

Using the absoluteness of F, we can prove by transfinite induction on $\alpha \in \mathrm{ON} \cap M$ that $G^M(\alpha) = G(\alpha)$. $\square$

Lemma 6.25. *The operation* trcl *is absolute for transitive models of ZF $-$ Power Set.*

Proof. The proof of the existence of $\mathrm{trcl}(x)$ goes through in ZF $-$ Power Set. That proof shows that $\mathrm{trcl}(x) = \mathcal{U}(\mathrm{range}\,(g_x))$ for some g_x defined by recursion from an absolute F_x (defined from x by a formula that is independent of x). Thus $\mathrm{trcl}(x) = \mathcal{U}(\mathrm{range}\,(G_x \restriction \omega))$ for the G_x defined by transfinite recursion from this same F_x. $\square$

For any set x, let $\mathrm{rank}\,(x)$ be the least ordinal α such that $x \in V_{\alpha+1}$. Since the V_α, $\alpha > \omega$, may not exist in models of ZF $-$ Power Set, let us adopt the following definition of $\mathrm{rank}\,(x)$ as our official definition. Given x, define by transfinite recursion a function $G_x : \mathrm{ON} \to V$ by

$$\begin{aligned} G_x(0) &= \emptyset \\ G_x(\alpha+1) &= \{y \in \mathrm{trcl}(x) \mid y \subseteq G_x(\alpha)\}\,; \\ G_x(\lambda) &= \mathcal{U}(\mathrm{range}\,(G_x \restriction \lambda)) \quad \text{for limit ordinals } \lambda. \end{aligned}$$

Then let rank(x) be the least α, such that $x \subseteq G_x(\alpha)$. In ZF, one can easily show that $G_x(\alpha) = V_\alpha \cap \mathrm{trcl}(x)$, and so that the two definitions of rank(x) are equivalent in ZF.

Lemma 6.26. *The operation* rank *is absolute for transitive models of ZF − Power Set.*

Lemma 6.27. *Let M be a transitive model of ZF. Then*

(a) $\mathcal{P}^M(x) = \mathcal{P}(x) \cap M$ *for* $x \in M$ *;*

(b) $V_\alpha^M = V_\alpha \cap M$ *for* $\alpha \in \mathrm{ON} \cap M$*.*

Proof. (a) follows from the absoluteness of $\subseteq$. (b) follows from the absoluteness of rank. □

The following lemma gives the relation between the relativization of a formula to a set and the satisfaction of that formula by the model determined by that set.

Lemma 6.28. *Let $\varphi(x_1, \dots, x_n)$ be a formula. For any non-empty set b and any $a_1, \dots, a_n$ belonging to b,*

$$\varphi^b(a_1, \dots, a_n) \leftrightarrow (b; \in) \models \varphi[a_1, \dots, a_n]\,.$$

Proof. We can show by induction on complexity that all instances of this schema are provable in ZF. □

Let FODO$(\emptyset) = \{\emptyset\}$. For any non-empty set x, let FODO(x) be the set of all $u \subseteq x$ such that, for some formula $\varphi(v_0, \dots, v_n)$ and some sequence $\langle y_1, \dots, y_n\rangle$ of elements of x (i.e., some $f : n \to x$),

$$u = \{y_0 \in x \mid (x; \in) \models \varphi[y_0, \dots y_n]\}\,.$$

Lemma 6.29. *For any set x,*

(a) $\mathrm{FODO}(x) \subseteq \mathcal{P}(x)$ *;*

(b) if x is transitive, then $x \subseteq \mathrm{FODO}(x)$ *;*

(c) every finite subset of x belongs to $\mathrm{FODO}(x)$ *;*

(d) (Uses Choice) *if* $\mathrm{card}\,(x) \geq \omega$ *then* $\mathrm{card}\,(\mathrm{FODO}(x)) = \mathrm{card}\,(x)$.

Proof. The case $x = \emptyset$ is easy, so assume that x is non-empty.

(a) is obvious.

(b) Assume that x is transitive and let $b \in x$. Let $\varphi(v_0, v_1)$ be the formula $v_0 \in v_1$. Then $\{a \in x \mid (x; \in) \models \varphi[a, b]\} \in \mathrm{FODO}(x)$. But

$$\begin{aligned} \{a \in x \mid (x; \in) \models \varphi[a, b]\} &= \{a \in x \mid a \in b\} \\ &= b\,, \end{aligned}$$

where the last equality holds because x is transitive.

(c) Let $n \in \omega$ and let $u \subseteq x$ with $\mathrm{card}\,(u) = n$. Let $f : n \to u$ be one-one and onto. Then the formula $\bigvee\!\!\!\bigvee_{1 \leq i \leq n} v_0 = v_i$ and $f(0) \ldots, f(n-1)$ witness that $u \in \mathrm{FODO}(x)$.

(d) Assume that $\mathrm{card}\,(x) \geq \omega$. By (c), $\{y\} \in \mathrm{FODO}(x)$ for every $y \in x$. Thus $\mathrm{card}\,(x) \leq \mathrm{card}\,(\mathrm{FODO}(x))$. But $\mathrm{card}\,(\mathrm{FODO}(x))$ is no greater than the cardinal of $u \times v$, where u is the set of all formulas and $v = {}^{<\omega}x$. Thus $\mathrm{card}\,(\mathrm{FODO}(x)) \leq \aleph_0 \cdot \mathrm{card}\,(x) = \mathrm{card}\,(x)$. □

Remark. Choice is needed for (d) only to get the existence of $\mathrm{card}\,(x)$.

By transfinite recursion, we define a function $\mathbf{L} : \mathrm{ON} \to V$. We write L_α for $\mathbf{L}(\alpha)$.

