CHAPTER 5

INTRODUCTION TO COMPUTABILITY THEORY

The class of recursive functions was originally introduced as a tool for establishing undecidability results (via the Church-Turing Thesis); but it is a very interesting class, it has been studied extensively since the 1930s, and its theory has found important applications in many mathematical areas. Here we will give only a brief introduction to some of its aspects.

5A. Semirecursive relations

It is convenient to introduce the additional notation

$$\{e\}(\vec{x}) = \varphi_e^n(\vec{x})$$

for the recursive n-ary partial function with code e, as in the Normal Form Theorem 4F.1, which puts the "program" e and the "data" $\vec{x}$ on equal footing and eliminates the need for double and triple subscripts in the formulas to follow.

We start with a Corollary to the proof of Theorem 4F.1, which gives some additional information about the coding of recursive partial functions and whose significance will become apparent in the sequel.

Theorem 5A.1 (S^m_n-Theorem, Kleene)**.** *For all $m, n \geq 1$, there is a one-to-one, $m+1$-ary primitive recursive function $S^m_n(e, y_1, \dots, y_m)$, such that for all $\vec{y} = y_1, \dots, y_m$, $\vec{x} = x_1, \dots, x_n$,*

$$\varphi_{S^m_n(e,\vec{y})}(\vec{x}) = \varphi_e(\vec{y}, \vec{x}), \ \textit{i.e.,}\ \{S^m_n(e, \vec{y})\}(\vec{x}) = \{e\}(\vec{y}, \vec{x}).$$

Proof. For each sequence of numbers $e, \vec{y} = e, y_1, \dots, y_m$, let

$$\theta' \equiv \Delta e = 0 \ \&\ \Delta y_1 = 0 \ \&\ \cdots \ \&\ \Delta y_m = 0 \ \&\ 0 = 1,$$

and for each full extended formula

$$\psi \equiv \psi(\mathrm{v}_0, \dots, \mathrm{v}_{m-1}, \mathrm{v}_m, \dots, \mathrm{v}_{m+n})$$

(as in the definition of $T_{n+m}(e, \vec{y}, \vec{x}, z)$ in Theorem 4F.1) let

$$\theta \equiv \phi(\mathrm{v}_0, \dots, \mathrm{v}_n) \equiv \psi(\Delta y_1, \dots, \Delta y_m, \mathrm{v}_0, \dots, \mathrm{v}_n)$$

and put

$$S^m_n(e, \vec{y}) = \begin{cases} \text{the code of } \theta, & \text{if } e \text{ is the code of some } \psi \text{ as above} \\ \text{the code of } \theta', & \text{otherwise.} \end{cases}$$

It is clear that each $S^m_n(e, \vec{y})$ is a primitive recursive function, and it is also one-to-one, because the value $S^m_n(e, \vec{y})$ codes all the numbers $e, y_1, \dots, y_m$—this was the reason for introducing the extra restriction on the variables in the definition of the T predicate. Moreover:

$$\begin{aligned} T_{m+n}(e, \vec{y}, \vec{x}, z) \iff & e \text{ is the code of some } \psi \text{ as above,} \\ & \text{and } (z)_1 \text{ is the code of a proof in } \mathsf{Q} \text{ of} \\ & \phi(\Delta y_1, \dots, \Delta y_m, \Delta x_1, \dots, \Delta x_n, \Delta(z)_0) \\ \iff & S^m_n(e, \vec{y}) \text{ is the code of the associated } \theta \\ & \text{and } (z)_1 \text{ is the code of a proof in } \mathsf{Q} \text{ of} \\ & \theta(\Delta x_1, \dots, \Delta x_n, \Delta(z)_0) \\ \iff & T_n(S^m_n(e, \vec{y}), \vec{x}, z). \end{aligned}$$

To see this, check first the implications in the direction $\Longrightarrow$, which are all immediate—with the crucial, middle implication holding because (literally)

$$\theta(\Delta x_1, \dots, \Delta x_n, \Delta(z)_0) \equiv \psi(\Delta y_1, \dots, \Delta y_m, \Delta x_1, \dots, \Delta x_n, \Delta(z)_0).$$

For the implications in the direction $\Longleftarrow$, notice that if $T_n(S^m_n(e, \vec{y}), \vec{x}, z)$ holds, then $S^m_n(e, \vec{y})$ is the code of a true sentence, since $(z)_0$ is the code of a proof of it in Q, and so it cannot be the code of θ', which is false; so it is the code of θ, which means that e is the code of some ϕ as above, and then the argument runs exactly as in the direction $\Longrightarrow$.

From this we get immediately, by the definitions, that

$$\{S^m_n(e, \vec{y})\}(\vec{x}) = \{e\}(\vec{y}, \vec{x}). \qquad \dashv$$

Example 5A.2. The class of recursive partial functions is "uniformly" closed for composition, for example there is a primitive recursive function $u^n(e, m_1, m_2)$ such that for all $\vec{x} = (x_1, \dots, x_n)$,

$$\{u^n(e, m_1, m_2)\}(\vec{x}) = \{e\}(\{m_1\}(\vec{x}), \{m_2\}(\vec{x})).$$

Proof. The partial function

$$f(e, m_1, m_2, \vec{x}) = \{e\}(\{m_1\}(\vec{x}), \{m_2\}(\vec{x}))$$

is recursive, and so for some number $\widehat{f}$ and by Theorem 5A.1,

$$\begin{aligned} f(e, m_1, m_2, \vec{x}) &= \{\widehat{f}\}(e, m_1, m_2, \vec{x}) \\ &= \{S^3_n(\widehat{f}, e, m_1, m_2)\}(\vec{x}), \end{aligned}$$

and it is enough to set

$$u^n(e, m_1, m_2) = S^3_n(\widehat{f}, e, m_1, m_2). \qquad \dashv$$

This is, obviously, a special case of a general fact which follows from the S^m_n-Theorem, in slogan form: *if the class of recursive partial function is closed for some operation, it is then closed uniformly* (in the codes) *for the same operation.*

To simplify the statements of several definitions and results in the sequel, we recall here and name the basic, "logical" operations on relations:

$(\neg)$	$P(\vec{x})$	$\iff \neg Q(\vec{x})$
$(\&)$	$P(\vec{x})$	$\iff Q(\vec{x})\ \&\ R(\vec{x})$
$(\vee)$	$P(\vec{x})$	$\iff Q(\vec{x}) \vee R(\vec{x})$
$(\Rightarrow)$	$P(\vec{x})$	$\iff Q(\vec{x}) \Longrightarrow R(\vec{x})$
$(\exists^{\mathbb{N}})$	$P(\vec{x})$	$\iff (\exists y)Q(\vec{x}, y)$
$(\exists_{\leq})$	$P(z, \vec{x})$	$\iff (\exists i \leq z)Q(\vec{x}, i)$
$(\forall^{\mathbb{N}})$	$P(\vec{x})$	$\iff (\forall y)Q(\vec{x}, y)$
$(\forall_{\leq})$	$P(z, \vec{x})$	$\iff (\forall i \leq z)Q(\vec{x}, i)$
(replacement)	$P(\vec{x})$	$\iff Q(f_1(\vec{x}), \dots, f_m(\vec{x}))$

For example, we have already shown that the class of primitive recursive relations is closed under all these operations (with primitive recursive $f_i(\vec{x})$), except for the (unbounded) quantifiers $\exists^{\mathbb{N}}, \forall^{\mathbb{N}}$, under which it is not closed by Theorem 4F.2.

Proposition 5A.3. *The class of recursive relations is closed under the propositional operations* $\neg$, $\&$, $\vee, \Rightarrow$, *the bounded quantifiers* $\exists_{\leq}, \forall_{\leq}$, *and substitution of* (total) *recursive functions, but it is not closed under the* (unbounded) *quantifiers* $\exists, \forall$.

Definition 5A.4. (1) A relation $P(\vec{x})$ is **semirecursive** if it is the domain of some recursive partial function $f(\vec{x})$, i.e.,

$$P(\vec{x}) \iff f(\vec{x})\downarrow .$$

(2) A relation $P(\vec{x})$ is Σ^0_1 if there is some recursive relation $Q(\vec{x}, y)$, such that

$$P(\vec{x}) \iff (\exists y)Q(\vec{x}, y).$$

Proposition 5A.5. *The following are equivalent, for an arbitrary relation* $P(\vec{x})$:

(1) $P(\vec{x})$ *is semirecursive.*

(2) $P(\vec{x})$ *is* Σ^0_1.

(3) $P(\vec{x})$ *satisfies the equivalence*

$$P(\vec{x}) \iff (\exists y)Q(\vec{x}, y)$$

with some primitive recursive $Q(\vec{x}, y)$.

Proof. (1) $\Longrightarrow$ (3) by the Normal Form Theorem; (3) $\Longrightarrow$ (2) trivially; and for (2) $\Longrightarrow$ (1) we set

$$f(\vec{x}) = \mu y Q(\vec{x}, y),$$

so that

$$(\exists y)Q(\vec{x}, y) \iff f(\vec{x})\downarrow . \qquad \dashv$$

Proposition 5A.6 (Kleene's Theorem)**.** *A relation* $P(\vec{x})$ *is recursive if and only if both* $P(\vec{x})$ *and its negation* $\neg P(\vec{x})$ *are semirecursive.*

Proof. If $P(\vec{x})$ is recursive, then the relations

$$Q(\vec{x}, y) \iff P(\vec{x}), \qquad R(\vec{x}, y) \iff \neg P(\vec{x})$$

are both recursive, and (trivially)

$$\begin{aligned} P(\vec{x}) &\iff (\exists y)Q(\vec{x}, y) \\ \neg P(\vec{x}) &\iff (\exists y)R(\vec{x}, y). \end{aligned}$$

For the other direction, if

$$\begin{aligned} P(\vec{x}) &\iff (\exists y)Q(\vec{x}, y) \\ \neg P(\vec{x}) &\iff (\exists y)R(\vec{x}, y) \end{aligned}$$

with recursive Q and R, then the function

$$f(\vec{x}) = \mu y[Q(\vec{x}, y) \vee R(\vec{x}, y)]$$

is total and recursive, and

$$P(\vec{x}) \iff Q(\vec{x}, f(\vec{x})). \qquad \dashv$$

Proposition 5A.7. *The class of semirecursive relations is closed under the "positive" propositional operations* $\&, \vee$*, under the bounded quantifiers* $\exists_{\leq}, \forall_{\leq}$*, and under the existential quantifier* $\exists^{\mathbb{N}}$*; it is not closed under negation* $\neg$ *and under the universal quantifier* $\forall^{\mathbb{N}}$*.*

Proof. Closure under (total) recursive substitutions is trivial, and the following transformations show the remaining positive claims of the proposition:

$$\begin{aligned}
(\exists y)Q(\vec{x}, y) \vee (\exists y)R(\vec{x}, y) &\iff (\exists u)[Q(\vec{x}, u) \vee R(\vec{x}, u)]\\
(\exists y)Q(\vec{x}, y) \;\&\; (\exists y)R(\vec{x}, y) &\iff (\exists u)[Q(\vec{x}, (u)_0) \;\&\; R(\vec{x}, (u)_1)]\\
(\exists z)(\exists y)Q(\vec{x}, y, z) &\iff (\exists u)R(\vec{x}, (u)_0, (u)_1)\\
(\exists i \leq z)(\exists y)Q(\vec{x}, y, i) &\iff (\exists y)(\exists i \leq z)Q(\vec{x}, y, i)\\
(\forall i \leq z)(\exists y)Q(\vec{x}, y, i) &\iff (\exists u)(\forall i \leq z)Q(\vec{x}, (u)_i, i).
\end{aligned}$$

On the other hand, the class of semirecursive relations is not closed under $\neg$ or $\forall^{\mathbb{N}}$, otherwise the basic Halting relation

$$H(e, x) \iff (\exists y)T_1(e, x, y)$$

would have a semirecursive negation and so would be recursive by 5A.6, which it is not. $\dashv$

The *graph* of a partial function $f(\vec{x})$ is the relation

$$G_f(\vec{x}, w) \iff f(\vec{x}) = w, \tag{111}$$

and the next restatement of Theorem 4E.10 often gives (with the closure properties of Σ^0_1) simple proofs of recursiveness for partial functions:

Proposition 5A.8 (The Σ^0_1-Graph Lemma)**.** *A partial function* $f(\vec{x})$ *is recursive if and only if its graph* $G_f(\vec{x}, w)$ *is a semirecursive relation.*

Proof. If $f(\vec{x})$ is recursive with code $\widehat{f}$, then

$$G_f(\vec{x}, w) \iff (\exists y)[T_n(\widehat{f}, \vec{x}, y) \;\&\; U(y) = w],$$

so that $G_f(\vec{x}, w)$ is semirecursive; and if

$$f(\vec{x}) = w \iff (\exists u)R(\vec{x}, w, u)$$

with some recursive $R(\vec{x}, w, u)$, then

$$f(\vec{x}) = \Big(\mu u R(\vec{x}, (u)_0, (u)_1)\Big)_0,$$

so that $f(\vec{x})$ is recursive. ⊣

The next Lemma is also very simple, but it simplifies many proofs.

Proposition 5A.9 (The Σ^0_1-Selection Lemma). *For each semirecursive relation $R(\vec{x}, w)$, there is a recursive partial function*

$$f(\vec{x}) = \nu w R(\vec{x}, w)$$

such that for all $\vec{x}$,

$$(\exists w)R(\vec{x}, w) \iff f(\vec{x})\downarrow$$
$$(\exists w)R(\vec{x}, w) \implies R(\vec{x}, f(\vec{x})).$$

Proof. By the hypothesis, there is a recursive relation $P(\vec{x}, w, y)$ such that

$$R(\vec{x}, w) \iff (\exists y)P(\vec{x}, w, y),$$

and the conclusion of the lemma follows if we just set

$$f(\vec{x}) = \Big(\mu u P(\vec{x}, (u)_0, (u)_1)\Big)_0. \quad ⊣$$

Problems for Section 5A

Problem 5A.1. Prove that if $R(x_1, \dots, x_n)$ is a recursive relation, then there exists a formula $\mathbf{R}(v_1, \dots, v_n)$ in the language of arithmetic such that

$$R(x_1, \dots, x_n) \Longrightarrow \mathsf{Q} \vdash \mathbf{R}(\Delta x_1 \dots, \Delta x_n)$$
$$\text{and } \neg R(x_1, \dots, x_n) \Longrightarrow \mathsf{Q} \vdash \neg\mathbf{R}(\Delta x_1 \dots, \Delta x_n).$$

Problem 5A.2. Prove that every axiomatizable, complete theory is decidable.

Problem 5A.3. Show that the class of recursive partial functions is *uniformly* closed under definition by primitive recursion in the following, precise sense: there is a primitive recursive function $u^n(e, m)$, such that if $f(y, \vec{x})$ is defined by the primitive recursion

$$f(0, \vec{x}) = \varphi_e(\vec{x}), \quad f(y + 1, \vec{x}) = \varphi_m(f(y, \vec{x}), y, \vec{x}),$$

then $f(y, \vec{x}) = \{u^n(e, m)\}(y, \vec{x})$.

Problem 5A.4. Define a total, recursive, one-to-one function $u^n(e, i)$, such that for all $e, i, \vec{x}$,

$$\{u^n(e, i)\}(\vec{x}) = \{e\}(\vec{x}).$$

(In particular, each recursive partial function has, effectively, an infinite number of distinct codes.)

Problem 5A.5. Show that the partial function

$$f(e,u) = \langle \varphi_e((u)_0), \varphi_e((u)_1), \dots, \varphi_e((u)_{\mathrm{lh}(u) \dot{-} 1}) \rangle$$

is recursive.

Problem 5A.6 (Craig's Lemma)**.** Show that a theory T has a primitive recursive set of axioms if and only if it has a semirecursive set of axioms.

Problem 5A.7. Prove that there is a recursive relation $R(x)$ which is not primitive recursive.

Problem 5A.8. Suppose $R(\vec{x}, w)$ is a semirecursive relation such that for each $\vec{x}$ there exist at least two distinct numbers $w_1 \neq w_2$ such that $R(\vec{x}, w_1)$ and $R(\vec{x}, w_2)$. Prove that there exist two, total recursive functions $g(\vec{x})$ and $h(\vec{x})$, such tht for all $\vec{x}$,

$$R(\vec{x}, g(\vec{x})) \ \& \ R(\vec{x}, h(\vec{x})) \ \& \ g(\vec{x} \neq h(\vec{x}).$$

Problem 5A.9. Suppose $R(\vec{x}, w)$ is a semirecursive relation such that for each $\vec{x}$, there exists at least one w such that $R(\vec{x}, w)$.

(a) Prove that there is a total recursive function $f(n, \vec{x})$ such that

$$(*) \qquad R(\vec{x}, w) \iff (\exists n)[w = f(n, \vec{x})].$$

(b) Prove that if (in addition), for each $\vec{x}$, there exist infinitely many w such that $R(\vec{x}, w)$, then there exists a total, recursive $f(n, \vec{x})$ which satisfies $(*)$ and which is one-to-one, i.e.,

$$m \neq n \Longrightarrow f(m, \vec{x}) \neq f(n, \vec{x}).$$

5B. Recursively enumerable sets

Some of the properties of semirecursive relations are easier to identify when we view unary relations as sets:

Definition 5B.1 (R.e. sets)**.** A set $A \subseteq \mathbb{N}$ is **recursively** or **computably enumerable** if either $A = \emptyset$, or some total, recursive function enumerates it, i.e.,

$$(112) \qquad A = \{f(0), f(1), \dots, \}.$$

The term "recursively enumerable" is unwieldy and it is always abbreviated by the initials "r.e." or "c.e.".

Proposition 5B.2. *The following are equivalent for any $A \subseteq \mathbb{N}$:*

(1) *A is r.e.*

(2) *The relation $x \in A$ is semirecursive.*

(3) *A is finite, or there exists a one-to-one recursive function $f : \mathbb{N} \rightarrowtail \mathbb{N}$ which enumerates it.*

PROOF. The implication (3) $\Longrightarrow$ (1) is trivial, and (1) $\Longrightarrow$ (2) follows from the equivalence

$$x \in A \iff (\exists n)[f(n) = x]$$

which holds for all non-empty r.e. sets A. To show (2) $\Longrightarrow$ (3), we suppose that A is infinite and

$$x \in A \iff (\exists y)R(x, y)$$

with a recursive $R(x, y)$, and set

$$B = \{u \mid R((u)_0, (u)_1) \ \&\ (\forall v < u)[R((v)_0, (v)_1) \Longrightarrow (v)_0 \neq (u)_0]\}.$$

It is clear that B is a recursive set, that

$$u \in B \Longrightarrow (u)_0 \in A,$$

and that if $x \in A$ and we let

$$t = (\mu u)[R((u)_0, (u)_1) \ \&\ (u)_0 = x],$$

then (directly from the definition of B),

$$t \in B \ \&\ (\forall u)[(u \in B \ \&\ (u)_0 = x) \iff u = t];$$

it follows that the projection

$$\pi(u) = (u)_0$$

is a one-to-one correspondence of B with A, and hence B is infinite. Now B is enumerated without repetitions by the recursive function

$$g(0) = (\mu u)[u \in B]$$
$$g(n + 1) = (\mu u)[u > g(n) \ \&\ u \in B],$$

and the composition

$$f(n) = (g(n))_0$$

enumerates A without repetitions. ⊣

The next fact shows that we cannot go any further in producing "nice" enumerations of arbitrary r.e. sets.

Proposition 5B.3. *A set $A \subseteq \mathbb{N}$ is recursive if and only if it is finite, or there exists an increasing, total recursive function which enumerates it,*

$$A = \{f(0) < f(1) < \dots,\}.$$

Proof. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if

$$f(n) < f(n+1) \qquad (n \in \mathbb{N}),$$

from which it follows (by an easy induction) that for all n

$$n \leq f(n);$$

thus, if some increasing, recursive f enumerates A, then

$$x \in A \iff (\exists n \leq x)[x = f(n)],$$

and A is recursive. For the opposite direction, if A is recursive and infinite, then the function

$$f(0) = (\mu x)[x \in A]$$
$$f(n+1) = (\mu x)[x > f(n) \ \&\ x \in A]$$

is recursive, increasing and enumerates A. ⊣

The simplest example of an r.e. non-recursive set is the "diagonal" set

(113) $$K = \{x \mid (\exists y)T_1(x,x,y)\} = \{x \mid \{x\}(x)\downarrow\},$$

and the next Proposition shows that (in some sense) K is the "most complex" r.e. set.

