MATH 33A Worksheet Week 1 Solutions

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Topic 1: Solving Systems of Linear Equations

Exercise 1.1. Describe all solutions to the following linear systems.

(a)

$$x + 3y - 2z = 4$$
$$2x - y + 3z = 15$$
$$x - z = 3$$

(b)

$$x + y - 2z = 1$$
$$2x - 3y + z = 1$$
$$x - z = 2$$

(c)

$$x - 2y = 2$$
$$2y + 3z = 4$$

(a) $\begin{bmatrix} 31 & 3 & -2 & 4 \\ 2 & -1 & 3 & 15 \\ 1 & 0 & -1 & 3 \end{bmatrix} \xrightarrow{(2)-2(1),(3)-(1)} \begin{bmatrix} 31 & 3 & -2 & 4 \\ 0 & -7 & 7 & 7 \\ 0 & -3 & 1 & -1 \end{bmatrix} \xrightarrow{(2)/-7} \begin{bmatrix} 31 & 3 & -2 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & 1 & -1 \end{bmatrix} \xrightarrow{(1)-3(2),(3)+3(2)}$ $\begin{bmatrix} 31 & 0 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -4 \end{bmatrix} \xrightarrow{(4)/-2} \begin{bmatrix} 31 & 0 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{(1)-(3),(2)+(3)} \begin{bmatrix} 31 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. This matrix is in RREF since every column with a leading one has zeroes in the remaining entries and every leading non-zero term is a one. Furthermore, since every column has a leading one, there are no free variables. This matrix represents the equations $\boxed{x = 5, y = 1, z = 2}$, so this is the only solution.

*Worksheet based on previous worksheets made with Emil Geisler

(b) $\begin{bmatrix} 31 & 1 & -2 & 1 \\ 2 & -3 & 1 & 1 \\ 1 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{(2)-2(1),(3)-(1)} \begin{bmatrix} 31 & 1 & -2 & 1 \\ 0 & -5 & 5 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{(2)/-5} \begin{bmatrix} 31 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1/5 \\ 0 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{(3)+(2)}$
$\begin{bmatrix} 31 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1/5 \\ 0 & 0 & 0 & 6/5 \end{bmatrix}$ This matrix represents the equation $0 = 6/5$. In other words, there are $\boxed{\text{no solutions}}$ to the linear system.

(c) $\begin{bmatrix} 31 & -2 & 0 & 2 \\ 0 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{(2)/2} \begin{bmatrix} 31 & -2 & 0 & 2 \\ 0 & 1 & 3/2 & 2 \end{bmatrix} \xrightarrow{(1)+2(2)} \begin{bmatrix} 31 & 0 & 3 & 6 \\ 0 & 1 & 3/2 & 2 \end{bmatrix}$. This matrix is in RREF. There are leading ones in the first and second columns, but not the third column, so the third variable z is free. This matrix represents the equations $x + 0y + 3z = 6$, $0x + y + 3/2z = 2$. Rewriting these in terms of the dependent variables, we have

solutions to the linear system are: $\quad x = 6 - 3z \quad y = 2 - 3/2z \quad z$ free

Another way of writing this is that the solutions to the linear system are the following set of vectors:

$$\left\{ \begin{bmatrix} 6 - 3z \\ 2 - 3/2z \\ z \end{bmatrix} \text{ for all } z \in \mathbb{R} \right\}$$

Exercise 1.2. Write down what it means for a matrix to be in row reduced echelon form (RREF). Which of the following matrices are in RREF? For each matrix, write its rank.

(a) $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A matrix is in row reduced echelon form if: 1) the leading non-zero entry in each row is a 1. These are the *leading ones* of the matrix. 2) In each column with a leading one, all the entries except for the leading one are zero.

(a) Not in RREF, the fourth column has a leading one but has other non-zero entries in the column.

(b) Yes this is in RREF. The rank is 2, since there are two leading ones.

(c) This matrix is in RREF. The rank is 2 since there are two leading ones.

(d) This matrix is not in RREF. The 4th column has a leading one but non-zero entries in the same column.

(e) This matrix is in RREF. Its rank is 3 since it has 3 leading ones.

Exercise 1.3. Find values a and b so that the ellipse $ax^2 + by^2 = 1$ goes through the points $(3, 2)$ and $(17, 12)$.

