# 4 The semantics of full first-order logic

In this section we make two additions to the languages $\mathcal{L}^*_{\mathrm{C}}$ of §3. The first is the addition of a symbol for identity. The second is the addition of symbols that are used to denote functions.

**The languages $\mathcal{L}^*_{=,\mathrm{C}}$ of predicate logic with identity.**

For each set C of constant symbols, we have a language $\mathcal{L}^*_{=,\mathrm{C}}$.

*Symbols of $\mathcal{L}^*_{=,\mathrm{C}}$:* All symbols of $\mathcal{L}^*_{\mathrm{C}}$ plus the symbol $=$.

*Formulas of $\mathcal{L}^*_{=,\mathrm{C}}$:* Modify the definition, given on page 28, of formulas of $\mathcal{L}^*_{\mathrm{C}}$ by renumbering clause (6) as clause (7) and adding the following clause.

(6) If $t_1$ and $t_2$ are variables or constants, then $t_1 = t_2$ is a formula.

*Remark.* Unique readability holds for $\mathcal{L}^*_{=,\mathrm{C}}$ by a proof very similar to the proof that it holds for $\mathcal{L}^*_{\mathrm{C}}$.

*Models for $\mathcal{L}^*_{=,\mathrm{C}}$: Models* for $\mathcal{L}^*_{=,\mathrm{C}}$ are the same as models for $\mathcal{L}^*_{\mathrm{C}}$.

*Satisfaction and truth for $\mathcal{L}^*_{=,\mathrm{C}}$:*

The notions of a *variable assignment* and of $\mathrm{den}^s_{\mathfrak{M}}$ are the same as for $\mathcal{L}^*_{\mathrm{C}}$. The definition of $v^s_{\mathfrak{M}}$ is the same as that for $\mathcal{L}^*_{=,\mathrm{C}}$, except that there is an extra subclause of the atomic clause (i):

$$\text{(c)}\quad v^s_{\mathfrak{M}}(t_1 = t_2) = \begin{cases} \mathbf{T} & \text{if } \mathrm{den}^s_{\mathfrak{M}}(t_1) = \mathrm{den}^s_{\mathfrak{M}}(t_2); \\ \mathbf{F} & \text{if } \mathrm{den}^s_{\mathfrak{M}}(t_1) \neq \mathrm{den}^s_{\mathfrak{M}}(t_2). \end{cases}$$

*Satisfaction* and *truth* are defined as for $\mathcal{L}^*_{\mathrm{C}}$.

*Logical implication for $\mathcal{L}^*_{=,\mathrm{C}}$: Logical implication*, *validity*, and *satisfiability* are defined as for $\mathcal{L}^*_{\mathrm{C}}$.

*Example.* The following formulas are valid.

(a) $v_1 = v_1$ (d) $v_1 = v_2 \rightarrow (P^1 v_1 \leftrightarrow P^1 v_2)$
(b) $\forall v_1\, v_1 = v_1$ (e) $\forall v_1 \forall v_2 (v_1 = v_2 \rightarrow (P^1 v_1 \leftrightarrow P^1 v_2))$
(c) $\exists v_1\, v_1 = v_1$ (f) $v_1 = c \rightarrow (c = v_2 \rightarrow v_1 = v_2)$

The proof of the Compactness Theorem for $\mathcal{L}^*_{=,\mathrm{C}}$ is similar to that for $\mathcal{L}^*_{\mathrm{C}}$, but there is one important difference, as we shall see.

**Lemma 4.1.** *Let* $\Gamma$ *be a finitely satisfiable set of sentences of* $\mathcal{L}^*_{=,\mathrm{C}}$ *and let* $A$ *be a sentence of* $\mathcal{L}^*_{=,\mathrm{C}}$. *Then either* $\Gamma \cup \{A\}$ *is finitely satisfiable or* $\Gamma \cup \{\neg A\}$ *is finitely satisfiable.*

**Proof**. The proof is exactly like that of Lemma 3.4. □

**Lemma 4.2.** *Let* $\Gamma$ *be a finitely satisfiable set of sentences of* $\mathcal{L}^*_{=,\mathrm{C}}$. *Let* $\mathrm{C}^*$ *be a set gotten from* C *by adding infinitely many new constants. There is a set* $\Gamma^*$ *of sentences of* $\mathcal{L}^*_{=,\mathrm{C}^*}$ *such that*

*(1)* $\Gamma \subseteq \Gamma^*$ *;*

*(2)* $\Gamma^*$ *is finitely satifiable ;*

*(3) for every sentence* $A$ *of* $\mathcal{L}^*_{=,\mathrm{C}^*}$, *either* $A$ *belongs to* $\Gamma^*$ *or* $\neg A$ *belongs to* $\Gamma^*$*;*

*(4)* $\Gamma^*$ *is Henkin in* $\mathcal{L}^*_{=,\mathrm{C}^*}$ .