Proposition 5B.4. *For each r.e. set A, there is a one-to-one recursive function $f : \mathbb{N} \rightarrowtail \mathbb{N}$ such that*

(114) $$x \in A \iff f(x) \in K.$$

Proof. By hypothesis, $A = \{x \mid g(x)\downarrow\}$ for some recursive partial function $g(x)$. We set

$$h(x,y) = g(x)$$

and we choose some code $\widehat{h}$ of h, so that for any y,

$$\begin{aligned} x \in A &\iff h(x,y)\downarrow \\ &\iff \{\widehat{h}\}(x,y)\downarrow \\ &\iff \{S^1_1(\widehat{h},x)\}(y)\downarrow; \end{aligned}$$

in particular, this holds for $y = S^1_1(\widehat{h},x)$ and it yields in that case

$$\begin{aligned} x \in A &\iff \{S^1_1(\widehat{h},x)\}(S^1_1(\widehat{h},x))\downarrow \\ &\iff S^1_1(\widehat{h},x) \in K, \end{aligned}$$

so that (114) holds with $f(x) = S^1_1(\widehat{h}, x)$. ⊣

Definition 5B.5 (Reducibilities)**.** A **reduction** of a set A to another set B is any (total) recursive function f, such that

$$x \in A \iff f(x) \in B, \tag{115}$$

and we set:

$$\begin{aligned} A \leq_m B &\iff \text{there exists a reduction of } A \text{ to } B, \\ A \leq_1 B &\iff \text{there exists a one-to-one reduction of } A \text{ to } B, \\ A \equiv B &\iff \text{there exists a reduction } f \text{ of } A \text{ to } B \\ &\qquad \text{which is a permutation,} \end{aligned}$$

where a permutation $f : \mathbb{N} \rightarrowtail\!\!\!\rightarrow \mathbb{N}$ is any one-to-one correspondence of $\mathbb{N}$ onto $\mathbb{N}$. Clearly

$$A \equiv B \Longrightarrow A \leq_1 B \Longrightarrow A \leq_m B.$$

Proposition 5B.6. *For all sets A, B, C,*

$$A \leq_m A \text{ and } [A \leq_m B \;\&\; B \leq_m C] \Longrightarrow A \leq_m C,$$

and the same holds for the stronger reductions $\leq_1$ and $\equiv$; in addition, the relation $\equiv$ of recursive isomorphism is symmetric,

$$A \equiv B \iff B \equiv A.$$

Definition 5B.7. A set B is **r.e. complete** if it is r.e., and every r.e. set A is one-one reducible to B, $A \leq_1 B$.

Proposition 5B.4 expresses precisely the r.e. completeness of K, and the next, basic result shows that up to recursive isomorphism, there is only one r.e. complete set.

Theorem 5B.8 (John Myhill)**.** *For any two sets A and B,*

$$A \leq_1 B \;\&\; B \leq_1 A \Longrightarrow A \equiv B.$$

PROOF. The argument is s constructive version of the classical Schröder-Bernstein in set theory, and it is based on the next Lemma, in which a sequence of pairs

$$W = (x_0, y_0), (x_1, y_1), \dots, (x_n, y_n) \tag{116}$$

is called *good* (as an approximation to an isomorphism) for A and B if

$$i \neq j \Longrightarrow [x_i \neq x_j \text{ and } y_i \neq y_j], \qquad x_i \in A \iff y_i \in B \qquad (i \leq n).$$

For any good sequence, we set

$$X = \{x_0, x_1, \dots , x_n\}, \quad Y = \{y_0, y_1, \dots , y_n\}.$$

Lemma X. *If there is a recursive one-to-one function* $f : \mathbb{N} \rightarrowtail \mathbb{N}$ *such that*

$$x \in A \iff f(x) \in B,$$

then for every good sequence (116) *and each* $x \notin X$*, we can find some* $y \notin Y$ *such that the extension*

$$W' = (x_0, y_0), (x_1, y_1), \dots , (x_n, y_n), (x, y) \tag{117}$$

is also good, i.e.,

$$x \in A \iff y \in B,$$

Proof of Lemma X. We set

$$z_0 = f(x)$$
$$z_{i+1} = \begin{cases} z_i & \text{if } z_i \notin Y, \\ f(x_j) & \text{otherwise, if } z_i = y_j, \end{cases}$$

and we verify two basic properties of the sequence $z_0, z_1, \dots$.

(1) For each i, $x \in A \iff z_i \in B$.

Proof. For $i = 0$, $x \in A \iff f(x) = z_0 \in B$, by the hypothesis on f. Inductively, if $z_i \notin Y$, then

$$x \in A \iff z_{i+1} = z_i \in B$$

by the induction hypothesis, and if $z_i \in Y$, then

$$\begin{aligned} x \in A &\iff z_i = y_j \in B && \text{(by the induction hypothesis)} \\ &\iff x_j \in A && \text{(because the given sequence is good)} \\ &\iff f(x_j) = z_{i+1} \in B. \end{aligned}$$

(2) For every i, $z_i \in Y \Longrightarrow (\forall k < i)[z_i \neq z_k]$.

Proof. The proposition is trivially true if $z_0 = f(x) \notin Y$, since, in this case, $z_i = f(x) \notin Y$ for every i, by the definition. The proposition is also trivially true for $i = 0$, and, inductively, we assume that it holds for i and that $z_{i+1} \in Y$. Notice that $z_i \in Y$, otherwise (by the definition) $z_{i+1} = z_i \notin Y$; hence, by the definition again, for some j,

$$z_i = y_j, \quad z_{i+1} = f(x_j). \tag{118}$$

Towards a contradiction, let k be the least counterexample for z_{i+1}, i.e.,

$$z_{i+1} = z_k \ \& \ (\forall l < k)[z_{i+1} \neq z_l].$$

Notice that $k \neq 0$, since $z_0 = f(x)$, $z_{i+1} = f(x_j)$, and hence,

$$z_{i+1} = z_0 \Longrightarrow f(x_j) = f(x) \Longrightarrow x_j = x,$$

which is absurd, since $x \notin X$ while $x_j \in X$; hence $k = t + 1$ for some $t < i$, and by the selection of k, $z_t \in Y$ (otherwise $z_{t+1} = z_t$ and $t + 1$ would not be a counterexample), and hence, for some s,

$$z_t = y_s, \quad z_{t+1} = f(x_s). \tag{119}$$

We compute:

$$\begin{aligned} z_{i+1} = z_{t+1} &\Longrightarrow f(x_j) = f(x_s) && \text{(from (118) and (119))} \\ &\Longrightarrow x_j = x_s && \text{(because } f \text{ is one-to-one)} \\ &\Longrightarrow y_j = y_s && \text{(because the sequence is good)} \\ &\Longrightarrow z_i = z_t && \text{(by (118) and (119)).} \end{aligned}$$

It follows from the inductive hypothesis that $t \geq i$, hence $t + 1 \geq i + 1$, and this contradicts the assumption $t + 1 < i + 1$.

Now (2) implies that for some $j < n+2$, $z_j \notin Y$ (since Y has $n+1$ members), and the Lemma holds if we choose $y = z_j$, $W' = W, (x, y)$. ⊣ (Lemma X)

The symmetric Lemma Y gives us for each good sequence W and each $y \notin Y$ some $x \notin X$ such that the extension (117) $W' = W, (x, y)$ is good, and the construction of the required recursive permutation proceeds by successive application of these two Lemmas starting with the good sequence

$$W_0 = \langle 0, f(0) \rangle, \quad X_0 = \{0\}, Y_0 = \{f(0)\}.$$

Odd step $2n+1$. Let $y = \min(\mathbb{N} \setminus Y_{2n})$ and extend W_{2n} by applying Lemma Y, so that $y \in Y_{2n+1}$.

Even step $2n + 2$. Let $x = \min(\mathbb{N} \setminus X_{2n+1})$ and extend W_{2n+1} by applying Lemma X so that $x \in X_{2n+2}$.

In the end, the union $\bigcup_n W_n$ is the graph of a permutation $h : \mathbb{N} \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathbb{N}$ which reduces A to B,

$$x \in A \iff h(x) \in B.$$

The recursiveness of h follows from the construction and completes the proof that $A \equiv B$. ⊣

Definition 5B.9 (Codes for r.e. sets)**.** For each $e \in \mathbb{N}$, let

$$W_e = \{x \mid \varphi_e(x)\downarrow\},$$

so that the relation $x \in W_e$ is semirecursive and the sequence

$$W_0, W_1, \ldots$$

enumerates all the r.e. sets.

Proposition 5B.10. *If $A \leq_m B$ and B is recursive, then A is also recursive; hence, if $A \leq_m B$ and A is not recursive, then B is not recursive either.*

With the r.e. completeness of K, this simple fact is the basic tool for proving non-recursiveness for sets and relations: because if we verify that $K \leq_m B$, then B is not recursive.

Example 5B.11. The set

$$A = \{e \mid W_e \neq \emptyset\}$$

is r.e. but not recursive.

Proof. The set A is r.e. because the relation

$$e \in A \iff (\exists x)[x \in W_e]$$

is Σ^0_1. To show that $K \leq_m A$, we let

$$g(e,x) = \mu y T_1(e,e,y),$$

so that the value $g(e,x)$ is independent of x, i.e.,

$$g(e,x) = \begin{cases} \mu y T_1(e,e,y) & \text{if } (\exists y)T_1(e,e,y) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It follows that for all e and x,

$$e \in K \iff g(e,x)\downarrow,$$

so that

$$e \in K \iff (\exists x)g(e,x)\downarrow;$$

and so, if $\widehat{g}$ is any code of $g(x,y)$,

$$\begin{aligned} e \in K &\iff (\exists x)[\{\widehat{g}\}(e,x)\downarrow] \\ &\iff (\exists x)[\{S^1_1(\widehat{g},e)\}(x)\downarrow] \\ &\iff W_{S^1_1(\widehat{g},e)} \neq \emptyset \\ &\iff S^1_1(\widehat{g},e) \in A, \end{aligned}$$

so that $K \leq_1 A$ and A is not recursive. ⊣

Notice that with this construction,

$$\begin{aligned} e \in K &\iff W_{S^1_1(\widehat{g},e)} = \mathbb{N} \\ &\iff W_{S^1_1(\widehat{g},e)} \text{ has at least 2 members,} \end{aligned}$$

so that the sets

$$B = \{e \mid W_e = \mathbb{N}\}, \quad C = \{e \mid W_e \text{ has at least 2 members}\}$$

are also not recursive.

Problems for Section 5B

Problem 5B.1. Prove that a relation $P(\vec{x})$ is Σ^0_1 if and only if it is definable by a Σ_1 formula, in the sense of Definition 4C.12.

Problem 5B.2. Show that there is a recursive, partial function $f(e)$ such that

$$W_e \neq \emptyset \Longrightarrow [f(e)\downarrow \ \& \ f(e) \in W_e].$$

Problem 5B.3. Prove or give a counterexample to each of the following propositions:

(a) There is a total recursive function $u_1(e, m)$ such that for all e, m,

$$W_{u_1(e,m)} = W_e \cup W_m.$$

(b) There is a total recursive function $u_2(e, m)$ such that for all e, m,

$$W_{u_2(e,m)} = W_e \cap W_m.$$

(c) There is a total recursive function $u_3(e, m)$ such that for all e, m,

$$W_{u_3(e,m)} = W_e \setminus W_m.$$

Problem 5B.4. Prove or give a counterexample to each of the following propositions, where $f : \mathbb{N} \to \mathbb{N}$ is a total recursive function and

$$f[A] = \{f(x) \mid x \in A\}, \quad f^{-1}[A] = \{x \mid f(x) \in A\}.$$

(a) If A is recursive, then $f[A]$ is also recursive.
(b) If A is r.e., then $f[A]$ is also r.e.
(c) If A is recursive, then $f^{-1}[A]$ is also recursive.
(d) If A is r.e., then $f^{-1}[A]$ is also r.e.

Problem 5B.5. Prove that there is a total recursive function $u(e, m)$ such that

$$W_{u(e,m)} = \{x + y \mid x \in W_e \ \& \ y \in W_m\}.$$

Problem 5B.6. Prove that every infinite r.e. set A has an infinite recursive subset.

Problem 5B.7. (The Reduction Property of r.e. sets.) Prove that for every two r.e. sets A, B, there exist r.e. sets A^*, B^* with the following properties:

$$A^* \subseteq A, \quad B^* \subseteq B, \quad A \cup B = A^* \cup B^*, \quad A^* \cap B^* = \emptyset.$$

Problem 5B.8. (The Separation Property for r.,e. complements.) Prove that if A and B are disjoint sets whose complements are r.e., then there exists a recursive set C such that

$$A \subseteq C, \quad C \cap B = \emptyset.$$

Problem 5B.9. (Recursively inseparable r.e. sets.) Prove that there exist two disjoint r.e. sets A and B such that there is no recursive set C satisfying

$$A \subseteq C, \quad C \cap B = \emptyset.$$

Problem 5B.10. Prove that for every two r.e. sets A, B, there is a formula $\phi(x)$ so that *whenever* $x \in (A \cup B)$,

$$\mathsf{Q} \vdash \phi(\mathbf{x}) \Longrightarrow x \in A, \tag{120}$$

$$\mathsf{Q} \vdash \neg\phi(\mathbf{x}) \Longrightarrow x \in B, \tag{121}$$

$$\mathsf{Q} \vdash \phi(\mathbf{x}) \quad \textit{or} \quad \mathsf{Q} \vdash \neg\phi(\mathbf{x}) \tag{122}$$

5C. Productive, creative and simple sets

Up until now, the only r.e non-recursive sets we have seen are r.e. complete, and the question arises whether that is all there is. The next sequence of definitions and results (due to Emil Post) shows that this simplistic picture of the class of r.e. sets is far from the truth.

Definition 5C.1. A function $p : \mathbb{N} \rightarrowtail \mathbb{N}$ is a **productive function** for a set B if it is recursive, one-to-one, and such that

$$W_e \subseteq B \Longrightarrow p(e) \in B \setminus W_e;$$

and a set B is **productive** if it has a productive function.

A set A is **creative** if it is r.e. and its complement

$$A^c = \{x \in \mathbb{N} \mid x \notin A\}$$

is productive.

Proposition 5C.2. *The complete set K is creative, with productive function for its complement the identity $p(e) = e$.*

Proof. We must show that

$$W_e \subseteq K^c \Longrightarrow e \in K^c \setminus W_e,$$

i.e.,

$$(\forall t)[t \in W_e \Longrightarrow t \notin K] \Longrightarrow [e \notin W_e \ \& \ e \notin K].$$

Spelling out the hypothesis of the required implication:

$$(\forall t)[\{e\}(t)\downarrow \Longrightarrow \{t\}(t)\uparrow];$$

and the conclusion simply says that

$$\{e\}(e)\uparrow,$$

because

$$e \notin W_e \iff e \notin K \iff \{e\}(e)\uparrow .$$

Finally, the hypothesis implies the conclusion because if $\{e\}(e)\downarrow$, then, setting $t = e$ in the hypothesis we get $\{e\}(e)\uparrow$, which is contradictory. ⊣

Corollary 5C.3. *Every r.e. complete is creative.*

Proof. It is enough to show that if A is productive and $A \leq_1 B$, then B is also productive, and then apply this to the complement X^c of the given, r.e. complete set X for which we have $K^c \leq_1 X^c$ (because $K \leq_1 X$). So suppose that

$$x \in A \iff f(x) \in B$$

with $f(x)$ recursive and 1 - 1, and that $p(e)$ is a productive function for A. Choose $u(e)$ (by appealing to the S^m_n-Theorem) such that it is recursive, 1 - 1, and for each e,

$$W_{u(e)} = f^{-1}[W_e],$$

and let

$$q(e) = f(p(u(e))).$$

To verify that $q(e)$ is a productive function for B, we compute:

$$\begin{aligned} W_e \subseteq B &\Longrightarrow W_{u(e)} = f^{-1}[W_e] \subseteq A \\ &\Longrightarrow p(u(e)) \in A \setminus f^{-1}[W_e] \\ &\Longrightarrow q(e) = f(p(u(e))) \in B \setminus W_e. \end{aligned}$$

⊣

Proposition 5C.4. *Every productive set B has an infinite r.e. subset.*

Proof. The idea is to define a function $f : \mathbb{N} \to \mathbb{N}$ by the recursion

$$f(0) = e_0, \text{ where } W_{e_0} = \emptyset$$
$$f(x+1) = \text{some code of } W_{f(x)} \cup \{p(f(x))\},$$

where $p(e)$ is a given productive function for B. If we manage this, then a simple induction will show that, for every x,

$$W_{f(x)} \subsetneq W_{f(x+1)} \subseteq B,$$

so that the set

$$A = W_{f(0)} \cup W_{f(1)} \cup \ldots = \{y \mid (\exists x)[y \in W_{f(x)}\}$$

is an infinite, r.e. subset of B. For the computation of the required function $h(w, x)$ such that

$$f(x+1) = h(f(x), x),$$

let first

$$R(e, y, x) \iff x \in W_e \vee x = y$$

and notice that this is a semirecursive relation, so that for some $\widehat{g}$,

$$x \in W_e \cup \{y\} \iff \{\widehat{g}\}(e, y, x)\downarrow$$
$$\iff \{S^2_1(\widehat{g}, e, y)\}(x)\downarrow .$$

This means that if we set

$$u(e, y) = S^2_1(\widehat{g}, e, y),$$

then

$$W_{u(e,y)} = W_e \cup \{y\}.$$

Finally, set

$$h(w, x) = u(w, p(w)),$$

and in the definition of f,

$$f(x+1) = h(f(x), x) = u(f(x), p(f(x))),$$

so that

$$W_{f(x+1)} = W_{f(x)} \cup \{p(f(x))\}$$

as required. ⊣

Definition 5C.5. A set A is **simple** if it is r.e., and its complement A^c is infinite and has no infinite r.e. subset, i.e.,

$$W_e \cap A = \emptyset \Longrightarrow W_e \text{ is finite.}$$

Theorem 5C.6 (Emil Post)**.** *There exists a simple set.*

PROOF. The relation

$$R(x,y) \iff y \in W_x \;\&\; y > 2x$$

is semirecursive, so that by the Σ^0_1-Selection Lemma 5A.9, there is a recursive partial function $f(x)$ such that

$$\begin{aligned}(\exists y)[y \in W_x \;\&\; y > 2x] &\iff f(x)\downarrow \\ &\iff f(x)\downarrow \;\; \& \; f(x) \in W_x \;\&\; f(x) > 2x.\end{aligned}$$

The required set is the image of f,

$$\begin{aligned}A &= \{f(x) \mid f(x)\downarrow\} \\ &= \{y \mid (\exists x)[f(x) = y]\} \\ &= \{y \mid (\exists x)[f(x) = y \;\&\; 2x < y]\},\end{aligned} \tag{123}$$

where the last, basic equality follows from the definition of the relation $R(x,y)$.

(1) A is r.e., from its definition, because the graph of $f(x)$ is Σ^0_1.

(2) The complement A^c of A is infinite, because

$$\begin{aligned}y \in A \;\&\; y \leq 2z &\Longrightarrow (\exists x)[y = f(x) \;\&\; 2x < y \leq 2z] \\ &\Longrightarrow (\exists x)[y = f(x) \;\&\; x < z],\end{aligned}$$

so that *at most* z *of the* $2z+1$ *numbers* $\leq 2z$ *belong to* A; it follows that some $y \geq z$ belongs to the complement A^c, and since this holds for every z, the set A^c is infinite.

(3) For every infinite W_e, $W_e \cap A \neq \emptyset$, because

$$\begin{aligned}W_e \text{ is infinite} &\Longrightarrow (\exists y)[y \in W_e \;\&\; y > 2e] \\ &\Longrightarrow f(e)\downarrow \;\; \& \; f(e) \in W_e \\ &\Longrightarrow f(e) \in W_e \cap A.\end{aligned}$$

⊣

Corollary 5C.7. *Simple sets are neither recursive nor r.e. complete, and so there exists an r.e., non-recursive set which is not r.e. complete.*

PROOF. A simple set cannot be recursive, because its (infinite, by definition) complement is a witness against its simplicity; and it cannot be r.e. complete, because it is not creative by Proposition 5C.4. ⊣

Problems for Section 5C

Problem 5C.1. Show that if A is simple and B is infinite r.e., then the intersection $A \cap B$ is infinite.

Problem 5C.2. Show that the intersection of two simple sets is simple.

Problem 5C.3. Prove or give a counterexample:

(a) For each infinite r.e. set A, there is a total, recursive function $f(x)$ such that for each x,

$$f(x) > x \;\&\; f(x) \in A.$$

(b) (a) For each r.e. set A with infinite complement, there is a total, recursive function $f(x)$ such that for each x,

$$f(x) > x \;\&\; f(x) \notin A.$$

Problem 5C.4. Prove that if Q is interpretable in a consistent, axiomatizable theory T, then the set

$$\#T = \{\#\theta \mid \theta \text{ is a sentence and } T \vdash \theta\}$$

of (codes of) theorems of T is creative.