The ellipse containing these points means that plugging (x, y) equal to each of these points satisfies the equation $ax^2 + by^2 = 1$. In particular,

$$9a + 4b = 1 \qquad 289a + 144b = 1$$

This is a linear system corresponding to the following augmented matrix: $\begin{bmatrix} 29 & 4 & 1 \\ 289 & 144 & 1 \end{bmatrix} \xrightarrow{(1)/9}$ $\begin{bmatrix} 21 & 4/9 & 1/9 \\ 289 & 144 & 1 \end{bmatrix} \xrightarrow{(2)-289(1)} \begin{bmatrix} 21 & 4/9 & 1/9 \\ 0 & \frac{140}{9} & -\frac{280}{9} \end{bmatrix} \xrightarrow{(2)/140*9} \begin{bmatrix} 21 & 4/9 & 1/9 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{(1)-4/9(2)} \begin{bmatrix} 21 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$. This matrix is in RREF, and represents the equations $a = 1$ and $b = -2$. Since there are no free variables (every column has a leading one) these are the only solutions, so the only values of a, b so the ellipse $ax^2 + by^2 = 1$ goes through the points $(3, 2)$ and $(17, 12)$ are $\boxed{a = 1, b = -2}$.

Topic 2: Matrix Algebra

Exercise 2.1. Compute the following in the order of the parentheses, and observe they are the same. This is an example of the rule $A \cdot (\vec{v} + \vec{w}) = A \cdot \vec{v} + A \cdot \vec{w}$ for any $m \times n$ matrix A and $\vec{v}, \vec{w} \in \mathbb{R}^n$, i.e., A is a linear transformation.

(a) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right)$

(b) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Both of them are equal to

$$\begin{bmatrix} -8 \\ -2 \\ 4 \end{bmatrix}$$

Exercise 2.2.
Compute the following in the order of the parentheses, and observe they are the same. This is an example of the rule $B \cdot (A \cdot \vec{v}) = (B \cdot A) \cdot \vec{v}$ for any $m \times n$ matrix B, $n \times q$ matrix A, and $\vec{v}$ a vector in $\mathbb{R}^q$.

(a) $\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$

(b) $\left(\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & 4 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

Exercise 2.3. Compute the following or state that it is not defined.

(a) $\begin{bmatrix} 4 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

(a) $\begin{bmatrix} 8 \\ 2 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} 13 & 7 \\ 0 & 0 \\ 3 & 2 \end{bmatrix}$

(c) Not defined, 3×3 and 2×3 can't be multiplied since $2 \neq 3$. Notice that the other way, $(2 \times 3) \cdot (3 \times 3)$ works.

(d) The 1×1 matrix $\begin{bmatrix} 5 \end{bmatrix}$. Sometimes we treat this as just a single number, 5.

Topic 3: Linear Transformations

Exercise 3.1. For each of the following linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$, find the corresponding matrix that represents T:

(a) Rotate any vector $\vec{v}$ counter-clockwise by an angle of $\frac{\pi}{2}$ radians

(b) Projection onto the x-axis

(c) Projection onto the y-axis

(d) First reflect a vector across the line $y = x$, then rotate it by $\frac{\pi}{2}$ radians. (We have matrices A_1 and A_2 that represent the first and second steps of this transformation respectively, and a single matrix A_3 that represents the whole transformation. What is the relationship between A_1, A_2 and A_3?)

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 3.2. Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ be the standard basis vectors of $\mathbb{R}^n$. Show that if A is an $m \times n$ matrix such that $A\vec{e}_1 = A\vec{e}_2 = \cdots = A\vec{e}_n = 0$, then A is the zero matrix.

Let A have column vectors $v_1, v_2, \ldots, v_m$. Then $A \cdot e_1 = v_1 = \vec{0}$. Moreover, $A \cdot e_i = v_i = 0$ for all the standard basis vectors. Therefore, the first column of A is zero, the second column of A is zero, and so on, so A is all zeroes.

Exercise 3.3. Compute the following for all $\theta \in \mathbb{R}$:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

What linear transformation do each of these matrices represent? What is the geometric interpretation of the matrix you get as their product?

$$= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that the first matrix is rotation by θ, and the second matrix is rotation by $-\theta$ since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. That is why their product is the identity matrix, since they are inverses as functions $\mathbb{R}^2 \to \mathbb{R}^2$.