MATH 33A Worksheet Week 3 Solutions

TA: Caleb Partin

July 11, 2024

Topic 1: Images, Kernels, Bases

Exercise 1.1. Let $A : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix}$. Find a basis for $\ker A$. Find a basis for $\operatorname{im} A$. Notice that $\dim \ker A + \dim \operatorname{im} A = 4$.

RREF:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so solutions to $Ax = 0$ are given by $x_1 = -2x_2 + x_3 - 3x_4$, x_2, x_3, x_4 free. So a basis for $\ker A$ is given by

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since only first column has a leading one in RREF, the first column $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ forms a basis for $\operatorname{im} A$.

Exercise 1.2. Let $A : \mathbb{R}^8 \to \mathbb{R}^7$ be given by the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & 2 \\ 2 & 3 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine $\dim \ker A$ (hint: use rank-nullity and find $\dim \operatorname{im} A$).

$\operatorname{im} A$ is two dimensional by observation or by noticing that the row reduced echelon form has exactly two leading ones. Therefore, since $\dim \ker A + \dim \operatorname{im} A = 8$ and $\dim \operatorname{im} A = 2$, $\dim \ker A = 6$.

Exercise 1.3. Find a basis for the following subspaces of $\mathbb{R}^3$:

(a) $V = \operatorname{span}\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \right)$

(b) $V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \;\middle|\; v_1 - 3v_2 = 0 \right\}$

(c) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be any linear transformation such that $\dim \ker A = 0$. Find a basis for $V = \operatorname{im} A$.

(a) $V = \operatorname{im} A$ for $A = \begin{bmatrix} 3 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \\ 2 & 1 & 1 & -4 \end{bmatrix}$. RREF of A has leading ones in 1,2, and 4th column, so the first, second, and fourth of these vectors form a basis for V.

(b) Notice that $V = \ker A$ for $A = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix}$. A is already row reduced, so writing the solutions to $A\vec{x} = \vec{0}$, we find that $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ must satisfy

$$x_1 - 3x_2 = 0 \iff x_1 = 3x_2$$

Since only the first column of (the row reduced echelon form) of A has a leading one, x_2, x_3 are free, so

$$\ker A = \left\{ \begin{bmatrix} 3x_2 \\ x_2 \\ x_3 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R} \right\} = \operatorname{span}\langle \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle$$

so a basis for $\ker A$ is $\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$

(c) Since $\dim \ker A = 0$, by rank nullity, $\dim \operatorname{im} A = 3$. Therefore, since $\dim \operatorname{im} A = 3$ and is a subspace of $\mathbb{R}^3$, $\operatorname{im} A = \mathbb{R}^3$. Thus, a basis for A is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Topic 2: Orthogonality

Exercise 2.1. For what values of $\lambda \in \mathbb{R}$ are the following pairs of vectors orthogonal?

(a) $\begin{bmatrix} \lambda \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} \lambda \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -\lambda \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$

(a) Orthogonal if and only if $-2\lambda + 3 = 0$, so orthogonal when $\lambda = 3/2$ and not otherwise.

(b) The dot product between these two vectors is 0, so orthogonal regardless of λ.

(c) Orthogonal if and only if $\lambda + \lambda = 0$, so only orthogonal when $\lambda = 0$.

Exercise 2.2. Determine whether the following sets of vectors are *orthonormal* (orthogonal and unit length):

(a) $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}.$

(b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(c) $\begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$

(a) Dot product is zero, and $(3/5)^2 + (4/5)^2 = 1$, so yes.

(b) No, since $| \begin{bmatrix} 1 \\ -1 \end{bmatrix} | = 2 \neq 1.$

(c) Yes, dot product between any pair is zero, and $| \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} | = 4/9+4/9+1/9 = 1$ (and similarly for the other two).

Exercise 2.3. Find the orthogonal projection of $\begin{bmatrix}5\\5\\5\end{bmatrix}$ onto the subspace

$$V = \text{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$$

CORRECTION: We neeed for our basis for V to be an orthonormal basis to apply the projection formula, which it is currently not! So to correct for this, we can do Gram-Schmidt!

$$v_1^\perp = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$v_2^\perp = \begin{bmatrix}-2\\1\\1\end{bmatrix} - \frac{\begin{bmatrix}-2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\1\end{bmatrix}}{\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\1\end{bmatrix}} \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}-2\\1\\1\end{bmatrix} - \frac{1}{6}\begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}-13\\4\\5\end{bmatrix}$$