(a) $L_0 = \emptyset$;

(b) $L_{\alpha+1} = \mathrm{FODO}(L_\alpha)$;

(c) $L_\lambda = \mathcal{U}(\{L_\alpha \mid \alpha < \lambda\})$ if λ is a limit ordinal.

Let $L = \mathcal{U}(\{L_\alpha \mid \alpha \in \mathrm{ON}\})$. Members of L are said to be *constructible.*

Lemma 6.30. *For each ordinal α,*

(a) L_α is transitive;

(b) $(\forall \beta \leq \alpha)\, L_\beta \subseteq L_\alpha$.

Moreover L is transitive.

Proof. We prove (a) by transfinite induction. The case $\alpha = 0$ is trivial. The case that α is a limit ordinal follows from the fact that the union of a set of transitive sets is transitive. The case α is a successor follows from part (b) of Lemma 6.29.

The proof of (b) is just like the proof of the corresponding fact for V_α.

L is transitive because it is a union of transitive sets. □

For each $x \in L$, let $\rho(x)$ (the *L-rank of x*) be the least ordinal α such that $x \in L_{\alpha+1}$.

Lemma 6.31. *(a)* $(\forall\alpha\in\mathrm{ON})(\alpha\in L\wedge\rho(\alpha)=\alpha)$.

(b) $(\forall\alpha\in\mathrm{ON})\,\mathrm{ON}\cap L_\alpha=\alpha$.

Proof. It is easy to see that (a) and (b) are equivalent. We prove (b) by transfinite induction.

The cases that α is 0, 1, or a limit ordinal are trivial.

Assume that α is $\beta+1$ with $\beta>0$. Note that the proof of part (a) of Lemma 6.20 establishes that "x is an ordinal" is equivalent in ZF $-$ Power Set to a Δ_0 formula. Calling this formula $\mathrm{Ord}(x)$, we have, for $y\in L_\beta$:

$$\begin{aligned} L_\beta\models\mathrm{Ord}[y] &\leftrightarrow \mathrm{Ord}^{L_\beta}(y)\\ &\leftrightarrow \mathrm{Ord}(y)\\ &\leftrightarrow y\text{ is an ordinal}\end{aligned}$$

Thus $\mathrm{Ord}(v_0)$ witnesses that $L_\beta\cap\mathrm{ON}\in L_\alpha$ so, by induction, that $\beta\in L_\alpha$. We have then that $\mathrm{ON}\cap L_\alpha\supseteq\alpha$. But if $\gamma\geq\alpha$ then $\gamma\not\subseteq\beta$ and so $\gamma\not\subseteq L_\beta$. Thus no $\gamma\geq\alpha$ belongs to L_α. □

Lemma 6.32. *For $\alpha\in\omega$, V_α and L_α are finite. For $\alpha\leq\omega$, $L_\alpha=V_\alpha$.*

Proof. The power set of a finite set is finite, so the first assertion is easily proved by induction. The second then follows using part (c) of Lemma 6.29 and induction. □

Lemma 6.33. (Uses Choice) *For $\alpha\geq\omega$,* $\mathrm{card}\,(L_\alpha)=\mathrm{card}\,(\alpha)$.

Proof. By Lemma 6.31, $\mathrm{card}\,(L_\alpha)\geq\mathrm{card}\,(\alpha)$ for every α.

By transfinite induction, we show that $\mathrm{card}\,(L_\alpha)\leq\mathrm{card}\,(\alpha)$ for every $\alpha\geq\omega$.

The case $\alpha=\omega$ follows from Lemma 6.32.

For limit $\alpha>\omega$,

$$\begin{aligned}\mathrm{card}\,(L_\alpha) &= \mathrm{card}\,(\bigcup_{\beta<\alpha}L_\beta)\\ &\leq \mathrm{card}\,(\alpha\times\sup_{\beta<\alpha}\mathrm{card}\,(L_\beta))\\ &\leq \mathrm{card}\,(\alpha\times\mathrm{card}\,(\alpha))\\ &= \mathrm{card}\,(\alpha)\,.\end{aligned}$$

The case that α is a successor follows from part (d) of Lemma 6.29. □

Remark. Choice is not really needed for Lemma 6.33, as the proof of Theorem 6.45 will show.

Lemma 6.34. *Extensionality, Foundation, Pairing, Union, and Replacement hold in L.*

Proof. Extensionality holds, since L is transitive.

Foundation is trivial.

To show that Pairing holds, we use Lemma 6.5. Suppose that x and y belong to L. By part (b) of Lemma 6.30, let α be such that both x and y belong to L_α. Using the formula $v_0 = v_0$, we see that $L_\alpha \in L_{\alpha+1}$ Thus $L_\alpha \in L$ and $\{x, y\} \subseteq L_\alpha$.

For Union, we use Lemma 6.6. Let $x \in L_\alpha$. Since L_α is transitive, $x \subseteq L_\alpha$. This fact and the transitivity of L_α imply that $\mathcal{U}(x) \subseteq L_\alpha$.

For Replacement, we use Lemma 6.7. Let z and $w_1, \ldots, w_n$ belong to L and assume that

$$(\forall x \in z \cap L)(\exists! y \in L)\, \varphi^L(x, y, z, w_1, \ldots, w_n)\,.$$

By the transitivity of L and by Replacement in V, there is an α such that

$$(\forall x \in z)(\exists y \in L_\alpha)\, \varphi^L(x, y, z, w_1, \ldots, w_n)\,.$$

Lemma 6.35. *If Comprehension holds in L, then all axioms of ZF hold in L.*

Proof. For Power Set, we use Lemma 6.13. Let $x \in L$. By Replacement in V, let α be such that $\mathcal{P}(x) \cap L \subseteq L_\alpha$.

