Mathematics 220B Winter 2010

FINAL EXAMINATION

Due Friday, March 19.

1. The functions f_0, f_1 are defined by *simultaneous primitive recursion* from w_0, w_1, h_0 and h_1 if they satisfy the identities:

$$\begin{aligned} f_0(0) &= w_0, & f_1(0) &= w_1, \\ f_0(x+1) &= h_0(f_0(x), f_1(x), x), & f_1(x+1) &= h_1(f_0(x), f_1(x), x). \end{aligned}$$

Prove that if h_0, h_1 are primitive recursive, then so are f_0 and f_1.

2. A sentence in the language of **PA** is Π_1 if it is of the form

$$\phi \equiv (\forall x_1) \cdots (\forall x_n)\theta$$

where θ has only bounded quantifiers of the form

$$(\exists x \leq y), \qquad (\forall x \leq y).$$

Prove that for every Π_1 sentence ϕ,

$$\mathbf{PA}, \mathrm{Con}_P(\ulcorner \phi \urcorner) \vdash \phi,$$

where $\mathrm{Con}_P(\ulcorner \phi \urcorner)$ expresses in a natural way the consistency of ϕ with Peano arithmetic, in other words it is the sentence $\neg(\exists y)\mathbf{Proof}(\ulcorner \neg\phi \urcorner, y)$.

3. Let A be the set of all $\#\phi$ such that ϕ is a sentence of the language of **PA** and $\mathbf{PA} \vdash \phi$. Let B be the set of all $\#\phi$ for ϕ such that ϕ is a sentence of the same language and $\mathbf{PA} \vdash \neg\phi$. Prove that A and B are recursively inseparable, i.e., that they are disjoint and there is no recursive C such that $A \subseteq C$ and $B \cap C = \emptyset$.

4. Suppose $A \subseteq \mathbb{N}$, $x \preceq y$ is a wellordering of A,

$$F : (\mathbb{N} \rightharpoonup \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$$

is a function(al) defined on partial functions on $\mathbb{N}$, and $g : A \to \mathbb{N}$ is defined by the transfinite recursion

$$g(x) = F(g \upharpoonright \{y \in A \mid y \prec x\}, x).$$

Suppose in addition, that there is a recursive partial function $\psi(e, x)$ satisfying the following, for every e, every x, and every partial function p:

$$\text{if } (\forall y \prec x)[p(y) = \varphi_e(y)], \text{ then } F(p, x) = \psi(e, x).$$

Prove that there is a recursive partial function g^* such that for all $x \in A$, $g(x) = g^*(x)$.

Note: It is not assumed that A is a recursive set, or that $x \preceq y$ is a recursive relation; the result holds for completely arbitrary A and $x \preceq y$.

5. Prove that $\{e \mid W_e \text{ has a finite complement}\}$ is Σ^0_3 complete.

6. Let A be a recursively enumerable set such that A^c is infinite. Let $f : \mathbb{N} \rightharpoonup A^c$ be one-one onto and order preserving. Assume that f eventually dominates every recursive partial function, i.e., that, for every recursive partial g,

$$(\exists m)(\forall n \geq m)(g(n) \text{ is defined} \Rightarrow g(n) \leq f(n))\,.$$

Prove that $\deg(A) = \deg(K)$.

7. Let a be a degree of unsolvability. Prove that there is a degree $\mathrm{d} \neq 0$ such that $\mathrm{a} \not\leq \mathrm{d}$.

Hint. Let $g : \mathbb{N} \to \mathbb{N}$ with $\deg(g) = \mathrm{a}$. Define $\emptyset = s_0 \subsetneq s_1 \subsetneq s_2 \subsetneq \dots$ and set $f = \bigcup_{e \in \mathbb{N}} s_e$. In defining s_{e+1} from s_e, make sure that $f \neq \{e\}$ and also make sure that $g \neq \{e\}^f$.