2 Deduction in Sentential Logic

Though we have not yet introduced any formal notion of deductions (i.e., of derivations or proofs), we can easily give a formal method for showing that formulas are tautologies: Construct the truth table of a given formula; i.e., compute the truth-value of the formulas for all possible assignments of truth-values to the sentence letters occurring in it. If all these truth values are **T**, then the formula is a tautology. This method extends to give a formal method for showing that $\Gamma \models A$, provided that Γ is finite. The method even extends to the case Γ is infinite, since the second form of Compactness guarantees that if $\Gamma \models A$ then $\Delta \models A$ for some finite $\Delta \subseteq \Gamma$.

Nevertheless we are now going to introduce a different system of formal deduction. This is because we want to gain experience with the metatheory of a more standard deductive system.

The system SL

Axioms: From now on we shall often adopt the convention of omitting outmost parentheses in formulas. For any formulas A, B, and C, each of the following is an *axiom* of our deductive sytem.

(1) $B \rightarrow (A \rightarrow B)$

(2) $(\neg A \rightarrow B) \rightarrow ((\neg A \rightarrow \neg B) \rightarrow A)$

(3) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

Remarks:

1. (1)–(3) are not axioms but *axiom schemas.* There are infinitely many instances of each of these schemas, since A, B, and C may be any formulas whatsoever.

2. We have used abbreviations in presenting these axiom schemas, in dropping parentheses.

3. If the consequent of a *material* conditional is true, then so is the conditional. → has this property, according to our definition of v^* from Chapter 1. Axiom Schema (1) is a proof-theoretic “reflection” of that fact; in a proof, we can write down any conditional whose consequent is itself a conditional, and whose antecedent is the consequent of the embedded conditional.

4. *Reductio ad absurdum* is the form of argument according to which you can infer a sentence if the sentence’s negation entails a contradiction. Axiom

Schema (2) is a proof-theoretic "reflection" of the validity that argument form.

5. Axiom Schema (3) governs the conditional's distribution over itself. Part of what (3) tells us is that *modus ponens* (described below) holds "within" the consequent of a conditional.

6. In describing the axioms, we used the vague word "reflection" to describe the axioms. That is because, as yet, we are not yet in a position to describe more precisely the relations among the axioms, the semantics for $\mathcal{L}$, and deductions in **SL**.

Rule of Inference:

$$\text{Modus Ponens (MP)} \qquad \frac{A\,,\ (A \to B)}{B}$$

For any formulas A and B, we say that B *follows by modus ponens* from A and $(A \to B)$.

Deductions: A *deduction* in **SL** from a set Γ of formulas is a finite sequence **D** of formulas such that whenever a formula A occurs in the sequence **D** then at least one of the following holds.

(1) $A \in \Gamma$.
(2) A is an axiom.
(3) A follows by modus ponens from two formulas occurring earlier in the sequence **D**.

If A is the nth element of the sequence **D**, then we say that *A is on line n of* **D** or even that *A is line n of* **D**.

A *deduction in* **SL** *of A from* Γ is a deduction **D** in **SL** from Γ with A on the last line of **D**. We write $\Gamma \vdash_{\mathbf{SL}} A$ and say A is *deducible* in **SL** from Γ to mean that there is a deduction in **SL** of A from Γ. Sometimes we may express this by saying Γ *proves A in* **SL**. We write $\vdash_{\mathbf{SL}} A$ for $\emptyset \vdash_{\mathbf{SL}} A$. We shall mostly omit the subscript "**SL**" and the phrase "in **SL**" during our study of sentential logic, since **SL** will be the only system we consider until we get to predicate logic.

Example 1. Let C and D be any formulas. Here is a very short deduction of $C \to (\neg D \to C)$ from $\emptyset$. This deduction shows that $\vdash C \to (\neg D \to C)$.

1. $C \to (\neg D \to C)$ Ax. 1

The formula C is the B of the axiom schema, and the formula $\neg D$ is the A of the axiom schema.

