APPENDIX 1

APPENDIX: SOME BASIC FACTS

This is a dynamic Appendix: it will be updated periodically, as we need more basic facts from mathematics which are known to most of the students in the class and are not properly part of logic.

Notations. The (cartesian) product of two sets A, B is the set of all ordered pairs from A and B,

$$A \times B = \{(x, y) \mid x \in A \ \&\ y \in B\};$$

for products of more than two factors, similarly,

$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) \mid x_1 \in A_1, \ldots, x_n \in A_n\}.$$

We write W^n for $A_1 \times \cdots \times A_n$ with $A_1 = A_2 = \cdots = A_n = W$, and W^* for the set of all finite sequences (words) from W.

We write $f : A \to W$ to indicate that f is a function on A to W, i.e.,

$$f \subseteq A \times W \ \&\ (\forall x \in A)(\exists! w \in W)[(x, w) \in f].$$

We also write $f : A \rightarrowtail W$ to indicate that f is an *injection* (one-to-one); $f : A \twoheadrightarrow W$ to indicate that f is a *surjection* (onto W); and finally, we write $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow W$ to indicate that f is a *bijection*, i.e., a one-to-one correspondence of A with W. If $f : A \to W$, $X \subseteq A$ and $Y \subseteq W$, we let

$$\begin{aligned} f[X] &= \{f(x) \mid x \in X\} \qquad \text{(the image of } X \text{ by } f) \\ f^{-1}[Y] &= \{x \in A \mid f(x) \in Y\}\text{(the inverse image of } Y \text{ by } f. \end{aligned}$$

Problem s1 (Definition by recursion)**.** For any two sets W, Y and any two functions $g : Y \to W$, $h : W \times Y \times \mathbb{N} \to W$, there is exactly one function $f : \mathbb{N} \times Y \to W$ which satisfies the following two equations, for all $n \in \mathbb{N}$ and $y \in Y$:

$$\begin{aligned} f(0, y) &= g(y), \\ f(n+1, y) &= h(f(n, y), y, n) \end{aligned} \tag{162}$$

HINT: To prove that such a function exists, define the relation

$$\begin{aligned} P(n,y,w) \iff &\text{(there exists a sequence } w_0 w_1 \cdots w_n \in W^*) \\ &\qquad \text{such that } \Big[w_0 = g(y) \\ &\qquad\qquad \& \text{ (for all } i < n)[w_{i+1} = h(w_i, y, i)]] \\ &\qquad\qquad\qquad \& \ w_n = w \Big], \end{aligned}$$

and prove by induction on n that for all $y \in Y$, there is exactly one $w \in W$ such that $P(n, y, w)$. We can then set

$$f(n, y) = \text{the unique } w \text{ such that } P(n, y, w).$$

To prove uniqueness, we assume that $f_1, f_2 : \mathbb{N} \times Y \to W$ both satisfy (162) and we show by induction that for all n, for all y, $f_1(n, y) = f_2(n, y)$.

Problem s2 (Definition by complete recursion)**.** For any set W, any point $w_0 \in W$ and any function $h : W^* \times \mathbb{N} \to W$, there is exactly one function $f : \mathbb{N} \to W$ such that for all n,

$$f(0) = w_0, \quad f(n+1) = h(f(0)f(1)\cdots f(n), n).$$

Suppose $F : U^m \to U$ is an m-ary function on a set U and $X \subseteq U$; we say that X *is closed under* F if

$$x_1, \ldots, x_m \in X \Longrightarrow F(x_1, \ldots, x_m) \in X.$$

Problem s3 (Functional closure)**.** For any set U, any collection of functions $\mathcal{F}$ on U, of any arity, and any $A \subseteq U$, let

$$\begin{aligned} A^{(0)} = A, \quad A^{(n+1)} = A^{(n)} \cup \{F(w_1, \ldots, w_m) \mid w_1, \ldots, w_m \in A^{(n)}, \\ F \in \mathcal{F}, \mathrm{arity}(F) = m\}, \end{aligned}$$

$$\overline{A}^{\mathcal{F}} = \bigcup_{n=0}^{\infty} A^{(n)}.$$

Prove that $\overline{A}^{\mathcal{F}}$ is the least subset of U which contains A and is closed under all the functions in $\mathcal{F}$, i.e.,

(1) $A \subseteq \overline{A}^{\mathcal{F}}$;
(2) $\overline{A}^{\mathcal{F}}$ is closed under every $F \in \mathcal{F}$;
(3) if $X \subseteq U$, $A \subseteq X$ and X is closed under every $F \in \mathcal{F}$, then $\overline{A}^{\mathcal{F}} \subseteq X$.

Note. We call $A^{\mathcal{F}}$ the set *generated by* A *and* $\mathcal{F}$. For a standard example, take U to be the set of all strings of symbols of $\mathbb{FOL}(\tau)$ for some signature τ; let A be the set of all the variables and the constants (viewed as strings of length 1); for any m-ary function symbol f in τ let

$$F_f(\alpha_1, \dots, \alpha_m) \equiv f(\alpha_1, \dots, \alpha_m);$$

and take $\mathcal{F}$ to be the collection of all F_f, one for each function symbol f of τ. The set $A^{\mathcal{F}}$ is then the set of terms of $\mathbb{FOL}(\tau)$.