Now we can normalize our vectors:

$$\left\|\begin{bmatrix}1\\2\\1\end{bmatrix}\right\| = \sqrt{1+2^2+1} = \sqrt{6}$$

$$\left\|\frac{1}{6}\begin{bmatrix}-13\\4\\5\end{bmatrix}\right\| = \frac{\sqrt{(-13)^2+4^2+5^2}}{6} = \frac{\sqrt{210}}{6}$$

$$V = \text{span}\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}, \frac{1}{\sqrt{210}}\begin{bmatrix}-13\\4\\5\end{bmatrix}\right\}$$

Then the projection formula tells us:

$$\begin{aligned}
\text{proj}_V\left(\begin{bmatrix}5\\5\\5\end{bmatrix}\right) &= \left(\begin{bmatrix}5\\5\\5\end{bmatrix} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} + \left(\begin{bmatrix}5\\5\\5\end{bmatrix} \cdot \frac{1}{\sqrt{210}}\begin{bmatrix}-13\\4\\5\end{bmatrix}\right)\frac{1}{\sqrt{210}}\begin{bmatrix}-13\\4\\5\end{bmatrix} \\
&= \left(\frac{20}{\sqrt{6}}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} + \left(\frac{-20}{\sqrt{210}}\right)\frac{1}{\sqrt{210}}\begin{bmatrix}-13\\4\\5\end{bmatrix} = \frac{10}{3}\begin{bmatrix}1\\2\\1\end{bmatrix} - \frac{2}{21}\begin{bmatrix}-13\\4\\5\end{bmatrix} \\
&= \frac{1}{21}\begin{bmatrix}96\\132\\60\end{bmatrix}
\end{aligned}$$

Exercise 2.4. Find a basis for $W^\perp$, where

$$W = \text{span}\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}\right\}$$

(Hint: How can we relate $W^\perp$ to subspaces where we know how to find a basis?)

A vector v is in $W^\perp$ if and only if $v \cdot \begin{bmatrix}1\\2\\3\\4\end{bmatrix} = 0$ and $v \cdot \begin{bmatrix}5\\6\\7\\8\end{bmatrix} = 0$ (to check if a vector is orthogonal to a subspace, we only need to check that it is orthogonal to the basis vectors). However, notice that if we let A be the matrix with the basis vectors as rows:

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}$$

that this is the same as asking that v is in $\ker A$. Thus, we need to find a basis for kernel of A:

$$\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix} \to \begin{bmatrix}1 & 2 & 3 & 4\\0 & -4 & -8 & -12\end{bmatrix} \to \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{bmatrix}$$

So letting $x_3 = t$, $x_4 = s$ for $t, s \in \mathbb{R}$, we see that $x_1 = x_3 + 2x_4 = t + 2s$ and $x_2 = -2x_3 - 3x_4 = -2t - 3s$, giving a general solution

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}t+2s\\-2t-3s\\t\\s\end{bmatrix} = t\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + s\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$$

And so we can see:

$$W^\perp = \ker(A) = \text{span}\left\{\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-3\\0\\1\end{bmatrix}\right\}$$

Exercise 2.5. For each of the following vectors $\vec{v}$, find the decomposition $v^{||} + v^{\perp}$ with respect to the subspace

$$V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

First we normalize all of our basis vectors to get

$$V = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$$

(a)

$$v^{||} = \operatorname{proj}_V\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$v^{\perp} = v - v^{||} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

(b)

$$v^{||} = \operatorname{proj}_V\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + 0\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + 0\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v^{\perp} = v - v^{||} = \vec{0}$$

(c)

$$v^{||} = \mathrm{proj}_V\left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right) = 0\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix} + 0\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} + 0\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$v^{\perp} = v - v^{||} = \vec{0}$$

Exercise 2.6. Let $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ be a subspace of $\mathbb{R}^n$ where the vectors $\vec{v}_1, \dots, \vec{v}_k$ give an orthonormal basis for V.

(a) If $\vec{w} \in V$, show that $\text{proj}_V(\vec{w}) = \vec{w}$.

(b) If $\vec{w} \in V^\perp$, show that $\text{proj}_{V^\perp}(\vec{w}) = 0$.

