PARTIAL CLASSICALITY OF HILBERT MODULAR FORMS

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Abstract. Let $F$ be a totally real field and $p$ a rational prime unramified in $F$. We prove a partial classicality theorem for overconvergent Hilbert modular forms: when the slope is small compared to certain but not all weights, an overconvergent form is partially classical. We use the method of analytic continuation.

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1. Introduction

Coleman [Col96] proved that a $p$-adic overconvergent modular form of weight $k \in \mathbb{Z}$ must be classical if its slope, i.e., the $p$-adic valuation of the $U_p$-eigenvalue, is less than $k - 1$. His proof involves analyzing the rigid cohomology of modular curves. On the other hand, Buzzard [Buz03] and Kassaei [Kas06] developed the alternate method of analytic continuation to prove classicality theorems. The key is to understand the dynamic of the $U_p$ Hecke operator.

Let $F$ be a totally real field of degree $g$ over $\mathbb{Q}$. In the situation of Hilbert modular forms associated to $F$, many results are known. Coleman’s cohomological method is developed by Tian–Xiao [TX16] to prove a classicality theorem, assuming $p$ is unramified in $F$. The method of analytic continuation is worked out first in the case when $p$ splits completely in $F$ by Sasaki [Sas10], then in the case when $p$ is unramified by Kassaei [Kas16] and Pilloni–Stroh [PS17], and finally when $p$ is allowed to be ramified by Bijakowski [Bij16].

Let $\Sigma$ be the set of archimedean embeddings of $F$, which we identify with the set of $p$-adic embeddings of $F$ through some fixed isomorphism $\mathbb{C} \cong \mathbb{C}_p$. For each prime $\mathfrak{p}$ of $F$ above $p$, denote by $\Sigma_\mathfrak{p} \subseteq \Sigma$ the subset of $p$-adic embeddings inducing $\mathfrak{p}$. Let $e_\mathfrak{p}$ be the ramification index, and $f_\mathfrak{p}$ the residue degree of $\mathfrak{p}$. Then the classicality theorem for overconvergent Hilbert modular forms proved by analytic continuation is as follows.

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Theorem 1 (Bijakowski). Let $f$ be an overconvergent Hilbert modular form of weight $k \in \mathbb{Z}^2 \cong \mathbb{Z}^g$. Assume that for all $p \mid p$, $U_p(f) = a_p f$ such that

$$\text{val}_p(a_p) < \frac{1}{e_p} \inf \{k_\tau\} - f_p.$$ 

Then $f$ is classical.

Remark 1.1. When $p$ is unramified in $F$, using the cohomological method of Tian–Xiao, the slope assumption can be improved to an optimal bound: $\text{val}_p(a_p) < \inf_{\tau \in \Sigma_p}\{k_\tau\} - 1$

In this paper, we prove a “partial” classicality theorem for overconvergent Hilbert modular forms. Let $I$ be a subset of $\Sigma$. Breuil defined the notion of I-classical overconvergent Hilbert modular forms ([?], p.3, Definition 2.4). When $I = \emptyset$, they are the usual overconvergent forms; when $I = \Sigma$, they are the classical forms.

Theorem 2 (Theorem 3.1). Assume $p$ is unramified in $F$. Let $f$ be an overconvergent Hilbert modular form of weight $k \in \mathbb{Z}^2$. Let $I \subseteq \Sigma$. Assume that for all $p \mid p$, $U_p(f) = a_p f$ such that

$$\text{val}_p(a_p) < \inf_{\tau \in I \cap \Sigma_p}\{k_\tau\} - f_p.$$ 

Then $f$ is I-classical.

We use the method of analytic continuation to prove Theorem 2. In the situation when $I = \Sigma$, this recovers the classicality theorem proven by Kassaei, who assumed $p$ is unramified. Although when $I = \Sigma$, Bijakowski proved a classicality theorem not assuming $p$ is unramified, it is Kassaei’s approach that is more suitable for partial classicality. Indeed, when studying the dynamic of $U_p$ operators, it is customary to use the degree function to parametrize regions on the Hilbert modular variety, and analyze how $U_p$ operators influence degrees. Kassaei made efforts to analyze how $U_p$ operators affect the more refined directional degrees, but only when $p$ is unramified. On the other hand, Bijakowski was able to use only the degree function to prove a classicality theorem allowing $p$ to be ramified. In the situation of partial classicality, the weight $k_\tau$ with $\tau \in \Sigma$ in the slope condition are independent of each other, while the $U_p$ operator intertwines all directional degrees inducing $p$. As a result, we cannot avoid analyzing the directional degrees like Bijakowski did.

We mention some related work on partial classicality theorems. Yiwen Ding [Din17, Appendix A] studied partial classicality from the perspective of Galois representations. Namely, let $\rho_f: \text{Gal}_F \to \GL_2(L)$ be the Galois representation associated to an overconvergent Hilbert modular form $f$. If $\text{val}_p(a_p) < \inf_{\tau \in \Sigma_p}k_\tau - 1$, then $\rho_f|_{\text{Gal}_p}$ is $I \cap \Sigma_p$-de Rham. Barrera Salazar and Williams [?] took the perspective of overconvergent cohomology for a general quasi-split reductive group $G$ over $\mathbb{Q}$ with respect to a parabolic subgroup $Q$ of $G = G/\mathbb{Q}_p$. Applying their work to the situation of Hilbert modular forms (i.e., $G = \text{Res}_{F/\mathbb{Q}}\GL_2$), we would recover the partial classicality theorem in the case of $I \subset \Sigma$ such that $I \cap \Sigma_p$ is either $\Sigma_p$ or $\emptyset$ for each $p \mid p$.