SOLUTION. We will show how to define for each m some $\overline{m}$ such that

$$W_m \cap \#T = \emptyset \Longrightarrow \overline{m} \notin \#T \;\&\; \overline{m} \notin W_m,$$

skipping the (easy) uniformity argument, that, in fact, $\overline{m} = f(m)$ for some recursive, total f. The idea is to imitate the proof of the Rosser Theorem 4C.4, replacing the refutation relation by membership in W_m. So put

$$R_m(e, y) \iff e \text{ codes a sentence } \phi \text{ of } \mathsf{PA} \text{ and } T_1(m, \#\pi(\phi), y).$$

We let $\mathbf{R}(\mathsf{e}, \mathsf{y})$ be a formula which numeralwise expresses this relation in Q, and choose by the Fixed Point Lemma 4B.14 a sentence σ of PA such that

$$\mathsf{Q} \vdash \sigma \leftrightarrow (\forall y)[\mathbf{Proof}_{\pi,T}(\ulcorner\sigma\urcorner, y) \rightarrow (\exists u \leq y)\mathbf{R}(\ulcorner\sigma\urcorner, u)].$$

We will show that $\overline{m} = \#\pi(\sigma)$ has the required property—and it can be computed recursively from m because the proof of the Fixed Point Lemma is effective.

So assume that $W_m \cap \#T = \emptyset$.

(a) $T \nvdash \pi(\sigma)$. The argument is almost exactly like that in the proof of Rosser's Theorem: if there were a proof in T of $\pi(\sigma)$ with code y, then we would know that

$$\mathsf{Q} \vdash \mathbf{Proof}_{\pi,T}(\ulcorner\sigma\urcorner, \Delta y);$$

and since (by the hypothesis) $\#\pi(\sigma) \notin W_m$, we know that for each u,

$$\mathsf{Q} \vdash \neg\mathbf{R}(\Delta m, \ulcorner\sigma\urcorner, \Delta u);$$

and using properties of Q as in the proof of the Rosser Theorem, we show that $\mathsf{Q} \vdash \neg\sigma$ so that $T \vdash \neg\pi(\sigma)$ contradicting the consistency of T.

(b) $\#\pi(\sigma) \notin W_m$. Suppose $\#\pi(\sigma) \in W_m$. This means that there is a u such that $R(\#\sigma, u)$, and so $\mathsf{Q} \vdash \mathbf{R}(\ulcorner\sigma\urcorner, \Delta u)$. On the other hand, no y codes a proof of $\pi(\sigma)$ in T by (a), and so for every y, $\mathsf{Q} \vdash \neg\mathbf{Proof}_{\pi,T}(\ulcorner\sigma\urcorner, \Delta y)$; and then the usual arguments about Q (as in the proof of the Rosser Theorem) show that $\mathsf{Q} \vdash \sigma$, which contradicts (a).

5D. The Second Recursion Theorem

In this section we will prove a very simple result of Kleene, which has surprisingly strong and unexpected consequences in many parts of definability theory, and even in analysis and set theory. Here we will prove just one, substantial application of the Second Recursion Theorem, but we will also use it later in the theory of *recursive functionals* and *effective operations*.

Theorem 5D.1 (Kleene)**.** *For each recursive partial function $f(z, \vec{x})$, there is a number z^*, such that for all $\vec{x}$,*

$$\varphi_{z^*}(\vec{x}) = \{z^*\}(\vec{x}) = f(z^*, \vec{x}). \tag{124}$$

In fact, for each n, there is a primitive recursive function $h_n(e)$, such that if $f = \varphi_e^{n+1}$, is $n+1$-ary, then equation (124) *holds with $z^* = h_n(e)$, i.e.,*

$$\varphi_{h_n(e)}(\vec{x}) = \{h_n(e)\}(\vec{x}) = \varphi_e(h_n(e), \vec{x}). \tag{125}$$

The theorem gives immediately several simple propositions which show that the coding of recursive partial functions has many unexpected (and even weird) properties.

Example 5D.2. There exist numbers $z_1 - z_4$, such that

$$\begin{aligned} \varphi_{z_1}(x) &= z_1 \\ \varphi_{z_2}(x) &= z_2 + x \\ W_{z_3} &= \{z_3\} \\ W_{z_4} &= \{0, \dots, z_4\}. \end{aligned}$$

PROOF. For z_1, we apply the Second Recursion Theorem to the function

$$f(z, x) = z$$

and we set $z_1 = z^*$; it follows that

$$\varphi_{z_1}(x) = f(z_1, x) = z_1.$$

The rest are similar and equally easy. ⊣

Proof of the Second Recursion Theorem 5D.1. The partial function

$$g(z, \vec{x}) = f(S^1_n(z, z), \vec{x})$$

is recursive, and so there some number $\widehat{g}$ such that

$$\{S^1_n(\widehat{g}, z)\}(\vec{x}) = \{\widehat{g}\}(z, \vec{x}) = f(S^1_n(z, z), \vec{x});$$

the result follows from this equation if we set

$$z^* = S^1_n(\widehat{g}, \widehat{g}).$$

For the stronger (uniform) version (125), let d be a number such that

$$\varphi_d(e, z, \vec{x}) = \varphi_e(S^1_n(z, z), \vec{x});$$

it follows that

$$\widehat{g} = S^1_{n+1}(d, e)$$

is a code of $\varphi_e(S^1_n(z, z), \vec{x})$, and the required function is

$$h(e) = S^1_n(\widehat{g}, \widehat{g}) = S^1_n(S^1_{n+1}(d, e), S^1_{n+1}(d, e)). \qquad \dashv$$

For a (much more significant) example of the strength of the Second Recursion Theorem, we show here the converse of 5C.3, that every creative set is r.e. complete (and a bit more).

Theorem 5D.3 (John Myhill). *The following are equivalent for every r.e. set A.*

(1) *There is a recursive partial function $p(e)$ such that*

$$W_e \cap A = \emptyset \Longrightarrow [p(e)\downarrow \ \ \& \ p(e) \in A^c \setminus W_e].$$

(2) *There is a total recursive function $q(e)$ such that*

$$W_e \cap A = \emptyset \Longrightarrow q(e) \in A^c \setminus W_e. \tag{126}$$

(3) *A is creative, i.e.,* (126) *holds with a one-to-one recursive function $q(e)$.*

(4) *A is r.e. complete.*

In particular, an r.e. set is complete if and only if it is creative.

Proof. (1) $\Rightarrow$ (2). For the given, recursive partial function $p(e)$, there exists (by the Second Recursion Theorem) some number z such that

$$\{S^1_1(z, e)\}(t) = \varphi_z(e, t) = \begin{cases} \varphi_e(t), & \text{if } p(S^1_1(z, e))\downarrow, \\ \uparrow, & \text{otherwise.} \end{cases}$$

We set $q(e) = p(S^1_1(z,e))$ with this z, and we observe that $q(e)$ is a total function, because

$$\begin{aligned} q(e) = p(S^1_1(z,e))\uparrow &\Longrightarrow W_{S^1_1(z,e)} = \emptyset \text{ by the definition} \\ &\Longrightarrow p(S^1_1(z,e))\downarrow . \end{aligned}$$

In addition, since $q(e)\downarrow$, $W_{S^1_1(z,e)} = W_e$, and hence

$$W_e \cap A = \emptyset \Longrightarrow q(e) = p(S^1_1(z,e)) \in A^c \setminus W_{S^1_1(z,e)} = A^c \setminus W_e$$

which is what we needed to show.

(2) $\Rightarrow$ (3) (This implication does not use the Second recursion Theorem, and could have been given in Section 5A.) For the given $q(e)$ which satisfies (126), we observe first that there is a recursive function $h(e)$ such that

$$W_{h(e)} = W_e \cup \{q(e)\};$$

and then we set, by primitive recursion,

$$\begin{aligned} g(0,e) &= e \\ g(i+1,e) &= h(g(i,e)), \end{aligned}$$

so that (easily, by induction on i),

$$W_{g(i+1,e)} = W_e \cup \{q(g(0,e)), q(g(1,e)), \dots, q(g(i,e))\}.$$

It follows that for each $i > 0$,

$$\text{(127)} \quad \begin{aligned} &W_e \cap A = \emptyset \\ &\Longrightarrow q(g(i,e)) \in A^c \setminus (W_e \cup \{q(g(0,e)), q(g(1,e)), \dots, q(g(i-1,e))\}), \end{aligned}$$

and, more specifically,

$$\text{(128)} \qquad W_e \cap A = \emptyset \Longrightarrow (\forall j < i)[q(g(i,e)) \neq q(g(j,e))].$$

Finally, we set

$$f(0) = q(0),$$

and for the (recursive) definition of $f(e+1)$, we compute first, in sequence, the values $q(g(0,e+1)), \dots, q(g(e+1,e+1))$ and we distinguish two cases.

Case 1. If these values are all distinct, then one of them is different from the values $f(0), \dots, f(e)$, and we just set

$$\begin{aligned} j &= (\mu i \leq (e+1))(\forall y \leq e)[q(g(i,e+1)) \neq f(y)] \\ f(e+1) &= q(g(j,e+1)). \end{aligned}$$

Case 2. There exist $i, j \le e+1$, $i \neq j$, such that $q(g(i, e+1)) = q(g(j, e+1))$. In this case we set

$$f(e+1) = \max\{f(0), \dots, f(e)\} + 1.$$

It is clear that $f(e)$ is recursive and one-to-one, and that it is a productive function for A^c follows immediately from (128) and (127).

(3) ⇒ (4). If $q(e)$ is a productive function for the complement A^c and B is any r.e. set, then (by the Second Recursion Theorem) there is some number z such that

$$\varphi_z(x, t) = \begin{cases} 1 & \text{if } x \in B \ \& \ t = q(S^1_1(z, x)) \\ \uparrow & \text{otherwise;} \end{cases}$$

the function

$$f(x) = q(S^1_1(z, x))$$

is one-to-one (as a composition of one-to-one functions), and it reduces B to A, as follows.

If $x \in B$, then $W_{S^1_1(z,x)} = \{q(S^1_1(z, x)\} = \{f(x)\}$, and

$$\begin{aligned} f(x) \notin A &\Longrightarrow W_{S^1_1(z,x)} \cap A = \emptyset \\ &\Longrightarrow q(S^1_1(z, x)) \in A^c \setminus W_{S^1_1(z,x)} \\ &\Longrightarrow f(x) \in A^c \setminus \{f(x)\}, \end{aligned}$$

which is a contradiction; hence $f(x) \in A$. On the other hand, if $x \notin B$, then $W_{S^1_1(z,x)} = \emptyset \subseteq A^c$, hence $f(x) = q(S^1_1(z, x)) \in A^c$. ⊣

Problems for Section 5D

Problem 5D.1. Prove that there is some number z such that

$$W_z = \{z, z+1, \dots\} = \{x \mid x \ge z\}.$$

Problem 5D.2. Prove that for some number t and all x, $\varphi_t(x) = t + x$.

Problem 5D.3. Prove that for each total, recursive function $f(x)$ one of the following holds:

(a) There is a number z such that $f(z)$ is odd and for all x,

$$\varphi_z(x) = f(z + x);$$

or

(b) there is a number w such that $f(w)$ is even and for all x,

$$\varphi_w(x) = f(2w + x + 1).$$

Problem 5D.4. Prove or give a counterexample: for each total, recursive function $f(x)$, there is some z such that

$$W_{f(z)} = W_z.$$

Problem 5D.5. Prove or give a counterexample: for every total, recursive function $f(x)$, there is a number z such that for all t,

$$\varphi_{f(z)}(t) = \varphi_z(t).$$

Problem 5D.6. (a) Prove that for every total, recursive function $f(x)$, there is a number z such that

$$W_z = \{f(z)\}.$$

(b) Prove that there is some number z such that

$$\varphi_z(z)\downarrow \text{ and } W_z = \{\varphi_z(z)\}.$$

Problem 5D.7. Suppose $A \subseteq \mathbb{N}$, $x \preceq y$ is a wellordering of A,

$$F : (\mathbb{N} \rightharpoonup \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$$

is a function(al) defined on partial functions on $\mathbb{N}$, and $g : A \to \mathbb{N}$ is defined by the transfinite recursion

$$g(x) = F(g \restriction \{y \in A \mid y \prec x\}, x).$$

Suppose in addition, that there is a recursive partial function $\psi(e, x)$ satisfying the following, for every e, every x, and every partial function p:

$$\text{if } (\forall y \prec x)[p(y) = \varphi_e(y)], \text{ then } F(p, x) = \psi(e, x).$$

Prove that there is a recursive partial function g^* such that for all $x \in A$, $g(x) = g^*(x)$.

Note: It is not assumed that A is a recursive set, or that $x \preceq y$ is a recursive relation; the result holds for completely arbitrary A and $x \preceq y$.

5E. The arithmetical hierarchy

The semirecursive (Σ^0_1) relations are of the form

$$(\exists y)Q(\vec{x}, y)$$

where $Q(\vec{x}, y)$ is recursive, and so they are just one *existential quantifier* "away" from the recursive relations in complexity. The next definition gives us a useful tool for the classification of complex, undecidable relations.

Definition 5E.1. The classes (sets) of relations Σ^0_k, Π^0_k, Δ^0_k are defined recursively, as follows:

$$\begin{aligned} \Sigma^0_1 &: \text{the semirecursive relations}\\ \Pi^0_k = \neg\Sigma^0_k &: \text{the negations (complements) of relations in } \Sigma^0_k\\ \Sigma^0_{k+1} = \exists^{\mathbb{N}}\Pi^0_k &: \text{the relations which satisfy an equivalence}\\ &\quad P(\vec{x}) \iff (\exists y)Q(\vec{x},y), \text{where } Q(\vec{x},y) \text{ is } \Pi^0_k\\ \Delta^0_k = \Sigma^0_k \cap \Pi^0_k &: \text{ the relations which are both } \Sigma^0_k \text{ and } \Pi^0_k. \end{aligned}$$

A set A is in one of these classes Γ if the relation $x \in A$ is in Γ.

5E.2. Canonical forms. These classes of the *arithmetical hierarchy* are (obviously) characterized by the following "canonical forms", in the sense that a given relation $P(\vec{x})$ is in a class Γ if it is equivalent with the canonical form for Γ, with some recursive Q:

$$\begin{array}{lcr} \Sigma^0_1 & : & (\exists y)Q(\vec{x},y)\\ \Pi^0_1 & : & (\forall y)Q(\vec{x},y)\\ \Sigma^0_2 & : & (\exists y_1)(\forall y_2)Q(\vec{x},y_1,y_2)\\ \Pi^0_2 & : & (\forall y_1)(\exists y_2)Q(\vec{x},y_1,y_2)\\ \Sigma^0_3 & : & (\exists y_1)(\forall y_2)(\exists y_3)Q(\vec{x},y_1,y_2,y_3)\\ & \vdots & \end{array}$$

A trivial corollary of these canonical forms is that:

Proposition 5E.3. *The relations which belong to some Σ^0_k or some Π^0_k are precisely the arithmetical relations.*

Proof. Each primitive recursive relation is arithmetical, by Theorem 4B.13 and Lemma 4B.2, and then (inductively) every Σ^0_k and every Π^0_k relation is arithmetical, because the class of arithmetical relations is closed under negation and quantification on $\mathbb{N}$. For the other direction, we notice that relations defined by quantifier-free formulas are (trivially) recursive, and that every arithmetical relation is defined by some formula in prenex form with quantifier-free matrix; and by introducing dummy quantifiers, if necessary, we may assume that the quantifiers in the prefix are alternating and start with an $\exists$, so that the relation defined by each formula is in some Σ^0_k. ⊣

Theorem 5E.4. (1) *For each $k \geq 1$, the classes Σ^0_k, Π^0_k, and Δ^0_k are closed for (total) recursive substitutions and for the operations &, $\vee$, $\exists_{\leq}$ and $\forall_{\leq}$. In addition*:

- *Each Δ^0_k is closed for negation $\neg$.*

- *Each Σ^0_k is closed for $\exists^{\mathbb{N}}$, existential quantification over $\mathbb{N}$.*
- *Each Π^0_k is closed for $\forall^{\mathbb{N}}$, universal quantification over $\mathbb{N}$.*

(2) *For each $k \geq 1$,*

$$\Sigma^0_k \subseteq \Delta^0_{k+1}, \tag{129}$$

and hence the arithmetical classes satisfy the following diagram of inclusions:

$$\begin{array}{ccccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & \Sigma^0_3 & & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & & & & \cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & & \Pi^0_3 & &
\end{array}$$

Proof. First we verify the closure of all the arithmetical classes for recursive substitutions, by induction on k; the proposition is known for $k = 1$ by 5A.7, and (inductively), for the case of Σ^0_{k+1}, we compute:

$$\begin{aligned}
P(\vec{x}) &\iff R(f_1(\vec{x}), \dots , f_n(\vec{x})) \\
&\iff (\exists y)Q(f_1(\vec{x}), \dots , f_n(\vec{x}), y) \\
&\qquad \text{where } Q \in \Pi^0_k, \text{ by definition} \\
&\iff (\exists y)Q'(\vec{x}, y) \\
&\qquad \text{where } Q' \in \Pi^0_k \text{ by the induction hypothesis.}
\end{aligned}$$

The remaining parts of (1) are shown directly (with no induction) using the transformations in the proof of 5A.7.

We show (2) by induction on k, where, in the basis, if

$$P(\vec{x}) \iff (\exists y)Q(\vec{x}, y)$$

with a recursive Q, then P is surely Σ^0_2, since each recursive relation is Π^0_1; but a semirecursive relation is also Π^0_2, since, obviously,

$$P(\vec{x}) \iff (\forall z)(\exists y)Q(\vec{x}, y)$$

and the relation

$$Q_1(\vec{x}, z, y) \iff Q(\vec{x}, y)$$

is recursive. The induction step of the proof is practically identical, and the inclusions in the diagram follow easily from (129) and simple computations. ⊣

More interesting is the next theorem which justifies the appellation "hierarchy" for the classes Σ^0_k, Π^0_k:

Theorem 5E.5 (Kleene).

(1) (Enumeration for Σ^0_k) *For each $k \geq 1$ and each $n \geq 1$, there is an $n+1$-ary relation $\widetilde{S}^0_{k,n}(e,\vec{x})$ in the class Σ^0_k which enumerates all the n-ary, Σ^0_k relations, i.e., $P(\vec{x})$ is Σ^0_k if and only if for some e,*

$$P(\vec{x}) \iff \widetilde{S}^0_{k,n}(e,\vec{x}).$$

(2) (Enumeration for Π^0_k) *For each $k \geq 1$ and each $n \geq 1$, there is an $n+1$-ary relation $\widetilde{P}^0_{k,n}(e,\vec{x})$ in Π^0_k which enumerates all the n-ary, Π^0_k relations, i.e., $P(\vec{x})$ is Π^0_k if and only if, for some e,*

$$P(\vec{x}) \iff \widetilde{P}^0_{k,n}(e,\vec{x}).$$

(3) (Hierarchy Theorem) *The inclusions in the Diagram of Proposition* 5E.4 *are all strict, i.e.,*

$$\begin{array}{ccccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & \Sigma^0_3 & & \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & & & & \cdots \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & & \Pi^0_3 & &
\end{array}$$

Proof. For (1) and (2) we set, recursively,

$$\begin{aligned}
\widetilde{S}^0_{1,n}(e,\vec{x}) &\iff (\exists y)T_n(e,\vec{x},y)\\
\widetilde{P}^0_{k,n}(e,\vec{x}) &\iff \neg\widetilde{S}^0_{k,n}(e,\vec{x})\\
\widetilde{S}^0_{k+1,n}(e,\vec{x}) &\iff (\exists y)\widetilde{P}^0_{k,n+1}(e,\vec{x},y),
\end{aligned}$$

and the proofs are easy, with induction on k. For (3), we observe that the "diagonal" relation

$$D_k(x) \iff \widetilde{S}^0_{k,1}(x,x)$$

is Σ^0_k but cannot be Π^0_k, because, if it were, then for some e we would have

$$\neg\widetilde{S}^0_{k,1}(x,x,) \iff \widetilde{S}^0_{k,1}(e,x)$$

which is absurd when $x = e$. It follows that for each k, there exist relations which are Σ^0_k but not Π^0_k, and from this follows easily the strictness of all the inclusions in the diagram. ⊣

Theorem 5E.5 gives an alternative proof—and a better understanding—of Tarscki's Theorem 4A.5, that the truth set of arithmetic $\mathrm{Truth}^{\mathbf{N}}$ is not arithmetical, cf. Problem 5E.1.

Definition 5E.6 (Classifications)**.** A (complete) **classification** of a relation $P(\vec{x})$ (in the arithmetical hierarchy) is the determination of "the least" arithmetical class to which $P(\vec{x})$ belongs, i.e., the proof of a proposition of the form

$$P \in \Sigma^0_k \setminus \Pi^0_k, \quad P \in \Pi^0_k \setminus \Sigma^0_k, \quad \text{or } P \in \Delta^0_{k+1} \setminus (\Sigma^0_k \cup \Pi^0_k);$$

for example, in 5B.11 we showed that

$$\{e \mid W_e \neq \emptyset\} \in \Sigma^0_1 \setminus \Pi^0_1.$$

The complete classification of a relation P is sometimes very difficult, and we are often satisfied with the computation of some "upper bound", i.e., some k such that $P \in \Sigma^0_k$ or $P \in \Pi^0_k$. The basic method for the computation of a "lower bound," when this can be done, is to show that the given relation is *complete* in some class Σ^0_k or Π^0_k as in the next result.

