Philosophy 136 Winter 2011 Tony Martin

Modal Logic

0 Review of Sentential Logic

The language $\mathcal{L}$ of sentential logic.

Symbols:

(i)	sentence letters	$p_0, p_1, p_2, \ldots$
(ii)	connectives	$\supset$, $\neg$
(iii)	parentheses	(,)

Formulas:

(i) Each sentence letter is a formula.

(ii) If A is a formula, then so is $\neg A$.

(iii) If A and B are formulas, then so is $(A \supset B)$.

(iv) Nothing is a formula unless its being one follows from (i)–(iii).

We may sometimes call formulas *sentences.*

Remarks:

(a) We shall mostly ignore the use-mention distinction, treating symbols and formulas as their own names.

(b) In the presence of the first three clauses of the definition of formula, Clause (iv) justifies *proof by formula induction* and *definition by recursion on formulas.* Using proof by formula induction, one can prove that all formulas of $\mathcal{L}$ have some property P by showing that all sentence letters have P, that $\neg A$ has P whenever A does, and that $(A \supset B)$ has P whenever both A and B have P. Using definition by recursion on formulas, one can define a function f whose domain is the set of all formulas by directly defining $f(p_i)$ for each i, defining $f(\neg A)$ in terms of A and $f(A)$ for each formula A, and defining $f(A \supset B)$ from A, B, $f(A)$, and $f(B)$ for all formulas A and B. In a similar manner, definition by recursion on formulas allows one to define a property of formulas. The proof of Lemma 0.1 is an example of proof by formula induction. The definition preceding that lemma is an example of definition by recursion on formulas. We will often say "induction" for "formula induction" and "recursion" for "recursion on formulas."

Abbreviations:

$$\begin{array}{lll}
(A \vee B) & \text{for} & (\neg A \supset B) \\
(A \wedge B) & \text{for} & \neg(\neg A \vee \neg B) \\
(A \equiv B) & \text{for} & ((A \supset B) \wedge (B \supset A)) \\
\top & \text{for} & (\neg p_0 \vee p_0) \\
\bot & \text{for} & (\neg p_0 \wedge p_0) \\
p & \text{for} & p_0 \\
q & \text{for} & p_1 \\
r & \text{for} & p_2 \\
s & \text{for} & p_3
\end{array}$$

We shall also write

$$\begin{array}{rcl}
(A_1 \vee A_2 \vee \ldots \vee A_n) & \text{for} & (A_1 \vee (A_2 \vee (\ldots \vee A_n)) \cdots) \\
(A_1 \wedge A_2 \wedge \ldots \wedge A_n) & \text{for} & (A_1 \wedge (A_2 \wedge (\ldots \wedge A_n)) \cdots) \\
(A_1 \supset A_2 \supset \ldots \supset A_n) & \text{for} & (A_1 \supset (A_2 \supset (\ldots \supset A_n)) \cdots)
\end{array}$$

Semantics.

A *valuation* (for $\mathcal{L}$) is a set of sentence letters. For valuations α and formulas A we define by recursion on formulas the relation $\alpha \models A$ ($\Leftrightarrow A$ is *true* under $\alpha \Leftrightarrow \alpha$ *satisfies* A):

(i) $\alpha \models p_i \;\Leftrightarrow\; p_i \in \alpha\,$;

(ii) $\alpha \models \neg A \;\Leftrightarrow\; \alpha \not\models A\,$;

(iii) $\alpha \models (A \supset B) \;\Leftrightarrow\; (\alpha \not\models A \text{ or } \alpha \models B)\,$.

The notion of a sentence letter *occurring* in a formula is defined by recursion in the obvious way.

Lemma 0.1. *Let α and β be valuations for $\mathcal{L}$. For any formula A, if α and β agree on all sentence letters occurring in A (i.e., if $p_i \in \alpha \Leftrightarrow p_i \in \beta$ for every p_i occurring in A), then*

$$\alpha \models A \;\Leftrightarrow\; \beta \models A.$$

Proof. We use formula induction.

If α and β agree on p_i, then this fact and clause (i) of the definition of $\models$ give us that

$$\alpha \models p_i \;\Leftrightarrow\; p_i \in \alpha \;\Leftrightarrow\; p_i \in \beta \;\Leftrightarrow\; \beta \models p_i.$$

Suppose that that our assertion holds for a formula A. Assume that α and β agree on all sentence letters occurring in $\neg A$. Then they agree on all sentence letters occurring in A (the same sentence letters). By our induction hypothesis for A, we get that $\alpha \models A \Leftrightarrow \beta \models A$. This and clause (ii) of the definition of $\models$ yield

$$\alpha \models \neg A \Leftrightarrow \alpha \not\models A \Leftrightarrow \beta \not\models A \Leftrightarrow \beta \models \neg A.$$

Finally suppose that our assertion holds for formulas A and B. Assume that α and β agree on all sentence letters occurring in $(A \supset B)$. Then they agree on all sentence letters occurring in A and on all sentence letters occurring in B. Using clause (iii) of the definition of $\models$ and our induction hypothesis for A and B, we get that

$$\alpha \models (A \supset B) \Leftrightarrow (\alpha \not\models A \text{ or } \alpha \models B) \Leftrightarrow (\beta \not\models A \text{ or } \beta \models B) \Leftrightarrow \beta \models (A \supset B).$$

□

The Implication Relation.

A formula A of $\mathcal{L}$ is *valid* ($\Leftrightarrow\ \models A$) if and only if A is true under every valuation for $\mathcal{L}$. A formula A *implies* a formula B ($\Leftrightarrow$ B is a *consequence* of $A \Leftrightarrow A \models B$) if and only if B is true under every valuation under which A is true. If α is a valuation and Φ is a set of formulas, then Φ is *true* under α ($\Leftrightarrow\ \alpha \models \Phi$) if and only if every member of Φ is true under α. If B is a formula and Φ is a set of formulas, then Φ *implies* B ($\Leftrightarrow$ B is a *consequence* of $\Phi \Leftrightarrow \Phi \models B$) if and only if B is true under every valuation under which Φ is true.

Remark. Note that the symbol "$\models$" is ambiguous. We use it to mean "satisfies" and also to mean "implies." This ambiguity is unfortunate, but both uses are standard.

Deduction.

We shall consider various *axiomatic systems* for $\mathcal{L}$ and for other languages. Each axiomatic system **S** will have *axioms* and *rules of inference.* A *deduction* in **S** from a set Φ of formulas is a finite sequence of formulas (the *lines* of the deduction) each of which is an axiom, a member of Φ, or a consequence of earlier lines by a rule of inference. The members of Φ are the *assumptions* or *premises* of the deduction. A *proof* in **S** is a deduction in **S** from the empty set of formulas. We say that A is *deducible* from Φ in **S**

($\Leftrightarrow\ \Phi \vdash_{\mathbf{S}} A$) if and only if there is a deduction from Φ in **S** with last line A (a *deduction of* A from Φ). A is *provable* in **S** ($\Leftrightarrow\ \vdash_{\mathbf{S}} A$) if and only if there is a proof in **S** with last line A (a *proof of* A). When there is no ambiguity, we shall usually write $\vdash$ for $\vdash_{\mathbf{S}}$.

The axiomatic system $\mathbf{SL}_0$ for $\mathcal{L}$ is as follows:

Axioms:

1) $B \supset (A \supset B)$
2) $(\neg A \supset B)\ \supset\ ((\neg A \supset \neg B)\ \supset\ A)$
3) $(A\ \supset\ (B \supset C))\ \supset\ ((A \supset B)\ \supset\ (A \rightarrow C))$

Remarks:

(a) Note that (1)–(3) are not axioms but *axiom schemas.* There are infinitely many instances of each of these schemas, since A, B, and C may be any formulas whatsoever.

We have used abbreviations in presenting these axiom schemas, in that we have dropped parentheses.

Rule of Inference: (MP) $\dfrac{A \quad A \supset B}{B}$

The axiomatic system **SL** for $\mathcal{L}$ has the rule of inference MP and has as axioms all valid formulas of $\mathcal{L}$.

The following theorem, which we shall neither prove nor make use of, asserts that the systems $\mathbf{SL}_0$ and **SL** are equivalent.

Theorem 0.2. *For all formulas A and sets Φ of formulas,*

$$\Phi \vdash_{\mathbf{SL}_0} A \ \Leftrightarrow\ \Phi \vdash_{\mathbf{SL}} A\,.$$

An axiomatic system **S** for $\mathcal{L}$ is *sound* if and only if, for all formulas A and sets Φ of formulas,

$$\Phi \vdash_{\mathbf{S}} A \ \Rightarrow\ \Phi \models A\,,$$

and **S** is *complete* if and only if, for all A and Φ,

$$\Phi \models A \ \Rightarrow\ \Phi \vdash_{\mathbf{S}} A\,.$$

If soundness or completeness holds for the special case that Φ is the empty set, then the system is *weakly sound* or *weakly complete* respectively.

We assume without proof the following basic fact.

Theorem 0.3 (Compactness). *If A is a formula, Φ is a set of formulas, and $\Phi \models A$, then there is a finite subset Δ of Φ such that $\Delta \models A$.*

Theorem 0.4 (Soundness and Completeness of SL). *The system* **SL** *is sound and complete.*

Proof. Soundness: Let Φ be a set of formulas. Let $\mathcal{D}$ be a deduction from Φ in **SL**. We prove by (complete mathematical) induction that every line of $\mathcal{D}$ is a consequence of Φ. This is evidently the case for lines that are axioms or members of Φ. Suppose that a line B comes by MP from earlier lines A and $A \supset B$. Assume that $\Phi \models A$ and that $\Phi \models A \supset B$. To see that $\Phi \models B$, let α be a valuation under which Φ is true. By our induction hypothesis, A and $A \supset B$ are both true under α. Hence B must be true under α.

Completeness: Suppose that $\Phi \models A$. By Compactness, let $\{B_1, \ldots, B_n\}$ be a finite subset of Φ such that $\{B_1, \ldots, B_n\} \models A$. Then

$$\models B_1 \supset \cdots \supset B_n \supset A\,.$$

Therefore this formula is an axiom of **SL**. We get a deduction of A from Φ as follows (with the justification of each line indicated):

1.	$B_1 \supset \cdots \supset B_n \supset A$	Axiom
2.	B_1	Premise
3.	$B_2 \supset \cdots \supset B_n \supset A$	$1, 2$ MP
..	...	...
..	...	...
..	...	...
$2n-1$.	$B_n \supset A$	$2n-3, 2n-2$ MP
$2n$.	B_n	Premise
$2n+1$.	A	$2n-1, 2n$ MP

□

Exercise 0.1. Which of the following are valid? Prove your answers.

(1) $((p_0 \supset (p_1 \supset p_2)) \supset (p_1 \supset p_2))\,.$

(2) $((p_0 \supset p_1) \vee (p_1 \supset p_2))\,.$

Exercise 0.2. Which of the following are statements are true? Prove your answers.

(1) $\{(p_0 \supset \neg p_1),\, ((p_2 \vee p_0) \supset (p_1 \vee p_2)),\, \neg p_2\} \models \neg p_0\,.$

(2) $\{((\neg p_3 \vee p_0) \vee p_1),\, (\neg p_1 \supset \neg p_2),\, (p_0 \supset (p_2 \wedge p_3))\} \models p_1\,.$

1 The System S5

The language $\mathcal{L}(\Box)$.

Symbols:

(i)	sentence letters	$p_0, p_1, p_2, \ldots$
(ii)	connectives	$\supset\,, \neg\,, \Box$
(iii)	parentheses	$(\,,\,)$

The symbol $\Box$ may be read "necessarily" or "it is necessary that."

Formulas:

(i) Each sentence letter is a formula.

(ii) If A is a formula, then so is $\neg A$.

(iii) If A and B are formulas, then so is $(A \supset B)$.

(iv) If A is a formula, then so is $\Box A$.

(v) Nothing is a formula unless its being one follows from (i)–(iv).

Remark. Formula induction and definition by recursion on formulas apply for $\mathcal{L}(\Box)$, but there are four steps rather than three steps as in the case of $\mathcal{L}$.

Additional abbreviations:

$\Diamond A$	for	$\neg\Box\neg A$
$\triangledown A$	for	$\Diamond A \wedge \neg\Box A$
$(A \rightarrow B)$	for	$\Box(A \supset B)$
$(A \leftrightarrow B)$	for	$((A \rightarrow B) \wedge (B \rightarrow A))$

The symbol $\Diamond$ is read as "possibly" or "it is possible that." Read $\triangledown$ as "it is contingent that." Read $\rightarrow$ as "strictly implies" and $\leftrightarrow$ as "is strictly equivalent with."

Semantics.

Our formal semantics is motivated by the conception of necessity as truth in all possible worlds. A *model* for $\mathcal{L}(\Box)$ is triple $\mathfrak{M} = (W, w_0, \varphi)$, where

(i) W is a non-empty set;

(ii) $w_0 \in W$;

(iii) φ is a function that assigns to each $w \in W$ a valuation φ_w for $\mathcal{L}$.

The set W represents the universe of all possible worlds, and we call its elements the *worlds* of the model. The object w_0 represents the actual world, and we shall call it the *actual world* of the model. We call φ a *modal valuation*, and from now on we speak of the valuations of Section 0 (and so also of the individual valuations φ_w) as *simple valuations*.