**Proof**. The only change we have to make in the proof of Lemma 3.5 is that we must specify an alphabetical order for the symbols of $\mathcal{L}^*_{=,\mathrm{C}^*}$. Let us do this by letting the new symbol = come immediately after $\forall$. □

In our proof of Compactness for $\mathcal{L}^*_{\mathrm{C}*}$, the next Lemma, Lemma 3.6, was that if a set $\Gamma$ of sentences in $\mathcal{L}^*_{\mathrm{C}}$ is finitely satisfiable, maximal, and Henkin, $\Gamma$ is also satisfiable. Lemma 4.4 is the same result for any such set of sentences of $\mathcal{L}^*_{=,\mathrm{C}}$. Our proof of Lemma 4.4 relies on facts about =, which we shall establish before proving Lemma 4.4.

A 2-place relation $R$ on a set $X$ is an *equivalence relation on* $X$ if and only if all three of the following conditions are satisfied.

(a) $R$ is *reflexive*: $Rxx$ holds for all $x \in X$.

(b) $R$ is *symmetric*: if $x \in X$ and $y \in X$ and $Rxy$ holds, then $Ryx$ holds.

(c) $R$ is *transitive*: if $x \in X$, $y \in X$, $z \in X$, and both $Rxy$ and $Ryz$ hold, then $Rxz$ holds.

If $R$ is an equivalence relation on $X$, then $R$ divides $X$ up into *equivalence classes.* For $x \in X$, let $[x]_R$, the *equivalence class of* $x$ *with respect to* $R$, be defined by

$$[x]_R = \{y \in X \mid Rxy \text{ holds}\}.$$

**Lemma 4.3.** *Let* $\Gamma^*$ *be a set of sentences of a language* $\mathcal{L}^*_{=,\mathrm{C}^*}$ *having properties (2) and (3) described in the statement of Lemma 4.2. Let* $R$ *be the relation on* $\mathrm{C}^*$ *defined by*

$$Rc_1c_2 \textit{ holds } \textit{ iff } \ c_1 = c_2 \in \Gamma^*.$$

*Then* $R$ *is an equivalence relation on* $\mathrm{C}^*$*.*

**Proof**. For reflexivity, we must show that $c = c$ belongs to $\Gamma^*$ for all members $c$ of $\mathrm{C}^*$. $t \neq t'$ abbreviates $\neg t = t'$. $\{c \neq c\}$ is finite and not satisfiable. By (2), $\{c \neq c\} \not\subseteq \Gamma^*$, so $c \neq c \notin \Gamma^*$. By (3), $c = c \in \Gamma^*$.

For symmetry, we must show that, for all members $c_1$ and $c_2$ of $\Gamma^*$, if $c_1 = c_2 \in \Gamma^*$, then $c_2 = c_1 \in \Gamma^*$. Assume that $c_1 = c_2 \in \Gamma^*$. $\{c_1 = c_2, c_2 \neq c_1\}$ is finite and unsatisfiable and so, by (2), not a subset of $\Gamma^*$. So, $c_2 \neq c_1 \notin \Gamma^*$. By (3), $c_2 = c_1 \in \Gamma^*$

For transitivity, we must show that, for all members $c_1$, $c_2$, and $c_3$ of $\Gamma^*$, if $c_1 = c_2 \in \Gamma^*$ and $c_2 = c_3 \in \Gamma^*$, then $c_1 = c_3 \in \Gamma^*$. Assume that $c_1 = c_2 \in \Gamma^*$ and $c_2 = c_3 \in \Gamma^*$. $\{c_1 = c_2, c_2 = c_3, c_1 \neq c_3\}$ is finite and unsatisfiable, so not a subset of $\Gamma^*$. By (2), $c_1 \neq c_3 \notin \Gamma^*$. By (3), $c_1 = c_3 \in \Gamma^*$. □