For Infinity, we use Lemma 6.14 and the fact that $\omega \in L$. □

Remark. An inspection of the proofs of Lemma 6.13 and Lemma 6.14 will show that they do not need the hypothesis that Comprehension holds. Thus Power Set and Infinity could have been included in Lemma 6.34.

A class C of ordinals is *closed* if the union of any subset of C belongs to C. If α is a limit ordinal, a *closed subset* of α is a subset C of α such that the union of any subset of C bounded in α belongs to C.

Theorem 6.36 (Reflection Schema). *Let $\mathbf{M} : \mathrm{ON} \to V$. (We write M_α for $\mathbf{M}(\alpha)$.) Let $M = \bigcup_{\alpha\in\mathrm{ON}} M_\alpha$. Assume that $M_\beta \subseteq M_\alpha$ whenever $\beta \leq \alpha \in \mathrm{ON}$ and that $M_\lambda = \bigcup_{\beta<\lambda} M_\beta$ for all limit λ. Let φ be a formula. Then there is a closed, unbounded class C of ordinals such that*

$$(\forall \alpha \in C)\, \varphi \textit{ is absolute for } (M_\alpha, M).$$

Proof. We proceed by induction on the complexity of φ (i.e., we show inductively how the instances of the schema can be proved).

If φ is atomic, then we can let $C = \mathrm{ON}$.

If φ is $\neg\psi$ and C witnesses that the theorem holds for ψ (and $\mathbf{M}$), then C witnesses that the theorem holds for φ.

If φ is $\psi \wedge \chi$ and C' and C'' respectively witness that the theorem holds for ψ and χ, then $C = C' \cap C''$ witnesses that the theorem holds for φ.

Assume that φ is $(\exists y)\, \psi(x_1, \ldots, x_n, y)$. Define $F : {}^nV \to \mathrm{ON}$ by

$$F(\langle x_1, \ldots, x_n\rangle) = \begin{cases} \mu\alpha\, (\exists y \in M_\alpha)\, \psi^M(x_1, \ldots, x_n, y) \\ \qquad\quad \text{if } (\exists y \in M)\, \psi^M(x_1, \ldots, x_n, y)\,; \\ 0 \quad \text{otherwise.} \end{cases}$$

For ordinals α let

$$G(\alpha) = \mathcal{U}(\{F(\langle x_1, \ldots, x_n\rangle) \mid \langle x_1, \ldots x_n\rangle \in {}^nM_\alpha\})\,.$$

Let

$$C' = \{\alpha \in \mathrm{ON} \mid \alpha \text{ is a limit ordinal } \wedge\ (\forall\beta < \alpha)\, G(\beta) < \alpha\}\,.$$

The class C' is obviously closed. To see that C' is unbounded, let $\beta \in \mathrm{ON}$. Let $\beta_0 = \beta$ and, for $i \in \omega$, let $\beta_{i+1} = \max\{\beta_i, G(\beta_i)\} + 1$. If $\alpha = \bigcup_{i\in\omega} \beta_i$ then $\beta < \alpha$ and $\alpha \in C'$. Let C'' witness that the theorem holds for ψ. Let $C = C' \cap C''$. The class C is closed and unbounded. Let $\alpha \in C$ and let $x_1, \ldots, x_n$ belong to M_α. Since α is a limit ordinal, there is a $\beta < \alpha$ such that $x_1, \ldots, x_n$ belong to M_β. We have

$$\begin{aligned} \varphi^M(x_1, \ldots, x_n) &\leftrightarrow (\exists y \in M)\, \psi^M(x_1, \ldots, x_n, y) \\ &\leftrightarrow (\exists y \in M_{F(\langle x_1,\ldots,x_n\rangle)})\, \psi^M(x_1, \ldots, x_n, y) \\ &\leftrightarrow (\exists y \in M_{G(\beta)})\, \psi^M(x_1, \ldots, x_n, y) \\ &\leftrightarrow (\exists y \in M_\alpha)\, \psi^M(x_1, \ldots, x_n, y) \\ &\leftrightarrow (\exists y \in M_\alpha)\, \psi^{M_\alpha}(x_1, \ldots, x_n, y) \\ &\leftrightarrow \varphi^{M_\alpha}(x_1, \ldots, x_n)\,. \end{aligned}$$

□

Exercise 6.4. Assume that κ is an inaccessible cardinal. Prove that there is an $\alpha < \kappa$ such that $(V_\alpha; \in) \models ZFC$.

Hint. Use part (5) of Exercise 6.3 and Reflection to prove that there is an $\alpha < \kappa$ such that $(V_\alpha; \in) \prec (V_\kappa; \in)$.

Theorem 6.37. *All axioms of ZF hold in L.*

Proof. By Lemmas 6.34 and 6.35, we need only show that Comprehension holds in L. Let $\varphi(v_1, \ldots, v_{n+2})$ be a formula and let z and $w_1, \ldots, w_n$ belong to L. By Theorem 6.36, let α be an ordinal such that φ is absolute for (L_α, L) and such that z and $w_1, \ldots, w_n$ belong to L_α. We have, suppressing the w_i for brevity,

$$\begin{aligned}
\{x \in z \mid \varphi^L(x, z)\} &= \{x \in L_\alpha \mid x \in z \wedge \varphi^L(x, z)\} \\
&= \{x \in L_\alpha \mid x \in z \wedge \varphi^{L_\alpha}(x, z)\} \\
&= \{x \in L_\alpha \mid (L_\alpha; \in) \models (v_1 \in v_2 \wedge \varphi(v_1, v_2))[x, z]\} \\
&\in L_{\alpha+1} \\
&\subseteq L\,.
\end{aligned}$$

By Lemma 6.4, we have shown that Comprehension holds in L. □

Our next task is to prove that $V = L$ holds in L.