In Section 2, we define the degree function and partially classical overconvergent forms. In Section 3, we prove Theorem 2.

Notations. Fix a totally real field $F$ of degree $g$ over $\mathbb{Q}$. Let $\Sigma$ denote the set of archimedean places of $F$; in particular $\# \Sigma = g$. Fix a rational prime $p$ which is unramified in $F$ and $(p) = p_1 \cdots p_r$ in $F$. For each prime $p$ of $F$ above $p$, let $f_p$ be the residue degree of $p$. Let $L$ be a finite unramified extension of $\mathbb{Q}_p$ containing $F_p$ for all primes $p$ of $F$ above $p$. Fix an isomorphism $i_p: \mathbb{C} \cong \overline{\mathbb{Q}}_p$, so we will identify an archimedean embedding $\tau: F \to \mathbb{C}$ with a $p$-adic embedding $t_p \circ \tau: F \to \overline{\mathbb{Q}}_p$. For each prime $p$ of $F$ above $p$, let $\Sigma_p \subseteq \Sigma$ be the subset of $p$-adic embeddings inducing $p$. Hence $\# \Sigma_i = f_i$. Let $\delta_F$ be the different ideal of $F$. 2


2. Partially classical overconvergent forms

2.1. Hilbert modular varieties. Let $N \geq 4$ be an integer. Let $c$ be a fractional ideal of $F$. Denote by $c^+ \subseteq c$ the cone of totally positive elements. Let $Y_c \to \Spec \mathcal{O}_L$ be the Hilbert modular scheme classifying $(A, H) = (A/S, i, \lambda, \alpha, H)$ where

- $A$ is an abelian scheme of relative dimension $g$ over a scheme $S$
- $i : \mathcal{O}_F \to \End_S(A)$ is a ring homomorphism, which provides a real multiplication on $A$
- $\lambda : \Hom_{\mathcal{O}_F}(A, A^\vee)^{\text{sym}} \to c$ is an $\mathcal{O}_F$-isomorphism identifying polarizations with $c^+$ and inducing an isomorphism $A \otimes_{\mathcal{O}_F} c \cong A^\vee$
- $\alpha : \mu_N \otimes \delta_F^{-1} \to A$ is a $\mu_N$ level structure
- $H \subseteq A[p]$ is a finite flat isotropic $\mathcal{O}_F$-subgroup scheme of rank $p^g$.

Let $Y = \coprod_{[\ell] \in \Cl(F)^+} Y_{\ell}$. Let $\mathcal{Y}$ denote the rigid analytic space associated to $Y$.

2.2. Directional degrees. We first recall the definition of the degree for a finite flat group scheme. See [Far10] for more detailed studies of the concept.

Let $K/\mathbb{Q}_p$ be a finite extension, and $G$ a finite flat group scheme over $\mathcal{O}_K$. Let $\omega_G$ be the $\mathcal{O}_K$-module of invariant differentials on $G$. Then [Far10, Définition 4]

$$\deg G := \ell(\omega_G)/e_K,$$

where $\ell(\omega_G)$ is the length of the $\mathcal{O}_K$-module $\omega_G$, and $e_K$ is the ramification index of $K$.

Recall that the height $\text{ht} G$ of $G$ is such that $|G| = p^{\text{ht} G}$. Hence $G$ is étale if and only if $\deg G = 0$, and $G$ is multiplicative if and only if $\deg G = \text{ht} G$.

We record some properties of $\deg$ which we will constantly use for computation.

**Lemma 2.1.** [Far10, lemme 4] Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of finite flat group schemes over $\mathcal{O}_K$. Then $\deg G = \deg G' + \deg G''$.

**Lemma 2.2.** [Far10, p.2] Let $\lambda : A \to B$ be an isogeny of $p$-power degree between abelian schemes over $S = \Spec \mathcal{O}_K$. Let $G := \ker \lambda$. Let $\omega_{A/S}$ and $\omega_{B/S}$ be the sheaves of invariant differentials of $A$ and $B$, respectively. Let $\lambda^* : \omega_{B/S} \to \omega_{A/S}$ be the induced pullback map. Then

$$\deg G = \text{val}_p(\det \lambda^*).$$

In particular, if $A$ is of dimension $g$, then $\deg A[p] = g$.

The degree of a finite flat group scheme can be used to define the degree function on $\mathcal{Y}$. Let $y = (A, H)$ be a rigid point of $\mathcal{Y}$. Let $\omega_H$ be the module of invariant differentials on $H$. Since $\omega_H$ is an $\mathcal{O}_F$-module, we have the decomposition

$$\omega_H = \bigoplus_{\tau \in \Sigma} \omega_{H, \tau},$$

where $\omega_{H, \tau}$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_L} \mathcal{O}_L$-module. Define the directional degree

$$\deg_{\tau} y = \deg_{\tau} H = \deg \omega_{H, \tau}.$$ 

This gives a map from $\mathcal{Y}$ to the $g$-dimensional hypercube in $\mathbb{Q}^g$

$$\deg : \mathcal{Y} \to ([0, 1] \cap \mathbb{Q})^\Sigma$$

which sends a point $y = (A, H)$ to $(\deg_{\tau} y)_{\tau \in \Sigma}$.