(a) Since $\vec{w} \in V$, we know that $\vec{w} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$ for some scalars $c_1, \dots, c_k$. Moreover, note that since $\vec{v_1}, \dots, \vec{v_k}$ form an orthonormal basis, we know that $\vec{v_i} \cdot \vec{v_j} = 0$ for $i \neq j$ and $\vec{v_i} \cdot \vec{v_j} = 1$ when $i = j$. Using this and the fact that the dot product distributes over sums, we can compute the projection:

$$\begin{aligned}
\text{proj}_V(\vec{w}) &= (\vec{w} \cdot \vec{v_1})\vec{v_1} + \cdots + (\vec{w} \cdot \vec{v_k})\vec{v_k} = ((c_1\vec{v_1} + \cdots + c_k\vec{v_k}) \cdot \vec{v_1})\vec{v_1} + \cdots + ((c_1\vec{v_1} + \cdots + c_k\vec{v_k}) \cdot \vec{v_k})\vec{v_k} \\
&= (c_1\vec{v_1} \cdot \vec{v_1} + \cdots + c_k\vec{v_k} \cdot \vec{v_1})\vec{v_1} + \cdots + ((c_1\vec{v_1} \cdot \vec{v_k} + \cdots + c_k\vec{v_k} \cdot \vec{v_k})\vec{v_k} \\
&= c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{w}
\end{aligned}$$

(b) If $\vec{w} \in V^\perp$, then $\vec{w}$ must be orthogonal to every basis vector for W:

$$\text{proj}_{V^\perp}(\vec{w}) = (\vec{w} \cdot \vec{v_1})\vec{v_1} + \cdots + (\vec{w} \cdot \vec{v_k})\vec{v_k} = 0\vec{v_1} + \cdots + 0\vec{v_k} = \vec{0}$$

Exercise 2.7. Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 2 & 1 & 1 & -4 \end{bmatrix}$$

Find an orthonormal basis $\mathcal{B} = \{u_1, u_2\}$ for $\ker A$.

RREF: $\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$, which results in $\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}\}$ being a basis for $\ker A$. Performing Gram-Schmidt with $v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, we have $v_1^\perp = v_1$ and

$$v_2^\perp = v_2 - \frac{(v_1^\perp \cdot v_2)}{||v_1^\perp||^2} v_1^\perp = v_2 + \frac{5}{3} v_1^\perp = \begin{bmatrix} 4/3 \\ -1/3 \\ 5/3 \\ 1 \end{bmatrix}$$

Thus, we get

$$u_1 = v_1^\perp / ||v_1^\perp|| = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, u_2 = v_2^\perp / ||v_2^\perp|| = \begin{bmatrix} 4/\sqrt{51} \\ -1/\sqrt{51} \\ 5/\sqrt{51} \\ 3/\sqrt{51} \end{bmatrix}$$

Exercise 2.8. Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

First we perform Gram Schmidt on the columns resulting in the vectors

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Thus, $Q = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$. Notice that R is a matrix which satisfies $A = QR$, so $Q^{-1}A = R$. Since Q is orthogonal we have $Q^{-1} = Q^T$, so $R = Q^T A$. Thus, we have

$$R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & 2 & 1 \\ 0 & 0 & \frac{3}{\sqrt{2}} \end{bmatrix}$$

True or False

Exercise TF. True or false: Explain your reasoning or find an example or counterexample.

(a) If V is a subspace of $\mathbb{R}^3$ that does not contain any of the elementary column vectors e_1, e_2, e_3, then $V = \{\vec{0}\}$.

(b) If v_1, v_2, v_3, v_4 are linearly independent vectors, then v_1, v_2, v_3 are linearly independent.

(c) If v_1, v_2, v_3 are linearly independent vectors, then v_1, v_2, v_3, v_4 are linearly independent.

(d) It is possible for a 4×4 matrix A to have $\ker A = \operatorname{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ and

$\operatorname{im} A = \operatorname{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} \right\rangle$

(e) There exists a 4×4 matrix A with $\ker A = \operatorname{span}\langle e_1, e_2, e_3 \rangle$ and $\operatorname{im} A = \operatorname{span}\langle e_3 + e_4 \rangle$

(f) There exists a 5×5 matrix A with $\ker A = \operatorname{im} A$.

(g) There exists a 4×4 matrix A with $\ker A = \operatorname{im} A$.

(h) If A is orthogonal then it is invertible.

(i) If A is symmetric ($A = A^T$) it is invertible.

(j) Let V be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{u_1, \ldots, u_m\}$, and let $\{v_1, \ldots, v_{n-m}\}$ be an orthonormal basis for $V^\perp$. Then $\{u_1, \ldots, u_m, v_1, \ldots, v_{n-m}\}$ is an orthonormal basis for $\mathbb{R}^n$.

(k) The entries of an orthogonal matrix are all less than or equal to 1 in absolute value.

(l) Let V be a subspace of $\mathbb{R}^n$ and B the matrix for orthogonal projection onto V. Then $B^2 = B$.