Lemma 6.38. *The functions $\langle m, n\rangle \mapsto m + n$ and $\langle m, n\rangle \mapsto m \cdot n$ are absolute for transitive models of ZF − Power Set.*

Proof. For each m, $n \mapsto m + n$ is defined by recursion from an absolute function defined by a formula independent of m. Thus addition is absolute. By a similar argument, multiplication is absolute. □

Lemma 6.39. *The functions $\langle f, g\rangle \mapsto f^\frown g$ and $h \mapsto \mathrm{concat}\,(h)$ are absolute for transitive models of ZF − Power Set.*

Proof. These functions are defined (on page 22) by recursion from absolute functions. □

Lemma 6.40. *The following are absolute for transitive models of ZF − Power Set:*

(a) *x is a variable;*

(b) $n \mapsto v_n$ *;*

(c) $x \in \mathrm{Formula}$*, i.e., x is a formula of the language of set theory;*

(d) $\langle x, y\rangle \in \mathrm{Free}$*, i.e., x is a formula and y is a variable occurring free in x.*

Proof. (a) and (b) follow from Lemma 6.38.

For (c), note that the function $n \mapsto \text{Formula}_n$ is defined (on page 23) by recursion from an absolute function. Formula is the union of the range of this function.

For (d), note that the function sending each n to $\{\langle x, y\rangle \in \text{Free} \mid x \in \text{Formula}_\text{n}\}$ is definable by recursion from an absolute function. $\square$

Lemma 6.41. *The 3-ary relation $\langle y, z\rangle \in \text{Sat}^{(x;\in)}$ is absolute for transitive models of ZF − Power Set.*

Proof. $n \mapsto \text{Sat}_n^{(x;\in)}$ is defined by recursion from an absolute function. $\square$

Lemma 6.42. *The operation* FODO *is absolute for transitive models of ZF − Power Set.*

Proof. FODO is defined in ZF − Power Set, since Replacement guarantees the existence of FODO(x). Thus it is enough to show that the relation $u \in \text{FODO}(x)$ is absolute. But $u \in \text{FODO}(x)$ if and only if $u \subseteq x$ and

$$\begin{aligned}(\exists\varphi)(\exists s)(&\varphi \in \text{Formula} \wedge s \in {}^{<\omega}x \\ &\wedge (\forall i \in \omega)(\langle \varphi, v_i\rangle \in \text{Free} \rightarrow i < \ell\text{h}(s) + 1) \\ &\wedge (\forall y \in x)(y \in u \leftrightarrow \langle \varphi, \langle y\rangle^\frown s\rangle \in \text{Sat}^{(x;\in)}))\,.\end{aligned}$$ $\square$

Lemma 6.43. *The function $\alpha \mapsto L_\alpha$ is absolute for transitive models of ZF − Power Set.*

Proof. This function is defined by transfinite recursion from an absolute function. $\square$

Theorem 6.44. *The Axiom of Constructibility $V = L$ holds in L.*

Proof. We have that

$$\begin{aligned}(V = L)^L \quad &\leftrightarrow \quad (\forall x \in L)(\exists \alpha \in L \cap \text{ON}^L)(x \in L_\alpha)^L \\ &\leftrightarrow \quad (\forall x \in L)(\exists \alpha \in \text{ON})\, x \in L_\alpha \\ &\leftrightarrow \quad (\forall x \in L)\, x \in L\,.\end{aligned}$$ $\square$

Theorem 6.45. *The Axiom of Choice holds in L.*

Proof. Fix a wellordering of Formula. By transfinite recursion, we define a function $\alpha \mapsto <_\alpha$. By induction we shall verify the following:

(i) $<_\alpha$ is a wellordering of L_α;

(ii) $(\forall x \in L_\alpha)(\forall y \in L_\alpha)(\rho(x) < \rho(y) \to x <_\alpha y)$;

(iii) $(\forall \beta < \alpha)\ <_\beta \subseteq <_\alpha$.

Set $<_0 = \emptyset$.

For α a limit ordinal, set $<_\alpha = \bigcup_{\beta<\alpha} <_\beta$. It is immediate that (iii) holds for α. The induction hypotheses that (ii) and (iii) hold for all ordinals $< \alpha$ guarantee that (ii) holds for α. Since (ii) holds for α, any failure of (i) for α would give a failure of (i) for some $\beta < \alpha$.

Assume $\alpha = \beta + 1$. For $n \in \omega$, wellorder ${}^n(L_\beta)$ lexicographically, using the ordering $<_\beta$ of L_β. (If s and t are distinct members of ${}^n(L_\beta)$, then s is less than t if, for the least m such that $s(m) \neq t(m)$, $s(m) <_\beta t(m)$.) Now order ${}^{<\omega}(L_\beta)$ by setting s less than t if $\ell\mathrm{h}(s) < \ell\mathrm{h}(t)$ or else $\ell\mathrm{h}(s) = \ell\mathrm{h}(t)$ and s is less than t in our ordering of ${}^{\ell\mathrm{h}(s)}(L_\beta)$. Finally order Formula $\times\ {}^{<\omega}(L_\beta)$ lexicographically. It is easy to check that this ordering is a wellordering. For x and y belonging to L_α, set $x <_\alpha y$ just in case one of the following holds:

(a) $x \in L_\beta \wedge y \in L_\beta \wedge x <_\beta y$;

(b) $x \in L_\beta \wedge y \notin L_\beta$;

(c) $x \notin L_\beta \wedge y \notin L_\beta$ and the least element of Formula $\times\ {}^{<\omega}(L_\beta)$ that witnesses $x \in L_\alpha$ is less than the least element that witnesses $y \in L_\alpha$.

Clearly (i), (ii), and (iii) hold for α.