Example. Let $\mathfrak{M} = (W, w_0, \varphi)$, with $W = \{w_0, w_1, w_2, w_3, w_4, w_5\}$ and, for $0 \leq k \leq 5$,

$$\varphi_{w_k} = \begin{cases} \{p_1, p_3\} & \text{if } k \text{ is odd;} \\ \{p_2, p_3\} & \text{if } k \text{ is even.} \end{cases}$$

We can also describe this model by a diagram as follows:

$$\begin{array}{rcl} w_0^* & : & p_2, p_3 \\ w_1 & : & p_1, p_3 \\ w_2 & : & p_2, p_3 \\ w_3 & : & p_1, p_3 \\ w_4 & : & p_2, p_3 \\ w_5 & : & p_1, p_3 \end{array}$$

The superscript * on w_0 indicates that it is the actual world of the model.

For models $\mathfrak{M} = (W, w_0, \varphi)$, worlds w of $\mathfrak{M}$, and formulas A, we define by recursion on formulas the relation $w \models_{\mathfrak{M}} A$ ($\Leftrightarrow$ A is *true* at w in $\mathfrak{M}$).

(i) $w \models_{\mathfrak{M}} p_i \;\Leftrightarrow\; p_i \in \varphi_w$;

(ii) $w \models_{\mathfrak{M}} \neg A \;\Leftrightarrow\; w \not\models_{\mathfrak{M}} A$;

(iii) $w \models_{\mathfrak{M}} (A \supset B) \;\Leftrightarrow\; (w \not\models_{\mathfrak{M}} A$ or $w \models_{\mathfrak{M}} B)$;

(iv) $w \models_{\mathfrak{M}} \Box A \;\Leftrightarrow\;$ for all $v \in W$, $v \models_{\mathfrak{M}} A$.

We say that a formula A is *true* in a model $\mathfrak{M} = (W, w_0, \varphi)$ ($\Leftrightarrow \models_{\mathfrak{M}} A \Leftrightarrow$ $\mathfrak{M}$ *satisfies* A) just in case $w_0 \models_{\mathfrak{M}} A$.

Example. Let $\mathfrak{M}$ be the model of the Example above. The following formulas are true in M: p_2, p_3, $\neg p_1$, $\Box p_3$, $\neg\Box p_1$, $\neg\Box p_2$, $\Diamond p_1$, $\Diamond p_2$, $\Diamond p_3$, $\Box(p_1 \vee p_2)$, $\neg(\Box p_1 \vee \Box p_2)$, $\neg\Diamond(p_1 \wedge p_2)$, $\Box\Diamond p_1$, $\neg\Diamond\Box p_1$, $\neg\Box\,\Box p_1$, $\Diamond\Diamond p_1$.

Lemma 1.1. *For any model $\mathfrak{M}$, any world w of $\mathfrak{M}$, and any formulas A and B,*

(1) $w \models_{\mathfrak{M}} \Diamond A$ if and only if A is true at some world of $\mathfrak{M}$;

(2) $w \models_{\mathfrak{M}} A \to B$ *if and only if* B *is true at every world of* $\mathfrak{M}$ *at which* A *is true;*

(3) $w \models_{\mathfrak{M}} A \leftrightarrow B$ *if and only if* A *and* B *are true at the same worlds of* $\mathfrak{M}$*;*

(4) $w \models_{\mathfrak{M}} \triangledown A$ *if and only if there is a world of* $\mathfrak{M}$ *at which* A *is true and there is a world of* $\mathfrak{M}$ *at which* A *is false.*

Exercise 1.1. Consider the model given by the following diagram:

$$\begin{array}{ll} w_0^* : & p_1, p_2 \\ w_1 : & p_1 \\ w_2 : & p_2 \\ w_3 : & \end{array}$$

Which of the following formulas are true in the model?

$$p_1 \vee p_2\,; \quad \Box(p_1 \supset \neg p_3)\,; \quad \Box\Diamond(p_1 \wedge p_2)\,; \quad \Box(p_1 \vee p_2) \vee \Box(\neg p_1 \vee \neg p_2)$$

Explain your answers.

Lemma 1.2. *Let* $\mathfrak{M} = (W, w_0, \varphi)$ *and* $\mathfrak{M}' = (W, w_0, \varphi')$ *be models. For any formula* A*, if for every* $w \in W$ *the simple valuations* φ_w *and* φ'_w *agree on all sentence letters occurring in* A*, then*

$$w \models_{\mathfrak{M}} A \;\Leftrightarrow\; w \models_{\mathfrak{M}'} A$$

for all $w \in W$*.*

Proof. We us formula induction, as in the proof of Lemma 0.1. All the steps except that of $\Box$ are similar to the corresponding steps of the proof of Lemma 0.1.

Suppose that our assertion holds for a formula A. Assume that, for every $w \in W$, φ_w and φ'_w agree on all sentence letters occurring in $\Box A$. Then, for every $w \in W$, φ_w and φ'_w agree on all sentence letters occurring in A. Let w be any member of W. By definition, $w \models_{\mathfrak{M}} \Box A$ if and only if $v \models_{\mathfrak{M}} A$ for all $v \in W$. By our induction hypothesis for A, this is true if and only if $v \models_{\mathfrak{M}'} A$ for all $v \in W$, and this in turn holds just in case $w \models_{\mathfrak{M}'} \Box A$. $\Box$

The Implication Relation.

A formula A of $\mathcal{L}(\Box)$ is *valid* ($\Leftrightarrow\ \models A$) if and only if A is true in every model for $\mathcal{L}(\Box)$. A formula A *implies* a formula B ($\Leftrightarrow$ B is a *consequence* of $A \Leftrightarrow A \models B$) if and only if B is true in every model in which A is true. If $\mathfrak{M}$ is a model and Φ is a set of formulas, then Φ is *true* in $\mathfrak{M}$ ($\Leftrightarrow\ \models_{\mathfrak{M}} \Phi$) if and only if every member of Φ is true in $\mathfrak{M}$; Φ is *true* at a world w in $\mathfrak{M}$ ($\Leftrightarrow\ w \models_{\mathfrak{M}} \Phi$) if and only if every member of Φ is true at w in $\mathfrak{M}$. If B is a formula and Φ is a set of formulas, then Φ *implies* B ($\Leftrightarrow$ B is a *consequence* of $\Phi \Leftrightarrow \Phi \models B$) if and only if B is true in every model in which Φ is true. Let us say that a formula or a set of formulas is *satisfiable* if there is a model in which it is true.

Example. The formula $\Box p \supset p$ is valid. To show this let $\mathfrak{M} = (W, w_0, \varphi)$ be a model. To prove that $\Box p \supset p$ is true in $\mathfrak{M}$, we assume that $\Box p$ is true in $\mathfrak{M}$ and argue that p is true in $\mathfrak{M}$. Since $\models_{\mathfrak{M}} \Box p$, we have by definition that $w \models_{\mathfrak{M}} p$ for every $w \in W$. In particular, $w_0 \models_{\mathfrak{M}} p$, i.e., $\models_{\mathfrak{M}} p$.

Example. $\{\Box(p_1 \vee p_2), \neg\Box p_1\} \not\models \Box p_2$. Consider the following model

$$\begin{array}{rl} w_0^* : & p_1 \\ w_1 : & p_2 \end{array}$$

Both $\Box(p_1 \vee p_2)$ and $\neg\Box p_1$ are true in the model, but $\Box p_2$ is false (not true) in it.

Exercise 1.2. For each of the following formulas, either prove that the formula is valid or give a model in which it is false.

$$\Diamond\Diamond p \equiv \Diamond p\,; \quad \Diamond\Box p \equiv \Diamond p\,; \quad \Box(p \vee \Box q) \supset (\Box p \vee \Box q)\,.$$

Extended valuations.

Say that a formula of $\mathcal{L}(\Box)$ is *sententially atomic* if it either is a sentence letter or is $\Box B$ for some formula B. An *extended valuation* is a set of sententially atomic formulas. Note that every formula of $\mathcal{L}(\Box)$ can be built from sententially atomic formulas by finitely many applications of the operations of negation ($A \mapsto \neg A$) and forming conditionals ($A, B \mapsto (A \supset B)$). For extended valuations α and formulas A, we define by recursion the relation $\alpha \models A$ ($\Leftrightarrow$ A is *true* under $\alpha \Leftrightarrow \alpha$ *satisfies* A):

(i) if A is sententially atomic, then $\alpha \models A \;\Leftrightarrow\; A \in \alpha\,;$

(ii) $\alpha \models \neg A \;\Leftrightarrow\; \alpha \not\models A\,;$

(iii) $\alpha \models (A \supset B) \;\Leftrightarrow\; (\alpha \not\models A \text{ or } \alpha \models B)\,.$

A formula A is a *tautology* ($\Leftrightarrow\ \models_{\mathrm{t}} A$) if and only if if A is true under every extended valuation. The notions A *tautologically implies* B ($\Leftrightarrow$ B is a *tautological consequence* of A $\Leftrightarrow$ $A \models_{\mathrm{t}} B$) and Φ *tautologically implies* B ($\Leftrightarrow$ B is a *tautological consequence* of Φ $\Leftrightarrow$ $\Phi \models_{\mathrm{t}} B$) are defined in the obvious way.

Lemma 1.3. *For any formula A and set Φ of formulas,*

- *(a)* $\Phi \models_{\mathrm{t}} A \ \Rightarrow\ \Phi \models A$*;*
- *(b) if A and all the members of Φ are formulas of $\mathcal{L}$, then*

$$\Phi \models_{\mathrm{t}} A \ \Leftrightarrow\ \Phi \models A\,.$$

Proof. (a) Assume that $\Phi \models_{\mathrm{t}} A$ and let $\mathfrak{M}$ be a model in which Φ is true. Let π be the set of all sententially atomic formulas that are true in $\mathfrak{M}$ at the actual world w_0 of $\mathfrak{M}$. Then π is an extended valuation, and it is easy to see by formula induction that, for every formula B,

$$\pi \models B \ \Leftrightarrow\ w_0 \models_{\mathfrak{M}} B\,.$$

This fact and our assumption that Φ is true in $\mathfrak{M}$ give us that $\pi \models \Phi$. Since $\Phi \models_{\mathrm{t}} A$, we get that $\pi \models A$ and so that $w_0 \models_{\mathfrak{M}} A$.

(b) Assume that A and the members of Φ are formulas of $\mathcal{L}$ and that $\Phi \not\models_{\mathrm{t}} A$. Let π be an extended valuation under which Φ is true and A is false. Let $\mathfrak{M} = (\{w_0\}, w_0, \varphi)$ where φ_{w_0} is the set of all sentence letters true under π. By induction one can show that the formulas of $\mathcal{L}$ true at w_0 in $\mathfrak{M}$ are the same as the formulas true in π. Thus $\mathfrak{M}$ witnesses that $\Phi \not\models A$. $\Box$

For formulas A of $\mathcal{L}$, Lemma 0.1 and part (b) of Lemma 1.3 imply that A is valid in the sense of §0 if and only if A is a tautology if and only if A is valid in our current sense. Note, however, that the formula $\Box(p \supset p)$ is valid and is not a tautology, so part (b) of Lemma 1.3 does not hold if the restriction to formulas of $\mathcal{L}$ is removed.

Two models (W, w_0, φ) and (V, v_0, ψ) are *isomorphic* if and only if there is a one-one onto function $f : W \to V$ such that $f(w_0) = v_0$ and such that $\varphi_w = \psi_{f(w)}$ for every $w \in W$. One consequence of the next exercise is that isomorphic models satisfy the same formulas.

Exercise 1.3. Let $\mathfrak{M} = (W, w_0, \varphi)$ and $\mathfrak{N} = (V, v_0, \psi)$ be models. Assume that (1) $\varphi_{w_0} = \psi_{v_0}$, (2) for every $w \in W$ there is a $v \in V$ such that $\varphi_w = \psi_v$, and (3) for every $v \in V$ there is a $w \in W$ such that $\varphi_w = \psi_v$. Prove that the same formulas are true in $\mathfrak{M}$ as are true in $\mathfrak{N}$.

Deduction.

The axiomatic system **S5** for $\mathcal{L}(\Box)$ is as follows:

Axioms:

(Taut) All tautologies

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

(T) $\Box A \supset A$

(5) $\Diamond A \supset \Box\Diamond A$

Rules of Inference:

(MP) $$\frac{A \quad A \supset B}{B}$$

(Nec) $$\frac{A}{\Box A}$$

The rule Nec is restricted in the following way. It applies to a line A of a deduction only when some subsequence of earlier lines, followed by A, is a proof of A (a deduction from no assumptions). Thus Nec is really a many-premise rule rather than a one-premise rule. If we did not restrict Nec in this way, the system **S5** would be unsound. Note that in proofs (as opposed to deductions) the restriction is vacuous.

Throughout the rest of §1, we shall write $\vdash$ for $\vdash_{\mathbf{S5}}$.

The definitions of *sound* and *complete* for axiom systems for $\mathcal{L}(\Box)$ are just like the definitions for axiomatic systems for $\mathcal{L}$.

Theorem 1.4. *The system* **S5** *is sound.*

Proof. We first prove that **S5** is *weakly sound*, that if $\vdash A$ then $\models A$. Let $\mathcal{P}$ be a proof in **S5**. We show that every line of $\mathcal{P}$ is valid. Assume that this is false, in order to derive a contradiction. Let C be the first non-valid line of $\mathcal{P}$.

We show that C cannot be an axiom by showing that every axiom is valid.