Proposition 5E.7. (1) *The set* $F = \{e \mid \varphi_e \textit{ is total }\}$ *is* Π^0_2 *but it is not* Σ^0_2.

(2) *The set* $\mathrm{Fin} = \{e \mid W_e \textit{ is finite}\}$ *is* $\Sigma^0_2 \setminus \Pi^0_2$.

Proof. (1) The upper bound is obvious, since

$$e \in F \iff (\forall x)(\exists y)T_1(e, x, y).$$

To show (by contradiction) that F is not Σ^0_2, suppose $P(x)$ is any Π^0_2 relation, so that

$$P(x) \iff (\forall u)(\exists v)Q(x, u, v)$$

with a recursive $Q(x, u, v)$, and set

$$f(x, u) = \mu v Q(x, u, v).$$

If $\widehat{f}$ is a code of this (recursive) partial function $f(x, u)$, then

$$\begin{aligned} P(x) &\iff (\forall u)[f(x, u)\downarrow] \\ &\iff (\forall u)[\{S^1_1(\widehat{f}, x)\}(u)\downarrow] \\ &\iff S^1_1(\widehat{f}, x) \in F; \end{aligned}$$

it follows that if F were Σ^0_2, then every Π^0_2 would be Σ^0_2, which contradicts the Hierarchy Theorem 5E.5 (3).

(2) The upper bound is again trivial,

$$e \in \mathrm{Fin} \iff (\exists k)(\forall x)[x \in W_e \Longrightarrow x \leq k].$$

For the lower bound, let $P(x)$ be any Σ^0_2 relation, so that

$$P(x) \iff (\exists u)(\forall v)Q(x, u, v)$$

with a recursive Q. We set

$$g(x,u) = \mu y(\forall i \leq u)\neg Q(x,i,(y)_i),$$

so that if $\widehat{g}$ is a code of g, then

$$\begin{aligned}(\exists u)(\forall v)Q(x,u,v) &\iff \{u \mid g(x,u)\downarrow\} \text{ is finite}\\ &\iff \{u \mid \{\widehat{g}\}(x,u)\downarrow\} \text{ is finite}\\ &\iff \{u \mid \{S^1_1(\widehat{g},x)\}(u)\downarrow\} \text{ is finite,}\end{aligned}$$

i.e.,

$$P(x) \iff S^1_1(\widehat{g},x) \in \text{Fin};$$

but this implies that Fin is not Π^0_2, because, if it were, then every Σ^0_2 relation would be Π^0_2, which it is not. ⊣

Problems for Section 5E

Problem 5E.1. Prove that for every arithmetical relation $P(\vec{x})$, there is a 1-1, total recursive function $f : \mathbb{N}^n \to \mathbb{N}$ such that

$$P(\vec{x}) \iff f(\vec{x}) \in \text{Truth}^{\mathbf{N}};$$

infer Tarski's Theorem 4A.5, that $\text{Truth}^{\mathbf{N}}$ is not arithmetical.

Problem 5E.2. Classify in the arithmetical hierarchy the set

$$A = \{e \mid W_e \subseteq \{0,1\}\}.$$

Problem 5E.3. Classify in the arithmetical hierarchy the set

$$A = \{e \mid W_e \text{ is a singleton}\}.$$

Problem 5E.4. Classify in the arithmetical hierarchy the set

$$A = \{e \mid W_e \text{ is finite and non-empty}\}.$$

Problem 5E.5. Classify in the arithmetical hierarchy the set

$$B = \{x \mid \text{there are infinitely many twin primes } p \geq x\},$$

where p is a twin prime if both p and $p+2$ are prime numbers.

Problem 5E.6. Classify in the arithmetical hierarchy the relation

$$\begin{aligned}Q(e,m) &\iff \varphi_e \sqsubseteq \varphi_m\\ &\iff (\forall x)(\forall w)[\varphi_e(x) = w \Longrightarrow \varphi_m(x) = w].\end{aligned}$$

Problem 5E.7. Classify in the arithmetical hierarchy the set

$$A = \{e \mid W_e \text{ has at least } e \text{ members}\}.$$

Problem 5E.8. Classify in the arithmetical hierarchy the set

$$B = \{e \mid \text{ for some } w \text{ and all } x, \text{ if } \varphi_e(x)\downarrow, \text{ then } \varphi_e(x) \leq w\}.$$

(This is the set of codes of bounded recursive partial functions.)

Problem 5E.9. For a fixed, unary, total recursive function f, classify in the arithmetical hierarchy the set of all the codes of f,

$$C_f = \{e \mid \varphi_e = f\}.$$

Problem 5E.10. Let A be some recursive set with non-empty complements, i.e., $A \subsetneq \mathbb{N}$. Classify in the arithmetical hierarchy the set

$$B = \{e \mid W_e \subseteq A\}.$$

Problem 5E.11. Show that the graph

$$G_f(\vec{x}, w) \iff f(\vec{x}) = w$$

of a total function is Σ^0_k if and only if it is Δ^0_k.

Is this also true of partial functions?

Problem 5E.12. A total function $f : \mathbb{N}^n \to \mathbb{N}$ is *limiting recursive* if there is a total, recursive function $g : \mathbb{N}^{n+1} \to \mathbb{N}$ such that for all $\vec{x}$,

$$f(\vec{x}) = \lim_{m\to\infty} g(m, \vec{x}).$$

Prove that a total $f(\vec{x})$ is limiting recursive if and only if the graph G_f of $f(\vec{x})$ is Δ^0_2.

5F. Relativization

The notions of μ-recursiveness in 4E.5 and reckonability in 4E.9 "relativize" naturally to a "given" partial function as follows.

Definition 5F.1. For a fixed partial function $p : \mathbb{N}^m \rightharpoonup \mathbb{N}$:

(1) A **μ-recursive derivation from** (or *relative to*) p is a sequence of partial functions on $\mathbb{N}$

$$f_1, f_2, \ldots , f_k,$$

where each f_i is S, or a constant C^n_q or a projection P^n_i, *or* p, or is defined by composition, primitive recursion or minimalization from functions before it in the sequence; and a partial function $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$ is **μ-recursive in** p if it occurs in a μ-recursive derivation from p.

(2) For each partial function p, let Q_p be Robinson's system in the extension of the language of arithmetic with a single m-ary function symbol p and with the additional axioms

$$D_p = \{\mathsf{p}(\Delta x_1, \dots, \Delta x_m) = \Delta w \mid p(x_1, \dots, x_m) = w\},$$

which express formally the graph of p. A partial function f is **reckonable in** p, if there is a formula $\mathbf{F}(v_1, \dots, v_n, y, \mathsf{p})$ of Q_p, such that for all $\vec{x}, w$,

$$f(\vec{x}) = w \iff Q_p \vdash \mathbf{F}(\Delta x_1, \dots, \Delta x_n, \Delta w, \mathsf{p}).$$

These notions express two different ways in which we can compute a function f *given access to the values of* p, and they behave best when the "given" p is total, in which case they coincide:

Proposition 5F.2. *If $p : \mathbb{N}^m \to \mathbb{N}$ is a total function, then, for every* (possibly partial) f,

$$f \text{ is } \mu\text{-recursive in } p \iff f \text{ is reckonable in } p.$$

Proof is a simple modification of the proof of 4E.10. ⊣

We will just say

$$f \text{ is recursive in } p \quad \text{ or } \quad f \text{ is Turing-recursive in } p$$

for this notion of reduction of a partial function to a total one, the reference to "Turing" coming from a third equivalent definition which involves "Turing machines with oracles".

The Enumeration Theorem

The Enumeration Theorem (4F.1) generalizes easily to the present context. For simplicity, we restrict ourselves to the case that p is a function of one variable.

Define $T_n^p(e, x_1, \dots, x_n, y)$ exactly as $T_n(e, x_1, \dots, x_n, y)$ was defined, except use Q_p in place of Q.

If $T_n^p(e, \vec{x}, y)$ and the number of $\mathsf{p}(\Delta x) = \Delta w$ is $((y)_1)_i$ for some $i < \text{lh(y)}$, then all of x and w are smaller than y. This allows us to define a single relation that does the work of all the T_n^p.

Set

$$\tilde{T}_n(a, e, \vec{x}, y) \Leftrightarrow (\exists p)(\overline{p}(y) = a \wedge T_n^p(e, \vec{x}, y)).$$

Notice that $(\exists p)$ could be replaced by $(\forall p)$ without affecting the relation defined. Define $\varphi_e^{p,n}(\vec{x}) = U(\mu y \tilde{T}_n(\overline{p}(y), e, \vec{x}, y)))$.

Theorem 5F.3. (1) *Each relation* $\tilde{T}_n(a, e, \vec{x}, y)$ *is primitive recursive.*

(2) *Each* $\varphi_e^{p,n}(\vec{x})$ *is a partial function recursive in* p*, and so is the partial function which "enumerates" all these,*

$$\varphi^{p,n}(e, \vec{x}) = \varphi_e^{p,n}(\vec{x}).$$

(3) *For each partial function* $f(x_1, \ldots, x_n)$ *of* n *arguments that is recursive in* p*, there exists some* e (a code of f) *such that*

$$f(\vec{x}) = \varphi_e^{p,n}(\vec{x}) = U(\mu y \tilde{T}_n(\overline{p}(y), e, x_1, \ldots, x_n, y)). \tag{130}$$

5F.4. Turing reducibility and Turing degrees. For any two total functions $f, g : \mathbb{N} \to \mathbb{N}$, we set

$$f \leq_T g \iff f \text{ is } \textit{recursive} \text{ in (or } \textit{Turing reducible to}\text{) } g$$

We also set

$$f \equiv_T g \iff f \leq_T g \;\&\; g \leq_T f,$$

and we assign to each set g its *degree* (of unsolvability)

$$\deg(f) = \{g \mid f \equiv_T g\}.$$

We set $\deg(f) \leq \deg(g) \Leftrightarrow f \leq_T g$.

We extend these definitions to subsets of $\mathbb{N}$ by, for example, letting $A \leq_T B$ just in case $\chi_A \leq_T \chi_B$, where χ_A, χ_B are the characteristic functions of A and B.

Proposition 5F.5. (1) $A \leq_m B \Longrightarrow A \leq_T B$*, but the converse is not always true.*
(2) *If* $f \leq_T g$ *and* $g \leq_T h$*, then* $f \leq_T h$*.*
(3) *If* g *is recursive, then, for every* f*,*

$$f \leq_T g \iff f \text{ is recursive},$$

and so

$$\deg(\emptyset) = \deg(\mathbb{N}) = \{f \mid g \text{ is recursive}\}.$$

Proposition 5F.6. *Any two Turing degrees have a least upper bound.*

PROOF. Given f and g, let $h(2n) = f(n)$ and $h(2n + 1) = g(n)$. ⊣

Proposition 5F.7. *For every Turing degree, there is a greater one.*

Proof. Since only countably many functions are recursive in f, there is a g that is not recursive in f. The least upper bound of $\deg(f)$ and $\deg(g)$ is greater than $\deg(f)$.

Another proof: It is easy to check that $\{e \mid \varphi_e^f(e)\downarrow\}$ is is not recursive in f. (In fact, it has greater degree than $\deg(f)$.) ⊣

Theorem 5F.8 (The Kleene-Post Theorem)**.** *There exist Turing incomparable functions $f, g : \mathbb{N} \to \mathbb{N}$, i.e.,*

$$f \not\leq_T g \text{ and } g \not\leq_T f. \tag{131}$$

Proof. There are numerous ways to prove the theorem. The following proof is essentially the original one, and it gives f and g that are recursive in the set K.

We define by recursion s_e, t_e s'_e, and t'_e for each $e \in \mathbb{N}$. Each will be a finite sequence of natural numbers. For each e, we shall have

$$s_e \subseteq s'_e \subsetneq s_{e+1} \text{ and } t_e \subsetneq t'_e \subseteq t_{e+1}.$$

We will then let f extend all the s_e and g extend all the t_e.

Let $s_0 = t_0 =$ the empty sequence.

Given s_e and t_e, let

$$a_e = \begin{cases} \mu a(a \text{ codes } s(a) \supseteq s_e \;\&\; (\exists y \leq \mathrm{lh}(a)\tilde{T}_1(a\restriction y, e, \mathrm{lh}(t_e), y)) & \text{if it exists;} \\ \# s_e & \text{otherwise.} \end{cases}$$

Let s'_e be the sequence coded by a_e. Let $\mathrm{lh}(t'_e) = \mathrm{lh}(t_e) + 1$. If a_e is defined by the first clause, let

$$t'_e(\mathrm{lh}(t_e)) = U(\mu y \tilde{T}_1(a_e\restriction y, e, \mathrm{lh}(t_e), y)) + 1.$$

If a_e is defined by the "otherwise" clause, let $t'_e(\mathrm{lh}(t_e)) = 0$. Note that the definitions assure that $\varphi_e^f(\mathrm{lh}(t_e))$ is either undefined or defined and different from $g(\mathrm{lh}(t_e))$.

Define t_{e+1} and s_{e+1} from t'_e and s'_e as s'_e and t'_e were defined from s_e and t_e. This definition guarantees that $f(\mathrm{lh}(t_e))$ is different from $\varphi_{e\cdot}^g(\mathrm{lh}(s'_e))$. ⊣

A set is *recursively enumerable (r.e.) in f* if it is the domain of a partial function recursive in f.

Proposition 5F.9. *A set is recursive in f just in case both it and its complement are r.e. in f.*

For each f, W_e^f be the domain of φ_e^f. Let $K^f = \{e \mid e \in W_e^f\}$.

Proposition 5F.10. *K^f is r.e. in f but not recursive in f.*

5F.11. Remark. Actually $\deg(f) < \deg(K^f)$, and every set r.e. in f is $\leq_1 K^f$. Moreover $f \leq_T g \Leftrightarrow K^f \leq_1 K^g$.

A degree of unsolvability is *r.e.* if it is the degree of an r.e. set.

Theorem 5F.12. *There is a simple set A such that $K^A \leq_T K$.* (Hence the remark above implies that $\deg(K^A) = \deg(K)$.)

Corollary 5F.13. *There is an r.e. degree d such that $0 < \deg(d) < \deg(K)$. (Here 0 is the degree of recursive functions and sets.)*

Proof. To prove the theorem, we will define by recursion a total recursive function $g(e, x)$ and a recursive relation $A(s, x)$. We let A be the r.e. set

$$\{x \mid (\exists s)A(s, x)\}.$$

Let $A^s = \{x \mid A(s, x)\}$. We will arrange that

$$(s < s' \ \&\ x \in A^s) \Rightarrow x \in A^{s'}.$$

Let $A^0 = \emptyset$.

Let a^s be the characteristic function of A^s and let a be the characteristic function of A. For each s, let

$$g(s, e) = \begin{cases} \mu y \leq s\, \tilde{T}_1(\bar{a^s}(y), e, e, y) & \text{if } (\exists y \leq s)\tilde{T}_1(\bar{a^s}(y), e, e, y); \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $g(s, e) \neq 0$ then (i) $e \in K^{A^s}$ and (ii) if $\bar{a}(g(s, e)) = \bar{a^s}(g(s, e))$ then $e \in K^A$.

Let $W_e^s = \{x \mid (\exists y \leq s)T_1(e, x, y)\}$.

Let $x \in A^{s+1}$ just in case $x \in A^s$ or there is an $e \leq s$ such that the following conditions are satisfied:

(1) $W_e^s \cap A^s = \emptyset$;
(2) $x \in W_e^s$;
(3) $x > 2e$;
(4) $(\forall e' < e)g(s, e') \leq x$;
(5) x is the smallest number satisfying (1)-(4).

Lemma 5F.14. *A^c is infinite.*

Proof. Suppose conditions (1)-(5) are satisfied by x for e and s. By condition (5), no $x' \neq x$ satisfies (1)-(5) for e and s. Since $x \in W_e^{s+1} \cap A^{s+1} \neq \emptyset$, no x' can satisfy condition (1) for e and any $s' > s$. Thus for each e there is at most one x that satisfies (1)-(5) for e and any s. By condition (3), for any e' there are at most $e' + 1$ numbers $x \leq 2e'$ that satisfy (1)-(5) for any e and s. Thus A has at most $e' + 1$ members that are $\leq 2e'$. ⊣

Lemma 5F.15. *For each e, $\lim_s g(s,e)$ exists.*

Proof. Fix e. Let s_0 be large enough that

$$(\forall e' \leq e)(W_e \cap A \neq \emptyset \Rightarrow W_e^s \cap A \neq 0).$$

We show that there is no $s \geq s_0$ such that $g(s,e) \neq 0$ and $g(s+1,e) \neq g(s,e)$. Assume that there is such an s. Some $x < g(s,e)$ must satisfy (1)-(5) for some e' and s. By condition (4), e' must be $\leq e$. Since $s \geq s_0$, this is impossible. ⊣

Lemma 5F.16. *A is simple.*

Proof. By Lemma 5F.14, it is enough to prove that $A \cap W_e \neq \emptyset$ for every e such that W_e is infinite. Fix e and assume that W_e is infinite. Let s_0 be such that

$$(\forall e' < e)(\forall s \geq s_0) g(s,e') = g(s_0,e').$$

Let $x_0 = \max\{g(s_0,e') \mid e' < e\}$. Since W_e is infinite, it has a member $\geq x_0$ and $> 2e$. Let x be the least such member. Let $s \geq s_0$ be such that $x \in W_e^s$. Either $W_e^s \cap A^s \neq \emptyset$ or else conditions (1)-(5) are satisfied by x for e and s. Whichever is the case, $W_e^{s+1} \cap A^{s+1} \neq \emptyset$. ⊣

Lemma 5F.17. *$K^A \leq_T K$.*

Proof. Define a function f by

$$f(e) = \mu s(\forall s' \geq s) g(s',e) = g(s,e).$$

Since

$$e \in K^A \Leftrightarrow g(f(e),e) > 0,$$

$K^A \leq_T f$. The relation $R(s,e) \Leftrightarrow (\forall s' \geq s) g(s',e) = g(s,e)$ is Π^0_1, and so it is recursive in K. Since f is recursive in R, $K^A \leq_T f \leq_T K$. ⊣

This completes the proof of the theorem. ⊣

Definition 5F.18 (Functionals)**.** A (partial) *functional* is any partial function $\alpha(x_1,\dots,x_n,p_1,\dots,p_m)$ of n number arguments and m partial function arguments, such that for $i = 1,\dots,m$, p_i ranges over the k_i-ary partial functions on $\mathbb{N}$, and such that (when it takes a value), $\alpha(x_1,\dots,x_n,p_1,\dots,p_m) \in \mathbb{N}$. We view every partial function $f(\vec{x})$ as a functional 0 partial function arguments. More interesting examples include:

$$\alpha_1(x,p) = p(x+1)$$

$$\alpha_2(x,p,q) = \textit{if } x = 0 \textit{ then } p(x) \textit{ else } q(x,p(x-1))$$

$$\alpha_3(p) = \begin{cases} 1 & \text{if } p(0)\downarrow \text{ or } p(1)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$$\alpha_4(p) = \begin{cases} 1 & \text{if } p(0)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_5(p) = \begin{cases} 1 & \text{if } (\forall x)[p(x)\downarrow] \\ \uparrow & \text{otherwise} \end{cases}$$

From these examples, we might say that α_1 and α_2 are "recursive", in the sense that we can see a direct method for computing their values if we have access to a "oracles" who can respond to questions of the form

$$\text{What is } p(x)? \qquad \text{What is } q(x, y)?$$

for specific x. To compute $\alpha_2(x, p)$, for example, if $x = 0$ we request of the oracle the value $p(0, x)$ and give it as output, while, if $x > 0$, then we first request the value $v = p(x - 1, 0)$, and then we request and give as output the value $p(x, v)$. On the other hand, there is no obvious way to compute the values of α_4 and α_5 in this way, unless we can ask the oracle questions about the domain of convergence of p, a conception which does not yield a natural and useful notion of computability. Finally, $\alpha_3(p)$ is a borderline case, which appears to be recursive if we can ask the oracle "non-deterministic" questions of the form

$$\text{what is } p(0) \text{ or } p(1)?,$$

which looks iffy—or, at the least, suggests on a different notion of "non-deterministic computability" for functionals.

Definition 5F.19 (Recursive functionals)**.** We make these two notions of functional computability precise, using the relativization process.