Define $<_L = \bigcup_{\alpha \in \mathrm{ON}} <_\alpha$. By (i)–(iii), $<_L$ is a wellordering of L. Thus $V = L$ implies that $<_L$ wellorders V, and so $V = L$ implies Choice. Since $V = L$ holds in L, Choice holds in L. □

Lemma 6.46 (Mostowski Collapse). *Let u be a set such that Extensionality holds in u. Then there is a unique transitive set v such that $(u; \in) \cong (v; \in)$. Moreover there is a unique isomorphism*

$$\pi : (u; \in) \cong (v; \in).$$

Proof. For $x \in u$ we define $\pi(x)$ by recursion on rank(x). Set

$$\pi(x) = \{\pi(y) \mid y \in x \cap u\}.$$

Note that this is the only possible choice of $\pi(x)$ if π is to be an isomorphism with range(π) transitive.

It is clear that

$$y \in x \to \pi(y) \in \pi(x).$$

To prove the converse, it is enough to show that π is one-one, and this will show that $\pi : (u; \in) \cong (\mathrm{range}\,(\pi); \in)$. By induction on the maximum of rank(x_1) and rank(x_2)}, we show that $\pi(x_1) = \pi(x_2) \to x_1 = x_2$. We have

$$\begin{aligned}\pi(x_1) = \pi(x_2) \quad &\to \quad \{\pi(y) \mid y \in x_1 \cap u\} = \{\pi(y) \mid y \in x_2 \cap u\}\\ &\to \quad \text{(by induction) } \{y \mid y \in x_1 \cap u\} = \{y \mid y \in x_2 \cap u\}\\ &\to \quad \text{(by Extensionality}^u\text{) } x_1 = x_2\,.\end{aligned}$$

□

Lemma 6.47. *Let κ be an uncountable regular cardinal. Then L_κ is a model of ZF − Power Set + $V = L$.*

Proof. Showing that L_κ is a model of ZF − Power Set will be part of a final examination problem. That $V = L$ holds in L_κ follows by Lemma 6.43. □

Lemma 6.48. *Let z be a transitive model of ZF − Power Set + $V = L$. There is an α such that $z = L_\alpha$.*

Proof. Let $\alpha = \mathrm{ON} \cap z$. Clearly α is a limit ordinal. The function $\gamma \mapsto L_\gamma$ is absolute for z. For $x \in z$ there is a $\gamma < \alpha$ such that $x \in L_\gamma$ holds in z. By absoluteness, every element of z belongs to L_α. For each $\gamma < \alpha$, $(L_\gamma)^z = L_\gamma$, and so every element of L_α belongs to z. □

Theorem 6.49. *The Generalized Continuum Hypothesis holds in L.*

Proof. Let α be an infinite ordinal number. We show that

$$\mathcal{P}(\alpha) \cap L \subseteq L_{\alpha^+}\,.$$

By Lemma 6.33, this implies that $\mathrm{card}\,(\mathcal{P}(\alpha) \cap L) \leq \alpha^+$. Hence $V = L$ implies that $2^{\mathrm{card}(\alpha)} = \alpha^+$. Since $V = L$ holds in L, the theorem will be proved.

Let $x \subseteq \alpha$ with $x \in L$. Let $\beta > \alpha$ be such that $x \in L_\beta$. By Lemma 6.47, L_{β^+} is a model of ZF − Power Set + $V = L$.

By the Löwenheim–Skolem Theorem, let y be such that

(i) $(y; \in) \prec (L_{\beta^+}; \in)$;

(ii) $\alpha \cup \{x\} \subseteq y$;

(iii) $\mathrm{card}\,(y) = \mathrm{card}\,(\alpha)$.

By Lemma 6.46, Let z and π be such that z is transitive and

$$\pi : (y; \in) \cong (z; \in) .$$

Since $(z; \in) \cong (y; \in) \prec (L_{\beta^+}; \in)$, z is a model of ZF $-$ Power Set $+$ $V = L$. By Lemma 6.48, there is an ordinal γ such that $z = L_\gamma$.

Since $\operatorname{card}(\gamma) \le \operatorname{card}(L_\gamma) = \operatorname{card}(z) = \operatorname{card}(y) \le \operatorname{card}(\alpha)$, we have that $\gamma < \alpha^+$.

It suffices then to prove that $x \in L_\gamma$. Since $x \in y$, we need only show that $\pi(x) = x$. First we show by induction on $\eta < \alpha$ that $\pi(\eta) = \eta$. We have

$$\begin{aligned} \pi(\eta) &= \{\pi(\xi) \mid \xi \in \eta \cap y\} \\ &= (\text{since } \alpha \subseteq y)\ \{\pi(\xi) \mid \xi \in \eta\} \\ &= (\text{by induction})\ \{\xi \mid \xi \in \eta\} \\ &= \eta . \end{aligned}$$

Finally we note that

$$\pi(x) = \{\pi(\eta) \mid \eta \in x \cap y\} = \{\pi(\eta) \mid \eta \in x\} = \{\eta \mid \eta \in x\} = x . \qquad \square$$

Remark. One can construct a sentence σ such that, for any transitive class M, σ holds in M if and only if $M = L$ or there is an ordinal α such that $M = L_\alpha$. Thus L_β rather than L_{β^+} could in principle have been used in the proof.

Theorem 6.50. *If ZF is consistent then so are*

(a) ZFC + $V = L$;

(b) ZFC + GCH.

Proof. Assume that ZF is consistent. Then (a) follows from Lemma 6.1 together with Lemmas 6.44 and 6.45. (b) then follows from (a) and Theorem 6.49. $\square$

The Axiom of Constructibility settles most interesting set-theoretic questions. A number of them can be answered using Jensen's combinatorial principle $\diamondsuit$. $\diamondsuit$ is the assertion that there is a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ (i.e., a function $\alpha \mapsto A_\alpha$ with domain ω_1) such that each $A_\alpha \subseteq \alpha$ and such that, for any $A \subseteq \omega_1$ and any closed, unbounded subset C of ω_1,

$$(\exists \alpha \in C)\, A \cap \alpha = A_\alpha .$$

Theorem 6.51. $V = L \rightarrow \Diamond$.