Example 2. Below we give a deduction of $X \to X$ from $\emptyset$. This deduction shows that $\vdash X \to X$.

1.	$(X \to ((X \to X) \to X)) \to ((X \to (X \to X)) \to (X \to X))$	Ax. 3
2.	$X \to ((X \to X) \to X)$	Ax. 1
3.	$(X \to (X \to X)) \to (X \to X)$	1,2; MP
4.	$X \to (X \to X)$	Ax. 1
5.	$X \to X$	3,4; MP

Line 1 of the derivation is an instance of Axiom Schema 3. A of the schema is replaced by X, B by $X \to X$, and C by X.

The main results of this chapter concern the relation between $\vdash_{\mathbf{SL}}$ and $\models$. We are going to prove that **SL** is *sound*:

$$\text{If } \Gamma \vdash_{\mathbf{SL}} A, \text{ then } \Gamma \models A.$$

And we are going to prove that **SL** is *complete*:

$$\text{If } \Gamma \models A, \text{ then } \Gamma \vdash_{\mathbf{SL}} A.$$

Before proving these results about the relation between $\vdash_{\mathbf{SL}}$ and $\models$, we shall prove some basic facts about $\vdash_{\mathbf{SL}}$.

In many of the proofs of this section, we are going to use a new form of mathematical induction. In Chapter 1, we saw that one form of proof by mathematical induction has two steps:

1. Prove that 0 has P.
2. Prove that, if n has P, then $n + 1$ has P.

It is a fact about the natural numbers that, if some n has P, then there is a least n that has P. To show that every n has Q, we proceed as follows. Assume, for *reductio*, that some n does not have Q. Then, there is a least n that does not have Q. We let n be the least number that does not have Q, and proceed to derive a contradiction.

Theorem 2.1 (Deduction Theorem). *Let* Γ *be a set of formulas and let* A *and* B *be formulas. If* $\Gamma \cup \{A\} \vdash B$ *then* $\Gamma \vdash (A \to B)$.

Proof. Assume that $\Gamma \cup \{A\} \vdash B$. Let **D** be a deduction of B from $\Gamma \cup \{A\}$. We prove that

$$\Gamma \vdash (A \to C)$$

for every line C of **D**. Assume that this is false. Consider the first line C of **D** such that $\Gamma \nvdash (A \to C)$.

Assume that C either belongs to Γ or is an axiom. The following gives a deduction of $(A \to C)$ from Γ.

$$\begin{array}{lll} 1. & C & \\ 2. & C \to (A \to C) & \text{Ax. 1} \\ 3. & A \to C & 1{,}2;\text{MP} \end{array}$$

Assume next that C is A. We have already shown that $\vdash (A \to A)$. Thus $\Gamma \vdash (A \to A)$.

Finally assume that C follows from formulas E and $(E \to C)$ by MP. These formulas are on earlier lines of **D** than C. Since C is the first "bad" line of **D**, let $\mathbf{D}_1$ be a deduction of $(A \to E)$ from Γ and let $\mathbf{D}_2$ be a deduction of $(A \to (E \to C))$ from Γ. We get a deduction of $(A \to C)$ from Γ by beginning with $\mathbf{D}_1$, following with $\mathbf{D}_2$, and then finishing with the lines

$$\begin{array}{ll} (A \to (E \to C)) \to ((A \to E) \to (A \to C)) & \text{Ax. 3} \\ (A \to E) \to (A \to C) & \text{MP} \\ A \to C & \text{MP} \end{array}$$

This contradiction completes the proof that the "bad" line C cannot exist. Applying this fact to the last line of **D**, we get that $\Gamma \vdash (A \to B)$. $\square$

Remarks:

(a) The converse of the Deduction Theorem is also true. Given a deduction of $(A \to B)$ from Γ, one gets a deduction of B from $\Gamma \cup \{A\}$ by appending the lines A and B, the latter coming by MP.

