Deterministic and Stochastic Nonlinear Partial Differential Equations and Applications

Department of Mathematics and The Institute for Scientific Computing and Applied Mathematics
Indiana University Bloomington, IN 47405, USA
email: wang211@indiana.edu
web: http://mypage.iu.edu/~wang211/

Deterministic and stochastic nonlinear partial differential equations with mixed features (e.g. partial hyperbolicity and anisotropicity) have played an important role in describing the models in physics, chemistry, finance and other real-world phenomena. Moreover, sitting at the interface between probability theory, mathematical analysis and theory of parabolic and hyperbolic partial differential equations, these problems provide interesting and challenging mathematical complications.

Motivated by this background, my research interests lie in the study of such models, especially those derived from mathematical physics like plasma physics, fluid dynamics and thermomechanics. The investigation emphasis has been on regularity behavior of solutions and the numerical implementations.

Specifically, my projects consist of three interrelated directions. One theme is to establish the basic mathematical theory for the deterministic and stochastic Zakharov-Kuznetsov (ZK) equation, a multi dimensional extension of the celebrated Korteweg-de Vries (KdV) equation. A second topic is to extend the time discrete numerical scheme and Courant-Friedrichs-Levy condition for the evolution equations of geophysical fluid dynamics from the deterministic to the stochastic case. Finally, a third subject is to investigate how to utilise noise perturbations to model the self-organized criticality for the atmospheric equations.

The Deterministic Zakharov-Kuznetsov Equation

My PhD thesis addresses the well-posedness and regularity of the Zakharov-Kuznetsov (ZK) equation both in the deterministic and stochastic cases (see [STW12], [Wan14a], [Wan14b] and [GHTW14]). The main focus has been on the handling of the mixed features of the partial hyperbolicity, nonlinearity, nonconventional boundary conditions, anisotropicity and stochasticity, which requires novel methods quite different than those of the classical models in fluid dynamics, such as the Navier-Stokes equation, Primitive Equation and related equations.

The deterministic ZK equation

\[
\frac{\partial u}{\partial t} + \Delta \frac{\partial u}{\partial x} + c \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} = f, \tag{1}
\]

is the long-wave small-amplitude limit of the Euler-Poisson system for cold plasmas uniformly magnetized in the z direction, where \( u \) represents the ion density and \( c > 0 \) is the sound velocity (see [ZK74], [LS82], and [LLS13] for details and [BPS81] and [BPS83] for general physical references).

Here \( u = u(x, x^+, t), \ x^+ = y \) or \( x^+ = (y, z) \), \( \Delta u = \Delta^+ u, \ \Delta^+ u = u_{yy} \) or \( u_{yy} + u_{zz} \) and \( f \) for now is a given deterministic forcing term. Moreover, the ZK equation is a natural multi-dimensional
extension of the classical KdV equation, quite different from the famous Kadomtsev-Petviashvili equation though; e.g., the ZK equation is not completely integrable but has a hamiltonian structure ([LLS13]).

A mixture of the periodic and Robin boundary conditions are assumed. The domain is a rectangle or parallelepiped in $\mathbb{R}^n$, $n = 2$ or $3$, denoted as $\mathcal{M} = (0,1)_x \times (-L,L)^d$, $d = 1$ or $2$. For the $y$ and $z$ boundaries, we assume either the Dirichlet or the periodic boundary conditions. For $x = 0, 1$, two types of boundary conditions will be assumed to serve different purposes (which will be elaborated subsequently):

(I) \[ u(0, x^+, t) = u(1, x^+, t) = u_x(1, x^+, t) = 0; \]

(II) \[ \frac{\partial^\ell u}{\partial x^\ell}(0, x^+, t) = \frac{\partial^\ell u}{\partial x^\ell}(1, x^+, t), \quad \ell = 0, 1, 2. \]

The interests and challenges of the ZK equation are inherent in the mixed attributions of the partial hyperbolicity with a preference in the $x$ direction, a nonlinear term that has the same structure as the nonlinear term in the 3D Navier-Stokes equations and interactions of these features with the boundary conditions.

We have established a series of results concerning the well posed-ness and the regularity of the deterministic ZK equation.

