Kernels, Linear Independence, Bases:

$$A, \quad m \times n, \quad \overset{\ker(A)}{\mathbb{R}^n} \longrightarrow \overset{\operatorname{im}(A)}{\mathbb{R}^m}$$

$$\ker(A) = \left\{ \begin{array}{c} \text{all solutions to} \\ Ax = 0 \end{array} \right\}$$

↳ Subspace: closed under scaler multiplication
closed under vector addition

Linear independence:

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

Linearly independent if the only solution to

$$x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{0}$$

$$x_1 = x_2 = \dots = x_n = 0$$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$

Basis for a subspace:

If V is a subspace of $\mathbb{R}^n$, (think $\text{im}(A), \ker(A)$)

then a generating set for V is a set of

vectors $\vec{v}_1, \dots, \vec{v}_k$ if

$$V = \underline{\text{span}}\{\vec{v}_1, \dots, \vec{v}_k\}$$

↳ all linear combinations of $\vec{v}_1, \dots, \vec{v}_k$

A generating set is a **basis** if it is also

linearly independent

→ The number of vectors in a basis, is called the **dimension** of the subspace.

Math 33A Worksheet Week 5

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Exercise 1. Determine whether the following sets of vectors are linearly independent or linearly dependent:

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Linearly dependent, $x \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for any $x \in \mathbb{R}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 3 & 0 & 2 \\ -1 & 1 & 2 \end{bmatrix}}_{A}$$

What are the sols. to $A\vec{x} = \vec{0}$?

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] /3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] + (I)$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 0 \\ 0 & 1 & 8/3 & 0 \end{array}\right]$$

$x_1 + 2/3\, x_3 = 0$

$x_2 + 8/3\, x_3 = 0$

$x_3 = t \quad , t \in \mathbb{R}$

$x_1 = -2/3\, x_3 = -2/3\, t$

$x_2 = -8/3\, x_3 = -8/3\, t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2/3 \\ -8/3 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Since $Ax = 0$ has non-zero solutions,

$\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are linearly dependant

$$\underline{-\frac{2}{3}} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \underline{-\frac{8}{3}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \frac{8}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Exercise 2. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $\begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 0 \\ -8 & 0 & 2 \end{bmatrix}$. Find a basis for $\ker A$. Find a basis for $\operatorname{im} A$. What are the dimensions of $\ker A$ and $\operatorname{im} A$?

$$\begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 0 \\ -8 & 0 & 2 \end{bmatrix}$$

Exercise 3. Find a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies each of the following conditions, or explain why such a linear transformation doesn't exist:

(a) $\ker A = \{\vec{0}\}$, $\operatorname{im} A = \{\vec{0}\}$.

(b) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and $\det A = 0$ non-invertible

(c) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and $\det A \neq 0$ invertible

(d) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$, $\operatorname{im} A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$.

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 0 \\ -8 & 0 & 2 \end{bmatrix}$$

How to find the basis for $\mathrm{im}(A)$ & $\ker(A)$

① Row reduce A to RREF

$$\begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 0 \\ -8 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ +2(\mathrm{I}) \end{matrix} \longrightarrow \begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \begin{matrix} /4 \\ \\ -4(\mathrm{II}) \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1/2 & -1/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

② Identify pivots

③

Find basis for $\mathrm{im}(A)$:

Take columns of A that correspond to cols with pivots in RREF of A

$$\mathrm{im}(A) = \mathrm{span}\left\{ \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

NOTE: YOU HAVE TO USE COLUMNS OF A, NOT RREF(A)

(4) Augment row-reduced matrix w/ $\vec{0}$, and identify free variables

$$\begin{array}{c}
\left[\begin{array}{ccc|c} 1 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
\begin{array}{ccc} x_1 & x_2 & \underbrace{\quad}_{x_3} \end{array}
\end{array}$$

free variable

$$x_1 + \overset{0}{\cancel{\tfrac{1}{2} x_2}} - \tfrac{1}{4} x_3 = 0$$

$$x_2 = 0$$

$$x_3 = t \quad , \; t \in \mathbb{R}$$

$$x_1 = \tfrac{1}{4} x_3 = \tfrac{1}{4} t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/4\, t \\ 0 \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix}}$$

$$\mathrm{Ker}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\dim(\mathrm{im}(A)) = \#$ of pivots

$\dim(\mathrm{Ker}(A)) = \#$ of free variables

$\dim(\mathrm{im}(A)) + \dim(\mathrm{Ker}(A)) = \#$ of columns of A

always true, for any A

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

basis vector

$$\operatorname{im}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variables

$$x_1 + x_2 + x_3 = 0$$

$$x_2 = t \ , \ t \in \mathbb{R} \quad \Rightarrow \quad x_1 = -x_2 - x_3 = -t - s$$

$$x_3 = s \ , \ s \in \mathbb{R}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix}$$

$$= t \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} + s \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

basis vectors for ker(A)

Exercise 4. Find the kernel of the following matrices:

(a) $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$

(c) A for $A : \mathbb{R}^n \to \mathbb{R}^n$ invertible.

(d) A with row reduced echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

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4. (c) A, $n \times n$

A invertible when # of pivots $= n$ $\left(A^{-1}A = I_n \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right)$

$\ker(A) = \vec{0}$

A has $\underbrace{\cancel{n-n}}_{0}$ free variables, so $\ker(A)$ only contains $\vec{0}$

A invertible $\iff$ # of pivots $= n$ $\iff$ $\ker(A) = 0$

$n \times n$ $\iff$ cols of A are linearly independent