All tautologies are valid by Lemma 1.3.

To see that K is a valid schema (i.e., that all its instances are valid), let A and B be formulas and let $\mathfrak{M}$ be a model in which $\Box(A \supset B)$ is true and $\Box A$ is true. Let w be any world of $\mathfrak{M}$. Then $A \supset B$ and A are both true at w in $\mathfrak{M}$. Hence B is true at w in $\mathfrak{M}$. This argument shows that B is true at every world in $\mathfrak{M}$, and so that $\Box B$ is true in $\mathfrak{M}$.

It is easy to see that T and 5 are valid schemas.

The conclusion of an instance of MP is a tautological consequence of its premises. By Lemma 1.3, it is a consequence of its premises. Hence it is valid if the premises are. Thus C cannot come from earlier lines of $\mathcal{P}$ by MP.

Suppose A is a valid formula. To see that $\Box A$ is also valid, let $\mathfrak{M} = (W, w_0, \varphi)$ be a model. Let $w \in W$. Consider the model $\mathfrak{M}' = (W, w, \varphi)$. Since A is valid, $\models_{\mathfrak{M}'} A$. Hence $w \models_{\mathfrak{M}} A$. Since w was arbitrary, $\models_{\mathfrak{M}} \Box A$.

The argument of the preceding paragraph shows that C cannot come from an earlier line of $\mathcal{P}$ by Nec.

We have ruled out all possibilities for C, so we have our desired contradiction.

Now let Φ be a set of formulas and let $\mathcal{D}$ be a deduction in **S5** from Φ. Assume for a contradiction that some line of $\mathcal{D}$ is not a consequence of Φ. Let C be the first such line.

Trivially every member of Φ is a consequence of Φ, so C is not a member of Φ.

Since all axioms are valid, they are all consequences of Φ. Hence C is not an axiom.

Observe that, for any sets Γ and Δ of formulas and for any formula B, if $\Gamma \models$ every member of Δ and if $\Delta \models B$, then $\Gamma \models B$. Thus if the premises of an instance of MP follow from Φ then so does the conclusion. Hence C cannot come from earlier lines by MP.

If $\Box A$ follows from earlier lines by Nec, then there is a proof in **S5** of $\Box A$. Hence $\Box A$ is valid; *a fortiori*, it is a consequence of Φ. Hence C does not come from an earlier line by Nec, and we have our contradiction. $\Box$

Example. $\vdash \Box p \supset \Box\neg\neg p$.

1.	$p \supset \neg\neg p$	Taut
2.	$\Box(p \supset \neg\neg p)$	1 Nec
3.	$\Box(p \supset \neg\neg p) \supset (\Box p \supset \Box\neg\neg p)$	K
4.	$\Box p \supset \Box\neg\neg p$	$2, 3$ MP

Lemma 1.5. *If $A \models_{\mathrm{t}} B$, then $\vdash \Box A \supset \Box B$.*

Proof. Assume that $A \models_{\mathrm{t}} B$. Then $A \supset B$ is a tautology. Thus the following is a proof of $\Box A \supset \Box B$:

$$\begin{array}{lll}
1. & A \supset B & \text{Taut} \\
2. & \Box(A \supset B) & 1 \text{ Nec} \\
3. & \Box(A \supset B) \supset (\Box A \supset \Box B) & K \\
4. & \Box A \supset \Box B & 2,3 \text{ MP}
\end{array}$$

$\Box$

Lemma 1.6. *If* $\vdash A \supset B$ *then* $\vdash \Box A \supset \Box B$.

Proof. Given a proof of $A \supset B$, one can get a proof of $\Box A \supset \Box B$ by appending by appending the three lines

$$\Box(A \supset B)\,; \quad \Box(A \supset B) \supset (\Box A \supset \Box B)\,; \quad \Box A \supset \Box B \qquad \Box$$

Lemma 1.6 justifies a *derived rule of inference*, which we shall cite by writing "Lemma 1.6." In using this rule to construct abbreviated deductions, we may enter a formula $\Box A \supset \Box B$ on a line if the formula $A \supset B$ is on an earlier line and some if subsequence of still earlier lines, followed by $A \supset B$, constitutes a proof of $A \supset B$.

Exercise 1.4. Assume that $A \models_{\mathrm{t}} B$. Give an (unabbreviated) proof of $\Diamond A \supset \Diamond B$. Indicate briefly how to show that if $\vdash A \supset B$ then $\vdash \Diamond A \supset \Diamond B$.

Hint. Any formula $C \supset D$ and its contrapositive are tautological consequences of each other.

Note that the last part of Exercise 1.4 justifies a derived rule of inference, a rule we shall cite by writing "Exercise 1.4.

Lemma 1.7 (Compactness of $\models_{\mathrm{t}}$). *If* $\Phi \models_{\mathrm{t}} A$ *then there is a finite subset* Δ *of* Φ *such that* $\Delta \models_{\mathrm{t}} A$.

Proof. This follows from Theorem 0.3. (Pretend that the sententially atomic formulas are sentence letters.) $\Box$

Lemma 1.8. *If* $\Phi \models_{\mathrm{t}} A$ *then* $\Phi \vdash A$.

Proof. The proof is just like that of the completeness part of Theorem 0.4. $\Box$

Lemma 1.8 justifies the derived rule of inference TC according to which a formula A follows from a set Φ of formulas if $\Phi \models_{\mathrm{t}} A$. In using this rule to construct abbreviated deductions, we may enter a formula as a line if the formula is a tautological consequence of earlier lines.

Lemma 1.9. *If* $\Phi \vdash A$ *and* $\Psi \cup \{A\} \vdash B$*, then* $\Phi \cup \Psi \vdash B$*.*

Lemma 1.9 justifies, in particular, allowing us to enter a formula A as a line of an abbreviated deduction when we know that $\vdash A$.

Lemma 1.10 (Deduction Theorem). *If* $\Phi \cup \{A\} \vdash B$ *then* $\Phi \vdash A \supset B$*.*

Proof. Let $\mathcal{D}$ be a deduction of B from $\Phi \cup \{A\}$. We show by complete induction that for every line C of $\mathcal{D}$ there is a deduction from Φ of $A \supset C$.

For any formula C, $C \models_{\mathrm{t}} A \supset C$. If C is an axiom or a member of Φ, then this plus TC shows that $\Phi \vdash A \supset C$. If C is A, then $A \supset C$ is the tautology $A \supset A$.

$A \supset C$ is a tautological consequence of $A \supset E$ and $A \supset (E \supset C)$. This takes care of the case that C comes in $\mathcal{D}$ from E and $E \supset C$ by MP.

Suppose that C is $\Box E$ and in $\mathcal{D}$ comes from E by Nec. Then $\vdash E$, and so $\vdash C$. TC gives $\vdash A \supset C$. $\Box$

Corollary 1.11. $\{A_1, \ldots, A_n\} \vdash B$ *if and only if* $\vdash A_1 \supset \cdots \supset A_n \supset B$*.*

Proof. Repeated applications of the Deduction Theorem give "only if." Repeated applicatins of MP give "if." (Literally the proof is by mathematical induction on n.) $\Box$

Lemma 1.12. *For all positive integers* n*,*

$$\Box(A_1 \supset \cdots \supset A_n) \vdash \Box A_1 \supset \cdots \supset \Box A_n\,.$$

Proof. The lemma is proved by induction on n, using K, TC, and the Deduction Theorem. $\Box$

For Φ a set of formulas, let $\Box\Phi = \{\Box A \mid A \in \Phi\}$.

Lemma 1.13. *If* $\Phi \vdash B$ *then* $\Box\Phi \vdash \Box B$*.*

Proof. Deductions are finite, so we may assume that $\{A_1, \ldots, A_n\} \vdash B$, where each $A_i \in \Phi$. By Corollary 1.11, $\vdash A_1 \supset \cdots \supset A_n \supset B$. Nec gives $\vdash \Box(A_1 \supset \cdots \supset A_n \supset$B). By Lemma 1.12, $\vdash \Box A_1 \supset \cdots \Box A_n \supset \Box B$. By Corollary 1.11, $\{\Box A_1, \ldots, \Box A_n\} \vdash \Box B$. Hence $\Box\Phi \vdash \Box B$. $\Box$

Note that we have not yet made any use of Axiom Schemas T and 5.

Lemma 1.14. $\vdash\ \Diamond\Box A \supset \Box A$.

Proof. Here is an abbreviated proof:

$$\begin{array}{lll}
1. & \Diamond\neg A \supset \Box\Diamond\neg A & 5 \\
2. & \neg\Box\Diamond\neg A \supset \neg\Diamond\neg A & 1\ \text{TC} \\
 & [\Diamond\Box\neg\neg A \supset \neg\neg\Box\neg\neg A] & \\
3. & \Diamond\Box\neg\neg A \supset \Box\neg\neg A & 2\ \text{TC} \\
4. & \Box\neg\neg A \supset \Box A & \text{Lemma 1.5} \\
5. & \Diamond\Box\neg\neg A \supset \Box A & 3, 4\ \text{TC} \\
6. & \Box A \supset \Box\neg\neg A & \text{Lemma 1.5} \\
7. & \Diamond\Box A \supset \Diamond\Box\neg\neg A & 6, \text{Exercise 1.4} \\
8. & \Diamond\Box A \supset \Box A & 5, 7\ \text{TC}
\end{array}$$

Exercise 1.5. Show that $\vdash\ \Box A \supset \Box\Box A$.

Hint. First use T to show that $\vdash \Box A \supset \Diamond\Box A$. Also show that $\vdash \Box\Diamond\Box A \supset \Box\Box A$.

Lemma 1.15. *If* $\Box\Phi \vdash A$ *then* $\Box\Phi \vdash \Box A$.

Proof. Assume that $\Box\Phi \vdash A$. Lemma 1.13 gives that $\Box\Box\Phi \vdash \Box A$. Let $\{B_1, \ldots, B_n\} \subseteq \Phi$ be such that $\{\Box\Box B_1, \ldots, \Box\Box B_n\} \vdash \Box A$. By Exercise 1.5, we have that $\Box B_i \vdash \Box\Box B_i$ for $1 \leq i \leq n$. By n applications of Lemma 1.9, we get that $\{\Box B_1, \ldots, \Box B_n\} \vdash \Box A$ and so that $\Box\Phi \vdash \Box A$. $\Box$

Say that a set Φ of formulas is *consistent* in an axiomatic system **S** if and only if $\Phi \nvdash_{\mathbf{S}} \bot$.

Lemma 1.16. **S5** *is complete if and only if every set of formulas consistent in* **S5** *is satisfiable.*

Proof. Suppose first that **S5** is complete. Let Φ be consistent in **S5**. By completeness, $\Phi \not\models \bot$. Thus there is a model in which Φ is true (and $\bot$ is false).

Now suppose that every set of formulas consistent in **S5** is satisfiable. Let Φ be a set of formulas and let A be a formula. Assume that $\Phi \models A$. Then $\Phi \cup \{\neg A\}$ is not satisfiable. Hence $\Phi \cup \{\neg A\}$ is inconsistent in **S5**, i.e., $\Phi \cup \{\neg A\} \vdash \bot$. By the Deduction Theorem, $\Phi \vdash (\neg A \supset \bot)$. But $(\neg A \supset \bot) \models_{\mathrm{t}} A$, and so $\Phi \vdash A$. $\Box$

In proving the completeness of **S5**, we shall consider models $\mathfrak{M}$ whose worlds are extended valuations. This will create a danger of confusing $\alpha \models A$ (A is true under α) with $\alpha \models_{\mathfrak{M}} A$ (A is true at α in $\mathfrak{M}$). To make sure that we avoid such confusion, let us us write $\alpha \models_{\mathrm{t}} A$ for $\alpha \models A$ when α is an extended valuation.

If **S** is an axiomatic system, an extended valuation α is **S**-*acceptable* if and only if all *theorems* of **S** are true under α, i.e., if and only if

$$\vdash_{\mathbf{S}} A \;\Rightarrow\; \alpha \models_{\mathrm{t}} A$$

for every formula A.

Exercise 1.6. Say that a set Φ of formulas is *maximal consistent* in **S5** if and only if both (i) and (ii) below hold.

(i) Φ is consistent in **S5**;

(ii) for every formula A, either $A \in \Phi$ or $\neg A \in \Phi$.

(a) Show that if α is an **S5**-acceptable extended valuation then the set of formulas true under α is maximal consistent in **S5**.

(b) Show that if Φ is maximal consistent in **S5** then the set of sententially atomic formulas belonging to Φ is an **S5**-acceptable extended valuation.

Lemma 1.17. *Let Φ be a set of formulas consistent in* **S5**. *Then there is an* **S5**-*acceptable extended valuation α such that $\alpha \models_{\mathrm{t}} \Phi$.*

Proof. Let Σ be the set of all theorems of **S5**. For any formula A, $\Phi \cup \Sigma \vdash A$ if and only if $\Phi \vdash A$. Since $\Phi \nvdash \bot$, $\Phi \cup \Sigma \nvdash \bot$. By Lemma 1.8, $\Phi \cup \Sigma \not\models_{\mathrm{t}} \bot$. Thus there is an extended valuation under which $\Phi \cup \Sigma$ is true (and $\bot$ is false). $\square$

An extended valuation β *conforms* to an extended valuation α if

$$\alpha \models_{\mathrm{t}} \Box B \;\Rightarrow\; \beta \models_{\mathrm{t}} B\,,$$

for every formula B.