(b) The proof of the Deduction Theorem would still go through if we added or dropped axioms, as long as we did not drop Axiom Schemas 1 and 3. It would not in general go through if we added rules of inference, and it would not go through if we dropped the rule of modus ponens.

A set Γ of formulas is *inconsistent* (in **SL**) if there is a formula B such that $\Gamma \vdash B$ and $\Gamma \vdash \neg B$. Otherwise Γ is *consistent.*

Theorem 2.2. *Let Γ and Δ be sets of formulas and let A, B, and $A_1, \ldots, A_n$ be formulas.*

(1) $\Gamma \cup \{A\} \vdash B$ *if and only if* $\Gamma \vdash (A \to B)$.

(2) $\Gamma \cup \{A_1, \ldots, A_n\} \vdash B$ *if and only if* $\Gamma \vdash (A_1 \to \ldots \to A_n \to B)$.

(3) Γ *is consistent if and only if there is some formula* C *such that* $\Gamma \nvdash C$.

(4) If $\Gamma \vdash C$ *for all* $C \in \Delta$ *and if* $\Delta \vdash B$*, then* $\Gamma \vdash B$.

Proof. We begin with (4). Let $\mathbf{D}$ be a deduction of B from Δ. We can turn $\mathbf{D}$ into a deduction of B from Γ as follows: whenever a formula $C \in \Delta$ is on a line of $\mathbf{D}$, replace that line with a deduction of C from Γ.

(1) is just the combination of the Deduction Theorem and its converse.

For (2), forget the particular Γ, $A_1, \ldots, A_n$, and B for the moment and let P be the property of being a positive integer n such that (2) holds for every choice of Γ, $A_1, \ldots, A_n$, and B. By a variant of mathematical induction (beginning with 1 instead of with 0) we show that every positive integer has P. The integer 1 has P by (1). Assume that n is a positive integer that has P. Let Γ, $A_1, \ldots, A_{n+1}$, and B be given. By (1) we have that

$$\Gamma \cup \{A_1, \ldots, A_{n+1}\} \vdash B \text{ if and only if } \Gamma \cup \{A_1, \ldots, A_n\} \vdash (A_{n+1} \to B)\,.$$

(2) then follows from the fact that, by the inductive assumption that n has P, we have

$$\Gamma \cup \{A_1, \ldots, A_n\} \vdash (A_{n+1} \to B) \text{ if and only if } \Gamma \vdash (A_1 \to \ldots \to A_{n+1} \to B).$$

For the "if" part of (3), assume that Γ is inconsistent. Let B be such that $\Gamma \vdash B$ and $\Gamma \vdash \neg B$. Let C be any formula. Using Axiom Schema 1 and MP, we get that $\Gamma \vdash (\neg C \to B)$ and $\Gamma \vdash (\neg C \to \neg B)$. The formula

$$(\neg C \to B) \;\to\; ((\neg C \to \neg B) \;\to\; C)$$

is an instance of Axiom Schema 2. Two applications of MP show that $\Gamma \vdash C$.

The "only if" part of (3) is obvious. □

Exercise 2.1. Show that the following hold for all formulas A and B.

(a) $\vdash (\neg A \to (A \to B))$;

(b) $\vdash (\neg\neg A \to A)$.

Exercise 2.2. Show that the following hold for all formulas A and B.

(a) $\vdash \neg(A \to B) \to A$

(b) $\vdash \neg(A \to B) \to \neg B$;

Hints: Exercise 2.1 is relevant to both proofs. Also remember that complex sentences can be substituted for A, B, and C in the Axiom Schemas.

Exercise 2.3. Use the Deduction Theorem and its converse to give a brief proof that $\vdash (B \to (A \to A))$. You may *not* use MP.

