MATH 33A Worksheet Week 5 Solutions

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Exercise 1. Find all the roots of the following polynomials: (Will need to possibly use factoring, the quadratic formula, rational roots theorem, and/or polynomial long division):

(a) $x^2 - 2x + 1$

(b) $x^2 - x - 1$

(c) $x^3 + 3x^2 - x - 3$

(d) $x^3 - 2x^2 - 2x + 4$

(Bonus: For each polynomial $p(x)$ above, can you construct a matrix A such that $\det(A - \lambda I)$)

(a) $x^2 - 2x + 1 = (x - 1)^2$, so $x = 1$ is the only root.

(b) $x = \frac{1 \pm \sqrt{6}}{2}$ by the quadratic formula

(c) By the rational root theorem, if this polynonial has a rational root, it will be ± 3 or ± 1. Checking these values, we will see that $x = -3$ is a root, thus $(x + 3)$ is a factor of this polynomial. Then we can also see that 1 and -1 are both roots, so $(x + 1)$ and $(x - 1)$ are also factors. Sine this is a degree 3 polynomial, it has at most 3 real roots, so we get that $-3, -1, 1$ are all the roots.

(d) By the rational root theorem, the possible rational roots of this polynomial are $\pm 1, \pm 2$ and ± 4. Checking each of these values, we see that only $x = 2$ is a root and thus $x - 2$ is a factor of this polynomial. We can then perform polynomial long division:

$$
\begin{array}{r}
x^2 \qquad\quad - 2 \\
x - 2 \overline{\big)\; x^3 - 2x^2 - 2x + 4} \\
\underline{- x^3 + 2x^2} \qquad\qquad \\
- 2x + 4 \\
\underline{2x - 4} \\
0
\end{array}
$$

And get a resulting quotient of $x^2 - 2$. Then as a difference of two squares, we get $\pm\sqrt{2}$ as the two other roots.

The bonus can be answered by constructing a matrix whose diagonal entries are the roots of each of these polynomials.

Exercise 2. Compute the characteristic polynomial for the following matrices:

(a) $\begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ with coefficients of the characteristic polynomial in terms of $\sin(\theta), \cos(\theta)$

(d) $\begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) $x^2 - 6x + 10$.

(b) $(x-3)(x-1)(x-4)$ since the matrix is upper triangular $\det A - xI_n$ is also upper triangular.

(c) $x^2 - 2\cos(\theta)x + 1$ (using that $\cos(\theta)^2 + \sin(\theta)^2 = 1$.

(d)

$$\det \begin{bmatrix} -x & 0 & 0 & 4 \\ 0 & -x & -2 & 0 \\ 0 & 1 & -x & 0 \\ 0 & 0 & 0 & -x \end{bmatrix} = -x \det \begin{bmatrix} -x & -2 & 0 \\ 1 & -x & 0 \\ 0 & 0 & -x \end{bmatrix} = x^2 \det \begin{bmatrix} -x & -2 \\ 1 & -x \end{bmatrix} = x^2(x^2+2)$$

Exercise 3. For what values of $a \in \mathbb{R}$ does the following matrix have an eigenvalue of 2?
$A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & a & 3 \\ 0 & a^2 & 1 \end{bmatrix}$

A has an eigenvalue of 2 if and only if $\ker A - 2I_3 \neq \{\vec{0}\}$, which is true if and only if $A - 2I_3$ is invertible. Thus, A has an eigenvalue of 2 if and only if $\det A - 2I_3 = 0$. Thus let us compute $\det A - 2I_3$ in terms of a.

$$\det A - 2I_3 = \det \begin{bmatrix} 2 & 0 & 2 \\ 2 & a-2 & 3 \\ 0 & a^2 & -1 \end{bmatrix} = 2\det \begin{bmatrix} a-2 & 3 \\ a^2 & -1 \end{bmatrix} - 0 + 2\det \begin{bmatrix} 2 & a-2 \\ 0 & a^2 \end{bmatrix}$$

$$= (-2a + 4 - 6a^2) + 4a^2 = -2a^2 - 2a + 4$$

This polynomial has roots of 1 and -2 by factoring or the quadratic formula. so the only values of a for which A has an eigenvalue of 2 are $a = 1$ and $a = -2$.

Exercise 4. Let $A = \begin{bmatrix} 19 & -12 \\ 30 & -19 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) Find bases for the eigenspaces of A.

(c) Using part (b), diagonalize A.

(d) Use diagonalization to find A^{100}.

(a) We compute that the characteristic polynomial of A is

$$(\lambda - 19)(\lambda + 19) + 12 * 30 = \lambda^2 - 361 + 360 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

Therefore, A has eigenvalues 1 and -1.