Given $I \subseteq \Sigma$, we define

$$\mathcal{F}_I := \prod_{\tau \in I} \mathcal{F}_{I, \tau},$$

where $\mathcal{F}_{I, \tau} = \begin{cases} [0, 1], & \tau \in I \\ [1, 1], & \tau \notin I. \end{cases}$
Then $\mathcal{F}_I$ is a closed $|I|$-dimensional hypercube in $([0,1] \cap \mathbb{Q})^{\Sigma} = \mathcal{F}_\Sigma$. We also define $x_I \in [0,1]^{\Sigma}$ to be the vertex

$$x_I,\tau = \begin{cases} 0, & \tau \in I \\ 1, & \tau \notin I. \end{cases}$$

Hence the vertices of $\mathcal{F}_I$ are exactly the $x_J$'s with $J \subseteq I$.

Let $\mathcal{F} \subseteq \mathcal{F}_\Sigma$ be a closed subset. Define $\mathcal{Y}\mathcal{F}$ to be the admissible open of $\mathcal{Y}$ whose points satisfy $\text{deg} \in \mathcal{F}$.

**Definition 2.3.** Let $p|p$ be a prime of $F$. For $\tau \in \Sigma_p$, define the twisted directional degree

$$\tilde{\text{deg}}_{\tau} := \sum_{j=0}^{f_p-1} p^{f_p-1-j} \text{deg}_{\sigma^j \tau} = p^{f_p-1} \text{deg}_{\tau} + p^{f_p-2} \text{deg}_{\sigma \tau} + \cdots + p^{1} \text{deg}_{\sigma^{f_p-1} \tau}.$$  

Here $\sigma$ is the Frobenius automorphism of the unramified extension $L$ over $\mathbb{Q}_p$, lifting $x \mapsto x^p$ modulo $p$.

We use the overhead tilde notation ($\tilde{\cdot}$) to denote the image under the linear transformation $\tilde{x}_\tau = \sum_{j=0}^{f_p-1} p^{f_p-1-j} x_{\sigma^j \tau}$ for $\tau \in \Sigma_p$. For example, $\tilde{x}_I$ is the vertex of $\tilde{\mathcal{F}}_\Sigma$ given by

$$\tilde{x}_I,\tau = \sum_{j=0}^{f_p-1} p^{f_p-1-j} x_{I,\sigma^j \tau}$$  

for $\tau \in \Sigma_p$. 

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2.3. Hilbert modular forms. Let \( \omega \) be the sheaf of relative differentials of the universal abelian scheme over \( \mathcal{Y} \). The \( \mathcal{O}_F \)-module structure on \( \omega \) provides the decomposition

\[
\omega = \bigoplus_{\tau \in \Sigma} \omega_{\tau}.
\]

Let \( k = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma \). Define

\[
\omega^k = \bigotimes_{\tau \in \Sigma} \omega_{\tau}^k,
\]

which is a line bundle on \( \mathcal{Y} \).

**Definition 2.4.**

1. The space of Hilbert modular forms of level \( \Gamma_1(N) \cap \Gamma_0(p) \) and weight \( k \) is \( H^0(\mathcal{Y}, \omega^k) \).
2. Let \( I \subseteq \Sigma \). The space of \( I \)-classical overconvergent Hilbert modular forms of level \( \Gamma_1(N) \cap \Gamma_0(p) \) and weight \( k \) is

\[
H^{0,\dagger}(I, \omega^k) := \lim_{\mathcal{V} \to k} H^0(\mathcal{V}, \omega^k),
\]

where \( \mathcal{V} \) runs through strict neighborhoods of \( \mathcal{Y} \mathcal{F}_I \) in \( \mathcal{Y} \).

When \( I = \emptyset \), \( I \)-classical simply means overconvergent, and when \( I = \Sigma \), \( I \)-classical means classical. Whenever \( J \subseteq I \), we have a map

\[
H^{0,\dagger}(I, \omega^k) \to H^{0,\dagger}(J, \omega^k)
\]
given by restriction. This is an injective map.

2.4. \( U_p \)-operators. Let \( p | p \) be a prime of \( F \) above \( p \) and \( f_p \) the residue degree of \( p \). Let \( U_p : \mathcal{Y} \to \mathcal{Y} \) be the correspondence sending \( (A, H) \) to \( \{A/D, A[p]/D : D \subseteq A[p]\} \) where \( D \) runs over all finite flat isotropic \( \mathcal{O}_F \)-subgroup scheme of \( A[p] \) of rank \( p^{\text{deg}} \) such that \( D \neq H[p] \).

For \( \mathcal{U}, \mathcal{V} \subseteq \mathcal{Y} \) such that \( U_p(\mathcal{V}) \subseteq \mathcal{U} \), we have \( U_p : \omega^k(\mathcal{U}) \to \omega^k(\mathcal{V}) \) defined by

\[
(U_p f)(A, H) = \frac{1}{p^{\text{deg}}} \sum_{\text{rank } D = p^{\text{deg}} \mathcal{V}} \text{pr}^* f(A/D, A[p]/D),
\]

where \( \text{pr} : A \to A/D \) is the natural projection.

**Proposition 2.5** ([Kas16, Proposition 2.9.7]). Let \( y = (A, H) \in \mathcal{Y} \). Let \( p | p \) be a prime of \( F \) above \( p \). If \( y' = (A/D, A[p]/D) \in U_p(y) \), then \( \text{deg}_{\tau}(y') \geq \text{deg}_{\tau}(y) \) for all \( \tau \in \Sigma_p \).

3. Partial classicality

The content of this section is to prove the following partial classicality theorem.

**Theorem 3.1.** Let \( f \) be an overconvergent Hilbert modular form of weight \( k \). Let \( I \subseteq \Sigma \). Assume that for all \( p | p \), \( U_p(f) = a_p f \) such that

\[
(1) \quad \text{val}_p(a_p) < \inf_{\tau \in I \cap \Sigma_p} \{k_\tau\} - f_p.
\]

Then \( f \) is \( I \)-classical.