Lemma 2.3. *For any formulas A and B,*

(a) $\{(\neg A \to B)\} \ \vdash \ (\neg B \to A)$*;*

(b) $\{(A \to B)\} \ \vdash \ (\neg B \to \neg A)$*.*

Proof. (a) By the Deduction Theorem, it is enough to show that

$$\{(\neg A \to B)\,,\ \neg B\} \ \vdash \ A\,.$$

Let $\Gamma = \{(\neg A \to B)\,,\ \neg B\}$. Axiom Schema 1 and MP give that $\Gamma \vdash (\neg A \to \neg B)$. The formula

$$(\neg A \to B) \ \to \ ((\neg A \to \neg B) \ \to \ A)$$

is an instance of Axiom Schema (2). Two applications of MP show that $\Gamma \vdash A$.

(b) We can use $\vdash (\neg\neg A \to A)$ and the Deduction Theorem to get that

$$\{(A \to B)\} \vdash (\neg\neg A \to B)\,.$$

And, $\{(\neg\neg A \to B)\} \vdash (\neg B \to \neg A)$ by part (a). □

Exercise 2.4. Show that

$$\{(A \to C)\,,\ (B \to C)\} \ \vdash \ ((\neg A \to B) \to C)\,.$$

Exercise 2.5. Exhibit a deduction of $(\neg p_2 \to p_1))$ from $\{(\neg p_1 \to p_2)$. Do not appeal to the deduction theorem.

Hint. First write out the deduction $\mathbf{D}$ of p_1 from $\{(\neg p_1 \to p_2)\,,\ \neg p_2\}$ that is implicitly given by the proof of part (a) of Lemma 2.3. Now use the proof of the Deduction Theorem to get the desired deduction. (The proof of the Deduction Theorem shows us how to put $\neg p_2 \to$ in front of all the lines of the given deduction and then to fix things up. There is one simplification here: If one puts $\neg p_2 \to$ in front of the formula $(\neg p_1 \to \neg p_2)$ that is on line 3 of $\mathbf{D}$, one gets an axiom. Thus one can forget about lines 1 and 2 of $\mathbf{D}$ and just begin with this axiom.)

Soundness and Completeness

A system $\mathbf{S}$ of deduction for $\mathcal{L}$ is *sound* if, for all sets Γ of formulas and all formulas A, if $\Gamma \vdash_{\mathbf{S}} A$ then $\Gamma \models A$.

An example of a system of deduction that is not sound can be gotten by adding to the axioms and rules for **SL** the extra axiom p_0. For this system $\mathbf{S}$, one has that $\emptyset \vdash_{\mathbf{S}} p_0$, but $\emptyset \not\models p_0$.

Theorem 2.4 (Soundness). *Let Γ be a set of formulas and let A be a formula. If $\Gamma \vdash_{\mathbf{SL}} A$ then $\Gamma \models A$. In other words,* **SL** *is sound.*

Proof. Let $\mathbf{D}$ be a deduction in **SL** of A from Γ. We shall show that, for every line C of $\mathbf{D}$, $\Gamma \models C$. Applying this to the last line of $\mathbf{D}$, this will give us that $\Gamma \models A$.

Assume that what we wish to show is false. Let C be the first line of $\mathbf{D}$ such that $\Gamma \not\models C$.

If $C \in \Gamma$ then trivially $\Gamma \models C$ (and so we have a contradiction).

It can easily be checked that all of our axioms are tautologies. If C is an axiom we have then that $\models C$ and so that $\Gamma \models C$.

Note that the rule of modus ponens is a *valid* rule, i.e., $\{D\,,\ (D \rightarrow E)\} \models E$ for any formulas D and E. Assume that C follows by MP from B and $(B \rightarrow C)$, where B and $(B \rightarrow C)$ are on earlier lines of $\mathbf{D}$. Since C is the first "bad" line of $\mathbf{D}$, $\Gamma \models B$ and $\Gamma \models (B \rightarrow C)$. By the validity of MP, it follows that $\Gamma \models C$. $\square$

A system $\mathbf{S}$ of deduction for $\mathcal{L}$ is *complete* if, for all sets Γ of formulas and all formulas A, if $\Gamma \models A$ then $\Gamma \vdash_{\mathbf{S}} A$.