**Existence and uniqueness of weak solutions.** In [STW12], assuming the boundary conditions in (2), the existence of the weak solutions in space dimensions two and three and uniqueness in space dimension two have been established by means of a parabolic regularizartion,

\[
\frac{\partial u^\epsilon}{\partial t} + \Delta \frac{\partial u^\epsilon}{\partial x} + c \frac{\partial u^\epsilon}{\partial x} + u^\epsilon \frac{\partial u^\epsilon}{\partial x} + \epsilon \left( \frac{\partial^4 u^\epsilon}{\partial x^4} + \frac{\partial^4 u^\epsilon}{\partial y^4} + \frac{\partial^4 u^\epsilon}{\partial z^4} \right) = f.
\]

Here one of the novelties is the derivation of a bound on $u^\epsilon$ in the space $L^2(0,T; H^1(\mathcal{M}))$ independent of $\epsilon$ by multiplying (4) by $xu^\epsilon$, integrating over $\mathcal{M}$ and integrating by parts. The idea is that intuitively the multiplication of the extra $x$ allows the compensation for the loss of an $x$ derivative in the linear term $\Delta \frac{\partial u^\epsilon}{\partial x}$, which, roughly speaking, produces coercivity as desired.

**Local existence of strong solutions in 3D.** Continuing assuming the boundary conditions in (2), the obstacles with the proof of local existence of strong solutions arise mainly from the anisotropicity of the nonlinear term in the $x$ direction. To deal with this issue, we make the following observation ([Wan14a]):

**Proposition 1.** Assume that $u_0 \in L^2(\mathcal{M})$ and $f \in L^\infty(0,T; L^2(\mathcal{M}))$. Then for every $T > 0$ we have

\[
|u^\epsilon_x(t)|^2 \leq |u^\epsilon_t(t)|^2 + \kappa, \quad 0 \leq t \leq T,
\]

where $\kappa$ is a constant depending only on $u_0$, $f$ and $T$.

With this result, we can differentiate (4) in time and then use a similar proof as that of the local existence for the Navier-Stokes equation to deduce the existence of $T_* = \min(T, \frac{1}{\mu})$, with $\mu$ only depending on the data, such that for every $t$, $0 \leq t \leq T_*$,

\[
|u^\epsilon_t(t)| \leq \mu, \quad \int_0^{T_*} |\nabla u^\epsilon_t(s)|^2 \, ds \leq \mu.
\]
Thus we have successfully circumvented the complications brought by the specific structure of the equation. Finally we obtain the main result informally stated as follows:

**Theorem 1.** With regular enough data and suitable compatibility conditions, in both dimensions two and three, there exists a local strong solution to the ZK equation on some time interval \([0, T_\ast)\), \(T_\ast > 0\) depending only on the data, such that

\[
\nabla u, u_{yy}, u_{zz}, u_t \in L^\infty(0, T_\ast; L^2(\mathcal{M})),
\]

\[
u_t \in L^2(0, T_\ast; H^1(\mathcal{M})),
\]

and all the spatial derivatives of the third order of \(u\) are in \(L^2(0, T_\ast; L^2(\mathcal{M}))\), except for \(u_{xxy}\) and \(u_{xxx}\).

**Global existence of strong solutions in 3D.** The global existence of strong solutions in 3D is a delicate issue with several difficulties: (i) As in the case of the 3D Navier-Stokes equation, the nonlinear term will pose a problem when the Sobolev imbedding is applied in 3D. (ii) Due to the partial hyperbolicity, the \(L^6\) estimations do not work. (iii) The trace \(u^\epsilon_{xx}|_{x=0}\) prevents the estimates from closing for the bound of \(\nabla u^\epsilon\) in the space \(L^\infty(0, T; L^2(\mathcal{M}))\) independent of \(\epsilon\).

To overcome these obstacles, firstly we utilize the anisotropic resonance of \(\Delta u^\epsilon_x\) and \(u^\epsilon u_x^\epsilon\) both in \(x\) to cancel \(u^\epsilon u_x^\epsilon\), that is, we multiply (4) with \(-\Delta u^\epsilon - \frac{1}{2}(u^\epsilon)^2\), integrate over \(\mathcal{M}\) and integrate by parts. Thus we can obtain a bound of the \(H^1\) norm over \((0, T)\) for \(u^\epsilon\). This step of canceling the nonlinear term may also be applied to other nonlinear equations with similar structures. Next, we are prompted to choose the periodic boundary conditions in (3) instead of (2) so that the trace \(u^\epsilon_{xx}|_{x=0}\) now vanishes as needed. Thirdly, we investigate a bound independent of \(\epsilon\) for \(u^\epsilon_{xxx}\) in \(L^{3/2}((0,1)_x; Y)\), with \(Y\) a Banach space in \(x^\perp\) and \(t\), by multiplying (4) with \(\epsilon u_{xxx}^\epsilon\), integrating over \(\mathcal{M}\) and integrating by parts. This then facilitates the passage to the limit on the traces of \(\frac{\partial^d u^\epsilon}{\partial x^d}\) at \(x = 0, 1, \ell = 1, 2\). Eventually we arrive at the following main result ([Wan14b]):

