Topic 1: Solving Systems of Linear Equations

Exercise 1.1. Describe all solutions to the following linear systems.

(a)  
\begin{align*}
x + 3y - 2z &= 4 \\
2x - y + 3z &= 15 \\
x - z &= 3
\end{align*}

(b)  
\begin{align*}
x + y - 2z &= 1 \\
2x - 3y + z &= 1 \\
x - z &= 2
\end{align*}

(c)  
\begin{align*}
x - 2y &= 2 \\
2y + 3z &= 4
\end{align*}

\[ \begin{bmatrix} 31 & 3 & -2 & 4 \\ 2 & -1 & 3 & 15 \\ 1 & 0 & -1 & 3 \end{bmatrix} \xrightarrow{(2) - 2(1), (3) - (1)} \begin{bmatrix} 31 & 3 & -2 & 4 \\ 0 & -7 & 7 & 7 \\ 0 & -3 & 1 & -1 \end{bmatrix} \xrightarrow{(2) / -7} \begin{bmatrix} 31 & 3 & -2 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & 1 & -1 \end{bmatrix} \xrightarrow{(1) - 3(2), (3) + 3(2)} \begin{bmatrix} 31 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \]

This matrix is in RREF since every column with a leading one has zeroes in the remaining entries and every leading non-zero term is a one. Furthermore, since every column has a leading one, there are no free variables. This matrix represents the equations \[ x = 5, y = 1, z = 2 \], so this is the only solution.

*Worksheet based on previous worksheets made with Emil Geisler*
(b) \[
\begin{bmatrix}
31 & 1 & -2 & 1 \\
2 & -3 & 1 & 1 \\
1 & 0 & -1 & 2 \\
31 & 1 & -2 & 1 \\
0 & 1 & -1 & 1/5 \\
0 & 0 & 0 & 6/5
\end{bmatrix} \rightarrow \begin{bmatrix}
31 & 1 & -2 & 1 \\
0 & -5 & 5 & -1 \\
0 & -1 & 1 & 1 \\
31 & 1 & -2 & 1 \\
0 & 1 & -1 & 1/5 \\
0 & -1 & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
31 & 1 & -2 & 1 \\
0 & -1 & 1 & 1 \\
0 & -1 & 1 & 1
\end{bmatrix}
\]
This matrix represents the equation \(0 = 6/5\). In other words, there are no solutions to the linear system.

(c) \[
\begin{bmatrix}
31 & -2 & 0 & 2 \\
0 & 2 & 3 & 4 \\
31 & -2 & 0 & 2 \\
0 & 1 & 3/2 & 2 \\
0 & 1 & 3/2 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
31 & -2 & 0 & 2 \\
0 & 1 & 3/2 & 2 \\
31 & 0 & 3 & 6 \\
0 & 1 & 3/2 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
31 & 0 & 3 & 6 \\
0 & 1 & 3/2 & 2
\end{bmatrix}
\]
This matrix is in RREF. There are leading ones in the first and second columns, but not the third column, so the third variable \(z\) is free. This matrix represents the equations \(x + 0y + 3z = 6\), \(0x + y + 3/2z = 2\). Rewriting these in terms of the dependent variables, we have

solutions to the linear system are: \(x = 6 - 3z\) \(y = 2 - 3/2z\) \(z\) free

Another way of writing this is that the solutions to the linear system are the following set of vectors:

\[
\left\{ \begin{bmatrix}
6 - 3z \\
2 - 3/2z \\
z
\end{bmatrix} \right\}
\]
Exercise 1.2. Write down what it means for a matrix to be in row reduced echelon form (RREF). Which of the following matrices are in RREF? For each matrix, write its rank.

(a) \[
\begin{bmatrix}
1 & -2 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
(d) \[
\begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
(e) \[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

A matrix is in row reduced echelon form if: 1) the leading non-zero entry in each row is a 1. These are the leading ones of the matrix. 2) In each column with a leading one, all the entries except for the leading one are zero.

(a) Not in RREF, the fourth column has a leading one but has other non-zero entries in the column.

(b) Yes this is in RREF. The rank is 2, since there are two leading ones.

(c) This matrix is in RREF. The rank is 2 since there are two leading ones.

(d) This matrix is not in RREF. The 4th column has a leading one but non-zero entries in the same column.

(e) This matrix is in RREF. Its rank is 3 since it has 3 leading ones.

Exercise 1.3. Find values \(a\) and \(b\) so that the ellipse \(ax^2 + by^2 = 1\) goes through the points \((3,2)\) and \((17,12)\).

