MATH 33A Worksheet Week 4

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Exercise 1. Compute the determinant of the following matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

(a) $1 \cdot 4 - 2 \cdot 3 = -2$

(b) $1 \cdot -2 \cdot 2 = 0$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{(II)-2(I)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{(III)-3(I)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Product of main diagonal is 0, so original matrix has determinant of 0 as well. Also, if we row-reduce a square matrix and get a row of 0s, we immediately know our matrix is not invertible and thus the determinant must be 0.

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \xrightarrow[(III)-(I)]{(II)-(I)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{(III)-2(II)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

And the row-reduced matrix has determinant 1, so our original matrix also has determinant 1.

Exercise 2. One important application of determinants is that they help us determine invertibility of a matrix. Use the determinant to determine for which values λ the following matrices are invertible:

(a) $\begin{bmatrix} \lambda & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & \lambda \\ 1 & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{bmatrix}$

(c) $\begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix}$

(d) $\begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 1 \\ 0 & -1 & 6-\lambda \end{bmatrix}$

(a) $\det\left(\begin{bmatrix} \lambda & 2 \\ 3 & 4 \end{bmatrix}\right) = 4\lambda - 6$

$4\lambda - 6 = 0 \implies \lambda = \frac{3}{2}$ is the only value when this matrix is not invertible.

(b) $\det\left(\begin{bmatrix} 1 & 1 & \lambda \\ 1 & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{bmatrix}\right) = (\lambda-1)(\lambda-\lambda^2) = (\lambda-1)^2\lambda$ So this matrix is not invertible when $\lambda = 0, 1$

(c) $\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix}\right) = (1-\lambda)(4-\lambda)$

So this matrix is not invertible when $\lambda = 1, 4$

(d) $\det\left(\begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 1 \\ 0 & -1 & 6-\lambda \end{bmatrix}\right) = (3-\lambda)[(4-\lambda)(6-\lambda)+1] = (3-\lambda)(\lambda^2 - 10\lambda + 25) =$ $(3-\lambda)(\lambda-5)^2$

So this matrix is not invertible when $\lambda = 3, 5$

Exercise 3. Solve the following linear systems using Cramer's rule:

(a)

$$\begin{aligned} 3x + 4y &= 11 \\ 4x + 11y &= 3 \end{aligned}$$

(b)

$$\begin{aligned} 2x + 3y &= 8 \\ 4y + 5z &= 3 \\ 6x + 7z &= -1 \end{aligned}$$

(a) Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 11 \end{bmatrix}$ and $b = \begin{bmatrix} 11 \\ 3 \end{bmatrix}$. Then let $A_1 = \begin{bmatrix} 11 & 4 \\ 3 & 11 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 3 & 11 \\ 4 & 3 \end{bmatrix}$. We can compute the determinant of all 3 of the previous matrices:

$$\begin{aligned} \det(A) &= 17 \\ \det(A_1) &= 109 \\ \det(A_2) &= -35 \end{aligned}$$

And then use Cramer's rule to conclude:

$$\begin{aligned} x &= \frac{\det(A_1)}{\det(A)} = \frac{109}{17} \\ y &= \frac{\det(A_2)}{\det(A)} = \frac{-35}{17} \end{aligned}$$

(b) Let $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$. Then let $A_1 = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}$. We can compute the determinant of all four of the previous matrices:

$$\begin{aligned} \det(A) &= 146 \\ \det(A_1) &= 146 \\ \det(A_2) &= 292 \\ \det(A_3) &= -146 \end{aligned}$$

And then use Cramer's rule to conclude:

$$\begin{aligned} x &= \frac{\det(A_1)}{\det(A)} = \frac{146}{146} = 1 \\ y &= \frac{\det(A_2)}{\det(A)} = \frac{292}{146} = 2 \\ z &= \frac{\det(A_3)}{\det(A)} = \frac{-146}{146} = -1 \end{aligned}$$

Exercise 4. Determine whether the following are true or false:

(a) $\det(A+B) = \det(A) + \det(B)$ for any two $n \times n$ matrices A and B.

(b) If B is the rref of A, then $\det(B) = \det(A)$.

(c) There exists a 3×3 matrix A with real valued entries such that $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(d) If A is an orthogonal matrix, then $\det(A) = \pm 1$.

(a) False. Here's a counter-example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1 + 1 \neq \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$$

(b) False, scaling a matrix or swapping rows changes the determinant!
(c) False. Such a matrix A would have $\det(A^2) = -1 \implies \det(A) = \sqrt{-1}$, however, if A is composed of all real entries, this cannot be the case.
(d) True. If A is orthogonal, we know that $A^T = A^{-1}$ and thus we have that $\det(A) = \det(A^T) = \det(A^{-1}) = \frac{1}{\det(A)} \implies \det(A)^2 = 1 \implies \det(A) = \pm 1$