Proof. Assume $V = L$. We define A_α by recursion. For α not a limit ordinal, set $A_\alpha = \emptyset$. Assume that α is limit ordinal and that A_β is defined for $\beta < \alpha$. Let ρ_α be the least ordinal ρ such that there are A and C belonging to L_ρ such that $A \subseteq \alpha$, C is a closed, unbounded subset of α, and

$$(\forall \beta \in C)\, A \cap \beta \neq A_\beta$$

if such a ρ exists. In this case let A_α and C_α be the lexicographically least A and C (using $<_L$). If ρ_α does not exist, let $A_\alpha = \emptyset$.

Suppose that $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ does not witness that $\Diamond$ holds. Let ρ be the least ordinal such that some counterexample sets A and C belong to L_ρ. Let A and C be the lexicographically least such pair (again using $<_L$). Note that $\rho < \omega_2$.

Let $(y; \in) \prec (L_{\omega_2}; \in)$ with y countable and with

$$\{\omega_1, \rho, A, C, \langle A_\alpha \mid \alpha < \omega_1 \rangle\} \subseteq y\,.$$

Let z and π be such that z is transitive and $\pi : (y; \in) \cong (z; \in)$. Let $\delta < \omega_1$ be such that $z = L_\delta$.

Let $\alpha = \pi(\omega_1)$. By part (a) of Exercse 6.5 below, we have that $\alpha \subseteq y$. It follows that

(i) $A \cap \alpha = \pi(A)$;

(ii) $C \cap \alpha = \pi(C)$;

(iii) $\langle A_\beta \mid \beta < \alpha \rangle = \pi(\langle A_\beta \mid \beta < \omega_1 \rangle)$.

Using (i)–(iii), the definitions of ρ, A, and C, and the fact that π^{-1} is an elementary embedding of $(L_\delta; \in)$ into $(L_{\omega_2}; \in)$, we get that $\pi(\rho)$, $A \cap \alpha$, and $C \cap \alpha$ satisfy in L_δ the definitions of ρ_α, A_α, and C_α respectively. Thus

(a) $\pi(\rho) = \rho_\alpha$;

(b) $A \cap \alpha = A_\alpha$;

(c) $C \cap \alpha = C_\alpha$.

Since $C \cap \alpha = \pi(C)$, $C \cap \alpha$ is an unbounded subset of α. Since C is closed, it follows that $\alpha \in C$. This fact and (b) contradict the definitions of A and C. □

Exercise 6.5. Assume $V = L$.

(a) Show that if $(y; \in) \prec (L_{\omega_2}; \in)$ then $y \cap \omega_1$ is transitive.

(b) Show that

$$\{\alpha < \omega_1 \mid L_{\alpha+1} \cap \mathcal{P}(\omega) \subseteq L_\alpha\}$$

has a subset that is closed and unbounded in ω_1.

Hint. If $\alpha < \omega_1$, then α is countable. For (a), begin with the definition of "α is countable." For (b), observe that $L_{\omega_1+1} \cap \mathcal{P}(\omega) \subseteq L_{\omega_1}$. Build an elementary chain of length ω_1 and then apply Mostowski collapse.

One of the earliest applications of $\diamondsuit$ was to show that *Souslin's Hypothesis* fails in L.

To state Souslin's Hypothesis, we need some definitions. Let R be a linear ordering of a set X. If every R-bounded subset of X has a least upper bound, then $(X; R)$ is said to be *complete*. If every set of disjoint open (in the obvious sense) R-intervals is countable, then $(X; R)$ is *ccc*: *satisfies the countable chain condition*. Give X the order topology: the basic open sets are the open intervals. If X has a countable dense subset then $(X; R)$ is *separable*.

The set $\mathbb{R}$ of reals, with its usual ordering, is—up to isomorphism—the unique separable, complete, dense linear ordering without endpoints. Souslin's hypothesis says this characterization continues to hold when "separable" is replaced by "ccc." Clearly the failure of Souslin's Hypothesis is equivalent to the existence of a *Souslin line*, a ccc, complete, dense linear ordering that is not separable.

The existence of a Souslin line is can be shown equivalent to the existence of a *Souslin tree*: a $(T; \lhd)$ such that

(1) $\lhd$ is a partial ordering of T;

(2) For all $x \in T$, $\{y \in T \mid x \lhd y\}$ is wellordered by $\lhd$;

(3) $\mathrm{card}(T) = \aleph_1$;

(4) $(T; \lhd)$ has no uncountable branches and no uncountable antichains.

Here a *branch* is a maximal subset of T linearly ordered by $\lhd$, and an *antichain* is a set of pairwise $\lhd$-incomparable elements of T.

Conditions (1) and (2) define the (set-theoretic) concept of a *tree*. Let us call a tree $(T; \lhd)$ *ultranormal* if

(i) $T \subseteq \omega_1$;

(ii) for β and $\gamma \in T$, $\beta \lhd \gamma \to \beta < \gamma$;

(iii) T has a $\lhd$-least element;

(iv) For each $\alpha < \omega_1$, the set of all $\beta \in T$ such that $\text{level}(\beta) = \alpha$ is countable, where $\text{level}(\beta)$ is the $\lhd$ order type of $\{\gamma \in T \mid \gamma \lhd \beta\}$;

(v) if $\beta \in T$ then β has infinitely many immediate successors with respect to $\lhd$;

(vi) for each $\beta \in T$ and each α such that $\text{level}(\beta) < \alpha < \omega_1$, there is a $\gamma \in T$ such that $\text{level}(\gamma) = \alpha$ and $\beta \lhd \gamma$;

(vii) if β and γ are elements of T with the same limit level and the same $\lhd$-predecessors, then $\beta = \gamma$.