(b) Since A has eigenvalues 1 and -1, we must find bases for $\ker I_2 - A$ and $\ker -I_2 - A$.
Eigenvalue of 1: $\ker I_2 - A = \ker \begin{bmatrix} -18 & 12 \\ -30 & 20 \end{bmatrix}$. Row reducing, we find that $\ker I_2 - A = \operatorname{span}\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix} \rangle$. So $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a basis for the subspace of eigenvectors of A with eigenvalue 1.
Eigenvalue of 2: $\ker -I_2 - A = \ker \begin{bmatrix} -20 & 12 \\ -30 & 18 \end{bmatrix}$. Row reducing, we find that $\ker -I_2 - A = \operatorname{span}\langle \begin{bmatrix} 3 \\ 5 \end{bmatrix} \rangle$. So $v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is a basis for the subspace of eigenvectors of A with eigenvalue -1.

(c) Therefore, letting $B = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and recalling that v_1 has eigenvalue 1 and v_2 has eigenvalue -1, we have:

$$A = B \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B^{-1}$$

We can explicitly compute B^{-1}, and it's not too bad since $\det B = 1$:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

(d) We have:

$$A^{100} = (BDB^{-1})^{100} = \overbrace{(BDB^{-1})(BDB^{-1})\dots(BDB^{-1})}^{100} = BD^{100}B^{-1}$$

We have

$$D^{100} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{100} = \begin{bmatrix} -1^{100} & 0 \\ 0 & 1^{100} \end{bmatrix} = I_2$$

Therefore,

$$A^{100} = BD^{100}B^{-1} = BI_2B^{-1} = BB^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 5. Diagonalize the following matrices or show that they cannot be diagonalized by showing that the geometric multiplicity of an eigenvalue is less than its algebraic multiplicity:

(a) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix}$ for $a \neq b \in \mathbb{R}$.

(d) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 1 & 2 & 2 \end{bmatrix}$

(a) Characteristic polynomial is $P(x) = x^2 - 5x + 6 = (x-3)(x-2)$.
Eigenvalue of 2:
$A - 2I_2 = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$, which has RREF $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and thus kernel of the form $\ker(A - 2I_2) = \left\{ \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \; x_2 \in \mathbb{R} \right\} = \text{span}\langle \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rangle$ so $\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \}$ is a basis for the eigenspace of A with eigenvalue 2.
Eigenvalue of 3:
$A - 3I_2 = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and thus has basis of the kernel given by $\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \}$.
Thus, we have $A = SDS^{-1}$ for $S = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. We can compute $S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(b) Characteristic polynomial is $P(x) = x^2 - 2x + 1 = (x-1)^2$.
Eigenvalue of 1:

$A - I_2 = \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ and thus has basis of the kernel given by $\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$. Thus, the geometric multiplicity of $(\lambda = 1)$ is 1 since the eigenspace has a basis with a single vector and is thus one dimensional, but the algebraic multiplicity of $(\lambda = 1)$ is 2 since $P(x) = (x-1)^2$. Thus, A is not diagonlizable.

(c) *(Note: if $a = b$, this matrix is not diagonalizable since the algebraic multiplicity of $(\lambda = a = b)$ is 2 but the geometric multiplicity is 1.*

The characteristic polynomial is $P(x) = (x-a)(x-b)$. Since $a \neq b$, P has two eigenvalues.

Eigenvalue of $\lambda = a$:

$A - aI_2 = \begin{bmatrix} 0 & 1 \\ 0 & b-a \end{bmatrix}$ which has RREF $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for $\ker A - aI_2$ is given by $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$.

Eigenvalue of $\lambda = b$:

$A - bI_2 = \begin{bmatrix} a-b & 1 \\ 0 & 0 \end{bmatrix}$. Since $a \neq b$, $a - b \neq 0$, so this matrix has RREF $\begin{bmatrix} 1 & \frac{1}{a-b} \\ 0 & 0 \end{bmatrix}$ and thus has basis for $\ker A - bI_2$ given by $\{\begin{bmatrix} 1 \\ b-a \end{bmatrix}\}$.

Thus, we have

$$S = \begin{bmatrix} 1 & 1 \\ 0 & b-a \end{bmatrix} \qquad D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

and $S^{-1} = \begin{bmatrix} 1 & -\frac{1}{b-a} \\ 0 & \frac{1}{b-a} \end{bmatrix}$.

(d)

Exercise 6. True or false:

(a) If 0 is an eigenvalue of a matrix A, then $\det(A) = 0$.

(b) If a matrix only has an eigenvalue of 1, then it is the identity matrix.

(c) All diagonalizable matrices are invertible

(d) All invertible matrices are diagonalizable

(a) True, if 0 is an eigenvalue, then A has a non-zero kernel and thus is not invertible, so it has determinant 0.

(b) False, consider the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, or any triangular matrix with all 1's on the diagonal.

(c) False, diagonalizability only depends on the geometric multiplicities of the eigenvalues, not what the eigenvalues are. A counter-example to this statement is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(d) False, matrices can be invertible and not diagonalizable! Consider $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.