**Remark 3.2.** In the case of \( I = \Sigma \), this is a theorem of Kassaei [Kas16]. Although when \( I = \Sigma \), Bijakowski [Bij16] proved a classicality theorem not assuming \( p \) is unramified, it is Kassaei’s approach that is more suitable for partial classicality. Both use the idea of analytic continuation. Kassaei made efforts to analyze how \( U_p \) operators affect \( \text{deg}_{\tau} \) for all \( \tau \in \Sigma_p \), but only when \( p \) is unramified. On the other hand, Bijakowski was able to use only \( \text{deg } H[p] \) to prove the classicality even when \( p \) is ramified. In the situation of partial classicality, the weight \( k_\tau \) with \( \tau \in \Sigma \) in the slope
condition are independent of each other, while the $U_p$ operator intertwines all directional degrees inducing $p$, so we do need to understand the directional degrees.

Throughout the section, we assume that $p$ is inert in $F$. For general unramified $p$, we can apply the argument to each prime $p | p$ to prove Theorem 3.1. We will prove that $f$ is $J$-classical for all $J \subseteq I$, by induction on $|J|$. For $|J| = 0$, it simply means that $f$ is overconvergent. Assume that $f$ is $J$-classical for all $J \subset I$, say $f$ is defined on a strict neighborhood of $\mathcal{Y} F_J$. In particular, $f$ is defined on a strict neighborhood of $\deg^{-1} x_J$ for all $J \subset I$.

3.1. Automatic analytic continuation. In the subsection, with the assumption that the slope of $f$ is finite (but not necessarily small), we can already show that $f$ can be analytically continued to a large region in $\mathcal{Y} F_I$.

Let $f$ be an overconvergent Hilbert modular form of weight $k$. Assume that $U_p(f) = a_p f$ with $\text{val}_p(a_p) < \infty$.

**Lemma 3.3.** Let $I \subseteq \Sigma$. Suppose that $f$ is defined on a strict neighborhood of $\deg^{-1} x_J = \tilde{\deg}^{-1} \tilde{x}_J$ for all $J \subset I$. Then $f$ can be extended to

$$U_f(\epsilon) = \{ y \in \mathcal{Y} : \sum_{\tau \in I} \deg_{\tau} y \geq \sum_{\tau \in I} \tilde{x}_{I,\tau} + \epsilon, \tilde{\deg}_{\tau} y \geq p^{g-2} + \cdots + 1 + \epsilon, \forall \tau \notin I \}.$$

for any rational number $\epsilon > 0$. Note that whenever $\epsilon' < \epsilon$, we have $U_f(\epsilon') \supseteq U_f(\epsilon)$.

**Proof.** We first note that $U_f(\epsilon)$ is $U_p$-stable because $U_p$ increases twisted directional degrees (Proposition 2.5).

If $\deg y = x_J$ is a tuple of integers, then for $\tau \in J$, $\tilde{\deg}_{\tau} y \leq p^{g-2} + \cdots + 1$. Hence the second condition of $U_f(\epsilon)$

$$\tilde{\deg}_{\tau} y \geq p^{g-2} + \cdots + 1 + \epsilon, \forall \tau \notin I$$

says that if $y \in U_f(\epsilon)$ is such that $\deg y = x_J$, then $\tau \notin I$ implies $\tau \notin J$, i.e., $J \subseteq I$. The first condition of $U_f(\epsilon)$

$$\sum_{\tau \in I} \tilde{\deg}_{\tau} y \geq \sum_{\tau \in I} \tilde{x}_{I,\tau} + \epsilon$$


Lemma 5.1.5. For the proof of Lemma 3.3. Further applying a power of $U$ that is a quasi-compact open disjoint from all $\tilde{\deg}^{-1}x_J$'s. By Proposition 2.5, $U_p$ increases the twisted directional degrees strictly when the degrees are not all integers, for example on $V$. Using the Maximum Modulus Principle, the quasi-compactness of $V$ implies that there is a positive lower bound for the increase of $\tilde{\deg}$ under $U_p$ on $V$. Because $U_f(\epsilon)$ is $U_p$-stable, there exists $M > 0$ such that $U_p^M V \subseteq \bigcup_{J \subseteq I} V_J$. We can then extend $f$ to $U_f(\epsilon)$ by $\left(\frac{U_p}{\epsilon}\right)^M f$.

3.2. Analytic continuation near vertices. In this subsection, we will make use of the small slope assumption (1) to extend $f$ to a strict neighborhood of $\deg^{-1}x_J$.

By (1), fix a rational number $\epsilon > 0$ such that

$$\text{val}(a_p) \leq \inf_{\tau \in I} k_{\tau} - g - \epsilon \sum_{\tau \in I} k_{\tau}.$$ We will choose a rational number $\delta > 0$ based on $\epsilon$, and define a sequence of strict neighborhoods

$$S_{f,0}(\delta) \supseteq S_{f,1}(\delta) \supseteq \cdots$$

deg^{-1}x_J. When $\delta' < \delta$ we will show that $S_{f,m}(\delta') \subseteq S_{f,m}(\delta)$. We have extended $f$ to $U_f(\delta)$ by Lemma 3.3. Further applying a power of $U_p$, we can extend $f$ to $S_{f,0}(\delta) \setminus S_{f,m}(\delta')$, named $f_m$. We will also define $F_m$ on $S_{f,m}(\delta)$. With the help of the estimates in Section 3.3, we can show that when $m \rightarrow \infty$, $f_m$ and $F_m$ glue to define an extension of $f$ on $S_{f,0}(\delta)$.