**Lemma 4.4.** *Let* $\Gamma^*$ *be a set of sentences of a language* $\mathcal{L}^*_{=,\mathrm{C}^*}$ *having properties (2), (3), and (4) described in the statement of Lemma 4.2. Then* $\Gamma^*$ *is satisfiable.*

**Proof**. We wish to begin, as we did in the proof of Lemma 3.6, by using $\Gamma^*$ to define a model for $\mathcal{L}^*_{=,\mathrm{C}^*}$. But we must not define the model as we did before (on page 40). To see this, assume that we did use the old definition. Let $c_1$ and $c_2$ be two distinct members of $\mathrm{C}^*$ and suppose that the sentence $c_1 = c_2$ belongs to $\Gamma^*$, as is possible. Since $\chi(c_1) = c_1$, $\chi(c_2) = c_2$, and $c_1$ and $c_2$ are distinct objects, the new clause (i)(c) in the definition of $v^s_{\mathfrak{M}}$ implies that $v_{\mathfrak{M}}(c_1 = c_2) = \mathbf{F}$. Thus it is not the case that, for all formulas $A$, $v_{\mathfrak{M}}(A) = \mathbf{T}$ if and only if $A \in \Gamma^*$. But for the proof of Lemma 3.6 it was critical that this *was* the case. If we are to define a model for which it is the case, then we must make sure that

$$\text{if } \ c_1 = c_2 \in \Gamma^*, \ \text{ then } \ \chi(c_1) = \chi(c_2).$$

To secure that result, our domain $\mathbf{D}$ will be composed, not of constants, but of the equivalence classes of constants defined in Lemma 4.3. Define a model $\mathfrak{M} = (\boldsymbol{D}, v, \chi)$ for $\mathcal{L}^*_{=,\mathrm{C}^*}$ as follows.

(i) $\boldsymbol{D} = \{[c]_R \mid c \in \mathrm{C}^*\}$.

(ii) (a) $v(p_i) = \mathbf{T}$ if and only if $p_i \in \Gamma^*$.

(b) $v((P^n_i, [c_1]_R, \dots, [c_n]_R)) = \mathbf{T}$ if and only if $P^n_i c_1 \dots c_n \in \Gamma^*$.

(iii) $\chi(c) = [c]_R$ for each $c \in \mathrm{C}^*$.

Consider how (ii)(b) defines $v$ for $(P^1_1,\ [c']_R)$. (ii)(b) says: take a representative $c'$ from $[c']_R$, and with it form the sentence $P^1_1 c'$. If $P^1_1 c' \in \Gamma^*$, $v((P^1_1,\ [c']_R) = \mathbf{T}$. If not, $v((P^1_1,\ [c']_R)) = \mathbf{F}$. Now, suppose that $[c']_R = \{c',\ c''\}$, where $c'$ and $c''$ are distinct constants. (ii)(b) is a genuine definition *if and only if* it does not matter which $c$ is chosen from $[c']_R$, *i.e.* if and only if: $P^1_1 c' \in \Gamma^*$ iff $P^1_1 c'' \in \Gamma^*$.

So, to show that (ii)(b) is a genuine definition, we are going to show that the truth-value (ii)(b) assigns does not depend on the choice of representatives from the equivalence classes. Assume that $[c_j]_R = [c'_j]_R$ for $1 \leq j \leq n$. By the definition of the equivalence classes, we have that $Rc_jc'_j$ holds for $1 \leq j \leq n$. By the definition of $R$, we get that the sentence $c_j = c'_j$ belongs to $\Gamma^*$ for $1 \leq j \leq n$. We must show that $P^n_i c_1 \dots c_n \in \Gamma^*$ if and only if $P^n_i c'_1 \dots c'_n \in \Gamma^*$. For the "only if" direction, assume that $P^n_i c_1 \dots c_n \in \Gamma^*$ and that $P^n_i c'_1 \dots c'_n \notin \Gamma^*$. By (3), $\neg P^n_i c'_1 \dots c'_n \in \Gamma^*$. Thus

$$\{P^n_i c_1 \dots c_n,\ P^n_i c'_1 \dots c'_n,\ c_1 = c'_1, \dots, c_n = c'_n\}$$

is a finite subset of $\Gamma*$. By (2) it is satisfiable. This is a contradiction. The "if" direction is similar.