Remark. Sometimes the word "complete" used to mean what we mean by "sound and complete."

We are now going to embark on the task of proving the completeness of **SL**. The proof will parallel the proof of the Compactness Theorem. In particular, the lemma that follows is the analogue of Lemma 1.4

Lemma 2.5. *Let Γ be a consistent (in* **SL***) set of formulas and let A be a formula. Then either $\Gamma \cup \{A\}$ is consistent or $\Gamma \cup \{\neg A\}$ is consistent.*

Proof. Assume for a contradiction neither $\Gamma \cup \{A\}$ nor $\Gamma \cup \{\neg A\}$ is consistent. It follows that there are formulas B and B' such that

(i) $\Gamma \cup \{A\} \vdash B\,;$

(ii) $\Gamma \cup \{A\} \vdash \neg B$;

(iii) $\Gamma \cup \{\neg A\} \vdash B'$;

(iv) $\Gamma \cup \{\neg A\} \vdash \neg B'$.

Using Axiom Schema (2) together with (iii), (iv), and the Deduction Theorem, we can show that

$$\Gamma \vdash A \,.$$

This fact, together with (i) and (ii), allows us to show that $\Gamma \vdash B$ and $\Gamma \vdash \neg B$. Thus we have the contradiction that Γ is inconsistent. ☐

Now we turn to the analogue of Lemma 1.5.

Lemma 2.6. *Let* Γ *be a consistent set of formulas. There is a set* Γ^* *of formulas such that*

(1) $\Gamma \subseteq \Gamma^*$ *;*

(2) Γ^* *is consistent;*

(3) for every formula A*, either* A *belongs to* Γ^* *or* $\neg A$ *belongs to* Γ^**.*

Proof. Let

$$A_0,\, A_1,\, A_2,\, A_3,\, \ldots$$

be the list (defined in the proof of Lemma 1.5) of all the formulas of $\mathcal{L}$. As in that proof we define, by recursion on natural numbers, a function that associates with each natural number n a set Γ_n of formulas.

Let $\Gamma_0 = \Gamma$.

Let

$$\Gamma_{n+1} \;=\; \begin{cases} \Gamma_n \cup \{A_n\} & \text{if } \Gamma_n \cup \{A_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

Let $\Gamma^* = \bigcup_n \Gamma_n$.

Because $\Gamma = \Gamma_0 \subseteq \Gamma^*$, Γ^* has property (1).

Γ_0 is consistent. By Lemma 2.5, if Γ_n is consistent then so is Γ_{n+1}. By mathematical induction, every Γ_n is consistent. Suppose, in order to obtain a contradiction, that Γ^* is inconsistent. Let B be a formula such that $\Gamma^* \vdash B$ and $\Gamma^* \vdash \neg B$. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be respectively deductions of B from Γ^* and of $\neg B$ from Γ^*. Let Δ be the set of all formulas belonging to Γ^* that are on lines of $\mathbf{D}_1$ or of $\mathbf{D}_2$. Then Δ is a finite subset of Γ^*, and so $\Delta \subseteq \Gamma_n$ for some n. But then $\Gamma_n \vdash B$ and $\Gamma_n \vdash \neg B$. This contradicts the consistency of Γ_n. Thus Γ^* has property (2).

Because either A_n or $\neg A_n$ belongs to Γ_{n+1} for each n and because each $\Gamma_{n+1} \subseteq \Gamma^*$, Γ^* has property (3). $\square$

Next comes the analogue of Lemma 1.6.