Let α be an **S5**-acceptable extended valuation. The *canonical model* on α is (W, α, φ), where W is the set of all **S5**-acceptable extended valuations that conform to α and where, for all $w \in W$ and all i,

$$p_i \in \varphi_w \;\Leftrightarrow\; p_i \in w \;\;(\Leftrightarrow\; w \models_{\mathrm{t}} p_i).$$

Since all instances of the schema T are theorems of **S5**, we have that $\alpha \models_{\mathrm{t}} (\Box B \supset B)$ for all formulas B. Thus α conforms to itself, and the canonical model on α is indeed a model.

Lemma 1.18. *Let* (W, α, φ) *be the canonical model on an* **S5***-acceptable extended valuation* α*. For all* $w \in W$ *and all formulas* A*,*

$$\alpha \models_{\mathrm{t}} \Box A \;\Leftrightarrow\; w \models_{\mathrm{t}} \Box A \,.$$

Proof. Given w and A, suppose first that $\alpha \models_{\mathrm{t}} \Box A$. Since $\vdash \Box A \supset \Box\Box A$, we have that $\alpha \models_{\mathrm{t}} \Box A \supset \Box\Box A$. Thus $\alpha \models_{\mathrm{t}} \Box\Box A$. Because w conforms to α, this gives us that $w \models_{\mathrm{t}} \Box A$.

Now suppose that $\alpha \not\models_{\mathrm{t}} \Box A$. Then $\alpha \models_{\mathrm{t}} \neg\Box A$. The formula $\neg\Box A \supset \Box\neg\Box A$ is (essentially) the contrapositive of $\Diamond\Box A \supset \Box A$, which is a theorem of **S5** by Lemma 1.14. Hence $\alpha \models_{\mathrm{t}} \neg\Box A \supset \Box\neg\Box A$. It follows that $\alpha \models_{\mathrm{t}} \Box\neg\Box A$. Since w conforms to α, $w \models_{\mathrm{t}} \neg\Box A$. $\Box$

Lemma 1.19. *Let* (W, α, φ) *be the canonical model on an* **S5***-acceptable extended valuation* α*. Let* $w \in W$ *and let* A *be a formula. Then*

$$w \not\models_{\mathrm{t}} \Box A \;\Rightarrow\; \text{for some } v \in W,\; v \not\models_{\mathrm{t}} A \,.$$

Proof. Assume that $w \not\models_{\mathrm{t}} \Box A$. By Lemma 1.18, this means that $\alpha \not\models_{\mathrm{t}} \Box A$.

Let

$$\Phi = \{B \mid \alpha \models_{\mathrm{t}} \Box B\} \cup \{\neg A\} \,.$$

Assume first that Φ is consistent in **S5**. By Lemma 1.17, we get an **S5**-acceptable extended valuation v under which Φ is true. Clearly v conforms to α. Hence v belongs to W. Moreover $v \not\models_{\mathrm{t}} A$.

Now assume, in order to derive a contradiction, that Φ is inconsistent in **S5**. By definition,

$$\{B \mid \alpha \models_{\mathrm{t}} \Box B\} \cup \{\neg A\} \;\vdash\; \bot \,.$$

By the Deduction Theorem,

$$\{B \mid \alpha \models_{\mathrm{t}} \Box B\} \;\vdash\; (\neg A \supset \bot) \,.$$

By TC,

$$\{B \mid \alpha \models_{\mathrm{t}} \Box B\} \;\vdash\; A \,.$$

But then $\{B_1, \ldots, B_n\} \vdash A$ for some $\{B_1, \ldots, B_n\} \subseteq \{B \mid \alpha \models_{\mathrm{t}} \Box B\}$. By Lemma 1.13,

$$\{\Box B_1, \ldots, \Box B_n\} \;\vdash\; \Box A \,.$$

Hence by Corollary 1.11,

$$\vdash\; \Box B_1 \supset \cdots \Box B_n \supset \Box A \,.$$

Since α is **S5**-acceptable,

$$\alpha \ \models_{\mathrm{t}} \ \Box B_1 \supset \cdots \Box B_n \supset \Box A\,.$$

Because $\alpha \models_{\mathrm{t}} \Box B_i$ for every i, $\alpha \models_{\mathrm{t}} \Box A$. This is a contradiction. $\Box$

Lemma 1.20. *Let $\mathfrak{M} = (W, \alpha, \varphi)$ be the canonical model on an* **S5***-acceptable extended valuation α. For every formula A and every $w \in W$,*

$$w \models_{\mathrm{t}} A \ \Leftrightarrow \ w \models_{\mathfrak{M}} A\,.$$

Proof. We prove the lemma by formula induction. The only non-trivial case is when A is $\Box B$ for some B.

Suppose first that $w \models_{\mathrm{t}} \Box B$. By Lemma 1.18, $\alpha \models_{\mathrm{t}} \Box B$. Since every $v \in W$ conforms to α, every $v \in W$ is such that $v \models_{\mathrm{t}} B$. By induction, $v \models_{\mathfrak{M}} B$ for every $v \in W$. By definition, $w \models_{\mathfrak{M}} \Box B$.

Now suppose that $w \not\models_{\mathrm{t}} \Box B$. By Lemma 1.19, there is a $v \in W$ such that $v \not\models_{\mathrm{t}} B$. By induction, such a $v \not\models_{\mathfrak{M}} B$. By definition, $w \not\models_{\mathfrak{M}} \Box B$. $\Box$

Theorem 1.21. *The axiomatic system* **S5** *is complete.*

Proof. Let Φ be a set of formulas consistent in **S5**. By Lemma 1.17, let α be an **S5**-acceptable extended valuation under which Φ is true. Let $\mathfrak{M}$ be the canonical model on α. By Lemma 1.20, the formulas true in $\mathfrak{M}$ are exactly the formulas true under α. Hence $\models_{\mathfrak{M}} \Phi$. $\Box$

Corollary 1.22 (Compactness). *If A is a formula of $\mathcal{L}(\Box)$, Φ is a set of formulas of $\mathcal{L}(\Box)$, and $\Phi \models A$, then there is a finite subset Δ of Φ such that $\Delta \models A$.*

Proof. Since deductions of A are finite, this follows from the soundness and completeness of **S5**. $\Box$

Theorem 1.23. *If A is a satisfiable formula (equivalently, if A is a formula consistent in* **S5***), then A is true in some finite model, i.e., in some model with finitely many worlds.*

Proof. Let $\mathfrak{M} = (W, , w_0, \varphi)$ be a model in which A is true. Let $\mathfrak{M}' = (W, w_0, \varphi')$, where φ' agrees with φ on all sentence letters occurring in A but all sentence letters not occurring in A are false at every world in $\mathfrak{M}'$. By Lemma 1.2, A is true in $\mathfrak{M}'$. Suppose there are n sentence letters occurring in A. Then there are at most 2^n distinct valuations φ'_w. Let $w_1, \ldots, w_k$ be such that every valuation that is φ'_w for some $w \in W$ is φ'_{w_i} for some i, $0 \leq i \leq k$. Let $V = \{w_0, w_1, \ldots, w_k\}$ and let $\mathfrak{N} = (V, w_0, \psi)$ where $\psi_{w_i} = \varphi'_{w_i}$ for each $i \leq k$. By Exercise 1.3, A is true in $\mathfrak{N}'$. $\Box$

Corollary 1.24. *The system* **S5** *is decidable, i.e., there is an effective algorithm for deciding whether or not a given formula is a theorem of* **S5**..

Proof. Given a formula A, simultaneously search for (1) a proof of A in **S5** and (2) (the relevant part of) a finite model in which A is false. By Theorem 1.23, one or the other exists. The proof of Theorem 1.23 actually yields a simpler alogorithm: Inspect the relevant parts of all models with $\leq 2^n$ worlds, where n is the number of distinct sentence letters occurring in A. The formula A is a theorem of **S5** if and only if A is true in all these models. $\Box$

Exercise 1.7. Suppose we redefine β *conforms to* α to mean that, for all formulas B, if $\beta \models_{\mathrm{t}} B$ then $\alpha \models_{\mathrm{t}} \Diamond B$. Show that the new definition and the old are equivalent for **S5**-acceptable valuations.

Exercise 1.8. Either prove the following assertion or describe a model witnessing that it is false. Doe not use the soundness of **S5**.

$$\{\Diamond p, \Diamond q\} \models \Diamond(p \wedge q) \vee \Box(p \vee q).$$

Exercise 1.9. Show the following, without using the completeness of **S5**.

(a) $\{\Box p\} \vdash \Diamond p\,;$

(b) $\{\Box(p \wedge q)\} \vdash (\Box p \wedge \Box q)\,;$

(c) $\vdash (\Diamond\Diamond p \supset \Diamond p)\,.$

Hint. For (c), it will help to show that $\vdash \Box\Box p \supset \Box\neg\neg\Box p$.

Exercise 1.10. Suppose we removed the axiom schema T of **S5** and replaced it by a rule of inference according to which a formula A follows from $\Box A$. Would the revised system be sound? Explain. Do you think it would be complete?

Exercise 1.11. Consider the following new sematics for $\mathcal{L}(\Box)$. A model is a $\mathfrak{M} = (W, w_0, \varphi, U_1, U_2)$ where (W, w_0, φ) is a model in the old sense and U_1 and U_2 are disjoint subsets of W whose union is all of W. Define truth as before, except that now, if $w \in U_i$, then $w \models_{\mathfrak{M}} \Box A$ if and only if $v \models_{\mathfrak{M}} A$ for all $v \in U_i$. Prove that the same sets of formulas are valid in the new and old semantics.

2 Relational Semantics

Although the semantics of Section 1 seems reasonable for the general notion of metaphysical necessity, some philosophers question its correctness. Moreover there are other kinds of necessity (such as physical, epistemic, or deontic necessity) for which this semantics may not be appropriate and for which some of the axioms of **S5** seem questionable or wrong. In addition there are other uses of the formal language of modal logic for which the semantics of Section 1 is clearly inappropriate and for which some of the axioms of **S5** are clearly wrong. An example is tense logic, in (one form of) which $\Box A$ is interpreted as "A will be true at every future time."

We now define a new notion of a model for $\mathcal{L}(\Box)$, along with new notions of truth and consequence. These notions will henceforth replace those of Section 1.

A *model* for $\mathcal{L}(\Box)$ is a quadruple (W, R, w_0, φ) such that

(a) (W, w_0, φ) is a model in the sense of Section 1;

(b) R is a binary relation on W.

We will call the relation R the *accessibility relation* of the model. We say that v is *accessible from* w just in case $w \, R \, v$. (So literally the converse of R is accessiblity.)

The word "accessible" should not be taken too seriously. In the context of physical necessity, the worlds accessible from a given world could be thought of as those worlds in which the physical laws of the given world are obeyed. In the context of tense logic, worlds should be thought of as times, with the times accessible from a given time being the later times.

For models $\mathfrak{M} = (W, R, w_0, \varphi)$ and formulas A, we define $w \models_{\mathfrak{M}} A$ by recursion on formulas. Clauses (i)–(iii) of this definition are the same as clauses (i)–(iii) of the corresponding definition in Section 1. (See page 7.) We replace the old clause (iv) by

(iv) $w \models_{\mathfrak{M}} \Box A \;\Leftrightarrow\;$ for all $v \in W$, if $w \, R \, v$ then $v \models_{\mathfrak{M}} A$.

As before, we say that A is *true* in $\mathfrak{M}$ ($\Leftrightarrow \models_{\mathfrak{M}} A$) if and only if $w_0 \models_{\mathfrak{M}} A$.

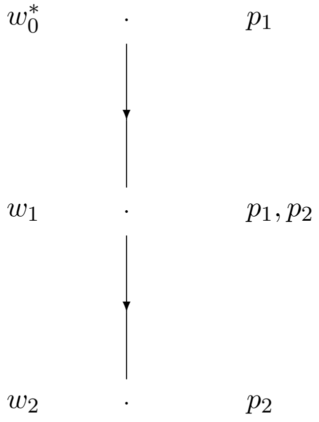
Example. Let $\mathfrak{M} = (W, R, w_0, \varphi)$, where

(i) $W = \{w_0, w_1, w_2\}$;

(ii) $w_i \, R \, w_j \;\Leftrightarrow\; j = i + 1$;

(iii) $\varphi_{w_0} = \{p_1\}$, $\varphi_{w_1} = \{p_1, p_2\}$, and $\varphi_{w_2} = \{p_2\}$.

We can diagram the model $\mathfrak{M}$ as follows:



If one of the worlds bore the relation R to itself, we would indicate this by putting a circle around the dot corresponding the world in question.

The following formulas are true in $\mathfrak{M}$:

$$p_1\,;\ \ \Box p_1\,;\ \ \Box p_2\,;\ \ \Diamond p_1\,;\ \ \Diamond p_2\,;\ \ \neg\Box\Box p_1\,;\ \ \neg\Box\Diamond p_1\,;\ \ \neg(\Box p_2 \supset p_2)\,;$$

Exercise 2.1. Consider the model $\mathfrak{M} = (W, R, w_0, \varphi)$, where

(i) $W = \{w_0, w_1, w_2, w_3\}$;

(ii) $w_i\, R\, w_j \iff (i < j \vee i = j = 0 \vee (i = 3 \wedge j = 0))$;

(iii) $\varphi_{w_0} = \varphi_{w_2} = \{\ \}$ and $\varphi_{w_1} = \varphi_{w_3} = \{p\}$.