Let $P$ be the property of being a sentence $A$ such that

$$v_{\mathfrak{M}}(A) = \mathbf{T} \text{ if and only if } A \in \Gamma^*\,.$$

We prove by induction on length that every sentence of $\mathcal{L}^*_{=,\mathrm{C}^*}$ has property $P$.

There are only two cases that are significantly different from the corresponding cases in the proof of Lemma 3.6, so we omit the other cases.

Case (i)(c). $A$ is $c_1 = c_2$. $A$ has $P$ because

$$v_{\mathfrak{M}}(c_1 = c_2) = \mathbf{T} \text{ iff } \chi(c_1) = \chi(c_2) \text{ iff } [c_1]_R = [c_2]_R \text{ iff } c_1 = c_2 \in \Gamma^*.$$

Case (iv). $A$ is $\forall xB$ for some formula $B$ and variable $x$. We prove that $\forall xB$ has $P$.

$$\begin{aligned} v_{\mathfrak{M}}(\forall xB) = \mathbf{T} \quad &\text{iff} \quad \text{for all } s,\ v^s_{\mathfrak{M}}(B) = \mathbf{T} \\ &\text{iff} \quad \text{for all } c \in \mathrm{C}^*, \text{ for all } s \text{ with } s(x) = [c]_R,\ v^s_{\mathfrak{M}}(B) = \mathbf{T} \\ &\text{iff} \quad \text{for all } c \in \mathrm{C}^*,\ v_{\mathfrak{M}}(B(x;c)) = \mathbf{T} \\ &\text{iff} \quad \text{for all } c \in \mathrm{C}^*,\ B(x;c) \in \Gamma^* \\ &\text{iff} \quad \forall xB \in \Gamma^* \end{aligned}$$

The first biconditional is by the definition of $v_{\mathfrak{M}}$ and the fact that no variable besides $x$ occurs free in $B$. The second biconditional is by the fact that no variable besides $x$ occurs free in $B$ and the fact that $\boldsymbol{D} = \{[c]_R \mid c \in \mathrm{C}^*\}$. The third biconditional is by the fact that $\chi(c) = [c]_R$ for each $c \in \mathrm{C}^*$. The fourth biconditional is by the fact that the sentences $B(x; c)$ have property $P$. The proof that the fifth biconditional holds is exactly the same as the corresponding step in the proof of Lemma 3.6.

Since, in particular, $v_{\mathfrak{M}}(A) = \mathbf{T}$ for every member of $A$ of $\Gamma^*$, we have shown that $\Gamma^*$ is satisfiable. □

The proof of the two Compactness Theorems that follow are just like the proofs Theorem 3.7 and Theorem 3.8.

**Theorem 4.5 (Compactness).** *Let $\Gamma$ be a finitely satisfiable set of sentences of $\mathcal{L}^*_{=,\mathrm{C}}$. Then $\Gamma$ is satisfiable, i.e., true in a model for $\mathcal{L}^*_{=,\mathrm{C}}$.*

**Corollary 4.6 (Compactness, Second Form).** *Let $\Gamma$ be a set of sentences of $\mathcal{L}^*_{=,\mathrm{C}}$ and let $A$ be a sentence such that $\Gamma \models A$. Then there is a finite subset $\Delta$ of $\Gamma$ such that $\Delta \models A$.*

**Exercise 4.1.** Exhibit a sentence of $\mathcal{L}^*_{=,\emptyset}$ that is true in every model with exactly three elements and is false in all other models.

**Exercise 4.2.** Tell which of the following sentences of $\mathcal{L}^*_{=,\{\mathrm{c}\}}$ are valid. If a sentence is valid, explain briefly why. If it is invalid, give a model in which it is false.

(a) $\forall v_1\, c = c$

(b) $\forall v_1 \forall v_2 P^2 v_1 v_2 \to \forall v_1 \forall v_2\, v_1 = v_2$

(c) $\forall v_1 (P^1 v_1 \to \exists v_2 (v_1 = v_2 \land P^1 v_2))$

(d) $\forall v_1 (P^1 v_1 \to \forall v_2 (P^1 v_2 \to v_1 = v_2))$

### The languages $\mathcal{L}^{\#}_{\mathrm{C}}$ of full first-order logic.

For each set C of constant symbols, we have a language $\mathcal{L}^{\#}_{\mathrm{C}}$.