**Theorem 2.** With regular enough data, in both dimensions two and three, the initial and boundary value problem for the ZK equation possesses at least a solution \(u\):

\[
u \in C([0,T]; H^1(\mathcal{M})) \cap W^{3,3/2}((0,1)_x; H^{-1}(0,T; H^{-4}(-L,L)^d))), \ d = 1, 2. \quad (7)
\]

However there are still two open problems. On the one hand, the uniqueness of strong solutions is still open in 3D, even with such a regularity. In particular, the methods in [ST10] and [STW12] can not be adapted to this case due to the absence of the boundary condition \(u_x = 0\) at \(x = 1\) in (2). On the other hand, it is not clear how to upgrade the strong regularity in (7) into higher smoothness. The essential obstacle seems to lie again in the lack of the same boundary condition.

**The Stochastic Zakharov-Kuznetsov Equation**

In order to establish an initial program of well posed-ness which will serve as the foundation for the investigation of the qualitative and quantitative properties of solutions in both the deterministic and stochastic cases and facilitate the comparison of these two models, we next extend the results of global existence and uniqueness of weak solutions in [STW12] to the stochastic case ([GHTW14])

\[
du + (\Delta u_x + cu_x + uu_x) \, dt = f \, dt + \sigma(u) \, dW(t), \quad (8)
\]
driven by a multiplicative white noise. Here we assume the boundary condition in (2) as in the deterministic case for the weak solutions. We also assume that $\sigma$ is a uniformly bounded and Lipschitz operator between suitable spaces and $f$ is deterministic.

**Martingale and pathwise solutions for the stochastic ZK equation.** Different notions of solutions are considered here, that is, the martingale and the pathwise solutions. In the former notion, the stochastic basis $\tilde{S}$ is not specified in the beginning and is viewed as part of the unknown, while in the latter case, the stochastic basis is fixed in advance as part of the assumptions. To pass from the martingale to the pathwise solutions, we apply the extension of the Yamada-Watanabe theorem ([YW71]) to the infinite dimension space (see [GK96]), that is, the pathwise solutions exist whenever there exists a pathwise unique martingale solution.

Our main result is the existence of martingale solutions in 2 and 3D and the existence and uniqueness of pathwise solutions in 2D.

One of the novelties here is again the treatment of the boundary conditions, more complicated now due to the addition of the randomness (within the domain). Firstly, it is not even clear whether all the boundary conditions of the parabolic regularization in (4) are still preserved after application of the Skorokhod embedding theorem ([DPZ92]) as the underlying stochastic basis has been shifted. To solve this problem, a measurability result concerning Hilbert spaces is developed based on the results in [Ond10]: Let $A_1$ be a separable Hilbert space. Assume that $A_2$ is a separable Hilbert space continuously injected into $A_1$. Then $A_2$ is a Borel set of $A_1$. Secondly, there remains to show that the boundary condition $u_{x} |_{x=1} = 0$ is satisfied almost surely. To achieve this we turn to analyze $U^\epsilon(t) := \int_0^t u^\epsilon ds$, which is more regular in time than $u^\epsilon$, so that we are able to extend the trace results in [STW12] to the stochastic setting by establishing the trace properties of the linearized ZK equation depending on the probabilistic parameters. This idea can be used to deal with the nontraditional boundary conditions in other circumstances.

A further novelty is contained in the proof of the pathwise uniqueness. Difficulties arise with the derivation of the energy inequality for the difference of the two weak solutions due to the lack of regularity. Moreover, the method in the deterministic case (see [STW12]) can not be adapted directly since the Gronwall lemma in [GHZ09] (see also [MR04]) brings issues with passage to the limit involving stopping times. We overcome this obstacle by establishing a variant of the stochastic Gronwall lemma, where in certain situations the stopping times can be avoided as much as possible. This idea of dealing with the insufficient regularity of the solutions to verify the pathwise uniqueness, we believe, is not only suited for this model but can also be applied to other equations.

**Numerical Analysis for the Stochastic Equations in Fluid Dynamics**

We aim to carry over the classical numerical methods for explicit schemes as analyzed in e.g. [Tem01] and [MT98] from the deterministic case to the stochastic case ([GTWb] and [GTWa]), where new challenges occur such as the adapted-ness issue, the stability condition and the rate of convergence of the scheme in an underlying probability space.