Which of the following formulas are true in $\mathfrak{M}$?

$$\Box p\,;\ \ \Diamond p\,;\ \ \Box\Diamond p\,;\ \ \Diamond\Box p\,;\ \ \Box(p \supset \Box\neg p)\,;\ \ \Box(p \vee \Box p)$$

The *implication relation* and *satisfiability* are defined in the same way as in Section 1. (See page 9). A formula or set of formulas is *valid* if and only if is true in every model. (In Section 1, we inadvertently defined validity only for formulas and not for sets of formulas.)

Exercise 2.2. For each of the following formulas, either prove that the formula is valid or give a model in which the formula is false.

$$\Box\Box p \supset \Box p\,;\ \ \Diamond p \vee \Diamond\neg p\,;\ \ \Diamond(p \supset q) \supset (\Diamond p \supset \Diamond q)\,;\ \ \Diamond\Box p \supset \Box p$$

Lemma 2.1. *For any model $\mathfrak{M} = (W, R, w_0, \varphi)$, any world w of $\mathfrak{M}$, and any formulas A and B,* *(1)* $w \models_{\mathfrak{M}} \Diamond A$ *if and only if* A *is true at some world* v *of* $\mathfrak{M}$ *such that* $w\,R\,v$*;*

(2) $w \models_{\mathfrak{M}} A \to B$ *if and only if, for every world* v *of* $\mathfrak{M}$ *such that* $w\,R\,v$*, if* A *is true at* v *then* B *is true at* v*;*

(3) $w \models_{\mathfrak{M}} A \leftrightarrow B$ *if and only if, for every world* v *of* $\mathfrak{M}$ *such that* $w\,R\,v$*,* A *is true at* v *if and only if* B *is true at* v*;*

(4) $w \models_{\mathfrak{M}} \triangledown A$ *if and only if there are worlds* v *and* u *of* $\mathfrak{M}$ *such that* $w\,R\,v$ *and* $w\,R\,u$ *and such that* A *is true at* v *and false at* u*.*

Lemma 2.2. *Let* $\mathfrak{M} = (W, R, w_0, \varphi)$ *and* $\mathfrak{M}' = (W, R, w_0, \varphi')$ *be models. For any formula* A*, if* φ_w *and* φ'_w *agree on all sentence letters occurring in* A *for every* $w \in W$*, then*

$$w \models_{\mathfrak{M}} A \;\Leftrightarrow\; w \models_{\mathfrak{M}'} A$$

for all $w \in W$*.*

Proof. By formula induction, as in the proofs of Lemmas 0.1 and 1.2. $\square$

Lemma 2.3. *For any formula* A *and set* Φ *of formulas,*

(a) $\Phi \models_{\mathrm{t}} A \;\Rightarrow\; \Phi \models A$*;*

(b) if A *and the members of* Φ *are formulas of* $\mathcal{L}$*, then* $\Phi \models_{\mathrm{t}} A \;\Leftrightarrow\; \Phi \models A$*.*

Proof. The proof of part (a) of Lemma 1.3 is word for word a proof of part (a) of the present lemma (with the new meaning of the word "model"). The proof of part (b) of Lemma 1.3 becomes a proof of part (b) of the present lemma if we now let the model $\mathfrak{M}$ of the earlier proof be $(\{w_0\}, R, w_0, \varphi)$ with φ as before and with R either of the two possible relations. $\square$

The axiomatic system K.

The axiomatic system **K** for $\mathcal{L}(\Box)$ is as follows:

Axioms: Taut and K.

Rules of Inference: MP and Nec.

Theorem 2.4. *The system* **K** *is sound.*

Proof. As in the proof of Theorem 1.4, we first show that **K** is weakly sound. Let $\mathcal{P}$ be a proof in **K**. By complete induction we show that every line of $\mathcal{P}$ is valid.

All tautologies are valid by Lemma 2.3.

To see that K is a valid schema, let A and B be formulas and let $\mathfrak{M} = (W, R, w_0, \varphi)$ be a model in which $\Box(A \supset B)$ is true and $\Box A$ is true. Let w be a world of $\mathfrak{M}$ such that $w_0 \, R \, w$. Then $A \supset B$ and A are both true at w in $\mathfrak{M}$. Hence B is true at w in $\mathfrak{M}$. Since B is thus true in every w such that $w_0 \, R \, w$ it follows that $\Box B$ is true in $\mathfrak{M}$.

The conclusion of an instance of MP is a tautological consequence of its premises. By Lemma 2.3, it is a consequence of its premises. Hence it is valid if the premises are.

Suppose A is a valid formula. To see that $\Box A$ is also valid, let $\mathfrak{M} = (W, R, w_0, \varphi)$ be a model. Let $w \in W$. (It is enough to consider only w such that $w_0 \, R \, w$.) Consider the model $\mathfrak{M}' = (W, R, w, \varphi)$. Since A is valid, $\models_{\mathfrak{M}'} A$. Hence $w \models_{\mathfrak{M}} A$. Since w was arbitrary, $\models_{\mathfrak{M}} \Box A$.

The rest of the proof is like that of Theorem 1.4, and we omit it. $\Box$

Exercise 2.3. Show that the following formulas are not theorems of **K**.

$$p \supset \Diamond p \ ; \quad \Box\Box p \supset p \ ; \quad p \supset \Box\Diamond p$$

If **S** and **S**$'$ are axiomatic systems for the same language, then **S** is an *extension* of **S**$'$ if and only if all axioms and rules of **S**$'$ are axioms and rules of **S**. **S** is an *axiomatic extension* of **S**$'$ if and only if **S** is an extension of **S**$'$ and **S** and **S**$'$ have the same rules.

The proofs of Lemmas 1.5–1.9 actually prove the following five lemmas. (Lemma 2.7 is *identical* with Lemma 1.7; we insert it again to keep the correspondence in numbering between the two sections.)

Lemma 2.5. *Let* **S** *be an extension of* **K**. *If* $A \models_{\mathrm{t}} B$, *then* $\vdash_{\mathbf{S}} \Box A \supset \Box B$.

Lemma 2.6. *Let* **S** *be an extension of* **K**. *If* $\vdash_{\mathbf{S}} A \supset B$ *then* $\vdash_{\mathbf{S}} \Box A \supset \Box B$.

Lemma 2.7 (Compactness of $\models_{\mathrm{t}}$). *If* $\Phi \models_{\mathrm{t}} A$ *then there is a finite subset* Δ *of* Φ *such that* $\Delta \models_{\mathrm{t}} A$.

Lemma 2.8. *Let* **S** *be an extension of* **K**. *If* $\Phi \models_{\mathrm{t}} A$ *then* $\Phi \vdash_{\mathbf{S}} A$.

Lemma 2.9. *Let* **S** *be any axiomatic system. If* $\Phi \vdash_{\mathbf{S}} A$ *and* $\Psi \cup \{A\} \vdash_{\mathbf{S}} B$, *then* $\Phi \cup \Psi \vdash_{\mathbf{S}} B$.

The proof of Lemma 1.10 contains a proof of the following lemma.

Lemma 2.10 (Deduction Theorem). *Let* $\mathbf{S}$ *be an axiomatic extension of* $\mathbf{K}$*. If* $\Phi \cup \{A\} \vdash_{\mathbf{S}} B$ *then* $\Phi \vdash_{\mathbf{S}} A \supset B$*.*

Corollary 2.11. *Let* $\mathbf{S}$ *be an axiomatic extension of* $\mathbf{K}$*.* $\{A_1, \ldots, A_n\} \vdash_{\mathbf{S}} B$ *if and only if* $\vdash_{\mathbf{S}} A_1 \supset \cdots A_n \supset B$*.*

The proofs of the next two lemmas are like those of Lemmas 1.12 and 1.13.

Lemma 2.12. *Let* $\mathbf{S}$ *be an extension of* $\mathbf{K}$*. For all positive integers* n*,*

$$\Box(A_1 \supset \cdots \supset A_n) \vdash_{\mathbf{S}} \Box A_1 \supset \cdots \supset \Box A_n\,.$$

Lemma 2.13. *Let* $\mathbf{S}$ *be an axiomatic extension of* $\mathbf{K}$*. If* $\Phi \vdash_{\mathbf{S}} A$ *then* $\Box\Phi \vdash_{\mathbf{S}} \Box A$*.*

The proofs of Lemmas 1.16 and 1.17 can easily be turned into proofs of the following two lemmas.

Lemma 2.14. *Let* $\mathbf{S}$ *be an axiomatic extension of* $\mathbf{K}$*.* $\mathbf{S}$ *is complete if and only if every set of formulas consistent in* $\mathbf{S}$ *is satisfiable.*

Lemma 2.15. *Let* $\mathbf{S}$ *be an extension of* $\mathbf{K}$*. Let* Φ *be a set of formulas consistent in* $\mathbf{S}$*. Then there is an* $\mathbf{S}$*-acceptable extended valuation* α *such that* $\alpha \models_{\mathrm{t}} \Phi$*.*

By a *structure* let us mean a triple (W, R, φ) such that there is a w with (W, R, w, φ) a model. In other words, a structure is a model without a distinguished actual world. For structures $\mathfrak{M} = (W, R, \varphi)$, worlds $w \in W$, and formulas A, we define $w \models_{\mathfrak{M}} A$ in the obvious way. A formula A is *valid* in a structure $\mathfrak{M}$ if and only if $w \models_{\mathfrak{M}} A$ for every world w of $\mathfrak{M}$. A formula A is *satisfiable* in $\mathfrak{M}$ if and only if there is some world w of $\mathfrak{M}$ such that $w \models_{\mathfrak{M}} A$. Similarly define the notion of a set Φ of formulas being *valid* or *satisfiable* in a structure.

Lemma 2.16. *A formula or set of formulas is valid if and only if it is valid in every structure. A formula or set of formulas is satisfiable if and only if it is satisfiable in some structure.*

If $\mathbf{S}$ is a *consistent* axiomatic system ($\not\vdash_{\mathbf{S}} \bot$) containing the axiom schema Taut and the rule MP, the *canonical structure* for $\mathbf{S}$ is (W, R, φ) where

(i) W is the set of all **S**-acceptable extended valuations;

(ii) for w and v belonging to W, $w\,R\,v$ holds if and only if v conforms to w;

(iii) for all i, $p_i \in \varphi_w \;\Leftrightarrow\; p_i \in w\; (\Leftrightarrow\; w \models_{\mathrm{t}} p_i)$.

Lemma 2.17. *Let* **S** *be a consistent axiomatic extension of* **K**. *Let* (W, R, φ) *be the canonical structure for* **S**. *Let* $w \in W$ *and let* A *be a formula. Then*

$$w \not\models_{\mathrm{t}} \Box A \;\Rightarrow\; \text{for some } v \in W,\; w\,R\,v \textit{ and } v \not\models_{\mathrm{t}} A\,.$$

Proof. Assume that $w \not\models_{\mathrm{t}} \Box A$.

Let

$$\Phi = \{B \mid w \models_{\mathrm{t}} \Box B\} \cup \{\neg A\}\,.$$

Assume first that Φ is consistent in **S**. By Lemma 2.15, there is an **S**-acceptable extended valuation (i.e., a member of W) in which Φ is true. Let v be such an extended valuation. Clearly v conforms to w. Hence $w\,R\,v$. Moreover $v \not\models_{\mathrm{t}} A$.

The rest of the proof is like that of Lemma 1.19. Assume, in order to derive a contradiction, that Φ is inconsistent in **S**. By definition,

$$\{B \mid w \models_{\mathrm{t}} \Box B\} \cup \{\neg A\} \;\vdash_{\mathbf{S}}\; \bot\,.$$

By the Deduction Theorem,

$$\{B \mid w \models_{\mathrm{t}} \Box B\} \;\vdash_{\mathbf{S}}\; (\neg A \supset \bot)\,.$$

By TC,

$$\{B \mid w \models_{\mathrm{t}} \Box B\} \;\vdash_{\mathbf{S}}\; A\,.$$

But then $\{B_1, \ldots, B_n\} \vdash_{\mathbf{S}} A$ for some $\{B_1, \ldots, B_n\} \subseteq \{B \mid w \models_{\mathrm{t}} \Box B\}$. By Lemma 2.13,

$$\{\Box B_1, \ldots, \Box B_n\} \;\vdash_{\mathbf{S}}\; \Box A\,.$$

Hence by Corollary 2.11,

$$\vdash_{\mathbf{S}}\; \Box B_1 \supset \cdots \Box B_n \supset \Box A\,.$$

Since w is **S**-acceptable,

$$w \;\models_{\mathrm{t}}\; \Box B_1 \supset \cdots \Box B_n \supset \Box A\,.$$

Because $w \models_{\mathrm{t}} \Box B_i$ for every i, $w \models_{\mathrm{t}} \Box A$. This is a contradiction. $\Box$

Lemma 2.18. *Let* **S** *be a consistent axiomatic extension of* **K**. *Let* $\mathfrak{M} = (W, R, \varphi)$ *be the canonical structure for* **S**. *For every formula* A *and every* $w \in W$,

$$w \models_{\mathrm{t}} A \;\Leftrightarrow\; w \models_{\mathfrak{M}} A\,.$$

Proof. The proof is by formula induction and is just like the proof of Lemma 1.20. □

Theorem 2.19. *The axiomatic system* **K** *is complete.*

Proof. Let Φ be a set of formulas consistent in **K**. By Lemma 2.15, let w be a **K**-acceptable extended valuation in which Φ is true. Let $\mathfrak{M} = (W, R, \varphi)$ be the canonical structure for **K**. By Lemma 2.18, the formulas true at w in $\mathfrak{M}$ are exactly the formulas A such that $w \models_{\mathrm{t}} A$. Hence $w \models_{\mathfrak{M}} \Phi$. This shows that Φ is satisfiable in a structure, hence that Φ is satisfiable. □

Corollary 2.20 (Compactness). *If* A *is a formula of* $\mathcal{L}(\Box)$, Φ *is a set of formulas of* $\mathcal{L}(\Box)$, *and* $\Phi \models A$, *then there is a finite subset* Δ *of* Φ *such that* $\Delta \models A$.