We first prove the following lemma, which will be used to decompose $U_p$ into the special part $U_p^{sp}$ and non-special part $U_p^{nsp}$.

**Lemma 3.4.** Let $y = (A, H) \in V$. Let $y_1 = (A/H_1, A[p]/H_1)$ and $y_2 = (A/H_2, A[p]/H_2)$ be in $U_p(y)$ and $y_1 \neq y_2$.

i. If $y, y_1 \in \tilde{\deg}^{-1}x_I$ for some $I \subseteq \Sigma$, then

$$\tilde{\deg}_I H_2 = \inf(\tilde{\deg}_I H, \tilde{\deg}_I H_1), \text{ for all } \tau \in \Sigma.$$  

ii. There exists arbitrarily small positive rational number $\epsilon$ so that if $|\tilde{\deg}_{\tau}(y) - \tilde{x}_{I,\tau}| \leq \epsilon$ and $|\tilde{\deg}_{\tau}(y_1) - \tilde{x}_{I,\tau}| \leq \epsilon$ for some $I \subseteq \Sigma$, then

$$\tilde{\deg}_{\tau} H_2 = \inf(\tilde{\deg}_{\tau} H, \tilde{\deg}_{\tau} H_1), \text{ for all } \tau \in \Sigma.$$  

In particular, $y_2 \in U_f(\epsilon)$.

**Proof.** For the proof of i., see [Kas16, Lemma 5.1.5 1.] The first statement of ii. follows from [Kas16, Lemma 5.1.5 2(a)].

The only statement remained to be proved is the one after “In particular”. By assumption,

$$\tilde{\deg}_I H_2 = \inf(\tilde{\deg}_I H, \tilde{\deg}_I H_1) = \begin{cases} \tilde{\deg}_I H & \text{if } \tau \in I \\ \tilde{\deg}_I H_1 & \text{if } \tau \notin I \end{cases}$$

and

$$\tilde{\deg}_{\tau} y_2 = (p^{g-1} + \cdots + 1) - \tilde{\deg}_{\tau} H_2 = \begin{cases} (p^{g-1} + \cdots + 1) - \tilde{\deg}_I H & \tau \in I \\ (p^{g-1} + \cdots + 1) - \tilde{\deg}_I H_1 & \tau \notin I \end{cases} \geq \begin{cases} (p^{g-1} + \cdots + 1) - \tilde{x}_{I,\tau} - \epsilon & \tau \in I \\ \tilde{x}_{I,\tau} - \epsilon & \tau \notin I \end{cases} \geq p^{g-1} - \epsilon.$$
If we further require that \( \epsilon < \frac{1}{2}(p^g - 1) \), then \( \deg_{\tau} y_2 \geq p^g + 1 + \epsilon \), i.e., \( y_2 \in U_\omega(\epsilon) \).

\[
\text{Remark 3.5. By comparing the twisted directional degree of } y_1 \text{ and } y_2, \text{ Lemma 3.4 implicitly says that if } I \neq \emptyset, \text{ then such } y_1 \in U_p(y), \text{ if exists, is unique.}
\]

For any rational \( \delta > 0 \), consider the strict neighborhood of \( \deg^{-1} x_I \):

\[
S_{I,0}(\delta) := \left\{ y \in \mathcal{Y} : \sum_{\tau \in I} \tilde{\deg}_\tau y \leq \sum_{\tau \in \mathcal{I}} \tilde{x}_{I,\tau} + \delta, \tilde{\deg}_\tau y \geq \tilde{x}_{I,\tau} - \delta, \forall \tau \notin I \right\}.
\]

Let \( S_{I,1}(\delta) \) be the special locus of order 1 in \( S_{I,0}(\delta) \), namely

\[
S_{I,1}(\delta) := \{ y \in S_{I,0}(\delta) : \exists y_1 \in U_p(y) \text{ also in } S_{I,0}(\delta) \}.
\]

We can then define

\[
U_p^{sp} : S_{I,1}(\delta) \to S_{I,0}(\delta)
\]

by \( y \mapsto y_1 \).

We also define \( U_p^{nsp} = U_p \setminus U_p^{sp} \). Note that the \( S_{I,0}(\delta) \)'s contain a fundamental system of strict neighborhoods of \( \deg^{-1} x_I \).

Then \( S_{I,0}(\delta) \cup V_I(\delta) \) is \( U_p \)-stable because \( U_p \) increases twisted directional degrees. Hence we also have

\[
U_p : S_{I,0}(\delta) \setminus S_{I,1}(\delta) \to V_I(\delta).
\]

Note that \( V_I(\delta) \subset U_I(\delta) \).

\[
\text{Lemma 3.6. Let } \delta' < \delta \text{ be two positive rational numbers. Then } S_{I,1}(\delta) \text{ is a strict neighborhood of } S_{I,1}(\delta').
\]

\[
\text{Proof. The proof has the same idea as [BPS16, Proposition 4.3.10]. By definition,}
\]

\[
S_{I,1}(\delta) = \{ y \in S_{I,0}(\delta) : \exists y_1 \in U_p(y), \sum_{\tau \in I} \tilde{\deg}_\tau y_1 \leq \sum_{\tau \in I} \tilde{x}_{I,\tau} + \delta, \tilde{\deg}_\tau y_1 \geq \tilde{x}_{I,\tau} - \delta, \forall \tau \notin I \}.
\]

Let \( Y(p) \) be the moduli space parametrizing \((A,H,H_1)\), where \( H \) and \( H_1 \) are distinct subgroups of \( A[p] \) of order \( p \). We have a finite étale morphism \( \mathcal{Y}(p) \to \mathcal{Y} \) given by forgetting \( H_1 \).