(1) A μ-**recursive** (functional) **derivation** (in one, m-ary partial function variable) is a sequence of functionals

$$\alpha_1(\vec{x}_1, p), \dots, \alpha_m(\vec{x}_m, p)$$

in which each α_i is S, C^n_q or P^n_j (not depending on p); an *evaluation functional*

$$\text{ev}^m(x_1, \dots, x_m, p) = p(x_1, \dots, x_m) \tag{132}$$

which introduces dependence on p; or it is defined from previously listed functionals by composition, primitive recursion or minimalization, which are defined as before, e.g,

$$\alpha_i(\vec{x}, p) = \mu y[\alpha_j(\vec{x}, y, p) = 0] \quad (j < i).$$

A functional is μ-**recursive**, or just **recursive**, if it occurs in some μ-recursive derivation.

(2) A functional $\alpha(\vec{x}, p)$ is **reckonable** (or *non-deterministically recursive*) if there is a formula $\mathbf{F}(v_1, \dots, v_n, y, \mathsf{p})$ in the system Q_p introduced in (2) of 5F.1, such that for all $\vec{x}, w$ and p,

$$\alpha(\vec{x}, p) = w \iff Q_p \vdash \mathbf{F}(\Delta x_1, \dots, \Delta x_n, \Delta w, \mathsf{p}). \tag{133}$$

5F.20. Remark. There is no formula $\mathbf{F}(v_1, \dots, v_n, y, \mathsf{p})$ such that, for all $\vec{x}$, w and p

$$p(\vec{x}) = w \iff \mathbf{F}(\Delta x_1, \dots, \Delta x_n, \Delta w) \tag{A}$$

$$Q_p \vdash (\exists! y)\mathbf{F}(\Delta x_1, \dots, \Delta x_n, y), \tag{B}$$

simply because if (A) holds for all p, then the formula on the right of (B) is not true, whenever p is not a totally defined function. This means that the simplest evaluation functional (132) is not "numeralwise representable" in Q, in the most natural extension of this notion to functionals, which is why we have not introduced it.

Proposition 5F.21. *Every recursive functional is reckonable.*

PROOF is a minor modification of the argument for (1) $\Longrightarrow$ (2) of Theorem 4E.10 (skipping the argument for the characteristic property of numeralwise representability which does not hold here), and we will skip it. ⊣

To separate recursiveness from reckonability for functionals, we need to introduce some basic notions, all of them depending on the following, partial ordering of partial functions of the same arity.

Definition 5F.22. For any two, m-ary partial functions p and q, we set

$$p \leq q \iff (\forall \vec{x}, w)[p(\vec{x}) = w \Longrightarrow q(\vec{x}) = w],$$

i.e., if the domain of convergence of p is a subset of the domain of convergence of q, and q agrees with p whenever they are both defined. For example, if $\emptyset$ is *the nowhere-defined* m-ary partial function, then, for every m-ary q, $\emptyset \leq q$; and, at the other extreme,

$$(\forall \vec{x}) p(\vec{x})\downarrow \ \& \ p \leq q \Longrightarrow p = q.$$

Proposition 5F.23. *For each m, $\leq$ is a partial ordering of the set of all m-ary partial functions, i.e.,*

$$p \leq p, \quad [p \leq q \ \& \ q \leq r] \Longrightarrow p \leq r, \quad [p \leq q \ \& \ q \leq p] \Longrightarrow p = q.$$

PROOF is simple and we will skip it. ⊣

Definition 5F.24. A functional $\alpha(\vec{x}, p)$ is:

1. **monotonic** (or monotone), if for all partial functions p, q, and all $\vec{x}$, w,

$$[\alpha(\vec{x}, p) = w \;\&\; p \le q] \Longrightarrow \alpha(\vec{x}, q) = w;$$

2. **continuous**, if for each p and al $\vec{x}$, w,

$$\alpha(\vec{x}, p) = w \Longrightarrow (\exists r)[r \le p \;\&\; \alpha(\vec{x}, r) = w \;\&\; r \text{ is finite}],$$

where a partial function is **finite** if its domain of convergence is finite; and

3. **deterministic**, if for each p and all $\vec{x}$, w,

$$\alpha(\vec{x}, p) = w \Longrightarrow (\exists! r \le p)[\alpha(\vec{x}, r) = w \;\&\; (\forall r' \le r)[\alpha(\vec{x}, r')\!\downarrow\; \Longrightarrow r' = r]].$$

5F.25. Exercise. Give counterexamples to show that no two of these properties imply the third.

Theorem 5F.26. (1) *Every reckonable functional is monotonic and continuous.*
(2) *Every recursive functional is monotonic, continuous and deterministic.*
(3) *There are reckonable functionals which are not deterministic.*

PROOF. (1) is immediate, using the (corresponding) properties of proofs: for example, if

$$Q_p \vdash \mathbf{F}(\Delta x_1, \dots, \Delta x_n, \Delta w, \mathsf{p})$$

for some p, $\vec{x}$ and w, then the proof can only use a finite number of the axioms in Q_p, which "fix" p only on a finite set of arguments—and if r is the (finite) restriction of p to this set, then

$$Q_r \vdash \mathbf{F}(\Delta x_1, \dots, \Delta x_n, \Delta w, \mathsf{r}),$$

so that $\alpha(\vec{x}, r) = w$.

(2) is proved by induction on a given μ-recursive derivation. There are several cases to consider, but the arguments are simple and we will skip them.

(3) The standard example is

$$\alpha_3(p) = \begin{cases} 1 & \text{if } p(0)\!\downarrow \text{ or } p(1)\!\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

as above, which is not deterministic because if $p(0) = p(1) = 0$ and $p(x)\!\uparrow$ for all $x > 1$, then there is no *least* $r \le p$ which determines the value $\alpha_3(p) = 1$.⊣

Part (1) of this theorem yields a simple normal form for reckonable functionals which characterizes them without reference to any formal systems. We need another coding.

5F.27. Coding of finite partial functions and sets. For each $a \in \mathbb{N}$ and each $m \geq 1$,

$$\begin{aligned} d(a,x) &= \begin{cases} (a)_x \dot{-} 1 & \text{if } x < \mathrm{lh}(a) \ \&\ (a)_x > 0, \\ \uparrow & \text{otherwise} \end{cases} \\ d_a(x) &= d(a,x), \\ D_x &= \{i \mid d_x(i)\downarrow\} \\ d_a^m(\vec{x}) &= d^m(a,\vec{x}) = d(a,\langle\vec{x}\rangle) \quad (\vec{x} = x_1,\dots,x_m). \end{aligned}$$

Note that, easily, *each partial function* $d^m(a,\vec{x})$ *is primitive recursive*; *the sequence*

$$d_0^m, d_1^m, \dots$$

enumerates all finite partial functions of m *arguments*; and *the sequence*

$$D_0, D_1, \dots$$

enumerates all finite sets, so that the binary relation of membership

$$i \in D_x \iff i < \mathrm{lh}(x) \ \&\ (x)_i > 0,$$

is primitive recursive.

Theorem 5F.28 (Normal form for reckonable functionals)**.** *A functional* $\alpha(\vec{x},p)$ *is reckonable if and only if there exists a semirecursive relation* $R(\vec{x},w,a)$*, such that for all* $\vec{x}$*,* w *and* p*,*

$$\alpha(\vec{x},p) = w \iff (\exists a)[d_a^m \leq p \ \&\ R(\vec{x},w,a)]. \tag{134}$$

Proof. Suppose first that $\alpha(\vec{x},p)$ is reckonable, and compute:

$$\begin{aligned} \alpha(\vec{x},p) = w &\iff (\exists \text{ finite } r \leq p)[\alpha(\vec{x},r) = w] \qquad \text{(by 5F.26)} \\ &\iff (\exists a)[d_a^m \leq p \ \&\ \alpha(\vec{x},d_a^m) = w)]. \end{aligned}$$

Thus, it is enough to prove that the relation

$$R(\vec{x},w,a) \iff \alpha(\vec{x},d_a^m) = w$$

is semirecursive; but if (133) holds with some formula $\mathbf{F}(v_1,\dots,v_m,y,\mathsf{p})$, then

$$\begin{aligned} R(\vec{x},w,a) &\iff Q_{d_a^m} \vdash \mathbf{F}(\Delta x_1,\dots,\Delta x_n,\Delta w,\mathsf{p}) \\ &\iff Q \vdash \sigma_{m,a,\mathsf{p}} \rightarrow \mathbf{F}(\Delta x_1,\dots,\Delta x_n,\Delta w,\mathsf{p}) \end{aligned}$$

where $\sigma_{m,a,\mathsf{p}}$ is the *finite* conjunction of equations

$$\mathsf{p}(\Delta u_1,\dots,\Delta u_m) = \Delta d_a^m(u_1,\dots,u_m),$$

one for each $u_1, \ldots, u_m$ in the domain of d_a^m. A code of the sentence on the right can be computed primitive recursively from $\vec{x}, w, a$, so that $R(\vec{x}, w, a)$ is reducible to the relation of provability in Q and hence semirecursive.

For the converse, we observe that with the same $\sigma_{m,a,\mathsf{p}}$ we just used and for any m-ary p,

$$d_a^m \leq p \iff Q_p \vdash \sigma_{m,a,\mathsf{p}},$$

and that, with some care, this $\sigma_{m,a,\mathsf{p}}$ can be converted to a formula $\sigma^*(a, \mathsf{p})$ with the free variable a, in which bounded quantification replaces the blunt, finite conjunction so that

$$d_a^m \leq p \iff Q_p \vdash \sigma^*(\Delta a, \mathsf{p}). \tag{135}$$

Assume now that $\alpha(\vec{x}, p)$ satisfies (134), choose a primitive recursive $P(\vec{x}, w, a, z)$ such that

$$R(\vec{x}, w, a) \iff (\exists z)P(\vec{x}, w, a, z),$$

choose a formula $\mathbf{P}(v_1, \ldots, v_n, v_{n+1}, v_{n+2}, z)$ which numeralwise expresses P in Q, and set

$$\mathbf{F}(v_1, \ldots, v_n, v_{n+1}, v_{n+2})) \equiv (\exists a)[\sigma^*(a, \mathsf{p}) \ \& \ (\exists z)\mathbf{P}(v_1, \ldots, v_n, v_{n+1}, a, z)]].$$

Now,

$$\begin{aligned} Q_p \vdash \mathbf{F}(\Delta x_1, \ldots, \Delta x_n, \Delta w, \mathsf{p}) & \\ \iff Q_p \vdash (\exists a)[\sigma^*(a, \mathsf{p}) & \\ \& \ (\exists z)[\mathbf{P}(\Delta x_1, \ldots, \Delta x_n, \Delta w, a, z)]] & \\ \iff \alpha(\vec{x}, p) = w, & \end{aligned}$$

with the last equivalence easy to verify, using the soundness of Q_p. ⊣

There is no simple normal form of this type for recursive functionals, because predicate logic is not well suitable for expressing "determinism".

5G. Effective operations

Intuitively, a functional $\alpha(\vec{x}, p)$ is *recursive* in either of the two ways that we made precise, if its values can be computed effectively and uniformly *for all partial functions* p, given access only to specific values of p—which simply means that the evaluation functional (132) is declared recursive. In many cases, however, we are interested in the values $\alpha(\vec{x}, p)$ only for *recursive partial functions* p, and then we might make available to the computation procedure some code of p, from which (perhaps) more than the values of p can be extracted.

Definition 5G.1. The *associate* of a functional $\alpha(\vec{x}, p)$ is the partial function

$$f_\alpha(\vec{x}, e) = \alpha(\vec{x}, \varphi_e), \tag{136}$$

and we call $\alpha(\vec{x}, p)$ an *effective operation* if its associate is recursive.

Note that this imposes no restriction on the values $\alpha(\vec{x}, p)$ for non-recursive p, and so, properly speaking, we should think of effective operations as (partial) functions on *recursive partial functions*, not on all partial functions—this is why the term "operation" is used. For purposes of comparison with recursive functionals, however, it is convenient to consider effective operations as functionals, with arbitrary values on non-recursive arguments, as we did in the precise definition.

Proposition 5G.2. *A recursive partial function $f(\vec{x}, e)$ is the associate of some effective operation if and only if it satisfies the invariance condition*

$$\varphi_e = \varphi_m \Longrightarrow f(\vec{x}, e) = f(\vec{x}, m); \tag{137}$$

and if f satisfies this condition, then it is the associate of the effective operation

$$\alpha(\vec{x}, \varphi_e) = f(\vec{x}, e),$$

(with $\alpha(\vec{x}, p)$ defined arbitrarily when p is not recursive).

Proof is immediate. ⊣

Theorem 5G.3. *Every reckonable functional* (and hence every recursive functional) *is an effective operation.*

Proof. This is immediate from the Normal Form Theorem for reckonable functionals 5F.28; because if $f(\vec{x}, e)$ is the associate of $\alpha(\vec{x}, p)$, then

$$f(\vec{x}, e) = w \iff (\exists a)[d_a^m \leq \varphi_e \ \& \ R(\vec{x}, w, a)]$$

with a semirecursive $R(\vec{x}, w, a)$ by 5F.28, and so the graph of f is semirecursive and f is recursive. ⊣

Definition 5G.4. A functional $\alpha(\vec{x}, p)$ is *operative* if $\vec{x} = x_1, \dots , x_n$ varies over n-tuples and p over n-ary partial functions, for the same n, so that the *fixed point equation*

$$p(\vec{x}) = \alpha(\vec{x}, p) \tag{138}$$

makes sense. Solutions of this equation are called *fixed points* of α.

Theorem 5G.5 (The Fixed Point Lemma)**.** *Every operative effective operation α has a recursive fixed point, i.e., there exists a recursive partial function p such that, for all $\vec{x}$,*

$$p(\vec{x}) = \alpha(\vec{x}, p).$$

Proof. The partial function

$$f(z, \vec{x}) = \alpha(\vec{x}, \varphi_z)$$

is recursive, and so, by the Second Recursion Theorem, there is some z^* such that

$$\begin{aligned} \varphi_{z^*}(\vec{x}) &= f(z^*, \vec{x}) \\ &= \alpha(\vec{x}, \varphi_{z^*}); \end{aligned}$$

and so $p = \varphi_{z^*}$ is a fixed point of α. ⊣

5G.6. Remark. By an elaboration of these methods (or different arguments), it can be shown that *every effective operation has a recursive least fixed point*: i.e., that for some recursive partial function p, the fixed point equation (138) holds, and in addition, for all q,

$$(\forall \vec{x})[q(\vec{x}) = \alpha(\vec{x}, q)] \Longrightarrow p \leq q.$$

The Fixed Point Lemma applies to all reckonable operative functionals, and it is a powerful tool for showing easily the recursiveness of partial functions defined by very general recursive definitions, for example by *double recursion*:

5G.7. Example. If g_1, g_2, g_3, π_1, π_2 are total recursive functions and $f(x, y, z)$ is defined by the *double recursion*

$$\begin{aligned} f(0, y, z) &= g_1(y, z) \\ f(x+1, 0, z) &= g_2(f(x, \pi_1(x, y, z), z), x, y, z) \\ f(x+1, y+1) &= g_3(f(x+1, y, \pi_2(x, y, z)), x, y, z), \end{aligned}$$

then $f(x, y, z)$ is recursive.

Proof. The functional

$$h(x, y, z, p) = \begin{cases} g_1(y, z) & \text{if } x = 0 \\ g_2(p(x \dot{-} 1, \pi_1(x \dot{-} 1, 0, z), z), x \dot{-} 1, 0, z) & \text{otherwise, if } y = 0 \\ g_3(p(x, y \dot{-} 1, \Pi^0_2(x, y \dot{-} 1, z)), x, y \dot{-} 1, z) & \text{otherwise} \end{cases}$$

is recursive, and so it has a recursive fixed point $f(x, y, z)$, which, easily, satisfies the required equations. It remains to show that $f(x, y, z)$ is a total function, and we do this by showing by an induction on x that $(\forall x) f(x, y, z)\downarrow$; both the basis case and the induction step require separate inductions on y. ⊣

The converse of Theorem 5G.3 depends on the following, basic result.

Lemma 5G.8. *Every effective operation is monotonic and continuous on recursive partial arguments.*

Proof. To simplify the argument we consider only effective operations of the form $\alpha(p)$, with no numerical argument and a unary partial function argument, but the proof for the general case is only notationally more complex.

To show monotonicity, suppose $p \le q$, where

$$p = \varphi_e \text{ and } q = \varphi_m,$$

and let $\widehat{f}$ be a code of the associate of α, so that for every z,

$$\alpha(\varphi_z) = \{\widehat{f}\}(z).$$

Suppose also that

$$\alpha(\varphi_e) = w;$$

we must show that $\alpha(\varphi_m) = w$.

The relation

$$R(z, x, v) \iff \varphi_e(x) = v \text{ or } [\{\widehat{f}\}(z) = w \;\&\; \varphi_m(x) = v]$$

is semirecursive; the hypothesis $\varphi_e \le \varphi_m$ implies that

$$R(z, x, v) \Longrightarrow \varphi_m(x) = v;$$

hence $R(z, x, v)$ is the graph of some recursive partial function $g(z, x)$; and so, by the Second recursion Theorem, there is some number z^* such that $\varphi_{z^*}(x) = g(z^*, x)$, so that

$$\varphi_{z^*}(x) = v \iff \varphi_e(x) = v \text{ or } [\{\widehat{f}\}(z^*) = w \;\&\; \varphi_m(x) = v]. \tag{139}$$

We now observe that:

(1a) $\alpha(\varphi_{z^*}) = \{\widehat{f}\}(z^*) = w$; because, if not, then $\varphi_{z^*} = \varphi_e$ from (139), and so $\alpha(\varphi_{z^*}) = \alpha(\varphi_e) = w$.

(1b) $\varphi_{z^*} = \varphi_m$, directly from the hypothesis $\varphi_e \le \varphi_m$ and (1a).

It follows that $\alpha(\varphi_m) = \alpha(\varphi_{z^*}) = w$.

The construction for the proof of continuity is a small variation, as follows. First, we find using the Second recursion Theorem some z^* such that

$$\varphi_{z^*}(x) = v \iff (\forall u \le x)\neg[T_1(\widehat{f}, z^*, u) \;\&\; U(u) = w] \;\&\; \varphi_e(x) = v, \tag{140}$$

and we observe:

(2a) $\alpha(\varphi_{z^*}) = w$. Because, if not, then

$$(\forall u)\neg[T_1(\widehat{f}, z^*, u) \;\&\; U(u) = w],$$

and hence, for every x,

$$(\forall u \le x)\neg[T_1(\widehat{f}, z^*, u) \ \&\ U(u) = w],$$

and so, from (140), $\varphi_{z^*} = \varphi_e$ and $\alpha(\varphi_{z^*}) = \alpha(\varphi_e) = w$.

(2b) $\varphi_{z^*} \le \varphi_e$, directly from (140).

(2c) The partial function φ_{z^*} is finite, because it converges only when

(141) $$x < (\mu u)[T_1(\widehat{f}, z^*, u) \ \&\ U(u) = w]. \quad \dashv$$

Theorem 5G.9 (Myhill-Shepherdson)**.** *For each effective operation $\alpha(\vec{x}, p)$, there is a reckonable functional $\alpha^*(\vec{x}, p)$ such that for all recursive partial functions p,*

(142) $$\alpha(\vec{x}, p) = \alpha^*(\vec{x}, p).$$

PROOF. By the Lemma,

$$\alpha(\vec{x}, \varphi_e) = w \iff (\exists a)[d_a^m \le \varphi_e \ \&\ \alpha(\vec{x}, d_a^m) = w],$$

and so (142) holds with

(143) $$\alpha^*(\vec{x}, p) = w \iff (\exists a)[d_a^m \le p \ \&\ \alpha(\vec{x}, d_a^m) = w].$$

To show that this α^* is reckonable, note that (by an easy application of the S_n^m-Theorem) there is a primitive recursive $u(a)$ such that

$$d_a^m = \varphi_{u(a)},$$

and so the partial function

$$\alpha(\vec{x}, d_a^m) = f_\alpha(\vec{x}, u(a))$$

is recursive, its graph is semirecursive, and (143) with Theorem 5F.28 imply that α^* is reckonable. $\dashv$

5G.10. Remark. It is natural to think of a functional $\alpha(\vec{x}, p)$ as interpreting a program A, which computes some function $f(\vec{x})$ but requires for the computations some unspecified partial function p—and hence, A must be "given" p in addition to the arguments $\vec{x}$. Now if p could be any partial function whatsoever, then the only reasonable way by which A can be "given" p is *through its values*: we imagine that A can look up a table or ask an "oracle" for $p(u)$, for any specific u, during the computation. We generally refer to this manner of "accessing" a partial function by a program as *call-by-value*, and it is modeled mathematically by recursive or reckonable functionals, depending on whether the program A is deterministic or not. On the other hand, if it is known that $p = \varphi_e$ is a recursive partial function, then some code e of it may be given to A, at the start of the computation, so that A can compute any $p(u)$ that it wishes, but also (perhaps) infer general properties of p from e, and use these properties in its computations; this manner of accessing a recursive partial function is (one version of) *call-by-name*, and it is modeled mathematically by effective operations.