Proof. Since deductions of A are finite, this follows from the soundness and completeness of **K**. □

Theorem 2.21. *If* A *is a satisfiable formula (equivalently, if* A *is a formula consistent in* **K***), then* A *is true in some finite model, i.e., in some model with finitely many worlds.*

Proof. (Sketch) Let $\mathcal{L}_A$ be the language whose formulas are all formulas of $\mathcal{L}(\Box)$ whose sententially atomic subformulas are sententially atomic subformulas of A. Let $\mathbf{K}_A$ be the axiomatic system for $\mathcal{L}_A$ whose axioms are those axioms of K that are formulas of $\mathcal{L}_A$ and whose rules are MP and Nec, except that Nec is restricted to formulas B such that $\Box B$ is a formula of $\mathcal{L}_A$. One can check that all our results from Theorem 2.4 through Theorem 2.19 go through for $\mathbf{K}_A$. (We must fully restrict to $\mathcal{L}_A$; e.g., in Lemma 2.13, all the formulas belonging to $\Box\Phi \cup \{\Box A\}$ must be formulas of $\mathcal{L}_A$.) Thus any satisfiable formula of $\mathcal{L}_A$ is satisfiable in the canonical structure for $\mathbf{K}_A$, which has only finitely many worlds (because there are only finitely many sententially atomic formulas of $\mathcal{L}_A$). The canonical structure for $\mathcal{L}_A$ can be extended to a structure for $\mathcal{L}(\Box)$ with the same worlds by making all sentence letters not occurring in A false in all the worlds. In this structure, A will be true at the same world as in the structure for $\mathcal{L}_A$. □

Corollary 2.22. *The system* **K** *is decidable.*

Exercise 2.4. Do the $\Box$ case of the proof of Lemma 2.18.

Axiomatic extensions of K.

A *frame* is a pair (W, R) such that W is a non-empty set and R is a binary relation on W. If $\mathfrak{M} = (W, R, w_0, \varphi)$ is a model, then (W, R) is the *frame* of $\mathfrak{M}$. Similarly (W, R) is the *frame* of the structure (W, R, φ).

A formula A is *valid* in a frame $\mathfrak{F}$ if and only if A is true in every model whose frame is $\mathfrak{F}$. A formula A is *satisfiable* in $\mathfrak{F}$ if and only if A is true in some model whose frame is $\mathfrak{F}$. Similarly define the notion of a set Φ of formulas being *valid* or *satisfiable* in a frame.

Genuinely proper extensions of **K** are not sound for the relational semantics. To get soundness and completeness results for them, we make the following definitions. If $\mathcal{F}$ is a class of frames, Φ is a set of formulas, and A is a formula, then we write $\Phi \models_{\mathcal{F}} A$ to mean that, for every model $\mathfrak{M}$ whose frame belongs to $\mathcal{F}$, if $\models_{\mathfrak{M}} \Phi$, then $\models_{\mathfrak{M}} A$. An axiomatic system **S** is *sound* for $\mathcal{F}$ if and only if, for all Φ and A,

$$\Phi \vdash_{\mathbf{S}} A \;\Rightarrow\; \Phi \models_{\mathcal{F}} A\,.$$

S is *complete* for $\mathcal{F}$ if and only if, for all Φ and A,

$$\Phi \models_{\mathcal{F}} A \;\Rightarrow\; \Phi \vdash_{\mathbf{S}} A\,.$$

Lemma 2.23. *Let $\mathcal{F}$ be a class of frames and let* **S** *be an axiomatic extension of* **K**. *The system* **S** *is sound for $\mathcal{F}$ if and only $\models_{\mathcal{F}} A$ for every axiom A of* **S** *that is not an axiom of* **K** *(i.e., if and only if every such axiom is valid in all frames in $\mathcal{F}$).*

Proof. If there is any axiom A of **S** such that $\not\models_{\mathcal{F}} A$, then A is a counterexample to soundness of **S** for $\mathcal{F}$.

For the other direction, assume that $\models_{\mathcal{F}} A$ for every axiom A of **S** that is not an axiom of **K**. In the proof of Lemma 2.4, replace“valid” by “valid in every frame in $\mathcal{F}$” and replace “**K**” by “**S**.” Our assumption provides the additional fact needed to make this into a proof of that **S** is sound for $\mathcal{F}$. $\Box$

Let **T** be the axiomatic extension of **K** resulting from adjoining to **K** the axiom schema T.

A frame (W, R) is *reflexive* if and only if $w\,R\,w$ for every $w \in W$.

Theorem 2.24. **T** *is sound for the class of all reflexive frames.*

Proof. By Lemma 2.23, it is enough to show that the schema T is valid in every reflexive frame. Let $\mathfrak{M} = (W, R, w_0, \varphi)$ be a model whose frame is reflexive and let A be a formula. Assume that $\Box A$ is true in $\mathfrak{M}$. Since $w_0 \, R \, w_0$, it follows that A is true in $\mathfrak{M}$. $\Box$

The proof of the following lemma is just like that of Lemmas 1.16 and 2.14.

Lemma 2.25. *Let* **S** *be an axiomatic extension of* **K** *and let* $\mathcal{F}$ *be a class of frames.* **S** *is complete for* $\mathcal{F}$ *if and only if every set of formulas consistent in* **S** *is satisfiable in some frame belonging to* $\mathcal{F}$*.*

The *canonical frame* for a consistent axiomatic system **S** containing Taut and MP is the frame of the canonical structure for **S**.

Theorem 2.26. **T** *is complete for the class of all reflexive frames.*

Proof. Let Φ be a set of formulas consistent in **T**. By Lemma 2.15, let w_0 be a **T**-acceptable extended valuation in which Φ is true. By Lemma 2.18, Φ is true at w_0 in the canonical structure for **T**. Therefore we need only show that the canonical frame $\mathfrak{F} = (W, R)$ for **T** is reflexive.

Let $w \in W$ and let B be a formula such that $w \models_{\mathrm{t}} \Box B$. Since

$$\vdash_{\mathbf{T}} \; \Box B \supset B \,,$$

the **T**-acceptability of w implies that

$$w \;\models_{\mathrm{t}}\; \Box B \supset B \,.$$

Hence $w \models_{\mathrm{t}} B$. This argument shows that w conforms to itself, and so that $w \, R \, w$. $\Box$

Exercise 2.5. *Show that the following formulas are not theorems of* **T***.*

$$\Box\Diamond p \supset \Diamond\Box p \; ; \quad \Diamond(p \wedge \Diamond\neg p) \supset \Diamond\neg p$$

Let **KB** be the axiomatic extension of **K** resulting from adjoining to **K** the axiom schema B given as follows:

$$\text{(B)} \qquad A \supset \Box\Diamond A$$

A frame (W, R) is *symmetric* if and only if, for all w and v belonging to W,

$$w \, R \, v \;\Rightarrow\; v \, R \, w \,.$$

Theorem 2.27. **KB** *is sound and complete for the class of all symmetric frames.*

Exercise 2.6. Prove Theorem 2.27.

Let **K4** be the axiomatic extension of **K** resulting from adjoining to **K** the axiom schema 4 given as follows:

$$(4) \qquad \Box A \supset \Box\Box A$$

Exercise 2.7. Show directly, without using the completeness and soundness theorems below, that the following formulas are theorems of K4.

$$\Diamond\Diamond A \supset \Diamond A\ ; \quad \Diamond(A \wedge \Diamond B) \supset \Diamond B$$

A frame (W, R) is *transitive* if and only if, for all w, v, and u belonging to W,

$$(w\,R\,v \,\text{and}\, v\,R\,u) \ \Rightarrow \ w\,R\,u\,.$$

Theorem 2.28. **K4** *is sound for the class of all transitive frames.*

Proof. By Lemma 2.23, it is enough to show that the schema 4 is valid in all transitive frames. Let $\mathfrak{M} = (W, R, w_0, \varphi)$ be a model whose frame is transitive and let A be a formula. Assume that $\models_{\mathfrak{M}} \Box A$. We must show that $\models_{\mathfrak{M}} \Box\Box A$. Let $w \in W$ be such that $w_0\,R\,w$. We must show that $w \models_{\mathfrak{M}} \Box A$. For this let $v \in W$ be such that $w\,R\,v$. We must show that $v \models_{\mathfrak{M}} A$. By the transitivity of the frame, we have that $w_0\,R\,v$. But $w_0 \models_{\mathfrak{M}} \Box A$, and so $v \models_{\mathfrak{M}} A$. $\Box$

Theorem 2.29. **K4** *is complete for the class of all transitive frames.*

Proof. As in the proof of Theorem 2.26, we need only show that the canonical frame (W, R) for **K4** is transitive.

Let w, v, and u be members of W such that $w\,R\,v$ and $v\,R\,u$. To prove that $w\,R\,u$, let B be a formula such that $w \models_{\mathrm{t}} \Box B$. Since w is **K4**-acceptable, we have that $w \models_{\mathrm{t}} \Box B \supset \Box\Box B$. Hence $w \models_{\mathrm{t}} \Box\Box B$. Since v conforms to w, $v \models_{\mathrm{t}} \Box B$. Since u conforms to v, $u \models_{\mathrm{t}} B$. This argument show that u conforms to w. $\Box$

Let **S4** (= **KT4**) be the the axiomatic extension of **K** resulting from adjoining to **K** the axiom schemas T and 4. Let **B** (= **KTB**) result from adjoining to **K** the axiom schemas T and B. Let **K4B** result from adjoining to **K** the axiom schemas 4 and B.

Theorem 2.30. *(a)* **S4** *is sound and complete for the class of all reflexive and transitive frames.*

(b) **B** *is sound and complete for the class of all reflexive and symmetric frames.*

(c) **K4B** *is sound and complete for the class of all transitive and symmetric frames.*

Proof. We indicate how to prove (a). (b) and (c) are similar. In the proofs of Theorems 2.24 and 2.28, we showed that the schema T is valid in all reflexive frames and that 4 is valid in all transitive frames. Thus both are valid in all reflexive and transitive frames. By Lemma 2.23, this gives the soundness part of (a). For completeness, we need only prove that the canonical frame for **S4** is reflexive and transitive. The proofs of Theorems 2.26 and 2.29 actually show that the canonical frame for any extension of **T** is reflexive and that the canonical frame for any extension of **K4** is transitive. $\square$

Let **KT4B** be the axiomatic extension of **K** resulting from adjoining to **K** the axiom schemas T, 4, and B.

An *equivalence* frame is a frame that is reflexive, transitive, and symmetric. In other language: (W, R) is an equivalence frame if and only if R is an *equivalence relation* on W.

Theorem 2.31. **KT4B** *is sound and complete for the class of all equivalence frames.*

Proof. The proof is like that of Theorem 2.30. $\square$

Corollary 2.32. **KT4B** *and* **S5** *are equivalent; i.e., for any set* Φ *of formulas and any formula* A*,*

$$\Phi \vdash_{\mathbf{KT4B}} A \;\Leftrightarrow\; \Phi \vdash_{\mathbf{S5}} A\,.$$

Proof. Let $\mathcal{E}$ be the class of all equivalence frames. By Theorems 1.4, 1.21, and 2.31, it is enough to prove that, for any Φ and A, $\Phi \models_{\mathcal{E}} A$ if and only if $\Phi \models A$ in the sense of Section 1.

Let $\mathfrak{M} = (W, R, w_0, \varphi)$ be any model such that (W, R) is an equivalence frame. Let $U = \{w \in W \mid w_0 \, R \, w\}$. Let $\mathfrak{N} = (U, w_0, \psi)$, where $\psi_w = \varphi_w$ for all $w \in U$. Then $\mathfrak{N}$ is a model in the sense of Section 1, and it is easy to show that the same formulas are true in $\mathfrak{M}$ and $\mathfrak{N}$. If $\Phi \models A$ in the sense of Section 1 and if $\models_{\mathfrak{M}} \Phi$, then $\models_{\mathfrak{N}} \Phi$, so $\models_{\mathfrak{N}} A$, and so $\models_{\mathfrak{M}} A$. This establishes the "if" part.

Let $\mathfrak{M} = (W, w_0, \varphi)$ be a model in the sense of Section 1. Let $\mathfrak{N} = (W, R, w_0, \varphi)$, where $w\,R\,v$ holds for all w and v belonging to W. The frame (W, R) is an equivalence frame, and the same formulas are true in $\mathfrak{M}$ and $\mathfrak{N}$. An argument analogous to that of the preceding paragraph establishes the "only if" part. $\Box$

Remarks:

(a) Corollary 2.32 can also be proved directly by showing that the axioms of each of the two systems are derivable in the other system.