(m) Let $\mathcal{B} = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}^3$. Then $\begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(n) If $v_1, \ldots, v_m$ is a basis of unit length vectors for a subspace V, there is an orthonormal basis of V containing the vectors v_1 and v_2.

(o) For all $v, w \in \mathbb{R}^n$, $\langle v, w \rangle^2 \leq ||v||^2 ||w||^2$ with equality if and only if v, w are perpendicular. **Notation:** $\langle v, w \rangle$ refers to the dot product $v \cdot w$.

(a) This is false. For instance, $V = \operatorname{span}\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rangle$ is not the zero subspace but doesn't contain any of e_1, e_2, e_3.

(b) This is true. Since v_1, v_2, v_3, v_4 are linearly independent, the only solution (x_1, x_2, x_3, x_4) to $x_1v_1+x_2v_2+x_3v_3+x_4v_4 = \vec{0}$ is $(0, 0, 0, 0)$. Therefore, the only solution to $x_1v_1+x_2v_2+x_3v_3 = \vec{0}$ is $(0, 0, 0)$, so v_1, v_2, v_3 are linearly independent.

(c) This is false. For instance, let $v_1 = e_1, v_2 = e_2, v_3 = e_3$ in $\mathbb{R}^3$, and let v_4 be any vector in $\mathbb{R}^3$.

(d) False. Suppose A was a matrix satisfying the conditions. Notice that the two vectors spanning $\ker A$ are linearly independent, so $\dim \ker A = 2$. Similarly, we show by RREF or inspection that the three vectors spanning $\operatorname{im} A$ are linearly independent, so $\dim \operatorname{im} A = 3$. So $\dim \ker A + \dim \operatorname{im} A = 5$, but this contradicts the rank-nullity theorem, so no such matrix A exists.

(e) Yes. Notice that in this case $\dim \ker A + \dim \operatorname{im} A = 3 + 1 = 4$, so there is no issue from rank nullity. Since $e_1, e_2, e_3 \in \ker A$, the first three columns of A must be zero, and so we find the following example:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(f) False. A 5×5 matrix must have $\dim \ker A + \dim \operatorname{im} A = 5$ by rank-nullity, and since 5 is not an even number, we cannot have $\dim \ker A = \dim \operatorname{im} A$.

(g) True. Notice that by rank nullity, we must have $\dim \ker A = \dim \operatorname{im} A = 2$, so suppose A is a 4×4 matrix with $\ker A = \operatorname{span}\langle e_1, e_2\rangle$. This implies its first two columns are the zero vector. In order for the image of A to also be $\operatorname{span}\langle e_1, e_2\rangle$, the last two columns must span $\operatorname{span}\langle e_1, e_2\rangle$, so the following matrix works:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(h) True. Since A is orthogonal, A^{-1} exists and is equal to A^T.

(i) False. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric but not invertible.

(j) True. Each of $u_1, \dots, u_m, v_1, \dots, v_{n-m}$ is unit length, and $u_i \cdot u_j = 0$ for $i \neq j$, $v_i \cdot v_j = 0$ for $i \neq j$, and $u_i \cdot v_j = 0$ for all i, j since $u_i \in V$ and v_j is in the perpendicular subspace $V^{\perp}$. Thus, the $u_1, \dots, u_m, v_1, \dots v_{n-m}$ are orthogonal

(k) True. Let $A = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}$ be orthogonal, so $\{u_1, \dots, u_n\}$ are orthonormal. Therefore for each of the u_i with entries $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have $1 = ||u_i|| = \sqrt{x_1^2 + \cdots + x_n^2}$, so $x_j^2 \leq 1$ for all j, so all the elements of A are bounded by 1 in absolute value.

(l) True. For any $v \in \mathbb{R}^n$, $B^2v = B(Bv)$. Since $\operatorname{im} B = V$ and B fixes V since B is orthogonal projection on to V, $B(Bv) = Bv$. Therefore, since $B^2v = Bv$ for all $v \in \mathbb{R}^n$, $B^2 = B$.

(m) No, since $\begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfies $av_1 + bv_2 + cv_3 = \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}$, but

$$2v_1 + v_2 - 2v_3 = \begin{bmatrix} 2 \\ -5 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}$$

(n) False. Consider

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

which are unit length and form a basis for $\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle$. Since $v_1 \cdot v_2 \neq 0$, we cannot form an orthonormal basis with v_1, v_2.

(o) False. If "perpendicular" was replaced with "parallel", this is true and is the Cauchy Schwarz inequality. But if v, w are perpendicular, then $v \cdot w = 0$ and thus $v \cdot w \neq ||v||^2||w||^2$ for any non-zero perpendicular v, w.