One might suspect that given access to a code of p, one might be able to compute effectively partial functions (depending on p) which cannot be computed when access to p is restricted in call-by-value fashion. The Myhill-Shepherdson Theorem tells us that, for non-deterministic programs, this cannot happen—knowledge of a code of p does not enlarge the class of partial functions which can be non-deterministically computed from it. Note that this is certainly false for deterministic computations, because of the basic example $\alpha_3(p)$ in 5F.18, which is reckonable but not recursive.

5H. Computability on Baire space

We will extend here the basic results about recursive partial functions and relations on $\mathbb{N}$, to partial functions and relations which can also take arguments in *Baire space*, the set

$$\mathcal{N} = (\mathbb{N} \to \mathbb{N}) = \{\alpha \mid \alpha : \mathbb{N} \to \mathbb{N}\}$$

of all total, unary functions on the natural numbers. For example, the total functions

$$f(\alpha) = \alpha(0), \quad g(x, \alpha, \beta, y) = \alpha(\beta(x)) + y,$$

will be deemed recursive by the definitions we will give, and so will the partial function

$$h(\alpha) = \mu t[\alpha(t) = 0],$$

which is defined only if $\alpha(t) = 0$ for some t. The relation

$$R(\alpha) \iff (\exists t)[\alpha(t) = 0]$$

will be semirecursive but not recursive.

5H.1. Notation. More precisely, in this section we will study partial functions

$$f : \mathbb{N}^n \times \mathcal{N}^\nu \rightharpoonup \mathbb{N},$$

with $n = 0$ or $\nu = 0$ allowed, so that the partial functions on $\mathbb{N}$ we have been studying are included. To avoid "too many dots", we set once and for all boldface abbreviations for vectors,

$$\boldsymbol{x} = (x_1, \dots, x_n),\ \boldsymbol{y} = (y_1, \dots, y_m),\ \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\nu),\ \boldsymbol{\beta} = (\beta_1, \dots, \beta_\mu),$$

so that the values of our partial functions will be denoted compactly by expression like

$$f(\boldsymbol{x}, \boldsymbol{\alpha}),\ g(t, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\alpha}, \gamma, \boldsymbol{\beta}),\ \text{etc.}$$

In defining specific functions we will sometimes mix the number with the Baire arguments, as above, the "official" reading always being the one in which all number arguments precede all the Baire ones, e.g.,

$$g(x, \alpha, \beta, y) = g(x, y, \alpha, \beta).$$

Codings of initial segments. Recall from Lemma 3I.13 the following notation, which will now be very useful:

$$\overline{\alpha}(0) = 1, \quad \overline{\alpha}(t) = \langle \alpha(0), \alpha(1), \dots, \alpha(t \dot{-} 1)\rangle \in \mathbb{N}$$

and for vectors of Baire points,

$$\overline{\boldsymbol{\alpha}}(t) = (\overline{\alpha}_1(t), \dots, \overline{\alpha}_\nu(t)) \in \mathbb{N}^\nu.$$

For any sequence code u and any s, put

$$u \restriction s = \begin{cases} \langle (u)_0, \dots, (u)_{s-1}\rangle, & \text{if } s \leq \text{lh}(u), \\ u, & \text{otherwise,} \end{cases}$$

so that

$$\text{lh}(u \restriction s) = \min\{\text{lh}(u), s\}, \quad \text{and} \quad s \leq t \Longrightarrow \overline{\alpha}(s) = \overline{\alpha}(t) \restriction s.$$

Similarly, for any tuple of sequence codes $\vec{u} = (u_1, \dots, u_\nu)$ and any s, put

$$\vec{u} \restriction s = (u_1 \restriction s, \dots, u_\nu \restriction s),$$

so that

$$\text{if } s \leq t, \text{ then } \overline{\boldsymbol{\alpha}}(s) = \overline{\boldsymbol{\alpha}}(t) \restriction s.$$

A relation $R(\boldsymbol{x}, \vec{u})$ is **monotone in** $\vec{u}$ if, for every $\boldsymbol{\alpha}$ and every s,

$$\text{if } R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(s)) \text{ and } s \leq t, \text{ then } R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t)).$$

λ-abstraction. We will also find useful Church's λ operation, by which, for any partial function $f : \mathbb{N}^{n+1} \times \mathcal{N}^\nu \rightharpoonup \mathbb{N}$,

$$\lambda(t) f(\boldsymbol{x}, t, \boldsymbol{\alpha}) = g_{\boldsymbol{x}, \boldsymbol{\alpha}} : \mathbb{N} \rightharpoonup \mathbb{N} \text{ where } g_{\boldsymbol{x}, \boldsymbol{\alpha}}(t) = f(\boldsymbol{x}, t, \boldsymbol{\alpha}),$$

For example, $\lambda(t)(xt+t^2)$ is that function $g_x : \mathbb{N} \to \mathbb{N}$ such that for all t, $g_x(t) = xt + t^2$, and (closer to the way we will use this),

$$\lambda(x)U(\mu y T_1(e,x,y)) = \varphi_e. \tag{144}$$

Definition 5H.2 (Semirecursive relations on $\mathcal{N}$)**.** A relation

$$P(\boldsymbol{x}, \boldsymbol{\alpha}) \iff P(x_1, \dots, x_n, \alpha_1, \dots, \alpha_\nu)$$

is *semirecursive* or Σ^0_1, if there is a semirecursive relation

$$R(\boldsymbol{x}, \vec{u}) \iff R(x_1, \dots, x_n, u_1, \dots, u_\nu)$$

on $\mathbb{N}$, such that

$$P(\boldsymbol{x}, \boldsymbol{\alpha}) \iff (\exists t) R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t)). \tag{145}$$

A relation $P(\boldsymbol{x}, \boldsymbol{\alpha})$ is *recursive* or Δ^0_1 if both $P(\boldsymbol{x}, \boldsymbol{\alpha})$ and its negation $\neg P(\boldsymbol{x}, \boldsymbol{\alpha})$ are semirecursive.

Notice that these definitions agree with the old ones for relations $R(\boldsymbol{x})$ which have no Baire arguments.

Lemma 5H.3. *The class of semirecursive relations with arguments in* $\mathbb{N}$ *and* $\mathcal{N}$ *is closed under permutations and identifications of variables: i.e., if*

$$\pi : \{1, \dots, n\} \to \{1, \dots, m\}, \quad \rho : \{1, \dots, \nu\} \to \{1, \dots, \mu\}$$

are any functions and $P(y_1, \dots, y_m, \beta_1, \dots, \beta_\mu)$ *is semirecursive, then so is the relation*

$$P'(x_1, \dots, x_n, \alpha_1, \dots, \alpha_\nu) \iff P(x_{\pi(1)}, \dots, x_{\pi(n)}, \alpha_{\rho(1)}, \dots, \alpha_{\rho(n)})$$

This justifies "explicit" definitions of the form

$$P'(x, y, \alpha, \beta) \iff P(y, x, x, \beta, \beta, \beta)$$

within Σ^0_1, and it is immediate from the definition.

Lemma 5H.4. (1) *If* $P(\boldsymbol{x}, \boldsymbol{\alpha}) \iff (\exists t) R(\boldsymbol{x}, t, \boldsymbol{\alpha})$ *with a semirecursive* $R(\boldsymbol{x}, t, \vec{u})$*, then* $P(\boldsymbol{x}, \boldsymbol{\alpha})$ *is semirecursive.*

(2) *If* $P(\boldsymbol{x}, \boldsymbol{\alpha})$ *is semirecursive and* $\nu \geq 1$*, then it satisfies* (145) *with some recursive relation* $R(\boldsymbol{x}, \vec{u})$ *which is monotone in* $\vec{u}$*.*

PROOF. (1) The claim is obvious when $\nu = 0$, since the assumed equivalence implies immediately that $R(\boldsymbol{x})$ is a semirecursive relation on $\mathbb{N}$.

If $\nu \geq 1$, so that $P(\boldsymbol{x}, \boldsymbol{\alpha})$ has at least one Baire argument, we set

$$R'(\boldsymbol{x}, \vec{u}) \iff (\exists s \leq \mathrm{lh}(u_1)) R(\boldsymbol{x}, s, \vec{u} \restriction s),$$

and compute:

$$\begin{aligned} P(\boldsymbol{x},\boldsymbol{\alpha}) &\iff (\exists t)R(\boldsymbol{x},t,\overline{\boldsymbol{\alpha}}(t))\\ &\iff (\exists t)(\exists s\le t)R(\boldsymbol{x},s,\overline{\boldsymbol{\alpha}}(s))\\ &\iff (\exists t)(\exists s\le \mathrm{lh}(\overline{\alpha}_1(t)))R(\boldsymbol{x},s,\overline{\boldsymbol{\alpha}}(t)\restriction s)\\ &\iff (\exists t)R'(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t)). \end{aligned}$$

(2) Assume that (145) holds with

$$R(\boldsymbol{x},\vec{u}) \iff (\exists y)Q(\boldsymbol{x},\vec{u},y)$$

where $Q(\boldsymbol{x},\vec{u},y)$ is recursive, and let

$$\begin{aligned} R'(\boldsymbol{x},\vec{u}) \iff &(\text{for } i=1,\dots,\nu)[\mathrm{lh}(u_i)=\mathrm{lh}(u_1)]\\ &\quad \& \ (\exists s\le \mathrm{lh}(u_1))(\exists y\le \mathrm{lh}(u_1))Q(\boldsymbol{x},\vec{u}\restriction s,y). \end{aligned}$$

Now $R'(\boldsymbol{x},\vec{u})$ is clearly recursive and monotone in $\vec{u}$, and

$$\begin{aligned} P(\boldsymbol{x},\boldsymbol{\alpha}) &\iff (\exists s)R(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(s))\\ &\iff (\exists s)(\exists y)Q(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(s),y)\\ &\iff (\exists t)(\exists s\le t)(\exists y\le t)Q(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(s),y)\\ &\iff (\exists t)R'(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t)). \end{aligned}$$ ⊣

Lemma 5H.5 (Closure properties of Σ^0_1 and Δ^0_1)**.** (1) *The class of semirecursive relations with arguments in* $\mathbb{N}$ *and* $\mathcal{N}$ *is closed under substitutions of total, recursive functions* $f:\mathbb{N}^n\to\mathbb{N}$*; the positive propositional operations* & *and* $\vee$*; bounded number quantification of both kinds; existential number quantification* $(\exists x)$*; and also existential quantification over* $\mathcal{N}$, $(\exists\alpha)$.

(2) *The class of recursive relations with arguments in* $\mathbb{N}$ *and* $\mathcal{N}$ *is closed under substitutions of total, recursive functions* $f:\mathbb{N}^n\to\mathbb{N}$*; the propositional operations* $\neg$, & *and* $\vee$*; and bounded number quantification of both kinds.*

Proof. We may assume in the proofs that $\nu\ge 1$ (i.e., Baire arguments are present), since otherwise these results are known.

For conjunction, assume that P_1 and P_2 satisfy (145) with a recursive, monotone matrix, by (2) of Lemma 5H.4, and compute:

$$\begin{aligned} P_1(\boldsymbol{x},\boldsymbol{\alpha}) \ \& \ P_2(\boldsymbol{x},\boldsymbol{\alpha}) &\iff (\exists t)R_1(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t)) \ \& \ (\exists t)R_2(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t))\\ &\iff (\exists t)[R_1(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t)) \ \& \ R_2(\boldsymbol{x},t,\overline{\boldsymbol{\alpha}}(t))], \end{aligned}$$

the last equivalence by the monotonicity.

For existential quantification over $\mathcal{N}$:

$$\begin{aligned} P(\boldsymbol{x},\boldsymbol{\alpha}) &\iff (\exists\beta)P_1(\boldsymbol{x},\boldsymbol{\alpha},\beta) \\ &\iff (\exists\beta)(\exists t)R_1(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t),\overline{\beta}(t)) \\ &\iff (\exists t)(\exists\beta)R_1(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t),\overline{\beta}(t)) \\ &\iff (\exists t)(\exists v)[\mathrm{Seq}(v)\ \&\ \mathrm{lh}(v)=t\ \&\ R_1(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t),v)]; \end{aligned}$$

the crucial "quantifier-drop" equivalence

$$(\exists\beta)R_1(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t),\overline{\beta}(t)) \iff (\exists v)[\mathrm{Seq}(v)\ \&\ \mathrm{lh}(v)=t\ \&\ R_1(\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t),v)]$$

is proved from left-to-right by setting $v=\overline{\beta}(t)$, and from right-to-left by taking β to be an arbitrary, infinite extension of the sequence v. Now set

$$R'(\boldsymbol{x},t,\vec{u}) \iff (\exists v)[\mathrm{Seq}(v)\ \&\ \mathrm{lh}(v)=t\ \&\ R(\boldsymbol{x},\vec{u},v)];$$

this is a semirecursive relation,

$$P(\boldsymbol{x},\boldsymbol{\alpha}) \iff (\exists\beta)P_1(\boldsymbol{x},\boldsymbol{\alpha},\beta) \iff (\exists t)R'(\boldsymbol{x},t,\overline{\boldsymbol{\alpha}}(t)),$$

and so $P(\boldsymbol{x},\boldsymbol{\alpha})$ is semirecursive by (1) of Lemma 5H.4.

The remaining arguments are similar. ⊣

Theorem 5H.6. (1) *For every n and every ν, there exists a Σ^0_1 relation*

$$\widetilde{S}^0_{n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha}) \iff \widetilde{S}^0_{n,\nu}(e,x_1,\dots,x_n,\alpha_1,\dots,\alpha_\nu)$$

such that an arbitrary relation $R(\boldsymbol{x},\boldsymbol{\alpha})$ is semirecursive if and only if there is some number e such that

$$R(\boldsymbol{x},\boldsymbol{\alpha}) \iff \widetilde{S}^0_{n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha}).$$

In fact, if $\nu\geq 1$, then

$$\widetilde{S}^0_{n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha}) \iff (\exists t)T^r_{n,\nu}(e,\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t)), \tag{146}$$

where $T^r_{n,\mu}(e,\boldsymbol{x},\vec{u})$ is a primitive recursive and monotone in $\vec{u}$ relation on $\mathbb{N}$.

(2) *If $n+\nu>0$, then there exists a semirecursive relation $P(\boldsymbol{x},\overline{\boldsymbol{\alpha}})$ which is not recursive.*

(3) *For every m, every n and every ν, there exists a primitive recursive function $S^{r,m}_{n,\nu}(e,\boldsymbol{y})$ such that for all $\boldsymbol{y},\boldsymbol{x},\overline{\boldsymbol{\alpha}}$,*

$$\widetilde{S}^0_{m+n,\nu}(e,\boldsymbol{y},\boldsymbol{x},\boldsymbol{\alpha}) \iff \widetilde{S}^0_{n,\nu}(S^{r,m}_{n,\nu}(e,\boldsymbol{y}),\boldsymbol{x},\boldsymbol{\alpha}).$$

PROOF. (1) The result is known when $\nu = 0$, so we assume $\nu \geq 1$, and with the notation of the Normal Form and Enumeration Theorem 4F.1, we let

$$T^r_{n,\nu}(e, \boldsymbol{x}, \vec{u}) \iff (\text{for } i = 1, \dots, \nu)[\text{Seq}(u_i) \ \& \ \text{lh}(u_i) = \text{lh}(u_1)] \\ \& \ (\exists s \leq \text{lh}(u_1))(\exists y \leq \text{lh}(u_1)) T_{n+\nu}(e, \boldsymbol{x}, \vec{u} \upharpoonright s, y).$$

This is clearly primitive recursive and monotone in $\vec{u}$, and if $\widetilde{S}^0_{n,\nu}$ is defined from it by (146), then it is semirecursive. For the converse, suppose $P(\boldsymbol{x}, \boldsymbol{\alpha})$ satisfies (145) with a semirecursive $R(\boldsymbol{x}, \vec{u})$. By the Enumeration Theorem 4F.1, there is some e such that

$$R(\boldsymbol{x}, \vec{u}) \iff (\exists y) T_{n+\nu}(e, \boldsymbol{x}, \vec{u}, y),$$

and then we compute:

$$\begin{aligned} P(\boldsymbol{x}, \boldsymbol{\alpha}) &\iff (\exists t) R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t)) \\ &\iff (\exists t)(\exists s \leq t) R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(s)) \\ &\iff (\exists t)(\exists s \leq t)(\exists y) T_{n+\nu}(e, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}(s), y) \\ &\iff (\exists t)(\exists y \leq t)(\exists s \leq t) T_{n+\nu}(e, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}(s), y) \\ &\iff (\exists t) T^r_{n,\nu}(e, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t)). \end{aligned}$$

(2) follows from (1) by the usual, diagonal method, and (3) follows from the S^m_n theorem for recursive partial functions on $\mathbb{N}$ by setting

$$S^{r,m}_{n,\nu}(e, \boldsymbol{y}) = S^m_{n+\nu}(e, \boldsymbol{y})$$

and chasing the definitions. ⊣

Definition 5H.7 (Recursive partial functions on $\mathcal{N}$ to $\mathbb{N}$)**.** A partial function $f(\boldsymbol{x}, \boldsymbol{\alpha})$ with values in $\mathbb{N}$ is *recursive* if its graph

$$G_f(\boldsymbol{x}, \boldsymbol{\alpha}, w) \iff f(\boldsymbol{x}, \boldsymbol{\alpha}) = w$$

is semirecursive. For example, the (total) *evaluation function*

$$\text{ev}(x, \alpha) = \alpha(x)$$

is recursive, because

$$\alpha(x) = w \iff (\exists t)[t > x \ \& \ (\overline{\alpha}(t))_x = w].$$

Lemma 5H.8 (Closure properties for recursive partial functions into $\mathbb{N}$)**.** *The class of recursive partial functions on Baire space with values in $\mathbb{N}$ contains all recursive partial functions $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$; it is closed under permutations and identifications of variables, i.e., if*

$$f(x_1, \dots, x_n, \alpha_1, \dots, \alpha_\nu) = g(x_{\pi(1)}, \dots, x_{\pi(n)}, \alpha_{\rho(1)}, \dots, \alpha_{\rho(n)})$$

with π, ρ as in Lemma 5H.3 and $g(\boldsymbol{y}, \boldsymbol{\beta})$ is recursive, then so is $f(\boldsymbol{x}, \boldsymbol{\alpha})$; and it is also closed under substitution (in its number arguments), primitive recursion, and minimalization.

PROOF. These claims all follow easily from the closure properties of the class of semirecursive relations, and we will consider just two of them, as examples.

For substitution (in a simple case), we are given that

$$f(\boldsymbol{x}, \boldsymbol{\alpha}) = g(h_1(\boldsymbol{x}, \boldsymbol{\alpha}), \boldsymbol{x}, \boldsymbol{\alpha}),$$

where $g(y, \boldsymbol{x}, \boldsymbol{\alpha})$ and $h(\boldsymbol{x}, \boldsymbol{\alpha})$ are recursive, and we compute the graph of $f(\boldsymbol{x}, \boldsymbol{\alpha})$:

$$f(\boldsymbol{x}, \boldsymbol{\alpha}) = w \iff (\exists y)[h_1(\boldsymbol{x}, \boldsymbol{\alpha}) = y \ \&\ g(y, \boldsymbol{x}, \boldsymbol{\alpha}) = w];$$

the result follows from the closure properties of Σ^0_1 in Lemma 5H.5.

For primitive recursion, we are given that

$$f(0, \boldsymbol{x}, \boldsymbol{\alpha}) = g(\boldsymbol{x}, \boldsymbol{\alpha}), \quad f(y+1, \boldsymbol{x}, \boldsymbol{\alpha}) = h(f(y, \boldsymbol{x}, \boldsymbol{\alpha}), y, \boldsymbol{x}, \boldsymbol{\alpha}).$$

Hence,

$$f(y, \boldsymbol{x}, \boldsymbol{\alpha}) = w \iff$$
$$(\exists u)[(u)_0 = g(\boldsymbol{x}, \boldsymbol{\alpha}) \ \&\ (\forall i \leq y)[(u)_{i+1} = h((u)_i, \boldsymbol{x}, \boldsymbol{\alpha})] \ \&\ (u)_y = w,$$

and so the graph of $f(y, \underset{\sim}{\mathrm{x}}, \boldsymbol{\alpha})$ is semirecursive. ⊣

Theorem 5H.9 (Normal Form and Enumeration)**.** (1) *For every n and every $\nu \geq 1$, there is a* (primitive) *recursive and monotone in $\vec{u}$ relation $T^1_{n,\nu}(e, \boldsymbol{x}, \vec{u})$ on $\mathbb{N}$, such that a partial function $f(\boldsymbol{x}, \boldsymbol{\alpha})$ into $\mathbb{N}$ is recursive if and only if there is a number e such that*

$$f(\boldsymbol{x}, \boldsymbol{\alpha}) = \{e\}(\boldsymbol{x}, \boldsymbol{\alpha}) = U(\mu t T^1_{n,\nu}(e, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t))), \tag{147}$$

with $U(t) = (t)_0$.