Let \( \omega_A \) be the sheaf of relative differentials of the universal abelian scheme over \( Y(p) \). Note that \( \omega_A \) is the pullback of \( \omega_A \sigma \) along \( Y(p) \to Y \). On \( Y(p) \), we have the line bundles \( L_\tau := \omega_A/_{\tau} \otimes \omega_{A/H_{\tau}}^{-1} \) and \( \mathcal{L}_{1,\tau} := \omega_A/_{\tau} \otimes \omega_{A/H_{1,\tau}}^{-1} \) for each \( \tau \in \Sigma \), and the natural pullback morphisms \( \omega_{A/H} \to \omega_A \) and \( \omega_{A/H_1} \to \omega_A \) gives a section \( \delta_\tau \) and \( \delta_{1,\tau} \) of \( L_\tau \) and \( \mathcal{L}_{1,\tau} \), respectively. Now \( |\delta_\tau(A,H,H_1)| = p^{-deg_\tau H} \) and \( |\delta_{1,\tau}(A,H,H_1)| = p^{-deg_\tau H_1} \). We also have \( \bar{\delta}_\tau := \prod_{j=0}^{g-1} \delta_{p^{g-1-j}} \) and \( \bar{\delta}_{1,\tau} := \prod_{j=0}^{g-1} \delta_{p^{g-1-j}} \), which are sections of \( \bar{\mathcal{L}}_\tau = \bigotimes_{j=0}^{g-1} \mathcal{L}_{p^{g-1-j}}^{-1} \) and \( \bar{\mathcal{L}}_{1,\tau} = \bigotimes_{j=0}^{g-1} \mathcal{L}_{1,\tau}^{p^{g-1-j}} \), respectively.

Since \( S_{I,1}(\delta) \) is the pushforward by the finite étale morphism \( \mathcal{Y}(p) \to \mathcal{Y} \) of the region cut out by \( \prod_{\tau \in I} |\delta_\tau| \geq p^{-\sum_{\tau \in I} \tilde{x}_{I,\tau} - \delta}, |\tilde{\delta}_\tau| \leq p^{-\tilde{x}_{I,\tau} - \delta}, \prod_{\tau \in I} |\delta_{1,\tau}| \geq p^{-\sum_{\tau \in I} \tilde{x}_{1,\tau} - \delta}, \text{ and } |\tilde{\delta}_{1,\tau}| \leq p^{-\tilde{x}_{1,\tau} - \delta} \), we conclude that \( S_{I,1}(\delta) \) is a strict neighborhood of \( S_{I,1}(\delta') \) whenever \( \delta' < \delta \). \qed
Lemma 3.6 shows that if $\delta' < \delta$ are two positive rational numbers, then $S_{I,1}(\delta)$ and $S_{I,0}(\delta) \setminus S_{I,1}(\delta')$ form an admissible covering of $S_{I,0}(\delta)$. Define

$$S_{I,m}(\delta) = (U_p^{sp})^{-m}S_{I,0}(\delta).$$

Then $S_{I,m}(\delta)$ and $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$ also form an admissible covering of $S_{I,0}(\delta)$.

By definition, $S_{I,m-1}(\delta) \supseteq S_{I,m}(\delta)$. In addition, $U_p^m : S_{I,0}(\delta) \setminus S_{I,m}(\delta) \to \mathcal{V}_I(\delta)$. By Lemma 3.3, we can extend $f$ to $\mathcal{V}_I(\delta) \subseteq U_I(\delta)$. Then we can further extend $f$ by $(U_p^{-m})\mathcal{V}_I(\delta) \supseteq S_{I,0}(\delta) \setminus S_{I,m}(\delta)$. Similarly, for any other rational number $\delta' < \delta$, we can extend $f$ by $(U_p^{-m})\mathcal{V}_I(\delta') \supseteq S_{I,0}(\delta) \setminus S_{I,m}(\delta')$. Because $S_{I,0}(\delta) \setminus S_{I,m}(\delta)$ and $S_{I,0}(\delta') \setminus S_{I,m}(\delta')$ form an admissible covering of $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$, we can actually extend $f$ to $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$.

We denote by $f_m$ the extension of $f$ to $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$.

On the other hand, by Lemma 3.3, we can extend $f$ to $U_\mathcal{V}(\epsilon)$. Then

$$F_m := \sum_{j=0}^{m-1} \left(\frac{1}{a_p}\right)^j U_p^{nsp}(U_p^{sp})^j f$$

can be defined on $(U_p^{sp})^{-(m-1)}(U_p^{nsp})^-(U_\mathcal{V}(\epsilon)) \supseteq S_{I,m}(\delta)$.

Assume the norm estimates in Proposition 3.7 in the next subsection. By (2), we can choose a subsequence so that $F_m$ and $f_m$ mod $p^m$ glue as $h_m$ (only defined modulo $p^m$) under the admissible covering $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$ and $S_{I,m}(\delta)$ of $S_{I,0}(\delta)$. We have $h_m \equiv f \mod p^m$ on $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$. By (3), we can further choose a subsequence so that $h_{m+1}$ mod $p^m$ agrees with $h_m$ mod $p^m$ on $S_{I,m+1}(\delta)$. Hence $h = \lim_{m \to \infty} h_m$ is defined on $S_{I,0}(\delta)$, and $h = f$ on $S_{I,0}(\delta) \setminus \bigcap_m S_{I,m}(\delta')$. Hence $h$ is the desired extension of $f$ to $S_{I,0}(\delta)$.