The ellipse containing these points means that plugging \((x, y)\) equal to each of these points satisfies the equation \(ax^2 + by^2 = 1\). In particular,

\[9a + 4b = 1 \quad 289a + 144b = 1\]

This is a linear system corresponding to the following augmented matrix:

\[
\begin{bmatrix}
29 & 4 & 1 \\
289 & 144 & 1 \\
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
21 & 4/9 & 1/9 \\
289 & 144 & 1 \\
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
21 & 4/9 & 1/9 \\
0 & 140/9 & -280/9 \\
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
21 & 4/9 & 1/9 \\
0 & 1 & -2 \\
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
21 & 0 & 1 \\
0 & 1 & -2 \\
\end{bmatrix}
\]

This matrix is in RREF, and represents the equations \(a = 1\) and \(b = -2\). Since there are no free variables (every column has a leading one) these are the only solutions, so the only values of \(a, b\) so the ellipse \(ax^2 + by^2 = 1\) goes through the points \((3,2)\) and \((17,12)\) are \(a = 1, b = -2\).
Topic 2: Matrix Algebra

Exercise 2.1.  Compute the following in the order of the parentheses, and observe they are the same. This is an example of the rule $A \cdot (\vec{v} + \vec{w}) = A \cdot \vec{v} + A \cdot \vec{w}$ for any $m \times n$ matrix $A$ and $\vec{v}, \vec{w} \in \mathbb{R}^n$, i.e., $A$ is a linear transformation.

(a) \[
\begin{bmatrix}
2 & -1 & 3 \\
0 & 1 & 1 \\
1 & 0 & -1
\end{bmatrix} \left( \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
-2
\end{bmatrix} \right)
\]

(b) \[
\begin{bmatrix}
2 & -1 & 3 \\
0 & 1 & 1 \\
1 & 0 & -1
\end{bmatrix} \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix} + \begin{bmatrix}
2 & -1 & 3 \\
0 & 1 & 1 \\
1 & 0 & -1
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
-2
\end{bmatrix}
\]

Both of them are equal to

\[
\begin{bmatrix}
-8 \\
-2 \\
4
\end{bmatrix}
\]

Exercise 2.2.
Compute the following in the order of the parentheses, and observe they are the same. This is an example of the rule $B \cdot (A \cdot \vec{v}) = (B \cdot A) \cdot \vec{v}$ for any $m \times n$ matrix $B$, $n \times q$ matrix $A$, and $\vec{v}$ a vector in $\mathbb{R}^q$.

(a) \[
\begin{bmatrix}
2 & -1 \\
0 & 1 \\
1 & 0
\end{bmatrix} \left( \begin{bmatrix}
0 & 1 & 2 \\
3 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix} \right)
\]

(b) \[
\left( \begin{bmatrix}
2 & -1 \\
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 & 2 \\
3 & 1 & 0
\end{bmatrix} \right) \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 & -1 \\
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
4
\end{bmatrix} = \begin{bmatrix}
-2 \\
4 \\
1
\end{bmatrix}
\]

4
\[
\begin{bmatrix}
-3 & 1 & 4 \\
3 & 1 & 0 \\
0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}
= \begin{bmatrix}
-2 \\
4 \\
1
\end{bmatrix}
\]
Exercise 2.3. Compute the following or state that it is not defined.

(a) \[
\begin{bmatrix}
4 & 2 & 0 \\
1 & -1 & 1 \\
0 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
4 & 2 & 3 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
4 & 2 \\
0 & 1 \\
-1 & -1
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
4 & 2 & 3 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
-1 & 0 & 1
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
0 & 1 & 3 & 2
\end{bmatrix}
\begin{bmatrix}
1 \\
-1 \\
0 \\
3
\end{bmatrix}
\]

(a) \[
\begin{bmatrix}
8 \\
2 \\
7
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
13 & 7 \\
0 & 0 \\
3 & 2
\end{bmatrix}
\]

(c) Not defined, $3 \times 3$ and $2 \times 3$ can’t be multiplied since $2 \neq 3$. Notice that the other way, $(2 \times 3) \cdot (3 \times 3)$ works.

(d) The $1 \times 1$ matrix $[5]$. Sometimes we treat this as just a single number, 5.
**Topic 3: Linear Transformations**

**Exercise 3.1.** For each of the following linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, find the corresponding matrix that represents $T$:

(a) Rotate any vector $\vec{v}$ counter-clockwise by an angle of $\frac{\pi}{2}$ radians

(b) Projection onto the $x$-axis

(c) Projection onto the $y$-axis

(d) First reflect a vector across the line $y = x$, then rotate it by $\frac{\pi}{2}$ radians. (We have matrices $A_1$ and $A_2$ that represent the first and second steps of this transformation respectively, and a single matrix $A_3$ that represents the whole transformation. What is the relationship between $A_1$, $A_2$ and $A_3$?)

(a) \[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

**Exercise 3.2.** Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $\vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ be the standard basis vectors of $\mathbb{R}^n$. Show that if $A$ is an $m \times n$ matrix such that $A\vec{e}_1 = A\vec{e}_2 = \ldots = A\vec{e}_n = 0$, then $A$ is the zero matrix.

Let $A$ have column vectors $v_1, v_2, \ldots, v_m$. Then $A \cdot e_1 = v_1 = 0$. Moreover, $A \cdot e_i = v_i = 0$ for all the standard basis vectors. Therefore, the first column of $A$ is zero, the second column of $A$ is zero, and so on, so $A$ is all zeroes.
Exercise 3.3. Compute the following for all $\theta \in \mathbb{R}$:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

What linear transformation do each of these matrices represent? What is the geometric interpretation of the matrix you get as their product?

\[
\begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Notice that the first matrix is rotation by $\theta$, and the second matrix is rotation by $-\theta$ since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. That is why their product is the identity matrix, since they are inverses as functions $\mathbb{R}^2 \to \mathbb{R}^2$. 