*Symbols of $\mathcal{L}^{\#}_{\mathrm{C}}$:* All symbols of all symbols of $\mathcal{L}^*_{=,\mathrm{C}}$ plus *n-place function letters*

$$F^n_0,\, F^n_1,\, F^n_2, \ldots,$$

for each $n \geq 1$.

*Terms of $\mathcal{L}^{\#}_{\mathrm{C}}$:*

(1) Each variable or constant is a term.

(2) For each $n$ and $i$, if $t_1, \ldots, t_n$ are terms, then $F_i^n t_1 \ldots t_n$ is a term.

(3) Nothing is a term unless its being one follows from (1)–(2).

*Example of a term:*

$$F_1^3 F_2^2 c F_0^1 v_4 v_6 F_0^1 c.$$

*Remarks:*

(a) As we shall see, terms are expressions that, in a model and under a variable assignment, denote a member of the domain of the model. The *terms* of $\mathcal{L}_{\mathrm{C}}^*$ and $\mathcal{L}_{=,\mathrm{C}}^*$ are—let us retroactively specify—the variables and constants. Variables and constants are the *atomic terms* of a language. The new ingredients of $\mathcal{L}_{\mathrm{C}}^{\#}$ are the complex terms given by clause (2) above.

(b) Proof by induction on length works for terms as well as for formulas. In proving by induction on length that every term $t$ has a property $P$, there are two cases: (1) $t$ is a variable or constant; (2) $t$ is $F_i^n t_1 \ldots, t_n$ for some numbers $n$ and $i$ and some terms $t_1, \ldots, t_n$.

(c) As we can define functions by recursion on formulas, we can use *definition by recursion on terms* to define functions whose domain is the set of all terms.

*Formulas of* $\mathcal{L}_{\mathrm{C}}^{\#}$*:* Replace clauses (2) and (6) in the definition of formulas of $\mathcal{L}_{=,\mathrm{C}}^*$ by the following clauses.

(2) For each $n$ and $i$, if $t_1, \ldots, t_n$ are terms, then $P_i^n t_1 \ldots t_n$ is a formula.

(6) If $t_1$ and $t_2$ are terms, then $t_1 = t_2$ is a formula.

Note that, with our retroactive definition of *term* for $\mathcal{L}_{\mathrm{C}}^*$ and $\mathcal{L}_{=,\mathrm{C}}^*$, the new clauses (2) and (6) have the same meaning as the old (2) and (6).

*Remark.* The proof of unique readability for formulas of $\mathcal{L}_{\mathrm{C}}^{\#}$ has a preliminary step. One first needs to prove *Unique Readability for Terms.* This states that every term is either a variable or constant or else is $F_i^n t_1 \ldots t_n$ for unique $n$, $i$, and $t_1, \ldots, t_n$. The rest of the proof of unique readability for formulas is similar to the proof for the other languages.

*Models for* $\mathcal{L}_{\mathrm{C}}^{\#}$*:*

A *model* for $\mathcal{L}_{\mathrm{C}}^{\#}$ is a triple $\mathfrak{M} = (\boldsymbol{D}, v, \chi)$ satisfying conditions (i) and (ii) in the definition of a model for $\mathcal{L}_{=,C}^*$ and satisfying the following condition:

(iii) $\chi$ is a function that assigns

(a) a member of $\boldsymbol{D}$ to each constant;

(b) a member of $\boldsymbol{D}$ to each $(n+1)$-tuple of the form $(F_i^n, d_1, \ldots, d_n)$ for $d_1, \ldots, d_n$ members of $\boldsymbol{D}$.

*Satisfaction, truth, and logical implication for $\mathcal{L}_{\mathrm{C}}^{\#}$:*

The notion of a *variable assignment* is the same as for $\mathcal{L}_{\mathrm{C}}^{*}$ and $\mathcal{L}_{=,\mathrm{C}}^{*}$.

The definition of $\mathrm{den}_{\mathfrak{M}}^{s}$ for the other languages has to be extended so that $\mathrm{den}_{\mathfrak{M}}^{s}(t)$ is defined for all terms $t$. The definition is by recursion on terms.

(1) $\mathrm{den}_{\mathfrak{M}}^{s}(t) = s(t)$ if $t$ is a variable, and $\mathrm{den}_{\mathfrak{M}}^{s}(t) = \chi(t)$ if $t$ is a constant.