Lemma 2.7. *Let Γ^* be a set of formulas having properties (2) and (3) described in the statement of Lemma 2.6. Then Γ^* is satisfiable.*

Proof. We first show that Γ^* is *deductively closed*: for any formula A, if $\Gamma^* \vdash A$ then $A \in \Gamma^*$. Assume that $\Gamma^* \vdash A$. If also $\neg A \in \Gamma^*$, then Γ^* is inconsistent, contradicting (2). By (3), $A \in \Gamma^*$.

Define a valuation v for $\mathcal{L}$ by setting

$$v(A) = \mathbf{T} \text{ if and only if } A \in \Gamma^*$$

for each sentence letter A. Let P be the property of being a formula A such that

$$v^*(A) = \mathbf{T} \text{ if and only if } A \in \Gamma^* .$$

We prove by induction on length that every formula has property P. Let A be a formula and assume that every formula shorter than A has P.

Case (i). A is a sentence letter. A has P by the definition of v.

Case (ii). A is $\neg B$ for some formula B. We want to show that $v^*(\neg B) = \mathbf{T}$ if and only if $\neg B \in \Gamma^*$. Consider the following biconditionals.

$$\begin{aligned} v^*(\neg B) = \mathbf{T} \quad & \text{iff} \quad v^*(B) = \mathbf{F} \\ v^*(B) = \mathbf{F} \quad & \text{iff} \quad B \notin \Gamma^* \\ B \notin \Gamma^* \quad & \text{iff} \quad \neg B \in \Gamma^* . \end{aligned}$$

These biconditionals imply that $v^*(\neg B) = \mathbf{T}$ if and only if $\neg B \in \Gamma^*$.

The first biconditional is true by definition of v^*. The second biconditional is true because B is shorter than A and so has P. To finish Case (ii), we need only prove the third biconditional.

For the "if" direction, assume that $\neg B \in \Gamma^*$. If $B \in \Gamma^*$, then Γ^* is inconsistent, so by (2) $B \notin \Gamma^*$. Now for for the "only if" direction, assume that $B \notin \Gamma^*$. By (3), $\neg B \in \Gamma^*$.

Case (iii). A is $(B \to C)$ for some formulas B and C. We want to show that $v^*((B \to C)) = \mathbf{T}$ if and only if $(B \to C) \in \Gamma^*$. Consider the following biconditionals.

$$\begin{aligned} v^*((B \to C)) = \mathbf{T} \quad & \text{iff} \quad \text{if } v^*(B) = \mathbf{T} \text{ then } v^*(C) = \mathbf{T} \\ \text{if } v^*(B) = \mathbf{T} \text{ then } v^*(C) = \mathbf{T} \quad & \text{iff} \quad \text{if } B \in \Gamma^* \text{ then } C \in \Gamma^* \\ \text{if } B \in \Gamma^* \text{ then } C \in \Gamma^* \quad & \text{iff} \quad (B \to C) \in \Gamma^* . \end{aligned}$$

These biconditionals imply that

$$v^*((B \to C)) = \mathbf{T} \text{ if and only if } (B \to C) \in \Gamma^*.$$

The first biconditional is true by definition of v^*. The second biconditional is true because B and C are shorter than $(B \to C)$, and so both have property P. To finish Case (iii), we need only prove the third biconditional.

For the "if" direction, assume that $(B \to C) \in \Gamma^*$ and $B \in \Gamma^*$. By MP, $\Gamma^* \vdash C$ and so $C \in \Gamma^*$ by deductive closure.

Now for the "only if" direction, assume that if $B \in \Gamma^*$ then $C \in \Gamma^*$. Either $B \in \Gamma^*$ or $B \notin \Gamma^*$. Assume first that $B \notin \Gamma^*$. By (3), $\neg B \in \Gamma^*$. By part (1) of Exercise 2.1, $\vdash (\neg B \to (B \to C))$. By deductive closure, $(\neg B \to (B \to C)) \in \Gamma^*$. By MP, $(B \to C) \in \Gamma^*$. Next assume that $B \in \Gamma^*$ By our assumption, $C \in \Gamma^*$. $(C \to (B \to C))$ is an instance of Axiom Schema 1. By MP, $\Gamma^* \vdash (B \to C)$. By deductive closure, $(B \to C) \in \Gamma^*$.