Problem s4 (Structural recursion)**.** Let $A, U, \mathcal{F}$ be as in Problem s3 and assume in addition:

1. Each $F : U^m \rightarrowtail U$ is one-to-one and never takes on a value in A, i.e., $F[U^m] \cap A = \emptyset$.
2. The functions in $\mathcal{F}$ have disjoint images, i.e., if $F_1, F_2 \in \mathcal{F}$, arity$(F_1) = m$, arity$(F_2) = n$ and $F_1 \neq F_2$, then for all $u_1, \dots, u_m, v_1 \dots, v_n \in U$,

$$F_1(u_1, \dots, u_m) \neq F_2(v_1, \dots, v_n).$$

Suppose W is any set, $G : W \to W$, and for each m-ary $F \in \mathcal{F}$, $H_F : W^m \to W$ is an m-ary function on W. Prove that there is a unique function

$$\phi : A^{\mathcal{F}} \to W$$

such that

$$\text{if } x \in A, \text{ then } \phi(x) = G(x),$$

and

$$\text{if } x_1, \dots, x_m \in A^{\mathcal{F}} \text{ and } F \text{ is } m\text{-ary in } \mathcal{F},$$
$$\text{then } \phi(F(x_1, \dots, x_m)) = H_F(\phi(x_1), \dots, \phi(x_m)).$$

Problem s5. Let U be the set of symbols of $\mathbb{FOL}(\tau)$, and specify $A \subseteq U$ and $\mathcal{F}$ so that the conditions in Problem s4 are satisfied and $A^{\mathcal{F}}$ is the set of formulas of $\mathbb{FOL}(\tau)$. Indicate how the definition of FO(χ) in Definition 1B.6 is justified by Problem s4.

A set A is **countable** if either A is empty, or A is the image of some function $f : \mathbb{N} \twoheadrightarrow A$, i.e.,

$$A = \{a_0, a_1, \dots\} \quad \text{with } a_i = f(i).$$

Problem s6 (Cantor)**.** If $A_0, A_1, A_2, \dots$ is a sequence of countable sets, then the union

$$\bigcup_{i=0}^{\infty} A_i = A_0 \cup A_1 \cup \cdots$$

is also countable. It follows that:

1. The union $A \cup B$ and the product $A \times B$ of two countable sets are countable.
2. If A is countable, then so is each finite power A^n.
3. If A is countable, then so is the set of all words A^*.

HINT: Assume (without loss of generality) that no A_i is empty; suppose for each $i \in \mathbb{N}$, $f_i : \mathbb{N} \twoheadrightarrow A_i$ enumerates A_i; choose some fixed $a_0 \in A_0$; and define $f : \mathbb{N} \to \bigcup_{i=0}^{\infty} A_i$ by

$$f(n) = \begin{cases} f_i(j), & \text{if } n = 2^i 3^j \text{, for some (necessarily) unique } i, j, \\ a_0, & \text{otherwise;} \end{cases}$$

now prove that f is onto $\bigcup_{i=0}^{\infty} A_i$.

The Corollary for the product $A \times B$ follows by noticing that

$$A \times B = \bigcup_{i=0}^{\infty} \{(a_i, x) \mid x \in B\},$$

with $A = \{a_0, a_1, \dots\}$.

Problem 1.5* (Cantor)**.** Prove that if $\mathbf{A} = (A, \leq_A)$ and $\mathbf{B} = (B, \leq_B)$ are both countable, dense in themselves linear orderings with no first or last element, then $\mathbf{A}$ and $\mathbf{B}$ are isomorphic. HINT: Let

$$A = \{a_0, a_1, \dots\}, \quad B = \{b_0, b_1, \dots\},$$

(with no repetitions) and construct by recursion a sequence of bijective mappings $\rho_n : A_n \rightarrowtail\!\!\!\!\!\twoheadrightarrow B_n$ such that:

(1) A_n, B_n are finite sets, $A_n \subseteq A, B_n \subseteq B$.
(2) $a_0, \dots, a_n \in A_n, b_0, \dots, b_n \in B_n$.
(3) $\rho_0 \subseteq \rho_1 \subseteq \cdots$.
(4) If $a, a' \in A_n$, then $a \leq_A a' \iff \rho_n(a) \leq_B \rho(a')$.

The required isomorphism is $\rho = \bigcup_n \rho_n$.

Problem s8. A binary relation $\sim$ on a set C is an equivalence relation if and only if there exists a surjection

$$\rho : C \twoheadrightarrow \overline{C} \tag{163}$$

of C onto a set $\overline{C}$, such that

$$x \sim y \iff \rho(x) = \rho(y) \qquad (x, y \in C). \tag{164}$$

When (163) and (164) hold we call $\overline{C}$ a *quotient* of C by $\sim$ and ρ a *determining homomorphism* of $\sim$.

HINT: For the non-trivial direction, define the *equivalence class* of each $x \in C$ by

$$\overline{x} = \{y \in C \mid y \sim x\} \subseteq \text{Powerset}(C),$$

let $\overline{C} = \{\overline{x} \mid x \in C\}$ and let $\rho(x) = \overline{x}$.

A **wellordering** or **well ordered set** is a linear ordering $(A, \leq)$ in which every non-empty subset X of A has a least element.

Problem s9. A linear ordering $(A, \leq)$ is a wellordering if and only if there is no infinite descending chain $x_0 > x_1 > \cdots$. HINT: This requires a mild form of the Axiom of Choice, the so-called *Axiom of Dependent Choices*. Use the fact that the image $\{x_0, x_1, \dots\}$ of an infinite descending chain is a non-empty set with no minimum.