(2) $\mathrm{den}_{\mathfrak{M}}^{s}(F_i^n t_1 \ldots t_n) = \chi((F_i^n, \mathrm{den}_{\mathfrak{M}}^{s}(t_1), \ldots, \mathrm{den}_{\mathfrak{M}}^{s}(t_n)))$.

The definitions of *satisfaction*, *truth*, *logical implication*, *validity*, and *satisfiability* are word for word the same as for $\mathcal{L}_{=,\mathrm{C}}^{*}$.

The proof of the Compactness Theorem for $\mathcal{L}_{\mathrm{C}}^{\#}$ is very much like that for $\mathcal{L}_{=,\mathrm{C}}^{*}$. We list the lemmas and indicate the ways the proofs of the analogous earlier lemmas are to be modified.

**Lemma 4.7.** *Let $\Gamma$ be a finitely satisfiable set of sentences of $\mathcal{L}_{\mathrm{C}}^{\#}$ and let $A$ be a sentence of $\mathcal{L}_{\mathrm{C}}^{\#}$. Then either $\Gamma \cup \{A\}$ is finitely satisfiable or $\Gamma \cup \{\neg A\}$ is finitely satisfiable.*

**Lemma 4.8.** *Let $\Gamma$ be a finitely satisfiable set of sentences of $\mathcal{L}_{\mathrm{C}}^{\#}$. Let $\mathrm{C}^*$ be a set gotten from $\mathrm{C}$ by adding infinitely many new constants. There is a set $\Gamma^*$ of sentences of $\mathcal{L}_{\mathrm{C}^*}^{\#}$ such that*

*(1) $\Gamma \subseteq \Gamma^*$;*

*(2) $\Gamma^*$ is finitely satifiable;*

*(3) for every sentence $A$ of $\mathcal{L}_{\mathrm{C}^*}^{\#}$, either $A$ belongs to $\Gamma^*$ or $\neg A$ belongs to $\Gamma^*$;*

*(4) $\Gamma^*$ is Henkin.*

**Proof**. The only change we have to make in the proof of Lemma 4.2 is that we must specify an alphabetical order for the symbols of $\mathcal{L}_{\mathrm{C}^*}^{\#}$. Let us do this by letting the new symbols $F_i^n$ come after the symbols of $\mathcal{L}_{=,\mathrm{C}^*}^{*}$, ordered first by superscript and then by subscript. $\square$

**Lemma 4.9.** *Let $\Gamma^*$ be a set of sentences of a language $\mathcal{L}^{\#}_{\mathbf{C}^*}$ having properties (2), (3), and (4) described in the statement of Lemma 4.8. Then $\Gamma^*$ is satisfiable.*

**Proof**. We need to make two changes in the proof of Lemma 4.4.

First we need to provide the second part of the definition of $\chi$. We do that by setting.

$$\chi((F^n_i, [c_1]_R, \ldots, [c_n]_R)) = [c]_R \quad \text{iff} \quad F^n_i c_1 \ldots c_n = c \in \Gamma^*.$$

We must show that this is a genuine definition. One thing doing that involves is proving that the definition does not depend on the choices of elements of equivalence classes. This proof is like the proof of the corresponding fact about the $P^n_i$ in the proof of Lemma 4.4, and we omit it. We also need to prove that for all $c_1, \ldots, c_n$, there is a $c$ such that $F^n_i c_1 \ldots c_n = c \in \Gamma^*$ and that all such $c$ have the same equivalence class. For the first fact, note that $\exists x F^n_i c_1 \ldots c_n = x$ is valid. By (2) and (3), it belongs to $\Gamma^*$. The desired conclusion follows by (4). For the second fact, one uses (2), (3), and the validity of $(F^n_i c_1 \ldots c_n = c \wedge F^n_i c_1 \ldots c_n = c') \rightarrow c = c'$.

The other change we have to make is in the atomic cases (i)(b) and (i)(c) of the proof that all formulas have property $P$.

Before considering the proofs of these facts, we first prove another fact that we will need in these proofs.

Say that a term $t$ containing no variables has property $Q$ if and only if, for every $c \in \mathbf{C}^*$,

$$\text{if } \mathrm{den}_{\mathfrak{M}}(t) = [c]_R \text{ then } c = t \in \Gamma^*,$$

where $\mathrm{den}_{\mathfrak{M}}(t)$ is the common value of the $\mathrm{den}^s_{\mathfrak{M}}(t)$. We prove by induction on length that all terms without variables have $Q$.

