MATH 33A Worksheet Week 1

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Topic 1: Solving Systems of Linear Equations

Exercise 1.1. Describe all solutions to the following linear systems.

(a)

$$x + 3y - 2z = 4$$
$$2x - y + 3z = 15$$
$$x - z = 3$$

(b)

$$x + y - 2z = 1$$
$$2x - 3y + z = 1$$
$$x - z = 2$$

(c)

$$x - 2y = 2$$
$$2y + 3z = 4$$

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*Worksheet based on previous worksheets made with Emil Geisler

Exercise 1.2. Write down what it means for a matrix to be in row reduced echelon form (RREF). Which of the following matrices are in RREF? For each matrix, write its rank.

(a) $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 1.3. Find values a and b so that the ellipse $ax^2 + by^2 = 1$ goes through the points $(3, 2)$ and $(17, 12)$.

Topic 2: Matrix Algebra

Exercise 2.1. Compute the following in the order of the parentheses, and observe they are the same. This is an example of the rule $A \cdot (\vec{v} + \vec{w}) = A \cdot \vec{v} + A \cdot \vec{w}$ for any $m \times n$ matrix A and $\vec{v}, \vec{w} \in \mathbb{R}^n$, i.e., A is a linear transformation.

(a) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right)$

(b) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Exercise 2.2.
Compute the following in the order of the parentheses, and observe they are the same. This is an example of the rule $B \cdot (A \cdot \vec{v}) = (B \cdot A) \cdot \vec{v}$ for any $m \times n$ matrix B, $n \times q$ matrix A, and $\vec{v}$ a vector in $\mathbb{R}^q$.

(a) $\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$

(b) $\left(\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Exercise 2.3. Compute the following or state that it is not defined.

(a) $\begin{bmatrix} 4 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

Topic 3: Linear Transformations

Exercise 3.1. For each of the following linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$, find the corresponding matrix that represents T:

(a) Rotate any vector $\vec{v}$ counter-clockwise by an angle of $\frac{\pi}{2}$ radians

(b) Projection onto the x-axis

(c) Projection onto the y-axis

Exercise 3.2. Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ be the standard basis vectors of $\mathbb{R}^n$. Show that if A is an $m \times n$ matrix such that $A\vec{e}_1 = A\vec{e}_2 = \cdots = A\vec{e}_n = 0$, then A is the zero matrix.

Exercise 3.3. Compute the following for all $\theta \in \mathbb{R}$:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

What linear transformation do each of these matrices represent? What is the geometric interpretation of the matrix you get as their product?