Since, in particular, $v^*(A) = \mathbf{T}$ for every member of A of Γ^*, we have shown that Γ^* is satisfiable. □

Theorem 2.8. *Let Γ be a consistent set of formulas. Then Γ is satisfiable.*

Proof. By Lemma 2.6, let Γ^* have properties (1)–(3) of that lemma. By Lemma 2.7, Γ^* is satisfiable. Hence Γ is satisfiable. □

Theorem 2.9 (Completeness). *Let Γ be a set of formulas and let A be a formula such that $\Gamma \models A$. Then $\Gamma \vdash_{\mathbf{SL}} A$. In other words,* **SL** *is complete.*

Proof. Since $\Gamma \models A$, $\Gamma \cup \{\neg A\}$ is not satisfiable. By Theorem 2.8, $\Gamma \cup \{\neg A\}$ is inconsistent. Let B be a formula such that $\Gamma \cup \{\neg A\} \vdash B$ and $\Gamma \cup \{\neg A\} \vdash \neg B$. By the Deduction Theorem, $\Gamma \vdash (\neg A \to B)$ and $\Gamma \vdash \neg A \to \neg B)$. Using Axiom Schema 4, we can use these facts to show that $\Gamma \vdash A$. □

Exercise 2.6. Derive Theorem 2.8 from Theorem 2.9.

Remark. Soundness and completeness imply compactness. To see this, assume that Γ is a set of formulas that is not satisfiable. By part (3) of Exercise 1.7, $\Gamma \models (p_0 \land \neg p_0)$. By completeness, $\Gamma \vdash (p_0 \land \neg p_0)$. Let $\mathbf{D}$ be a deduction of $(p_0 \land \neg p_0)$ from Γ. Let Δ be the set of all formulas $C \in \Gamma$ such that C is on a line of $\mathbf{D}$. Then Δ is a finite subset of Γ and $\Delta \vdash (p_0 \land \neg p_0)$. By soundness, $\Delta \models (p_0 \land \neg p_0)$. By part (3) of Exercise 1.7, Δ is not satisfiable. Thus Γ is not finitely satisfiable.

Exercise 2.7. Prove that $\{\neg(\neg A \wedge \neg B)\} \vdash (A \vee B)$ and that $\{(A \vee B)\} \vdash \neg(\neg A \wedge \neg B)$. You may use *any* of our theorems, lemmas, etc.

Exercise 2.8. We define by recursion on natural numbers a function that assigns to each natural number n a set $\mathbf{Formula}_n$ of formulas. Let $\mathbf{Formula}_0$ be the set of all sentence letters. Let A belong to $\mathbf{Formula}_{n+1}$ if and only if at least one of the following holds:

- (i) $A \in \mathbf{Formula}_n$;
- (ii) there is a $B \in \mathbf{Formula}_n$ such that A is $\neg B$;
- (iii) there are $B \in \mathbf{Formula}_n$ and $C \in \mathbf{Formula}_n$ such that A is $(B \vee C)$.

It is not hard to prove that A is a formula if and only if A belongs to $\mathbf{Formula}_n$ for some n. (You may assume this.)

Use mathematical induction to prove that every formula has an even number of parentheses.

Exercise 2.9. Suppose we changed our system of deduction by replacing the Axiom Schema 1 by the rule MC.

$$\frac{B}{(A \to B)}$$

Would the resulting system be sound? Would it be complete?

Hint: You need to consider two issues. First, there are places in our proofs where we use instances of Schema 1. Can those proofs be rewritten, using MC instead? Second, there are places in our proofs where we have to prove that each inferred line of a deduction has some property P. Can those proofs be rewritten, to take account of the new inference rule MC?