(1) If $t$ is a constant, then $\mathrm{den}_{\mathfrak{M}}(t) = [t]_R$. By definition of $[c]_R$, $c = t$ belongs to $\Gamma^*$ if and only if $[t]_R = [c]_R$. Thus $t$ has $Q$.

(2) Assume that $t$ is $F^n_i t_1 \ldots t_n$. Let $\mathrm{den}_{\mathfrak{M}}(t_i) = [c_i]_R$ for $1 \leq i \leq n$. All the $t_i$ have $Q$, so the sentence $c_i = t_i$ belongs to $\Gamma^*$ for each $i$. Let $\mathrm{den}_{\mathfrak{M}}(t) = [c]_R$. By the definition of $\mathrm{den}_{\mathfrak{M}}$, it follows that

$$\begin{aligned}
\mathrm{den}_{\mathfrak{M}}(F^n_i c_1 \ldots c_n) &= \chi((F^n_i, [c_1]_R, \ldots, [c_n]_R)) \\
&= \mathrm{den}_{\mathfrak{M}}(F^n_i t_1 \ldots t_n) \\
&= \mathrm{den}_{\mathfrak{M}}(t) \\
&= [c]_R.
\end{aligned}$$

By the definition of $\chi((F_i^n, [c_1]_R, \ldots, [c_n]_R))$, we have that $F_i^n c_1 \ldots c_n = c$ belongs to $\Gamma^*$. Assume that $c = F_i^n t_1 \ldots t_n$ does not belong to $\Gamma^*$. By property (3) of $\Gamma^*$, $c \neq F_i^n t_1 \ldots t_n$ belongs to $\Gamma^*$. Since $c_i = t_i \in \Gamma^*$ for every $i$, the set

$$\{c_1 = t_1, \ldots, c_n = t_n, F_i^n c_1 \ldots c_n = c, c \neq F_i^n t_1 \ldots t_n\}$$

is a finite subset of $\Gamma^*$. This set is not satisfiable, and so we have contradicted property (2) of $\Gamma^*$. In doing so, we have shown that $t$ has $Q$.

Now we are ready for cases (i)(b) and (i)(c) of the property $P$ proof. Let $\text{den}_{\mathfrak{M}}(t_i) = [c_i]_R$ for $1 \leq i \leq n$. Since the $t_i$ have $Q$, $c_i = t_i \in \Gamma^*$ for each $i$.

$$\begin{aligned} v_{\mathfrak{M}}(P_i^n t_1 \ldots t_n) = \mathbf{T} \quad &\text{iff} \quad v((P_i^n, \text{den}_{\mathfrak{M}}(t_1), \ldots, \text{den}_{\mathfrak{M}}(t_n))) = \mathbf{T} \\ &\text{iff} \quad v((P_i^n, [c_1]_R, \ldots, [c_n]_R)) = \mathrm{T} \\ &\text{iff} \quad P_i^n c_1 \ldots c_n \in \Gamma^* \\ &\text{iff} \quad P_i^n t_1 \ldots t_n \in \Gamma^*, \end{aligned}$$

where the last iff is by properties (2) and (3) of $\Gamma^*$.

The proof for case (i)(c) is similar, and we omit it. □

**Theorem 4.10 (Compactness).** *Let $\Gamma$ be a finitely satisfiable set of sentences of $\mathcal{L}_{\mathrm{C}}^{\#}$. Then $\Gamma$ is satisfiable, i.e., true in a model for $\mathcal{L}_{\mathrm{C}}^{\#}$.*

**Corollary 4.11 (Compactness, Second Form).** *Let $\Gamma$ be a set of sentences of $\mathcal{L}_{\mathrm{C}}^{\#}$ and let $A$ be a sentence such that $\Gamma \models A$. Then there is a finite subset $\Delta$ of $\Gamma$ such that $\Delta \models A$.*

**Exercise 4.3.** Which of the following sentences are valid? For each one, explain or give (the relevant part of) a counter-model.

(a) $\exists v_1 F^3 v_2 c v_3 = v_1$

(b) $\forall v_1 \forall v_2 (v_1 \neq v_2 \rightarrow F^1 v_1 \neq F^1 v_2) \rightarrow \forall v_1 \exists v_2 F^1 v_2 = v_1$

**Exercise 4.4.** Give the omitted case (i)(c) in the proof of Lemma 4.9.