MATH 33A Worksheet Week 3

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Topic 1: Images, Kernels, Bases

Exercise 1.1. Let $A : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix}$. Find a basis for $\ker A$. Find a basis for $\operatorname{im} A$. Notice that $\dim \ker A + \dim \operatorname{im} A = 4$.

Exercise 1.2. Let $A : \mathbb{R}^8 \to \mathbb{R}^7$ be given by the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & 2 \\ 2 & 3 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine $\dim \ker A$ (hint: use rank-nullity and find $\dim \operatorname{im} A$).

Exercise 1.3. Find a basis for the following subspaces of $\mathbb{R}^3$:

(a) $V = \operatorname{span}\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \right)$

(b) $V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \middle| \; v_1 - 3v_2 = 0 \right\}$

(c) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be any linear transformation such that $\dim \ker A = 0$. Find a basis for $V = \operatorname{im} A$.

Topic 2: Orthogonality

Exercise 2.1. For what values of $\lambda \in \mathbb{R}$ are the following pairs of vectors orthogonal?

(a) $\begin{bmatrix} \lambda \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} \lambda \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -\lambda \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$

Exercise 2.2. Determine whether the following sets of vectors are *orthonormal* (orthogonal and unit length):

(a) $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}.$

(b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(c) $\begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$

Exercise 2.3. Find the orthogonal projection of $\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ onto the subspace

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Exercise 2.4. Find a basis for $W^{\perp}$, where

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}$$

(Hint: How can we relate $W^{\perp}$ to subspaces where we know how to find a basis?)

Exercise 2.5. For each of the following vectors $\vec{v}$, find the decomposition $v^{||} + v^{\perp}$ with respect to the subspace

$$V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Exercise 2.6. Let $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ be a subspace of $\mathbb{R}^n$ where the vectors $\vec{v}_1, \dots, \vec{v}_k$ give an orthonormal basis for V.

(a) If $\vec{w} \in V$, show that $\text{proj}_V(\vec{w}) = \vec{w}$.

(b) If $\vec{w} \in V^\perp$, show that $\text{proj}_{V^\perp}(\vec{w}) = 0$.

Exercise 2.7. Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 2 & 1 & 1 & -4 \end{bmatrix}$$

Find an orthonormal basis $\mathcal{B} = \{u_1, u_2\}$ for $\ker A$.

Exercise 2.8. Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

True or False

Exercise TF. True or false: Explain your reasoning or find an example or counterexample.

(a) If V is a subspace of $\mathbb{R}^3$ that does not contain any of the elementary column vectors e_1, e_2, e_3, then $V = \{\vec{0}\}$.

(b) If v_1, v_2, v_3, v_4 are linearly independent vectors, then v_1, v_2, v_3 are linearly independent.

(c) If v_1, v_2, v_3 are linearly independent vectors, then v_1, v_2, v_3, v_4 are linearly independent.

(d) It is possible for a 4×4 matrix A to have $\ker A = \operatorname{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ and $\operatorname{im} A = \operatorname{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} \right\rangle$

(e) There exists a 4×4 matrix A with $\ker A = \operatorname{span}\langle e_1, e_2, e_3 \rangle$ and $\operatorname{im} A = \operatorname{span}\langle e_3 + e_4 \rangle$

(f) There exists a 5×5 matrix A with $\ker A = \operatorname{im} A$.

(g) There exists a 4×4 matrix A with $\ker A = \operatorname{im} A$.

(h) If A is orthogonal then it is invertible.

(i) If A is symmetric ($A = A^T$) it is invertible.

(j) Let V be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{u_1, \dots, u_m\}$, and let $\{v_1, \dots, v_{n-m}\}$ be an orthonormal basis for $V^\perp$. Then $\{u_1, \dots, u_m, v_1, \dots, v_{n-m}\}$ is an orthonormal basis for $\mathbb{R}^n$.

(k) The entries of an orthogonal matrix are all less than or equal to 1 in absolute value.

(l) Let V be a subspace of $\mathbb{R}^n$ and B the matrix for orthogonal projection onto V. Then $B^2 = B$.

(m) Let $\mathcal{B} = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}^3$. Then $\begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(n) If $v_1, \dots, v_m$ is a basis of unit length vectors for a subspace V, there is an orthonormal basis of V containing the vectors v_1 and v_2.

(o) For all $v, w \in \mathbb{R}^n$, $\langle v, w \rangle^2 \leq ||v||^2 ||w||^2$ with equality if and only if v, w are perpendicular. **Notation:** $\langle v, w \rangle$ refers to the dot product $v \cdot w$.