3.3. Norm estimates. Assume that $\operatorname{val}_p a_p \leq \inf_{\tau \in I} k_\tau - g - \epsilon \sum_{\tau \in I} k_\tau$. Choose a rational number $\delta > 0$ so that $S_{I,0}(\delta) \subseteq \{ y \in \mathcal{Y} : \deg y - x_{I,\tau} < \epsilon \}$ and $S_{I,0}(\delta) \subseteq \{ y \in \mathcal{Y} : \deg y - x_{I,\tau} < \epsilon \}$. Also let $\delta' < \delta$ be another positive rational number.

Let $f_m$ defined on $S_{I,0}(\delta)$ and $F_m$ defined on $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$ as in the previous section. The following proposition records the norm estimates used to glue $f_m$ and $F_m$ in the previous section.

Proposition 3.7.

(1) $|F_m|_{S_{I,m}(\delta)}$ and $|f_m|_{S_{I,0}(\delta) \setminus S_{I,m}(\delta')}$ are bounded.

(2) $|F_m - F_m|_{S_{I,0}(\delta) \setminus S_{I,m}(\delta')} \to 0$.

(3) $|F_{m+1} - F_m|_{S_{I,m+1}(\delta)} \to 0$.

We need the following two lemmas to prove Proposition 3.7.

Lemma 3.8. Let $\mathcal{V} \subseteq S_{I,1}(\delta)$ and $h \in \omega^{\mathcal{V}}(U_p^{sp}(\mathcal{V}))$. Then

$$|U_p^{sp}(h)|_{\mathcal{V}} \leq p^\mu - \sum_{\tau \in I} k_\tau (1 - \epsilon)|h|_{U_p^{sp}(\mathcal{V})}.$$

In particular, if $\operatorname{val}_p a_p < \inf_{\tau \in I} k_\tau - g - \epsilon \sum_{\tau \in I} k_\tau$, then

$$|U_p^{sp}(h)|_{\mathcal{V}} \leq p^{-\mu}|h|_{U_p^{sp}(\mathcal{V})}$$

for some small enough $\mu > 0$.

Proof. Recall that $U_p$ is defined by

$$H^0(U_p(\mathcal{V}), \omega^{\mathcal{V}}) \to H^0(p_1^{-1}(\mathcal{V}), p_2^0 \omega^{\mathcal{V}}) \xrightarrow{\pi^*} H^0((p_1^{-1}(\mathcal{V})), p_1^0 \omega^{\mathcal{V}}) \xrightarrow{\frac{1}{p^0} \operatorname{Tr}_p} H^0(\mathcal{V}, \omega^{\mathcal{V}}).$$
Let \( y \in \mathcal{V} \) and \( y_1 \in U_p^{sp}(y) \). Then
\[
| (U_p^{sp} h)(y) | = \left| \frac{1}{p^g}(\pi^* h)(y_1) \right| = p^{g-\sum_{\tau \in \Sigma} k_{\tau} \deg_{\tau} H_1} | h(y_1) |.
\]
By assumption, \( y_1 \in \mathcal{V} \subseteq S_{I,0}(\delta) \), i.e., \( \sum_{\tau \in I} \deg_{\tau} y_1 \leq \sum_{\tau \in I} \bar{x}_I + \delta, \deg_{\tau} y_1 \geq \bar{x}_I - \delta \). Hence by our choice of \( \delta \), we have
\[
| \deg_{\tau} y_1 - x_{I,\tau} | < \epsilon,
\]
namely
\[
| \deg_{\tau} H_1 - x_{I,\tau} | < \epsilon.
\]
Then
\[
g - \sum_{\tau \in \Sigma} k_{\tau} \deg_{\tau} H_1 \leq g - \sum_{\tau \in I} (1 - \epsilon).
\]

**Lemma 3.9.** For \( 1 \leq j \leq m \), \( f_m - (U_p^{sp}/a_p) f_m = F_j \) on \( S_{I,j}(\delta) \setminus S_{I,m}(\delta') \).

**Proof.** Recall that we have fixed \( \delta' < \delta \), and \( f_m \) is defined on \( S_{I,0}(\delta) \setminus S_{I,m}(\delta') \). In particular, \( (U_p^{sp}/a_p)^j f_m \) is defined on \( (U_p^{sp})^{-j} S_{I,0}(\delta) \setminus S_{I,m}(\delta') = S_{I,j}(\delta) \setminus S_{I,j+m}(\delta') \).

By definition, \( F_j = \sum_{\ell=0}^{j-1}(\frac{1}{a_p})^{\ell+1} U_p^{sp}(U_p^{sp})^{\ell} f \) on \( S_{I,j}(\delta) \). Hence \( F_j + (U_p^{sp}/a_p)^j f_m \) is defined on \( S_{I,j}(\delta) \setminus S_{I,m}(\delta') \). A simple calculation using the fact that \( U_p = U_p^{sp} + U_p^{reg} \) yields the claimed equality \( F_j + (U_p^{sp}/a_p)^j f_m = f_m \). \( \Box \)

**Proof of Proposition 3.7.**

(1) Because \( f \) is defined on the quasi-compact open \( \mathcal{V}(\delta) \), \( |f|_{\mathcal{V}(\delta)} \) is bounded. Since \( U_p \) is a compact operator, \( |f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta)} \leq \frac{U_p}{a_p} f \) is also bounded. Similarly, \( |f_1|_{S_{I,0}(\delta') \setminus S_{I,1}(\delta')} \) is bounded, and hence \( |f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')} \) is bounded.