Lemma 6.52. *If there is an ultranormal Souslin tree, then there is a Souslin line.*

Proof. We first observe that it is enough to construct a ccc, dense, linear ordering $(X; R)$ that is not separable. If we have such an $(X; R)$, then we can let X' be the set of all *Dedekind cuts* in $(X; R)$, i.e., the set of all bounded initial segments of $(X; R)$ without R-greatest elements, and we can let $x' \, R' \, y' \leftrightarrow x' \subseteq y'$. Clearly $(X'; R')$ a linear ordering. The function $x \mapsto \{y \in X \mid y \, R \, x\}$ embeds $(X; R)$ into $(X'; R')$ and has dense range. Therefore $(X'; R')$ is dense, ccc, and not separable. If A is an R'-bounded subset of X', then $\bigcup A$ is the R'-least upper bound of A; hence $(X'; R')$ is complete.

Let $(T; \lhd)$ be an ultranormal Souslin tree. Let

$$X = \{b \mid b \text{ is a branch of } T\}\,.$$

To define an ordering R on X, let us first fix, for each $\beta \in T$, an ordering $<_\beta$ of the the immediate successors of β with respect to $\lhd$. By (iv) and (v), we can—and do—make $<_\beta$ isomorphic to the standard ordering of the rationals. Let b and b' be distinct branches of $(T; \lhd)$. By (vii), there is a $\lhd$-greatest β that belongs to both b and b'. Let γ and γ' be the immediate $\lhd$-successors of β that belong to b and b' respectively. Define

$$b \, R \, b' \;\leftrightarrow\; \gamma <_\beta \gamma'.$$

It is easy to see that R is a linear ordering of X. Suppose that I is an open interval of $(X; R)$. let $I = (b, b')$. Define β, γ, and γ' as in the preceding paragraph. Let δ_I be such that $\gamma <_\beta \delta_I <_\beta \gamma'$. Observe that every branch containing δ_I belongs to the interval I. Observe also that if I_1 and I_2 are disjoint intervals, then δ_{I_1} and δ_{I_2} are $\lhd$-incomparable. The first fact implies that the $(X;R)$ is a dense ordering, and the second fact implies that $(X;R)$ has the ccc. For non-separability, let B be any countable subset of X. Since every member of B is countable, $\bigcup_{b\in B} b$ is countable. Let $\alpha \in T$ be $>$ every member of this countable set. Then the set of branches containing α is a neighborhood witnessing that B is not dense. $\square$

Theorem 6.53. *If $\diamondsuit$ holds, then there is an ultranormal Souslin tree.*

Proof. Let $\langle A_\alpha \mid \alpha < \omega_1\rangle$ witness that $\diamondsuit$ holds.

We shall define an ultranormal tree $(T;\lhd)$ by transfinite recursion. More precisely, we shall define for each $\alpha < \omega_1$ a tree $(T_\alpha;\lhd_\alpha)$, and we shall arrange that

(a) for $\alpha' < \alpha < \omega_1$, $T_{\alpha'}$ is the set of all elements of T_α of $\lhd_\alpha$-level $\leq \alpha'$, and $\lhd_{\alpha'}$ is the restriction of $\lhd_\alpha$ to T'_α;

(b) for $\alpha < \omega_1$, (i)-(vii) hold with $(T_\alpha;\lhd_\alpha)$ replacing $(T;\lhd)$ and with the $\alpha+1$ replacing ω_1..

We shall then let $T = \bigcup_{\alpha<\omega_1} T_\alpha$ and $\lhd = \bigcup_{\alpha<\omega_1} \lhd_\alpha$. The only task that will remain to us is the verification that $(T;\lhd)$ satisfies condition (4) in the definition of a Souslin tree.

Let $\alpha < \omega_1$ and assume that $(T_{\alpha'};\lhd_{\alpha'})$ is defined for $\alpha' < \alpha$ in such a way that (a) and (b) are not violated.

If $\alpha = 0$ let $T_0 = \{0\}$ and stipulate that 0 does not bear $\lhd_0$ to itself.

If $\alpha = \alpha' + 1$ for some α', then assign to the ordinals $\beta \in T_{\alpha'}$ of level α' disjoint countable infinite sets $B_\beta \subseteq \omega_1$. Do this so that $\beta < \gamma \notin T_{\alpha'}$ for each $\gamma \in B_\beta$. Let

$$T_\alpha = T_{\alpha'} \cup \bigcup\{B_\beta \mid \beta \in T_{\alpha'} \wedge \text{level}(\beta) = \alpha'\}\,.$$

Let

$$\lhd_\alpha = \lhd'_\alpha \cup \{\langle\beta,\gamma\rangle \mid \beta \in T_{\alpha'} \wedge \text{level}(\beta) = \alpha' \wedge \gamma \in B_\beta\}\,.$$

Assume that α is a limit ordinal. Let $\langle \alpha_i \mid i \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum α. Let

$$\begin{array}{rcll} T^*_\alpha & = & \bigcup_{\alpha'<\alpha} T_{\alpha'} & (= \bigcup_{i\in\omega} T_{\alpha_i}); \\ \lhd^*_\alpha & = & \bigcup_{\alpha'<\alpha} \lhd_{\alpha'} & (= \bigcup_{i\in\omega} \lhd_{\alpha_i}). \end{array}$$