(b) All the axiomatic extensions of **K** considered in this section can be shown to have the finite model property and so to be decidable.

Exercise 2.8. Which of the following formulas are valid (i.e., valid in all frames)? Explain your answers.

(a) $(p \supset \Box\Diamond p) \supset (\Diamond\Box\neg p \supset \neg p)$.

(b) $\Diamond(p \supset \Diamond p)$.

(c) $\Box\bot \supset \bot$.

Exercise 2.9. Is formula (b) of Exercise 2.8 valid in all reflexive frames? Explain.

Exercise 2.10. Is the following formula a theorem of the system **B**? Explain.

$$\Box p \supset \Box\Box p$$

Exercise 2.11. Is the following formula a theorem of the system **S4**? Explain.

$$p \supset \Box\Diamond p$$

Exercise 2.12. Is $\Diamond\Diamond p \supset \Diamond p$ valid in all transitive frames? Explain.

3 Modal Predicate Logic

The language $\mathcal{L}^*$ of predicate logic.

For each set C of symbols (not containing any of the symbols under (i)–(iv) below), we define a language $\mathcal{L}^*_C$. ($\mathcal{L}^*$ is $\mathcal{L}^*_{\{\ \}}$.)

Symbols:

(i)	all symbols of $\mathcal{L}$	
(ii)	for $n \geq 1$, n-ary predicate letters	P^n_0, P^n_1, $P^n_2, \ldots$
(iii)	quantifier	$\forall$
(iv)	variables	$v_0, v_1, v_2, \ldots$
(v)	constants	all members of C

Formulas:

(i) Atomic formulas:

  (a) Each sentence letter is a formula.

  (b) For each n and i, if $t_1, \ldots, t_n$ are variables or constants, then $P^n_i t_1 \ldots t_n$ is a formula.

(ii) If A is a formula, then so is $\neg A$.

(iii) If A and B are formulas, then so is $(A \supset B)$.

(iv) If A is a formula and x is a variable, then $\forall x\, A$ is a formula.

(v) Nothing is a formula unless its being one follows from (i)–(iv).

Abbreviations:

$\exists v_i$	for	$\neg\forall v_i \neg$
P^1	for	P^1_0
P^2	for	P^2_0

Semantics.

A *model* for $\mathcal{L}^*_C$ is triple $\mathfrak{M} = (D, \varphi, \chi)$, where

(i) D is a non-empty set;

(ii) φ is a set whose members are sentence letters and $n+1$-tuples of the form $(P^n_i, d_1, \ldots, d_n)$ with each $d_j \in D$;

(iii) χ is a function that assigns to each constant $c \in C$ an element $\chi(c)$ of D.

An occurrence of a variable v_i in a formula A is *free* if the occurrence is not within any subformula of A of the form $\forall v_i B$. A *sentence* is a formula with no free occurrences of variables.

Let $\mathfrak{M} = (D, \varphi, \chi)$ be a model for $\mathcal{L}^*_C$. We define truth in $\mathfrak{M}$ as follows. Let $C' = \{c_d \mid d \in D\}$ be a set of symbols distinct from those of of $\mathcal{L}^*_C$. Let $\mathfrak{M}' = (D, \varphi, \chi')$ be the model for $\mathcal{L}^*_{C \cup C'}$ given by setting $\chi'(c) = \chi(c)$ for $c \in C$ and $\chi'(c_d) = d$ for $d \in D$. For sentences A of $\mathcal{L}^*_{C \cup C'}$, we define by recursion on formulas the notion $\models_{\mathfrak{M}'} A$ (A is *true* in $\mathfrak{M}'$).

(i) The case of A atomic:

(a) $\models_{\mathfrak{M}'} p_i \ \leftrightarrow \ p_i \in \varphi\,;$

(b) $\models_{\mathfrak{M}'} P^n_i c_1, \ldots, c_n \ \leftrightarrow \ (P^n_i, \chi'(c_1), \ldots, \chi'(c_n)) \in \varphi\,;$

(ii) $\models_{\mathfrak{M}'} \neg A \ \leftrightarrow \not\models_{\mathfrak{M}'} A\,;$

(iii) $\models_{\mathfrak{M}'} (A \supset B) \ \leftrightarrow \ (\not\models_{\mathfrak{M}'} A$ or $\models_{\mathfrak{M}'} B)\,;$

(iv) $\models_{\mathfrak{M}'} \forall x\, A(x) \ \leftrightarrow$ for all $d \in D$, $\models_{\mathfrak{M}'} A(c_d)$, where $A(c_d)$ is the result of replacing each free occurrence of x in $A(x)$ by an occurrence of the constant c_d.

If A is a sentence of $\mathcal{L}^*_C$, then A is *true* in $\mathfrak{M}$ ($\Leftrightarrow \models_{\mathfrak{M}} A$) if and only if A is true in $\mathfrak{M}'$.

Remark. Note that for models $\mathfrak{M}'$ such as that used in the definition, we have *two* definitions of truth in $\mathfrak{M}'$. It is easy to verify that these definitions are equivalent.

If Φ is a set of sentences and A is a sentence, then Φ *implies* A ($\Leftrightarrow A$ is a *consequence* of $\Phi \Leftrightarrow \Phi \models A$) if and only if A is true in every model in which Φ is true (in which all members of Φ are true). Define validity and satisfiability in the obvious way.

Deduction.

For each set C, the axiomatic system $\mathbf{PL}_C$ for $\mathcal{L}^*_C$ is as follows.

Axioms:

(Taut) All tautologies

(Dist) $\forall x(A \supset B) \supset (\forall x\, A \supset \forall x\, B)$

(VacQ) $A \supset \forall x\, A$ (x not free in A)

(UI) $\forall x\, A(x) \supset A(t)$ ($A(t)$ comes by replacing each free occurrence of x in $A(x)$ by a free occurrence of the variable or constant t)

Rules of Inference:

(MP) $$\frac{A \quad A \supset B}{B}$$

(Gen) $$\frac{A}{\forall x\, A}$$

We consider only deductions from sets Φ of sentences, and we define $\Phi \vdash_{\mathbf{PL}_C} A$ only for Φ a set of sentences and A a sentence. The definition is the obvious one.

Theorem 3.1. *For each set C, the axiomatic system $\mathbf{PL}_C$ is sound.*

Proof. For each formula A let the *universal closure* of A be $\forall v_{i_1} \ldots \forall v_{i_k} A$, where $i_1, \ldots, i_k$ are, in order, the numbers i such that the variable v_i occurs free in A. Sentences are thus their own universal closures.

Let $\mathcal{D}$ be a deduction in $\mathbf{PL}_C$ from a set Φ of sentences. By complete induction, we can show that for each line A of $\mathcal{D}$, $\Phi \models A^*$, where A^* is the universal closure of A. We omit the details. □

As we did with formulas of $\mathcal{L}(\Box)$, say that a formula of $\mathcal{L}^*_C$ is *sententially atomic* if and only if it does not have either the form $\neg A$ or the form $(A \supset B)$. Let us define an *extended valuation* for $\mathcal{L}^*_C$ to be a set of sententially atomic sentences of $\mathcal{L}^*_C$. Define in the obvious way $\alpha \models_{\mathrm{t}} A$ for extended valuations α and sentences A. For axiomatic systems $\mathbf{S}$, say that an extended valuation

α is **S**-*acceptable* if and only if $\alpha \models_{\mathrm{t}} A$ for every theorem A of **S**, i.e., for every sentence A such that $\vdash_{\mathbf{S}} A$. Say that an extended valuation α is *Henkin* if and only if, for all sentences $\forall x\, A(x)$, if $\alpha \not\models_{\mathrm{t}} \forall x\, A(x)$, then there is a constant $c \in C$ such that $\alpha \not\models_{\mathrm{t}} A(c)$, where $A(c)$ comes from $A(x)$ by replacing all free occurrences of x by occurrences of c.

Lemma 3.2. *Let Φ be a set of sentences consistent in $\mathbf{PL}_C$. Then there is a set $C^* \supseteq C$ and there is a Henkin, $\mathbf{PL}_{C^*}$-acceptable extended valuation α such that $\alpha \models_{\mathrm{t}} \Phi$ (i.e., $\alpha \models_{\mathrm{t}} A$ for every $A \in \Phi$).*

Proof. We give only the barest sketch. Choose C^* such that there are as many new constants as there are sentences of $\mathcal{L}^*_C$. (It follows that there are as many new constants as there are sentences of $\mathcal{L}^*_{C^*}$.) One can show that Φ is still consistent in $\mathbf{PL}_{C^*}$.

Construct as follows a set of $\Phi^* \supseteq \Phi$ of sentences of $\mathcal{L}^*_{C^*}$. Consider in turn each sentence A of $\mathcal{L}^*_{C^*}$. For such an A, let Φ' be the set of sentences already put into Φ^* by the time A is considered. If $\Phi' \cup \{A\}$ is consistent in $\mathbf{PL}_{C^*}$, then put A into Φ^*. Otherwise put $\neg A$ into Φ^*. In the latter case, if A is $\forall x\, B(x)$, choose a constant $c \in C^*$ that does not occur in Φ' or in A. Add $\neg B(c)$ to Φ^*.

Now let α be the set of sententially atomic formulas that belong to Φ^*. $\square$

If α is a Henkin, $\mathbf{PL}_C$-acceptable extended valuation for $\mathcal{L}^*_C$, then the *canonical model* on α is (C, φ, χ), where

$$\begin{aligned} p_i \in \varphi \quad &\leftrightarrow \quad p_i \in \alpha\,; \\ (P^n_i, c_1, \ldots, c_n) \in \varphi \quad &\leftrightarrow \quad P^n_i c_1 \ldots c_n \in \alpha\,; \\ \chi(c) \quad &= \quad c\,. \end{aligned}$$

Lemma 3.3. *Let $\mathfrak{M}$ be the canonical model on a Henkin, $\mathbf{PL}_C$-acceptable extended valuation α for $\mathcal{L}^*_C$. For every sentence A of $\mathcal{L}^*_C$,*

$$\models_{\mathfrak{M}} A \;\leftrightarrow\; \alpha \models_{\mathrm{t}} A\,.$$

Proof. We proceed by induction of A. The case A atomic follows directly from the definitions. The cases of negations and conjunctions are routine.

Suppose that A is $\forall x\, B(x)$.

Assume first that $\alpha \models_{\mathrm{t}} A$. Since by UI

$$\vdash_{\mathbf{PL}_C} \forall x B(x) \supset B(c)\,,$$

for all $c \in C$, it follows by $\mathbf{PL}_C$-acceptability that $\alpha \models_{\mathrm{t}} B(c)$ for all $c \in C$. By induction, $\models_{\mathfrak{M}} B(c)$ for all $c \in C$. Hence $\models_{\mathfrak{M}} A$.

Next assume that $\alpha \not\models_{\mathrm{t}} A$. Since α is Henkin, there is a $c \in C$ such that $\alpha \not\models_{\mathrm{t}} B(c)$. By induction, $\not\models_{\mathfrak{M}} B(c)$. Hence $\not\models_{\mathfrak{M}} A$. $\square$

Theorem 3.4. *For each set C, the axiomatic system $\mathbf{PL}_C$ is complete.*

Proof. As for earlier systems, one can show that $\mathbf{PL}_C$ is complete if and only if every set of sentences consistent in $\mathbf{PL}_C$ is satsifiable. Let then Φ be consistent in $\mathbf{PL}_C$. By Lemma 3.2, let α be, for some $C^* \supseteq C$, a Henkin, $\mathbf{PL}_{C^*}$-acceptable extended valuation such that $\alpha \models_{\mathrm{t}} \Phi$. By Lemma 3.3, Φ is true in the canonical model $\mathfrak{M}'$ on α. Let $\mathfrak{M}$ be the *reduct* of $\mathfrak{M}'$ to $\mathcal{L}^*_C$ (i.e., get $\mathfrak{M}$ by removing the extra constants and the restricting χ to C). Then Φ is true in $\mathfrak{M}$. $\square$

The language $\mathcal{L}^*(\Box)$ of modal predicate logic.

For each set C we define a language $\mathcal{L}^*_C(\Box)$. ($\mathcal{L}^*(\Box)$ is $\mathcal{L}^*_{\{\ \}}(\Box)$.)

Symbols: All symbols of $\mathcal{L}^*_C$ plus $\Box$.

Formulas: Add to the recursive definition of the formulas of $\mathcal{L}^*_C$ the following clause (and renumber the old clause (v)):

(v) If A is a formula, then so is $\Box A$.

Constant domain semantics.

A *model* for $\mathcal{L}^*_C(\Box)$ is a quintuple $(W, D, w_0, \varphi, \chi)$ where

(1) W is a non-empty set;

(2) D is a non-empty set;

(3) $w_0 \in W$;

(4) φ is a function that assigns to each $w \in W$ a set φ_w;

(5) χ is a function that assigns to each $c \in C$ an element $\chi(c)$ of D;

(6) for each $w \in W$, (D, φ_w, χ) is a model for $\mathcal{L}^*_C$.