(2) *For every m, every n and every ν, there exists a primitive recursive function $S^m_{n,\nu}(e, \boldsymbol{y})$ such that for all $\boldsymbol{y}, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}$,*

$$\{e\}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\alpha}) = \{(S^m_{n,\nu}(e, \boldsymbol{y})\}(\boldsymbol{x}, \boldsymbol{\alpha}).$$

PROOF. (1) Every partial function defined by (147) with a recursive $T^1_{n,\nu}$ is recursive, by the closure properties. To define a suitable $T^1_{n,\nu}$, we note that by the definitions and Theorem 5H.6, for each recursive $f(\boldsymbol{x}, \boldsymbol{\alpha}) = w$, there is some e such that

$$f(\boldsymbol{x}, \boldsymbol{\alpha}) = w \iff (\exists t) T^r_{n+1,\nu}(e, \boldsymbol{x}, w, \overline{\boldsymbol{\alpha}}(t)). \tag{148}$$

We set

$$T^1_{n,\nu}(e, \boldsymbol{x}, \vec{u}) \iff (\text{for } i = 1, \dots, \nu)[\text{Seq}(u_i) \ \&\ \text{lh}(u_i) = \text{lh}(u_1)] \\ \&\ (\exists s \le \text{lh}(u_1)) T^r_{n+1,\nu}(e, \boldsymbol{x}, (s)_0, \vec{u} \restriction s).$$

This is obviously primitive recursive and monotone in $\vec{u}$, and to complete the proof, we need only show that if (148) holds for some e, then

$$f(\boldsymbol{x}, \boldsymbol{\alpha}) = \Big(\mu t T^1_{n,\nu}(e, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t))\Big)_0. \tag{149}$$

So fix $\boldsymbol{x}, \boldsymbol{\alpha}$, and first check that

$$s = \mu t T^1_{n,\nu}(e, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t)) \Longrightarrow f(\boldsymbol{x}, \boldsymbol{\alpha}) = (s)_0; \tag{150}$$

this holds because the hypothesis implies $T^r_{n+1,\nu}(e, \boldsymbol{x}, (s)_0, \overline{\boldsymbol{\alpha}}(s))$, which by (148) yields the conclusion. Conversely, if $f(\boldsymbol{x}, \boldsymbol{\alpha}) = w$, then

$$T^r_{n+1,\nu}(e, \boldsymbol{x}, w, \overline{\boldsymbol{\alpha}}(t))$$

holds for some t, and then taking $s = \langle w, t \rangle > t$ and using the monotonicity of $T^r_{n+1,\nu}(e, \boldsymbol{x}, w, \vec{u})$, we have

$$T^r_{n+1,\nu}(e, \boldsymbol{x}, (s)_0, \overline{\boldsymbol{\alpha}}(s));$$

so $(\exists t) T^1_{n,\nu}(e, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t))$, and then (150) gives

$$f(\boldsymbol{x}, \boldsymbol{\alpha}) = w \Longrightarrow \Big(\mu t T^1_{n,\nu}(e, \boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t))\Big)_0 = w, \tag{151}$$

which together with (150) yield (149).

(2) follows from (3) of Theorem 5H.6 by setting

$$S^m_{n,\nu}(e, \boldsymbol{y}) = S^{r,m}_{n+1,\nu}(e, \boldsymbol{y})$$

and chasing the definitions. ⊣

Definition 5H.10 (Recursive partial functions with values in $\mathcal{N}$)**.** A partial function $f : \mathbb{N}^n \times \mathcal{N}^\nu \rightharpoonup \mathcal{N}$ is recursive, if the following, associated, *unfolding* partial function $f^* : \mathbb{N}^{n+1} \times \mathcal{N} \rightharpoonup \mathbb{N}$ is recursive:

$$f^*(\boldsymbol{x}, \boldsymbol{\alpha}, t) = f(\boldsymbol{x}, \boldsymbol{\alpha})(t);$$

or, equivalently, if

$$f(\boldsymbol{x}, \boldsymbol{\alpha}) = \lambda(t) f^*(\boldsymbol{x}, \boldsymbol{\alpha}, t),$$

with some recursive $f^* : \mathbb{N}^{n+1} \times \mathcal{N}^\nu \rightharpoonup \mathbb{N}$. Thus

$$f(\boldsymbol{x}, \boldsymbol{\alpha}) = \beta \iff (\forall t)[f^*(\boldsymbol{x}, \boldsymbol{\alpha}, t) = \beta(t)],$$
$$f(\boldsymbol{x}, \boldsymbol{\alpha})\downarrow \iff (\forall t)[f^*(\boldsymbol{x}, \boldsymbol{\alpha}, t)\downarrow],$$

which suggests that *neither the graph nor the domain of convergence of a recursive partial functions with values in $\mathcal{N}$ need be semirecursive.* In fact, if we view (144) as a definition of a partial function $h : \mathbb{N} \rightharpoonup \mathcal{N}$, then

$$h(e) = \lambda(x)U(\mu y T_1(e,x,y))\downarrow \iff (\forall x)(\exists y)T_1(e,x,y),$$

so that by Proposition 5E.7, the domain of convergence of h is $\Pi^0_2 \setminus \Sigma^0_2$.

Lemma 5H.11. *The class of recursive partial functions with arguments in $\mathcal{N}$ is not closed under substitution.*

Proof. If $g(\alpha) = 0$, $h(e) = \lambda(x)U(\mu y T_1(e,x,y))$, and $f(e) = g(h(e))$, then $f : \mathbb{N} \rightharpoonup \mathbb{N}$,

$$f(e) = w \iff (\forall t)[\{e\}(t)\downarrow] \;\&\; w = 0,$$

and the graph of f is not semirecursive, so that f is not recursive. ⊣

However, $f(e)$ agrees with a recursive partial function (the constant 0) for values of e for which $h(e)\downarrow$; this is a general and useful fact:

Theorem 5H.12. (1) *Suppose $g : \mathbb{N}^n \times \mathcal{N} \times \mathcal{N}^\nu \rightharpoonup \mathbb{N}$ and $h : \mathbb{N}^n \times \mathcal{N}^\nu \rightharpoonup \mathcal{N}$ are recursive partial functions, and let*

$$f(\boldsymbol{x},\boldsymbol{\alpha}) = g(\boldsymbol{x}, h(\boldsymbol{x},\boldsymbol{\alpha}),\boldsymbol{\alpha});$$

then there exists a recursive partial function $\widetilde{f} : \mathbb{N}^n \times \mathcal{N}^\nu \rightharpoonup \mathbb{N}$ such that

$$\text{if } h(\boldsymbol{x},\boldsymbol{\alpha})\downarrow, \text{ then } f(\boldsymbol{x},\boldsymbol{\alpha}) = \widetilde{f}(\boldsymbol{x},\boldsymbol{\alpha}).$$

(2) *If $g(z_1,\dots,z_m), h_1(\boldsymbol{x},\boldsymbol{\alpha}),\dots,h_m(\boldsymbol{x},\boldsymbol{\alpha})$ are recursive partial functions such that for $i = 1,\dots,m$, if $h_i : \mathbb{N}^n \times \mathcal{N}^\nu \rightharpoonup \mathcal{N}$ then h_i is total, then the substitution*

$$f(\boldsymbol{x},\boldsymbol{\alpha}) = g(h_1(\boldsymbol{x},\boldsymbol{\alpha}),\dots,h_m(\boldsymbol{x},\boldsymbol{\alpha}))$$

is a recursive partial function.

Proof. (1) By the Normal Form Theorem 5H.9,

$$g(\boldsymbol{x},\beta,\boldsymbol{\alpha}) = U(\mu t R(\boldsymbol{x},\overline{\beta}(t),\overline{\boldsymbol{\alpha}}(t)))$$

with a recursive relation $R(\boldsymbol{x},v,\vec{u})$. Let also

$$\widetilde{h}(t,\boldsymbol{x},\boldsymbol{\alpha}) = \langle h(\boldsymbol{x},\boldsymbol{\alpha})(0),\dots,h(\boldsymbol{x},\boldsymbol{\alpha})(t-1)\rangle;$$

this is a recursive partial function, such that

$$\text{if } h(\boldsymbol{x},\boldsymbol{\alpha}) = \beta, \text{ then } \widetilde{h}(t,\boldsymbol{x},\boldsymbol{\alpha}) = \overline{\beta}(t).$$

Finally, put

$$\widetilde{f}(\boldsymbol{x},\boldsymbol{\alpha}) = U(\mu t R(\boldsymbol{x},\widetilde{h}(t,\boldsymbol{x},\boldsymbol{\alpha}),\boldsymbol{x},\boldsymbol{\alpha})),\overline{\boldsymbol{\alpha}}(t))).$$

Now this is a recursive partial function, and if $h(\boldsymbol{x}, \boldsymbol{\alpha}) = \beta$, then

$$\widetilde{f}(\boldsymbol{x}, \boldsymbol{\alpha}) = U(\mu t R(\boldsymbol{x}, \overline{\beta}(t), \boldsymbol{x}, \boldsymbol{\alpha})) = g(\boldsymbol{x}, h(\boldsymbol{x}, \boldsymbol{\alpha}), \boldsymbol{\alpha}),$$

as claimed.

(2) follows immediately from (1). ⊣

Corollary 5H.13. *The classes of semirecursive and recursive relations with arguments in Baire space are closed under substitution of total, recursive functions with values in $\mathcal{N}$.*

Among the most useful such substitutions are those which use the following codings of finite and infinite tuples of Baire points:

Definition 5H.14 (Sequence codings for Baire space)**.** We set

$$\langle \alpha_0, \dots, \alpha_{\nu-1} \rangle = \lambda(t) \begin{cases} \alpha_i(s), & \text{if } t = \langle i, s \rangle \text{ for some } i < \nu \text{ and some } s, \\ 0, & \text{otherwise,} \end{cases}$$

and similarly for an infinite sequence of Baire points,

$$\langle \alpha_0, \alpha_1, \dots \rangle = \lambda(t) \begin{cases} \alpha_i(s), & \text{if } t = \langle i, s \rangle \text{ for some } i \text{ and some } s, \\ 0, & \text{otherwise.} \end{cases}$$

We also set,

$$(\alpha)_i = \lambda(s)\alpha(\langle i, s \rangle).$$

Lemma 5H.15. (1) *The function*

$$(\alpha_0, \dots, \alpha_{\nu-1}) \mapsto \langle \alpha_0, \dots, \alpha_{\nu-1} \rangle,$$

is recursive and one-to-one on $\mathcal{N}^\nu \to \mathcal{N}$; and the function

$$(\alpha, i) \mapsto (\alpha)_i$$

is recursive and an inverse of the tuple functions, in the sense that

$$(\langle \alpha_0, \dots, \alpha_{n-1} \rangle)_i = \alpha_i \quad (i < n).$$

(2) *The function*

$$(\alpha_0, \alpha_1, \dots) \mapsto \langle \alpha_0, \alpha_1, \dots \rangle$$

is one-to-one on $\mathcal{N}^\infty \to \mathcal{N}$.

5H.16. The arithmetical hierarchy with arguments in Baire space. We can now define and establish the basic properties of the arithmetical hierarchy for relations with arguments in Baire space, exactly as we did for relations with arguments in $\mathbb{N}$ in Section 5E, starting with the Σ^0_1 relations on

Baire space and using recursive functions with arguments and values in Baire space. We comment briefly on the changes that must be made, which involve only the results needed to justify the theorems.

The basic definition of the classes Σ^0_k, Π^0_k and Δ^0_k is exactly that in 5E.1, starting with the definition 5H.2 of semirecursive relations on Baire space. The canonical forms of these classes are those in 5E.2, whose Table we repeat to emphasize the form of the dependence on the Baire arguments when these are present, i.e., with $\nu \geq 1$:

$$\begin{array}{lcr}
\Sigma^0_1 & : & (\exists y)Q(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(y)) \\
\Pi^0_1 & : & (\forall y)Q(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(y)) \\
\Sigma^0_2 & : & (\exists y_1)(\forall y_2)Q(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(y_2), y_1) \\
\Pi^0_2 & : & (\forall y_1)(\exists y_2)Q(\boldsymbol{x},, \overline{\boldsymbol{\alpha}}(y_2), y_1) \\
\Sigma^0_3 & : & (\exists y_1)(\forall y_2)(\exists y_3)Q(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(y_3), y_2, y_3) \\
 & \vdots &
\end{array}$$

The closure properties are again those in Theorem 5A.7, with the closure under substitutions of total recursive functions with values either in $\mathbb{N}$ or in $\mathcal{N}$ depending on Corollary 5H.13, and the quantifier contractions justified by the sequence codings in 5H.14 and 5H.15. Finally, the Enumeration and Hierarchy Theorems 5E.5 are proved as before, starting with Theorem 5H.6 now, and the proper inclusions diagram in that theorem still holds, with the "properness" witnessed by the same relations on $\mathbb{N}$.

5H.17. Baire codes of subsets of $\mathbb{N}$ and relations on $\mathbb{N}$. With each Baire point γ, we associate the set of natural numbers

$$A_\gamma = \{s \in \mathbb{N} \mid \gamma(s) = 1\}, \tag{152}$$

and for each $n \geq 2$, the n-ary relation

$$R^n_\gamma(x_1, \ldots, x_n) \iff \gamma(\langle x_1, \ldots, x_n \rangle) = 1; \tag{153}$$

we say that γ *is a code of* A if $A_\gamma = A$, and *a code of* $R \subseteq \mathbb{N}^n$ if $R = R^n_\gamma$. If $n = 2$, we often use "infix notation",

$$xR_\gamma y \iff R_\gamma(x, y) \iff \gamma(\langle x, y \rangle) = 1.$$

These trivial codings allow us to classify subsets of $\mathcal{N}$ and relations on $\mathcal{N}$ in the arithmetical hierarchy, as in the following, simple example.

Lemma 5H.18. *The set of codes of linear orderings*

$$LO = \{\gamma \mid \gamma \text{ is a code of a linear ordering (of a subset of } \mathbb{N})\} \tag{154}$$

is $\Pi^0_1 \setminus \Sigma^0_1$.

PROOF. To simplify notation, set

$$x \leq_\gamma y \iff \gamma(\langle x, y\rangle) = 1, \quad x \in D_\gamma \iff x \leq_\gamma x,$$

and compute:

$$\begin{aligned}\gamma \in LO \iff\; & \leq_\gamma \text{ is a linear ordering of } D_\gamma \\ \iff\; & (\forall x, y)[x \leq_\gamma y \Longrightarrow [x \in D_\gamma \;\&\; y \in D_\gamma]] \\ & \;\&\; (\forall x, y)[[x \leq_\gamma y \;\&\; y \leq_\gamma x] \Longrightarrow x = y] \\ & \;\&\; (\forall x, y, z)[[x \leq_\gamma y \;\&\; y \leq_\gamma z] \Longrightarrow x \leq_\gamma z] \\ & \;\&\; (\forall x, y)[[x \in D_\gamma \;\&\; y \in D_\gamma] \Longrightarrow x \leq_\gamma y \vee y \leq_\gamma x].\end{aligned}$$

For the converse, suppose that

$$P(x) \iff (\forall u)R(x, u)$$

with $R(x, u)$ recursive, and let

$$f^*(x, t) = \begin{cases} 1, & \text{if } R(x, \min((t)_0, (t)_1)) \;\&\; (t)_0 \leq (t)_1, \\ 0, & \text{if } R(x, \min((t)_0, (t)_1)) \;\&\; (t)_0 > (t)_1, \\ 1, & \text{if } \neg R(x, \min((t)_0, (t)_1)). \end{cases}$$

The function $f^*(x, t)$ is recursive and total, and hence so is the function

$$f(x) = \lambda(t)f^*(x, t);$$

moreover, if $(\forall t)R(x, t)$, then

$$f(x)(\langle u, v\rangle) = f^*(x, \langle u, v\rangle) = \begin{cases} 1, & \text{if } u \leq v, \\ 0, & \text{otherwise,} \end{cases}$$

so that $f(x) \in LO$, in fact $f(x)$ is a code of the natural ordering on $\mathbb{N}$. On the other hand, if, for some t, $\neg R(x, t)$, then

$$f(x)(\langle t, t+1\rangle) = f(x)(\langle t+1, t\rangle) = 1, \qquad \text{i.e., } t \leq_{f(x)} (t+1) \;\&\; (t+1) \leq_{f(x)} t,$$

so that $f(x) \notin LO$. This establishes the reduction.

$$P(x) \iff (\forall u)R(x, u) \iff f(x) \in LO.$$

It follows that if LO were Σ^0_1, then every Π^0_1 relation on $\mathbb{N}$ would be in Σ^0_1, which it is not, and hence LO is not Σ^0_1. ⊣

Problems for Section 5H

Problem 5H.1. Prove that the class of recursive partial functions on Baire space to $\mathbb{N}$ is closed under minimalization.

Problem 5H.2. Prove that a relation $P(\boldsymbol{x}, \boldsymbol{\alpha})$ is Σ^0_1 if and only if it is the domain of some recursive $f : \mathbb{N}^n \times \mathcal{N}^\nu \rightharpoonup \mathbb{N}$,

$$P(\boldsymbol{x}, \boldsymbol{\alpha}) \iff f(\boldsymbol{x}, \boldsymbol{\alpha})\downarrow .$$

Problem 5H.3. (Σ^0_1-Selection). Prove that for every semirecursive relation $P(\boldsymbol{x}, y, \boldsymbol{\alpha})$, there is a recursive partial function $f(\boldsymbol{x}, \boldsymbol{\alpha})$ such that:

(1) $f(\boldsymbol{x}, \boldsymbol{\alpha})\downarrow \iff (\exists y)P(\boldsymbol{x}, y, \boldsymbol{\alpha})$.

(2) If $(\exists y)P(\boldsymbol{x}, y, \boldsymbol{\alpha})$, then $P(\boldsymbol{x}, f(\boldsymbol{x}, \boldsymbol{\alpha}), \boldsymbol{\alpha})$.

Problem 5H.4. Prove that a relation $P(\boldsymbol{x}, \boldsymbol{\alpha})$ is recursive if and only if its characteristic (total) function $\chi_R(\boldsymbol{x}, \boldsymbol{\alpha})$ is recursive.

Problem 5H.5. Prove that the set

$$A = \{\alpha \in \mathcal{N} \mid \text{ for infinitely many } x, \alpha(x) = 0\}$$

is in $\Pi^0_2 \setminus \Sigma^0_2$.

5I. The analytical hierarchy

Once we have relations with arguments in Baire space, we can apply quantification over $\mathcal{N}$ on them to define more complex (and more interesting) relations. The resulting *analytical hierarchy* resembles in structure the arithmetical structure, which it extends, but it contains many of the fundamental relations of analysis and set theory.

Definition 5I.1. The *Kleene classes* of relations Σ^1_k, Π^1_k, Δ^1_k with arguments in $\mathbb{N}$ and $\mathcal{N}$ are defined recursively, for $k \geq 0$, as follows:

$$\begin{aligned}
\Pi^1_0 = \Pi^0_1 &: \text{the negations of semirecursive relations}\\
\Sigma^1_{k+1} = \exists^{\mathcal{N}}\Pi^1_k &: \text{the relations which satisfy an equivalence of the form}\\
&\qquad P(\boldsymbol{x}, \boldsymbol{\alpha}) \iff (\exists\beta)Q(\boldsymbol{x}, \boldsymbol{\alpha}, \beta),\\
&\qquad\qquad \text{where } Q(\boldsymbol{x}, \boldsymbol{\alpha}, \beta) \text{ is } \Pi^1_k\\
\Pi^1_k = \neg\Sigma^1_k &: \text{the negations (complements) of relations in } \Sigma^1_k\\
\Delta^1_k = \Sigma^1_k \cap \Pi^1_k &: \text{ the relations which are both } \Sigma^1_k \text{ and } \Pi^1_k.
\end{aligned}$$

A set of numbers $A \subseteq \mathbb{N}$ or of Baire points $A \subseteq \mathcal{N}$ is in one of these classes Γ if the relation $x \in A$ or $\alpha \in A$ is in Γ.