For $\beta \in T^*_\alpha$, define $\langle \beta_i \mid i \in \omega \rangle$ by recursion as follows. If A_α is not a maximal antichain in the tree $(T^*_\alpha; \lhd^*_\alpha)$ or if there is a $\xi \in A_\alpha$ such that $\xi \lhd^*_\alpha \beta$, then set $\beta_0 = \beta$. Otherwise there is a $\xi \in A_\alpha$ such that $\beta \lhd^*_\alpha \xi$. Let β_0 be some such ξ. If level$(\beta_i) \geq \alpha_i$, then let $\beta_{i+1} = \beta_i$. If level$(\beta_i) < \alpha_i$, let $\beta_{i+1} \in T_{\alpha_i}$ be such that $\beta_i \lhd_{\alpha_i} \beta_{i+1}$ and level$(\beta_{i+1}) = \alpha_i$. (Such a β_{i+1} exists by condition (vi) on $(T_{\alpha_i}; \lhd_{\alpha_i})$.) Let b_β be the unique branch containing all the β_i. Let $\mathcal{B}_\alpha$ be the set of all the b_β for $\beta \in T^*_\alpha$. For each $b \in \mathcal{B}_\alpha$, let γ_b be a countable ordinal γ such that $\gamma \notin T^*_\alpha$ and $\gamma >$ every member of b. Make sure that the function $b \mapsto \gamma_b$ is one-one. Let

$$T_\alpha = T^*_\alpha \cup \{\gamma_b \mid b \in \mathcal{B}_\alpha\}\,.$$

Let

$$\lhd_\alpha = \lhd^*_\alpha \cup \{\langle \delta, \gamma_b \rangle \mid (b \in \mathcal{B}_\alpha \wedge \delta \in b)\}\,.$$

To verify that $(T; \lhd)$ satisfies condition (4), we first show that if $(T; \lhd)$ has an uncountable branch then it has an uncountable antichain. Let b be an uncountable branch. By condition (v), each $\beta \in b$ has an immediate $\lhd$-successor that does not belong to b. Let

$$A = \{\gamma \mid \gamma \notin b \;\wedge\; (\exists \beta \in b)\, \gamma \text{ is an immediate } \lhd\text{-successor of } \beta\}\,.$$

The uncountable set A is clearly an antichain of $(T; \lhd)$.

Since every antichain can be extended to a maximal antichain, it suffices to prove that $(T; \lhd)$ has no uncountable maximal antichains.

Let A be a maximal antichain of $(T; \lhd)$. For limit $\alpha < \omega_1$, let $(T^*_\alpha; \lhd^*_\alpha)$ be defined as above. Note that T^*_α is the set of $\beta \in T$ such that, with respect to $\lhd$, level$(\beta) < \alpha$. Note also that $\lhd^*_\alpha$ is just the restriction of $\lhd$ to T^*_α.

Let C be the set of all limit $\alpha < \omega_1$ such that

(a) $T^*_\alpha = T \cap \alpha$;

(b) $A \cap \alpha$ is a maximal antichain of $(T^*_\alpha; \lhd^*_\alpha)$.

We shall prove that C is closed and unbounded in ω_1.

By the definition of T^*_α, it is clear that $\{\alpha \mid T^*_\alpha = T \cap \alpha\}$ is closed in ω_1. To show that C is closed, it is therefore enough to show that the set of all α that satisfy (b) is closed in ω_1. Suppose that $\langle \alpha_i \mid i \in \omega \rangle$ is a strictly increasing sequence of countable ordinals such that for each i, $A \cap \alpha_i$ a maximal antichain of $(T^*_{\alpha_i}; \lhd^*_{\alpha_i})$. Let $\alpha = \bigcup_{i \in \omega} \alpha_i$. Let $\beta \in T^*_\alpha$. For any sufficiently large $i \in \omega$, $\beta \in T^*_{\alpha_i}$. Thus β is comparable with some $\gamma \in A \cap \alpha_i \subseteq A \cap \alpha$. This shows that $A \cap \alpha$ is a maximal antichain in $(T^*_\alpha; \lhd^*_\alpha)$.

For $\alpha < \omega_1$, let

$$\begin{aligned} f(\alpha) &= \mu\delta\,(\forall\beta \in T^*_\alpha)\,\beta < \delta; \\ g(\alpha) &= \mu\delta\,(\forall\beta \in T^*_\alpha)(\exists\gamma \in A \cap \delta)\,\gamma \text{ is } \lhd\text{-comparable with } \beta. \end{aligned}$$

That $f(\alpha)$ and $g(\alpha)$ are defined for every α follows from the fact that T^*_α is countable (by (iv)) and the fact that A is an maximal antichain of $(T; \lhd)$. By an argument like one in the proof of Theorem 6.36, the set C' of all countable ordinals closed under f and g is an unbounded subset of ω_1. By (ii), $T \cap \alpha \subseteq T^*_\alpha$ for every $\alpha < \omega_1$. Therefore every $\alpha \in C'$ satisfies (a) and (b).

Since $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ witnesses the truth of $\Diamond$, let $\alpha \in C$ be such that $A \cap \alpha = A_\alpha$. By (b), A_α is a maximal antichain of T^*_α. By the definition of $\mathcal{B}_\alpha$, every $b \in \mathcal{B}_\alpha$ contains a member of A_α. For $b \in \mathcal{B}_\alpha$, every member of b is $\lhd_\alpha \gamma_b$ and so is $\lhd \gamma_b$. Hence for each $b \in \mathcal{B}_\alpha$ there is a $\xi \in A_\alpha$ such that $\xi \lhd \gamma_b$. If $\beta \in T \setminus T_\alpha$, then there is a b such that $\gamma_b \lhd \beta$. Putting all these facts together, we get that every element of T is $\lhd$-comparable with some element of A_α. In other words, A_α—i.e., $A \cap \alpha$—is a maximal antichain of T. But this means that $A = A \cap \alpha$. Hence A is countable. $\square$