Let $\mathfrak{M} = (W, D, w_0, \varphi, \chi)$ be a model for $\mathcal{L}^*_C(\Box)$. We define truth in $\mathfrak{M}$ as follows. Let $C' = \{c_d \mid d \in D\}$ be a set of symbols distinct from those of of $\mathcal{L}^*_C$. Let $\mathfrak{M}' = (W, D, w_0, \varphi, \chi')$ be the model for $\mathcal{L}^*_{C \cup C'}(\Box)$ given by setting $\chi'(c) = \chi(c)$ for $c \in C$ and $\chi'(c_d) = d$ for $d \in D$. For worlds $w \in W$ and sentences A of $\mathcal{L}^*_{C \cup C'}$, we define by recursion on A the notion $w \models_{\mathfrak{M}'} A$ (A is *true* in $\mathfrak{M}'$ at w).

(i) The case of A atomic:

(a) $w \models_{\mathfrak{M}'} p_i \;\leftrightarrow\; p_i \in \varphi_w$;
(b) $w \models_{\mathfrak{M}'} P^n_i c_1, \ldots, c_n \;\leftrightarrow\; (P^n_i, \chi'(c_1), \ldots, \chi'(c_n)) \in \varphi_w$;

(ii) $w \models_{\mathfrak{M}'} \neg A \;\leftrightarrow\; w \not\models_{\mathfrak{M}'} A$;

(iii) $w \models_{\mathfrak{M}'} (A \supset B) \;\leftrightarrow\; (w \not\models_{\mathfrak{M}'} A$ or $w \models_{\mathfrak{M}'} B)$;

(iv) $w \models_{\mathfrak{M}'} \forall x\, A(x) \;\leftrightarrow\;$ for all $d \in D$, $w \models_{\mathfrak{M}'} A(c_d)$, where $A(c_d)$ is the result of replacing each free occurrence of x in $A(x)$ by an occurrence of the constant c_d;

(v) $w \models_{\mathfrak{M}'} \Box A \;\leftrightarrow\;$ for all $v \in W$, $v \models_{\mathfrak{M}'} A$.

If A is a sentence of $\mathcal{L}^*_C(\Box)$, then $w \models_{\mathfrak{M}} A$ if and only if $w \models_{\mathfrak{M}'} A$, and $\models_{\mathfrak{M}} A$ ($\Leftrightarrow$ A is *true* in $\mathfrak{M}$) if and only if $w_0 \models_{\mathfrak{M}} A$.

If Φ is a set of sentences and A is a sentence, then we say that Φ *implies* A ($\Leftrightarrow \Phi \models A$) if and only if A is true in every model in which Φ is true (in which all members of Φ are true). Define validity and satisfiability in the obvious way.

Example. Consider the following model $\mathfrak{M} = (W, D, w_0, \varphi, \chi)$ for the language with $C = \{c\}$.

$$\begin{aligned} W &= \{w_0, w_1\} \\ D &= \{d_1, d_2\} \\ \varphi_0 &= \{(P^1, d_1)\} \\ \varphi_1 &= \{(P^1, d_2)\} \\ \chi(c) &= d_1 \end{aligned}$$

Then $\models_{\mathfrak{M}} \Box\exists v_1 P^1 v_1$, but $\not\models_{\mathfrak{M}} \exists v_1 \Box P^1 v_1$. Hence the formula

$$\Box\exists v_1 P^1 v_1 \supset \exists v_1 \Box P^1 v_1$$

is not valid.

Exercise 3.1. *Show that the following formulas are not valid (where x is v_1, and y is v_2).*

(a) $\forall x \Diamond P^1 x \supset \Diamond \forall x\, P^1 x$

(b) $(\Diamond \exists x\, P_1^1 x \wedge \Diamond \exists x\, P_2^1 x) \supset \Diamond \exists x (P_1^1 x \wedge P_2^1 x)$

Exercise 3.2. *Show that the following formulas are valid.*

(a) $(\Diamond \exists x\, P_1^1 x \wedge \Box \forall x\, P_2^1 x) \supset \Diamond \exists x (P_1^1 x \wedge P_2^1 x)$

(b) $\exists x \Box \forall y P^2 xy \supset \Box \forall y \exists x P^2 xy$

Remarks:

(a) The constant domain semantics and the variable domain semantics that we shall introduce later both seem committed to a somewhat controversial view about modality and objects: the acceptance of *de re modality*. This view holds that it is legitimate to attribute modal properties directly to an object: e.g., to say that the object necessarily has some property.

(b) The constant domain semantics has an obvious defect. It seems true of the intuitive notion of necessity that some objects in the actual world exist only contingently and also that there might have been objects that are not among the actually existing ones. Constant domain semantics does not allow for this. It does not allow one to talk of existing objects whose existence is not necessary or of possible objects that do not actually exist. One way to take care of this would be to keep the constant domain semantics but to add a new logical unary predicate letter E, interpreting Ex as "x exists." We shall say a little more about this possibility later. Another solution to the problem is the variable domain semantics that we introduce later.

Deduction

For each set C, the axiomatic system $\mathbf{S5B}_C$ for $\mathcal{L}_C^*(\Box)$ is as follows. (The "**B**" is for "Barcan," i.e., for Ruth Barcan Marcus.)

Axioms: The schemas Taut, Dist, VacQ, UI, K, T, and 5

Rules of Inference: MP, Gen, and Nec.

Theorem 3.5. *For each set C, the axiomatic system* $\mathbf{S5B}_C$ *is sound.*

Proof. We give only the plan of the proof. One first shows by induction that for every line A of a proof in $\mathbf{S5B}_C$, the universal closure of A is valid. Then one shows by induction that, for every line A of a deduction from a set Φ of sentences, $\Phi \models A^*$, where A^* is the universal closure of A. $\square$

Define *sententially atomic*, *extended valuation*, $\mathbf{S5B}_C$*-acceptable*, *Henkin*, *conforms*, etc. in the obvious ways. Say that a set W of extended valuations (for $\mathcal{L}^*_C(\Box)$) is *modally Henkin* if and only if, for all $\alpha \in W$ and for all sentences A, if $\alpha \not\models_{\mathrm{t}} \Box A$ then there is a $\beta \in W$ such that $\beta \not\models_{\mathrm{t}} A$. Say that a set W of extended valuations is *nice* if if and only if W is non-empty and modally Henkin, its members are Henkin and $\mathbf{S5B}_C$-acceptable, and any two members of W conform to one another.

Lemma 3.6. *Let Φ be a set of sentences of $\mathcal{L}^*_C(\Box)$ that is consistent in* $\mathbf{S5B}_C$*. Then there is a $C^* \supseteq C$ and there is a nice set W of extended valuations for $\mathcal{L}^*_{C^*}(\Box)$ such that, for some $w \in W$, $w \models_{\mathrm{t}} \Phi$.*

Proof. We omit the proof, which is a two-dimensional version of the proof of Lemma 3.2. $\square$

If W is a nice set of extended valuations and $\alpha \in W$, then the *canonical model* on (W, α) is $(W, C, \alpha, \varphi, \chi)$ where, for $\beta \in W$,

$$\begin{aligned} p_i \in \varphi_\beta \quad &\leftrightarrow \quad p_i \in \beta\,; \\ (P^n_i, c_1, \ldots, c_n) \in \varphi_\beta \quad &\leftrightarrow \quad P^n_i c_1 \ldots c_n \in \beta\,; \\ \chi(c) \quad &= \quad c\,. \end{aligned}$$

Lemma 3.7. *Let $\mathfrak{M}$ be the canonical model on (W, α), where W is a nice set of extended valuations for $\mathcal{L}^*_C(\Box)$ and $\alpha \in W$. For every $\beta \in W$ and every sentence A of $\mathcal{L}^*_C$,*

$$\beta \models_{\mathfrak{M}} A \;\leftrightarrow\; \beta \models_{\mathrm{t}} A\,.$$

Proof. By formula induction. $\square$

Theorem 3.8. *For every set C, the axiomatic system $\mathbf{S5B}_C$ is complete.*

Proof. The proof is like that of Theorem 3.4. $\square$

The Language $\mathcal{L}^*(\Box, E)$.

For each set C, we define $\mathcal{L}^*_C(\Box, E)$.

Symbols: All symbols of $\mathcal{L}^*_C(\Box)$ plus the symbol E.

The symbol E should be thought of as an existence predicate.

Formulas: Add to clause (i) of the definition of the formulas of $\mathcal{L}^*_C(\Box)$ the following:

(c) If t is a constant or a variable, then Et is a formula.

Variable domain semantics.

A *model* for $\mathcal{L}^*_C(\Box, E)$ is a sextuple $(W, D, \bar{E}, w_0, \varphi, \chi)$, where

(1) $(W, D, w_0, \varphi, \chi)$ is a model for $\mathcal{L}^*_C(\Box)$;

(2) $\bar{E}$ is a function that assigns to each $w \in W$ a subset $\bar{E}_w$ of D.

One way to define truth in a model for $\mathcal{L}^*_C(\Box, E)$ would be to modify the definition of truth in a model for $\mathcal{L}^*_C(\Box)$ just by adding to clause (i), the atomic case, the following:

(c) $w \models_{\mathfrak{M}'} Ec \;\leftrightarrow\; \chi'(c) \in \bar{E}_w$.

Quantification would still be over D, which could be thought of as the set of all possible objects (but see the remark on the following page). In each world w, $\bar{E}_w$ could be thought of as the set possible objects that exist in w.

In what we shall call the variable domain semantics, the definition of truth in a model for $\mathcal{L}^*_C(\Box, E)$ is gotten by making two modifications in the definition of truth in models for $\mathcal{L}^*_C(\Box)$. First we add (c) above to clause (i). Second, we replace clause (iv) by

(iv) $w \models_{\mathfrak{M}'} \forall x\, A(x) \;\leftrightarrow\;$ for all $d \in \bar{E}_w$, $w \models_{\mathfrak{M}'} A(c_d)$, where $A(c_d)$ is the result of replacing each free occurrence of x in $A(x)$ by an occurrence of the constant c_d;

Remark. If one thinks of D as the set of all possible objects, then it would be natural to require of models that $D = \bigcup_w \bar{E}_w$. Then one would add to $\mathbf{S5}\pi_C$ below axioms $\Diamond Et$ for variables and constants t.

Example. Let C be the empty set. Let $\mathfrak{M} = (W, D, \bar{E}, w_0, \varphi, \chi)$, where $W = \{w_0, w_1\}$, $D = \{d_0, d_1\}$, $\bar{E}_{w_0} = \{d_0\}$, $\bar{E}_{w_1} = \{d_1\}$, $\varphi_{w_0} = \{\ \}$, and $\varphi_{w_1} = \{(P^1, d_1)\}$.

Let x be a variable. Then $\models_{\mathfrak{M}} \Diamond\exists x P^1 x$, but $\not\models_{\mathfrak{M}} \exists x \Diamond P^1 x$. Thus the sentence $\Diamond\exists x P^1 x \supset \exists x \Diamond P^1 x$ is false in $\mathfrak{M}$ and so is not valid.

We would still have a model, and with $\Diamond\exists x P^1 x \supset \exists x \Diamond P^1 x$ still false, if we dropped d_0 from D, making $\bar{E}_{w_0}$ the empty set.

Suppose we modify $\mathfrak{M}$ by making $\varphi_{w_1} = \{(P^1, d_0), (P^1, d_1)\}$. Let $\mathfrak{N}$ be the resulting model. Then $\models_{\mathfrak{N}} \exists x \Diamond P^1 x$. To see this, let $\mathfrak{N}'$ come from $\mathfrak{N}$ by adjoining constants c_{d_0} and c_{d_1} as in the definition of truth. Then $w_1 \models_{\mathfrak{N}'} P^1 c_{d_0}$, even though Ec_{d_0} is false at w_1. Consequently $w_0 \models_{\mathfrak{N}'} \Diamond P^1 c_{d_0}$ and so $w_0 \models_{\mathfrak{N}} \exists x \Diamond P^1 x$.

If we modify $\mathfrak{N}$ by removing d_1 from $\bar{E}_{w_1}$, then we get a model in which $\exists x \Diamond P^1 x$ is true but $\Diamond \exists x P^1 x$ is false.

Remarks:

(a) The example $\mathfrak{N}$ of the preceding paragraph and the modification of it point up a questionable property of our variable domain semantics: One might want $\Diamond A(c)$ to be true only if there is a world in which $A(c)$ is true and $\chi(c)$ exists. Correspondingly, one might want $\Box A(c)$ to be true if $A(c)$ is true in every world in which $\chi(c)$ exists (i.e., where $\chi(c) \in \bar{E}_w$. Thus one might want to modify clause (v) in the definition of truth. A disadvantage of this move is that $\Box Ec$ would always be true, even if $\chi(c)$ existed in no possible world.

(b) Although we have demanded that D be non-empty, we have made not required that the sets E_w be non-empty. In fact, our definition permits all of the E_w to be non-empty, although this would be ruled out if we added the requirement, discussed earlier, that $D = \bigcup_w E_w$. A disadvantage of allowing empty sets E_w is that, for such w, (E_w, D, ϕ_w, χ) is not a model for $\mathcal{L}^*_C$.

Deduction.

For each set C, the axiomatic system $\mathbf{S5}\pi_C$ is as follows:

Axioms: Taut, Dist, VacQ, K, T, 5, and RUI, where RUI is the schema:

$$\forall x\, A(x) \supset (Et \supset A(t)),$$

with t a constant or variable subject to the same restrictions as in the rule UI.

Rules of Inference: MP, Nec, and RGen, where RGen is the rule: $\dfrac{Ex \supset A}{\forall x\, A}$.

Theorem 3.9. *For each set C, the axiomatic system $\mathbf{S5}\pi_C$ is sound and complete for the variable domain semantics.*