**Implicit Euler Scheme for the stochastic geophysical fluid models.** As a first step towards the numerical analysis of the stochastic Primitive Equations of the atmosphere and the oceans, we study their time discretization by an implicit Euler scheme. We consider the following abstract stochastic evolution equation with a wide class of nonlinear, state dependent, white noise
forcing:

\[ dU + (AU + B(U) + EU) \, dt = (\ell + \xi(U)) \, dt + \sigma(U) \, dW, \]  

(9)

where \( EU \) accounts for the Coriolis forces, and \( \xi \) arises when converting from a Stratonovich into an Itô type noise. This abstract equation covers the equations for the oceans, the atmosphere, the coupled oceanic-atmospheric system as well as other related geophysical equations. The implicit Euler time discretization scheme we choose to approximate (9) is as follows:

\[
\frac{U^n_N - U^{n-1}_N}{\Delta t} + AU^n_N + B(U^n_N) + EU^n_N = \ell^n_N + \xi(t^n, U^n_N) + \sigma_N(t^{n-1}, U^{n-1}_N) \eta^n_N / \Delta t, 
\]

(10)

where

\[ \Delta t = T/N, \quad t^n = t^n_N = n \Delta t, \quad \text{for } n = 0, 1, \ldots, N, \]

\[ \eta^n = \eta^n_N = W(t_n) - W(t_{n-1}), \quad \text{for } n = 1, \ldots, N. \]

Besides the numerical analysis of the Euler scheme, one significant theoretical result is the existence of solutions which are weak in both the PDE and probabilistic sense, a result which is new by itself to the best of our knowledge ([GTWb]).

The main challenges here are two folds. Firstly, while classical arguments involving the Brouwer fixed point theorem can be used to establish the existence of sequences satisfying the implicit scheme, it is crucial that these sequences are adapted to the driving noise. To address this concern we rely on a specifically chosen filtration. Consider the canonical Wiener space ([KS91]) equipped with the Borel \( \sigma \)-algebra and the Wiener measure \( \mathbb{P} \), then the evaluation map becomes a cylindrical Wiener process \( W(t) \) with its natural filtration, that is, the completion of the \( \sigma \)-algebra generated by the \( W(s) \) for \( s \in [0, t] \) with respect to \( \mathbb{P} \). Combining these elements we obtain a stochastic basis suitable for applying a measurable selection theorem from [BT73] (see also [KRN65], [Cas67]), with the additional application of universal Radon measurability ([Sch73] and [DS88]).

A second difficulty appears when we associate continuous time processes with the discrete time schemes. In contrast to the deterministic case, [Tem01] and [MT98], we must introduce processes which are lagged by a time step. While these processes are indeed adapted, we obtain a time evolution equation with bothersome error terms, for which it is not clear how to obtain the estimates on fractional derivatives in time. In turn these error terms prevent us from addressing compactness directly from the equations, instead, a series of interrelated auxiliary approximation sequences are introduced, for which we are able to carry out the compactness arguments step by step.

▶ **Stability and rate of convergence for the stochastic Navier-Stokes equation.** Now we use the techniques developed above to study the stability and consistency of a class of numerical schemes (both explicit and semi-implicit) for the 2D and 3D stochastic Navier-Stokes equations driven by a multiplicative noise ([GTWa]). Here the space and time are both discretized as in [Tem01] and [MT98]. The same issue occurs with the appearance of troublesome error terms in the schemes, which we resolve by the application of the estimates for higher moments and introduction of only one auxiliary process to facilitate the compactness arguments.

**Current and Future Projects**
Global existence of smooth solutions for the ZK equation. The difficulty lies again in the mixed feature of the structure of the ZK equation. In fact this result can be achieved if one can show that once an initial data is compactly supported then so is the corresponding solution at any later time, so that the boundary condition $u_x = 0$ at $x = 1$ can be obtained directly.

Stochastic Atmospheric equations driven by Lévy noise. The relation between precipitation and column water vapour (CWV) is indispensable for predictions of precipitation. Particularly, as discovered in [NS14] and [NS11], one feature of the transition to strong moist convection is a rise in precipitation with increasing CWV. Another feature is the occurrence of extreme events. Thirdly, interesting temporal variability arises, specifically suggesting that precipitation might not always initiate simultaneously when CWV surpasses some critical value, as might be expected from the concept of conditional instability. Our aim is to propose a model that captures these aspects of self-organized criticality. One idea is to replace the Heaviside function in the deterministic Primitive Equations of atmosphere by a suitable Levy noise, and analyze if the randomness injected in this way may produce the phenomena observed above.

References