We will show that \( |f_m|_{S_{I,0}(\delta) \setminus S_{I,m}(\delta')} \leq \sup(|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}, |F_1|_{S_{I,m}(\delta)')} \) for all \( m \geq 1 \). Because \( f_m \)'s are compatible, it suffices to show that
\[
|f_m|_{S_{I,m}(\delta) \setminus S_{I,m}(\delta')} \leq \sup(|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}, |F_1|_{S_{I,m}(\delta)}), |F_1|_{S_{I,m}(\delta)}
\]
for all \( m \geq 1 \). We do this by induction on \( m \). By Lemma 3.9, \( f_m - U_p^{sp}/a_p f_m = F_1 \) on \( S_{I,1}(\delta) \setminus S_{I,m}(\delta') \). Then it suffices to show that
\[
|U_p^{sp}/a_p f_m|_{S_{I,m}(\delta) \setminus S_{I,m+1}(\delta')} \leq \sup(|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}, |F_1|_{S_{I,m}(\delta)}).
\]
By Lemma 3.8,
\[
|U_p^{sp}/a_p f_m|_{S_{I,m}(\delta) \setminus S_{I,m+1}(\delta')} \leq |f_m|_{S_{I,m}(\delta) \setminus S_{I,m+1}(\delta')}
\]
Hence
\[
|U_p^{sp}/a_p f_m|_{S_{I,m}(\delta) \setminus S_{I,m+1}(\delta')} \leq \sup(|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}, |F_1|_{S_{I,m}(\delta)}).
\]

by induction hypothesis.
As for $|F_m|_{S_{I,m}(\delta)}$, by Lemma 3.8,

$$|F_m|_{S_{I,m}(\delta)} \leq \sup_{0 \leq j \leq m-1} \left| \left( \frac{1}{a_p} \right)^{j+1} U_p^{nsp} \left( U_p^{sp} \right)^j f \right|_{S_{I,m}(\delta)}$$

$$= \sup_{0 \leq j \leq m-1} \left| \left( \frac{U_p^{sp}}{a_p} \right)^j F_1 \right|_{S_{I,m}(\delta)}$$

$$\leq \sup_{0 \leq j \leq m-1} |F_1|_{S_{I,m-j}(\delta)}$$

$$= |F_1|_{S_{I,1}(\delta)}.$$  

(2) By Lemma 3.9 and Lemma 3.8,

$$|F_m - f_m|_{S_{I,m}(\delta) \setminus S_{I,m}(\delta')} = \left| \left( \frac{U_p^{sp}}{a_p} \right)^m f_m \right|_{S_{I,m}(\delta) \setminus S_{I,m}(\delta')}$$

$$\leq p^{-m\mu} |f_m|_{S_{I,0}(\delta) \setminus S_{I,0}(\delta')}$$

$$= p^{-m\mu} |f_0|_{S_{I,0}(\delta) \setminus S_{I,0}(\delta')}$$

$$\to 0 \text{ as } m \to \infty.$$  

(3) By Lemma 3.8,

$$|F_{m+1} - F_m|_{S_{I,m+1}(\delta)} = \left| \left( \frac{1}{a_p} \right)^{m+1} U_p^{nsp} \left( U_p^{sp} \right)^m f \right|_{S_{I,m+1}(\delta)}$$

$$= \left| \left( \frac{U_p^{sp}}{a_p} \right)^m F_1 \right|_{S_{I,m+1}(\delta)}$$

$$\leq p^{-m\mu} |F_1|_{S_{I,1}(\delta)}$$

$$\to 0 \text{ as } m \to \infty.$$  

3.4. **Finishing the proof of Theorem 3.1.** Assuming that the overconvergent form $f$ is defined on a strict neighborhood of $\deg^{-1} x_J$ for all $J \subseteq I$ and that $f$ satisfies the small slope condition (1), in Section 3.2 we have extended $f$ to a strict neighborhood $S_{I,0}(\delta)$ of $\deg^{-1} x_I$ for any small enough $\delta > 0$.

Note that the vertices in $F_I$ are exactly the $x_J$’s with $J \subseteq I$. We can then extend $f$ to a strict neighborhood of $\mathcal{Y} F_I$ again arguing by the fact that $U_p$ increases twisted directional degrees strictly when the deg is not one of the vertices.

Let $U$ be a strict neighborhood of $\mathcal{Y} F_I$ which is stable under $U_p$. Write $U = S_{I,0}(\delta) \cup \bigcup_{J \subseteq I} V_J \cup \mathcal{V}$, where $\mathcal{V}_J$ is a strict neighborhood of $\deg^{-1} x_J$ on which $f$ is defined, and $\mathcal{V}$ is a quasi-compact open disjoint from all $\deg^{-1} x_J$’s with $J \subseteq I$. By Proposition 2.5, $U_p$ increases the twisted directional degrees strictly when the degrees are not all integers, for example on $\mathcal{V}$. Using the Maximum Modulus Principle, the quasi-compactness of $\mathcal{V}$ implies that there is a positive lower bound for the increase of $\deg_{\tau}$ under $U_p$ on $\mathcal{V}$. Because we chose $U$ to be $U_p$-stable, there exists $M > 0$ such that $U_p^M \mathcal{V} \subseteq S_{I,0}(\delta) \cup \bigcup_{J \subseteq I} V_J$. We can then extend $f$ to $U$ by $(\frac{U_p}{a_p})^M f$.

**References**


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