MATH 33A Worksheet Week 2

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Topic 1: Invertibility

Exercise 1.1. Determine whether the following matrices are invertible, and if they are, find the inverse.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

Exercise 1.2. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) Compute A^{-1}.

(b) Use the inverse to find all solutions to $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, and all solutions to $A\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Exercise 1.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of projection onto the line $y = x$. Is T invertible? Argue both (1) geometrically, and (2) by finding the matrix representation for T and computing its determinant.

Exercise 1.4. Find the inverse of the following matrix (in terms of $c \in \mathbb{R}$. Verify your answer with matrix multiplication: $A = \begin{bmatrix} 1 & c & c^3 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

Exercise 1.5. True or False?

(a) There exists an invertible $n \times n$ matrix with two identical rows.

(b) There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Topic 2: Subspaces and Linear Independence

Exercise 2.1. Show that the following subsets are *not* subspaces of $\mathbb{R}^2$:

(a) $V = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\}$

(b) $V = \{\begin{bmatrix} 3s+1 \\ 2-s \end{bmatrix} \mid s \in \mathbb{R}\}$

Show that the following subsets *are* subspaces of $\mathbb{R}^2$:

(c) $V = \{\begin{bmatrix} t \\ 3s \end{bmatrix} \mid s, t \in \mathbb{R}\}$.

(d) $V = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$

Exercise 2.2. Determine whether the following sets of vectors are linearly independent or linearly dependent:

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

Topic 3: Image and Kernel

Exercise 3.1. Find a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies each of the following conditions, or explain why such a linear transformation doesn't exist:

(a) $\ker A = \{\vec{0}\}$, $\operatorname{im} A = \{\vec{0}\}$.

(b) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and A is not invertible.

(c) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and A is invertible.

(d) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$, $\operatorname{im} A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$.

Exercise 3.2. Find the kernel of the following matrices:

(a) $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$

(c) A for $A : \mathbb{R}^n \to \mathbb{R}^n$ invertible.

(d) A with row reduced echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

Exercise 3.3.
Let A be any matrix, and B be the RREF for A.

(a) Is $\ker A = \ker B$?

(b) Is $\operatorname{im} A = \operatorname{im} B$?
