Solutions for Practice Problems

Exercise 2.3 This was not on the practice problem list, and the problem is not clearly formulated, but here is what I had in mind:

By Axiom Schema 1, $\vdash (A \to (B \to A))$. By two applications of the converse of the Deduction Theorem, $\{A, B\} \vdash A$. By two applications of the Deduction Theorem, $\vdash (B \to (A \to A))$.

Exercise 2.4. By the Deduction Theorem, it is enough to show that

$$\{(A \to C), (B \to C), (\neg A \to B)\} \vdash C.$$

Here is an abbreviated deduction showing this:

1.	$A \to C$	Premise
2.	$\neg C \to \neg A$	1; Lemma 2.3(b)
3.	$\neg A \to B$	Premise
4	$\neg C \to B$	$2, 3$; (†)
5.	$B \to C$	Premise
6.	$\neg C \to \neg B$	5; Lemma 2.3(b)
7.	$(\neg C \to B) \to ((\neg C \to \neg B) \to C)$	Ax. 2
8.	$(\neg C \to \neg B) \to C$	$4, 7$; MP
9.	C	$6, 8$; MP

Remarks:

(†) is the statement $\{(A \to B), (B \to C)\} \vdash (A \to C)$. It is proved in the solution for Exercise 2.2(b). You may cite (†) in abbreviated deductions in future homework and on exams.

Notice that $(\neg A \to B)$ is the same as $(A \vee B)$. Hence $\{(A \to C), (B \to C)\} \vdash ((\neg A \to B) \to C)$ is essentially the rule of separation of cases.

Exercise 2.5. We begin with a deduction of p_1 from $\{(\neg p_1 \to p_2), \neg p_2\}$:

1.	$\neg p_2$	Premise
2.	$\neg p_2 \to (\neg p_1 \to \neg p_2)$	Ax. 1
3.	$\neg p_1 \to \neg p_2$	$1, 2$; MP
4	$(\neg p_1 \to p_2) \to ((\neg p_1 \to \neg p_2) \to p_1)$	Ax. 2
5.	$\neg p_1 \to p_2$	Premise
6.	$(\neg p_1 \to \neg p_2) \to p_1$	5; $4, 5$; MP
7.	p_1	$3, 6$; MP

Next we prefix each line of this deduction by "$\neg p_2 \to$," obtaining something whose justifications are false and which is not a deduction:

1.	$\neg p_2 \to \neg p_2$	Premise
2.	$\neg p_2 \to (\neg p_2 \to (\neg p_1 \to \neg p_2))$	Ax. 1
3.	$\neg p_2 \to (\neg p_1 \to \neg p_2)$	$1, 2$; MP
4	$\neg p_2 \to ((\neg p_1 \to p_2) \to ((\neg p_1 \to \neg p_2) \to p_1))$	Ax. 2
5.	$\neg p_2 \to (\neg p_1 \to p_2)$	Premise
6.	$\neg p_2 \to ((\neg p_1 \to \neg p_2) \to p_1)$	$4, 5$; MP
7.	$\neg p_2 \to p_1$	$3, 6$; MP

Now we begin to turn this into a deduction. If we were fully following the proof of the Deduction Theorem, then we would use Example 2 to preface line 1 with four lines get a deduction of $\neg p_2 \to \neg p_2$. But lines 1 and 2 of the original deduction were used only to get line 3, and the new line 3 is an axiom. So we omit lines 1 and 2 and add a justification of line 3:

3.	$\neg p_2 \to (\neg p_1 \to \neg p_2)$	Ax, 1
4.	$\neg p_2 \to ((\neg p_1 \to p_2) \to ((\neg p_1 \to \neg p_2) \to p_1))$	Ax. 2
5.	$\neg p_2 \to (\neg p_1 \to p_2)$	Premise
6.	$\neg p_2 \to ((\neg p_1 \to \neg p_2) \to p_1)$	$4, 5$; MP
7.	$\neg p_2 \to p_1$	$3, 6$; MP

Finally, we follow the proof of the Deduction Theorem and get the desired deduction. I have left lines 5a, 5b, 6a, and 6b to be filled in. To do this, use Ax. 3 as in the MP step of the proof of the Deduction Theorem.

3.	$\neg p_2 \to (\neg p_1 \to \neg p_2)$	Ax, 1
3a.	$(\neg p_1 \to p_2) \to ((\neg p_1 \to \neg p_2) \to p_1)$	Ax.2.
3b.	$((\neg p_1 \to p_2) \to ((\neg p_1 \to \neg p_2) \to p_1)) \to$ $(\neg p_2 \to ((\neg p_1 \to p_2) \to ((\neg p_1 \to \neg p_2) \to p_1)))$	Ax. 1
4.	$\neg p_2 \to ((\neg p_1 \to p_2) \to ((\neg p_1 \to \neg p_2) \to p_1))$	3a,3b;MP
4a.	$\neg p_1 \to p_2$	Premise
4b.	$(\neg p_1 \to p_2) \to (\neg p_2 \to (\neg p_1 \to p_2))$	Ax. 1
5.	$\neg p_2 \to (\neg p_1 \to p_2)$	4a, b; MP
5a.		Ax. 3
5b.		4,5a;MP
6.	$\neg p_2 \to ((\neg p_1 \to \neg p_2) \to p_1)$	5, 5b; MP
6a.		Ax. 3
6b		6,6a; MP
7.	$\neg p_2 \to p_1$	3, 6b; MP

Exercise 2.6. Assume that Γ is a consistent set of formulas. Let B be any formula. By consistency of Γ, either $\Gamma \nvdash B$ or $\Gamma \nvdash \neg B$. By Completeness, either $\Gamma \not\models B$ or $\Gamma \not\models \neg B$. By Exercise 1.8, either $\Gamma \cup \{\neg B\}$ is satisfiable or $\Gamma \cup \{\neg\neg B\}$ is satisfiable. Hence Γ is satisfiable.

Exercise 2.7. By Completeness, it is enough to show that each of the two formulas logically implies the other. We omit the details of doing this.

Exercise 2.9. Any deduction in the new system can be turned into a deduction in the old one by using Axiom Schema 1 and MP to get the effect of the new rule. Thus

$$\Gamma \vdash_{\text{new}} A \;\Rightarrow\; \Gamma \vdash_{\text{old}} A \;\Leftrightarrow\; \Gamma \models A.$$

This means that Γ is sound.

The new system is incomplete. The proof of incompleteness is not something the class would be expected to find.