5I.2. Canonical forms. Using the canonical form for Σ^0_1 relations in (2) of Lemma 5H.4, we obtain immediately the following canonical forms for the

Kleene classes, (with $k \geq 1$), where $R(\boldsymbol{x}, \vec{u}, v)$ is recursive on $\mathbb{N}$ and monotone in $\vec{u}, v$:

$$
\begin{array}{lcr}
\Pi^1_1 & : & (\forall\beta)(\exists t)R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t), \overline{\beta}(t)) \\
\Sigma^1_1 & : & (\exists\beta)(\forall t)R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t), \overline{\beta}(t)) \\
\Pi^1_2 & : & (\forall\beta_1)(\exists\beta_2)(\forall t)R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t), \overline{\beta}_1(t), \overline{\beta}_2(t)) \\
\Sigma^1_2 & : & (\exists\beta_1)(\forall\beta_2)(\exists t)R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t), \overline{\beta}_1(t), \overline{\beta}_2(t)) \\
\Pi^1_3 & : & (\forall\beta_1)(\exists\beta_2)(\forall\beta_3)(\exists t)R(\boldsymbol{x}, \overline{\boldsymbol{\alpha}}(t), \overline{\beta}_1(t), \overline{\beta}_2(t), \overline{\beta}_3(t)) \\
 & \vdots &
\end{array}
$$

It is worth singling out the form of Π^1_1 subsets of $\mathbb{N}$ and $\mathcal{N}$ which include some of the most interesting examples:

$$
(\Pi^1_1) \quad x \in A \iff (\forall\beta)(\exists t)R(x, \overline{\beta}(t)),
$$
$$
\alpha \in A \iff (\forall\beta)(\exists t)R(\overline{\alpha}(t), \overline{\beta}(t)),
$$

Theorem 5I.3. (1) *For each $k \geq 1$, the classes Σ^1_k, Π^1_k, and Δ^1_k are closed for (total) recursive substitutions with values in $\mathbb{N}$ or $\mathcal{N}$, and for the operations $\&$, $\vee$, $\exists_\leq$, $\forall_\leq$, $\exists^{\mathbb{N}}$ and $\forall^{\mathbb{N}}$. In addition:*

- *Each Δ^1_k is closed for negation $\neg$.*
- *Each Σ^1_k is closed for $\exists^{\mathcal{N}}$, existential quantification over $\mathcal{N}$.*
- *Each Π^1_k is closed for $\forall^{\mathcal{N}}$, universal quantification over $\mathcal{N}$.*

(2) *Every arithmetical relation is Δ^1_1.*

(3) *For each $k \geq 1$,*

$$
\Sigma^1_k \subsetneq \Delta^1_{k+1}, \tag{155}
$$

and hence the Kleene classes satisfy the following diagram of proper inclusions:

$$
\begin{array}{ccccccccccccc}
 & & & \Sigma^1_1 & & & & \Sigma^1_2 & & & & \Sigma^1_3 & & \\
 & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
\bigcup_k \Sigma^0_k \subseteq \Delta^1_1 & & & & & \Delta^1_2 & & & & \Delta^1_3 & & & & \cdots \\
 & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
 & & & \Pi^1_1 & & & & \Pi^1_2 & & & & \Pi^1_3 & &
\end{array}
$$

PROOF. (1) The closure of all Kleene classes under total, recursive substitutions follows from the canonical forms and the closure of Σ^0_1 and Π^0_1 under total recursive substitutions, Corollary 5H.13. The remaining closure properties are proved by induction on $k \geq 1$, using the recursiveness of the projection functions and the function

$$\gamma \mapsto \gamma' = \lambda(t)\gamma(t);$$

the known closure properties of Π^0_1; and, to contract quantifiers, the following equivalences and their duals, where we abbreviate $\boldsymbol{z} = \boldsymbol{x}, \boldsymbol{\alpha}$.

$$\begin{array}{lrcl}
\text{(E1)} & (\exists\beta)P(\boldsymbol{z},\beta) \vee (\exists\beta)Q(\boldsymbol{z},\beta) & \iff & (\exists\beta)[P(\boldsymbol{z},\beta) \vee Q(\boldsymbol{z},\beta)] \\
\text{(E2)} & (\exists\beta)P(\boldsymbol{z},\beta) \ \&\ (\exists\beta)Q(\boldsymbol{z},\beta) & \iff & (\exists\gamma)[P(\boldsymbol{z},(\gamma)_0) \ \&\ Q(\boldsymbol{z},(\gamma)_1)] \\
\text{(E3)} & (\exists s \le t)(\exists\beta)P(\boldsymbol{z},s,\beta) & \iff & (\exists\beta)(\exists s \le t)P(\boldsymbol{z},s,\beta) \\
\text{(E4)} & (\forall s \le t)(\exists\beta)P(\boldsymbol{z},s,\beta) & \iff & (\exists\gamma)(\forall s \le t)P(\boldsymbol{z},s,(\gamma)_s) \\
\text{(E5)} & (\exists s)(\exists\beta)P(\boldsymbol{z},s,\beta) & \iff & (\exists\beta)(\exists s)P(\boldsymbol{z},s,\beta) \\
\text{(E6)} & (\forall s)(\exists\beta)P(\boldsymbol{z},s,\beta) & \iff & (\exists\gamma)(\forall s)P(\boldsymbol{z},s,(\gamma)_s) \\
\text{(E7)} & (\exists\delta)(\exists\beta)P(\boldsymbol{z},\delta,\beta) & \iff & (\exists\gamma)P(\boldsymbol{z},(\gamma)_0,(\gamma)_1)
\end{array}$$

These are all either trivial, or direct expressions of the countable Axiom of Choice.

In some more detail:

(1a) The closure properties of Σ^1_1 follow from those of Π^0_1 and (E1)-(E7). To show closure under $\forall^{\mathbb{N}}$, for example, suppose

$$P(\boldsymbol{z},t) \iff (\exists\beta)Q(\boldsymbol{z},\beta,t) \quad \text{with } Q \in \Pi^0_1,$$

and compute:

$$\begin{aligned}
(\forall t)P(\boldsymbol{z},t) &\iff (\forall t)(\exists\beta)Q(\boldsymbol{z},\beta,t) \\
&\iff (\exists\gamma)(\forall t)Q(\boldsymbol{z},(\gamma)_t,t) \text{by (E6)}.
\end{aligned}$$

This is enough, because $Q(\boldsymbol{z},(\gamma)_t,t)$ is Π^0_1 by the closure of this class under recursive substitutions.

(1b) The closure properties of Π^1_k follow from those of Σ^1_k taking negations and pushing the negation operator through the quantifier prefix.

(1c) The closure properties of Σ^1_{k+1} follow from those of Π^1_k using (E1)-(E7).

(2) follows from (1), since $\Delta^0_1 \subsetneq \Delta^1_1$ and Δ^1_1 is closed under both number quantifiers. (And we will see in 5I.5 that the arithmetical relations are contained properly in Δ^1_1.)

(3) We notice first that the non-strict version of the diagram (with $\subseteq$ in place of $\subsetneq$) is trivial, using dummy quantification, and closure under recursive substitutions, e.g.,

$$(\exists\beta)P(\boldsymbol{z},\beta) \iff (\exists\beta)(\forall\alpha)P(\boldsymbol{z},\beta) \iff (\forall\alpha)(\exists\beta)P(\boldsymbol{z},\beta).$$

To show that the inclusions are strict, we need to define enumerating (universal) sets

$$\widetilde{S}^1_{k,n,\nu} \text{ for } \Sigma^1_k \text{ and } \widetilde{P}^1_{k,n,\nu} \text{ for } \Pi^1_k.$$

We start with the fact that the relation

$$\widetilde{P}^0_{n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha}) \iff (\forall t)\neg T^r_{n,\nu}(e,\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t)) \tag{156}$$

enumerates all Π^0_1 relations with arguments $\boldsymbol{x},\boldsymbol{\alpha}$, by (146) in Theorem 5H.6, and set recursively:

$$\begin{aligned}
\widetilde{S}^1_{1,n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha}) &\iff (\exists\beta)(\forall t)\neg T^r_{n,\nu+1}(e,\boldsymbol{x},\overline{\boldsymbol{\alpha}}(t),\overline{\beta}(t)),\\
\widetilde{P}^1_{k,n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha}) &\iff \neg\widetilde{S}^1_{k,n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha})\\
\widetilde{S}^1_{k+1,n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha}) &\iff (\exists\beta)\widetilde{P}^1_{k,n,\nu+1}(e,\boldsymbol{x},\boldsymbol{\alpha},\beta).
\end{aligned}$$

It is easy to verify (chasing the definitions) that a relation $R(\boldsymbol{x},\boldsymbol{\alpha})$ is Σ^1_k if and only if there is some e such that

$$R(\boldsymbol{x},\boldsymbol{\alpha}) \iff \widetilde{S}^1_{k,n,\nu}(e,\boldsymbol{x},\boldsymbol{\alpha}),$$

and then, as usual, the diagonal relation

$$D(x) \iff \widetilde{S}^1_{k,1,0}(x,x)$$

is in $\Sigma^1_k \setminus \Pi^1_k$. ⊣

We now place in the analytical hierarchy some important relations on Baire space, starting with the basic *satisfaction relation* on (codes of) structures.

5I.4. Codes of structures. Consider (for simplicity) $\mathbb{FOL}(\tau)$, where the signature τ has K relation symbols $P_1,\dots,P_K$ where P_i is n_i-ary. We code τ by its *characteristic*

$$u = \langle n_1,\dots,n_K\rangle.$$

With each $\alpha \in \mathcal{N}$ and each u, we associate the tuple

$$\mathbf{A}(u,\alpha) = (A_\alpha, R_{1,\alpha},\dots,R_{K,\alpha}),$$

where

$$\begin{gathered}
A_\alpha = \{n \in \mathbb{N} \mid (\alpha)_0(n) = 1\},\\
R_{i,\alpha}(x_1,\dots,x_{n_i}) \iff x_1,\dots,x_{n_i} \in A \;\&\; (\alpha)_i(\langle x_1,\dots,x_{n_i}\rangle) = 1.
\end{gathered}$$

This is a structure when $A_\alpha \neq \emptyset$, and so the semirecursive relation

$$(u,\alpha) \iff A_\alpha \neq \emptyset \iff (\exists n)[(\alpha)_0(n) = 1]$$

determines which Baire points code structures. Let

$$\begin{aligned}\text{Assgn}(u,x,\alpha) &\iff x \text{ codes an (ultimately 0) assignment into } A_\alpha\\ &\iff (u,\alpha)\ \&\ (\forall i < \text{lh}(x))[(x)_i \in A_\alpha];\end{aligned}$$

this, too, is a semirecursive relation. Finally, using the codings of Section 4A, we set

$$\begin{aligned}\text{(157)}\qquad \text{Sat}(u,\alpha,m,x) \iff &(u,\alpha)\\ &\ \&\ m \text{ codes a formula } \phi \text{ of } \mathbb{FOL}(\tau)\\ &\ \&\ \mathbf{A}(u,\alpha),(x)_0,(x)_1,\dots \models \phi\end{aligned}$$

This relation codes the basic satisfaction relation between structures (on subsets of $\mathbb{N}$), assignments and formulas.

Theorem 5I.5. (1) *The relation* Sat(u,α,m,x) *is* Δ^1_1.

(2) *The set* Truth *of codes of true arithmetical sentences is* Δ^1_1*, and so the arithmetical relations are properly contained in* Δ^1_1.

Proof. (1) Let

$$\begin{aligned}&P(u,\alpha,\beta)\\ &\qquad\iff (\forall m,x)[\beta(\langle m,x\rangle) \le 1\ \&\ \beta(\langle m,x\rangle)=1 \iff \text{Sat}(u,\alpha,m,x)];\end{aligned}$$

the equivalence determines β completely, once u and α are fixed, and so

$$\begin{aligned}\text{Sat}(u,\alpha,m,x) &\iff (\exists\beta)[P(u,\alpha,\beta)\ \&\ \beta(\langle m,x\rangle)=1]\\ &\iff (\forall\beta)[P(u,\alpha,\beta) \Longrightarrow \beta(\langle m,x\rangle)=1].\end{aligned}$$

Thus the proof will be complete once we show that $P(u,\alpha,\beta)$ is arithmetical, which is not too hard to do, since for it to hold, β must satisfy the "Tarski conditions" for satisfaction: $\beta(\langle m,x\rangle)$ must give the correct value when m codes a prime formula, and for complex formulas, the correct value of $\beta(\langle m,x\rangle)$ can be computed in terms of $\beta(\langle s,y\rangle)$ for codes s of shorter formulas.

(2) follows by first translating the formulas of number theory to $\mathbb{FOL}(\tau)$ with a signature τ which has only relation symbols (for the graphs of $0, S, +$ and $\cdot$), and then reformulating questions of truth to quries about satisfaction—which for sentences are one and the same thing.) ⊣

Next we turn to the classification of relation on countable ordinal numbers via their Baire codes. We set

$$\text{(158)}\qquad WO = \{\alpha \in LO \mid \le_\alpha \text{ is a well ordering}\},$$

where the set LO of codes of linear orderings is defined in (154), and for each $\alpha \in WO$, we set

$$|\alpha| = \text{the ordinal similar with } \leq_\alpha \quad (\alpha \in WO). \tag{159}$$

Each $|\alpha|$ is a countable ordinal, and each countable ordinal is $|\alpha|$ with some (in fact many) $\alpha \in WO$.

Theorem 5I.6. *The set WO of ordinal codes is Π^1_1. Moreover, there are relations $\leq_\Pi$, $\leq_\Sigma$ in Π^1_1 and Σ^1_1 respectively, such that*

if $\beta \in WO$, then for all α,

$$\alpha \leq_\Pi \beta \iff \alpha \leq_\Sigma \beta \iff [\alpha \in WO \;\&\; |\alpha| \leq |\beta|].$$

Proof. To see that WO is Π^1_1, we compute:

$$\begin{aligned}\alpha \in WO \iff &\alpha \in LO \\ &\& \; (\forall\beta)\Big[(\forall t)[\beta(t+1) \leq_\alpha \beta(t)] \Longrightarrow (\exists t)[\beta(t+1) = \beta(t)]\Big].\end{aligned}$$

To prove the second assertion, take first

$$\begin{aligned}\alpha \leq_\Sigma \beta \iff &\alpha \in LO \;\&\; (\exists\gamma)[\gamma \text{ maps } \leq_\alpha \text{ into } \leq_\beta \text{ in a one-to-one} \\ &\qquad\qquad \text{order-preserving manner}] \\ \iff &\alpha \in LO \;\&\; (\exists\gamma)(\forall n)(\forall m)[n <_\alpha m \Longrightarrow \gamma(n) <_\beta \gamma(m)].\end{aligned}$$

It is immediate that $\leq_\Sigma$ is Σ^1_1 and for $\beta \in WO$,

$$\alpha \leq_\Sigma \beta \iff [\alpha \in WO \;\&\; |\alpha| \leq |\beta|].$$

For the relation $\leq_\Pi$, take

$$\begin{aligned}\alpha \leq_\Pi \beta \iff &\alpha \in WO \;\&\; \text{there is no order-preserving map of } \leq_\beta \\ &\qquad\qquad \text{onto a proper initial segment of } \leq_\alpha \\ \iff &\alpha \in WO \\ &\& \; (\forall\gamma)\neg(\exists k)(\forall n)(\forall m)\big[n \leq_\beta m \iff [\gamma(n) \leq_\alpha \gamma(m) <_\alpha k]\big],\end{aligned}$$

where of course we abbreviate

$$s <_\alpha t \iff s \leq_\alpha t \;\&\; s \neq t. \qquad \dashv$$

That WO is not Σ^1_1 is a consequence of the following, basic result of definability theory, established (in various forms) by Lusin-Sierpinski and Kleene.

Theorem 5I.7 (The Basic Representation Theorem for Π^1_1). *If $P(\boldsymbol{x}, \boldsymbol{\alpha})$ is Π^1_1, then there exists a total recursive function $f : \mathbb{N}^n \times \mathcal{N}^\nu \to \mathcal{N}$, that for all $\boldsymbol{x}, \boldsymbol{\alpha}$, $f(\boldsymbol{x}, \boldsymbol{\alpha}) \in LO$, and*

$$P(\boldsymbol{x}, \boldsymbol{\alpha}) \iff f(\boldsymbol{x}, \boldsymbol{\alpha}) \in WO. \tag{160}$$

Proof. We set $\boldsymbol{z} = \boldsymbol{x}, \boldsymbol{\beta}$ to save some typing, and by 5I.2 we choose a recursive and monotone in u relation $R(\boldsymbol{z}, u)$ such that

$$P(\boldsymbol{z}) \iff (\forall \beta)(\exists t) R\big(\boldsymbol{z}, \overline{\beta}(t)\big).$$

For each $\boldsymbol{z}$, put

$$T(\boldsymbol{z}) = \{(u_0, \ldots, u_{t-1}) \mid \neg R(\boldsymbol{z}, \langle u_0, \ldots, u_{t-1} \rangle)\}$$

so that $T(\boldsymbol{z})$ is a tree on $\mathbb{N}$ (by the monotonicity or R) and clearly

$$P(\boldsymbol{z}) \iff T(\boldsymbol{z}) \text{ is wellfounded.}$$

What we must do is replace $T(\boldsymbol{z})$ by a linear ordering on a subset of $\mathbb{N}$ which will be wellfounded precisely when $T(\boldsymbol{z})$ is. Put

$$\begin{aligned}
&(v_0, \ldots, v_{s-1}) >^{\boldsymbol{z}} (u_0, \ldots, u_{t-1}) \\
&\qquad \iff (v_0, \ldots, v_{s-1}), (u_0, \ldots, u_{t-1}) \in T(\boldsymbol{z}) \\
&\qquad\qquad \& \Big[v_0 > u_0 \lor [v_0 = u_0 \;\&\; v_1 > u_1] \\
&\qquad\qquad\qquad \lor [v_0 = u_0 \;\&\; v_1 = u_1 \;\&\; v_2 > u_2] \\
&\qquad\qquad\qquad \lor \cdots \\
&\qquad\qquad\qquad \lor [v_0 = u_0 \;\&\; v_1 = u_1 \;\&\; \cdots \;\&\; v_{s-1} = u_{s-1} \;\&\; s < t] \Big]
\end{aligned}$$

where $>$ on the right is the usual "greater than" in $\mathbb{N}$.

It is immediate that if $(v_0, \ldots, v_{s-1})$, $(u_0, \ldots, u_{t-1})$ are both in $T(\boldsymbol{z})$ and $(v_0, \ldots, v_{s-1})$ is an initial segment of $(u_0, \ldots, u_{t-1})$, then $(v_0, \ldots, v_{s-1}) >^{\boldsymbol{z}} (u_0, \ldots, u_{t-1})$; thus if $T(\boldsymbol{z})$ has an infinite branch, then $>^{\boldsymbol{z}}$ has an infinite descending chain.

Assume now that $>^{\boldsymbol{z}}$ has an infinite descending chain, say

$$v^0 >^{\boldsymbol{z}} v^1 >^{\boldsymbol{z}} v^2 >^{\boldsymbol{z}} \cdots,$$

where

$$v^i = (v^i_0, v^i_1, \ldots, v^i_{s_i - 1}),$$

and consider the following array:

$$\begin{aligned} v^0 &= (v^0_0, v^0_1, \dots, v^0_{s_0-1}) \\ v^1 &= (v^1_0, v^1_1, \dots, v^1_{s_1-1}) \\ \dots & \quad \dots \\ v^i &= (v^i_0, v^i_1, \dots, v^i_{s_i-1}) \\ \dots & \quad \dots \end{aligned}$$

The definition of $>^{\boldsymbol{z}}$ implies immediately that

$$v^0_0 \geq v^1_0 \geq v^2_0 \geq \cdots,$$

i.e., the first column is a non-increasing sequence of natural numbers. Hence after a while they all are the same, say

$$v^i_0 = k_0 \quad \text{for } i \geq i_0.$$

Now the second column is non-increasing below level i_0, so that for some pair i_1, k_1,

$$v^i_1 = k_1 \quad \text{for } i \geq i_1.$$

Proceeding in the same way we find an infinite sequence

$$k_0, k_1, \dots$$

such that for each s, $(k_0, \dots, k_{s-1}) \in T(\boldsymbol{z})$, so $T(\boldsymbol{z})$ is not wellfounded. Thus we have shown,

$$\begin{aligned} P(\boldsymbol{z}) &\iff T(\boldsymbol{z}) \text{ is wellfounded} \\ &\iff >^{\boldsymbol{z}} \text{ has no infinite descending chains.} \end{aligned}$$

Finally put

$$\begin{aligned} u \leq^{\boldsymbol{z}} v \iff (\exists t \leq u)(\exists s \leq v)\Big[&\mathrm{Seq}(u) \ \&\ \mathrm{lh}(u) = t \ \&\ \mathrm{Seq}(v) \ \&\ \mathrm{lh}(v) = s \\ &\&\ \big[u = v \vee \big((v)_0, \dots, (v)_{s-1}\big) >^{\boldsymbol{z}} \big((u)_0, \dots, (u)_{t-1}\big)\big]\Big] \end{aligned}$$

and notice that $\leq^{\boldsymbol{z}}$ is always a linear ordering, and

$$P(\boldsymbol{z}) \iff \leq^{\boldsymbol{z}} \text{ is a wellordering.}$$

Moreover, the relation

$$P(\boldsymbol{z}, u, v) \iff u \leq^{\boldsymbol{z}} v$$

is easily recursive. The proof is completed by taking

$$f(\boldsymbol{z})(n) = \begin{cases} 1, & \text{if } (n)_0 \leq^{\boldsymbol{z}} (n)_1, \\ 0, & \text{otherwise.} \end{cases}$$

⊣

Corollary 5I.8. *The set WO of ordinal codes is not* Σ_1^1.