MATH 33A (Extra) Final Practice

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Disclaimer: These questions may not reflect what will appear on the final.

Exercise 1. Compute the characteristic polynomial for the following matrices:

(a) $\begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ with coefficients of the characteristic polynomial in terms of $\sin(\theta), \cos(\theta)$

(d) $\begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) $x^2 - 6x + 10$.

(b) $(x-3)(x-1)(x-4)$ since the matrix is upper triangular $\det A - xI_n$ is also upper triangular.

(c) $x^2 - 2\cos(\theta)x + 1$ (using that $\cos(\theta)^2 + \sin(\theta)^2 = 1$.

(d)

$$\det \begin{bmatrix} -x & 0 & 0 & 4 \\ 0 & -x & -2 & 0 \\ 0 & 1 & -x & 0 \\ 0 & 0 & 0 & -x \end{bmatrix} = -x \det \begin{bmatrix} -x & -2 & 0 \\ 1 & -x & 0 \\ 0 & 0 & -x \end{bmatrix} = x^2 \det \begin{bmatrix} -x & -2 \\ 1 & -x \end{bmatrix} = x^2(x^2+2)$$

Exercise 2. For what values of $a \in \mathbb{R}$ does the following matrix have an eigenvalue of 2?

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & a & 3 \\ 0 & a^2 & 1 \end{bmatrix}$$

A has an eigenvalue of 2 if and only if $\ker A - 2I_3 \neq \{\vec{0}\}$, which is true if and only if $A - 2I_3$ is invertible. Thus, A has an eigenvalue of 2 if and only if $\det A - 2I_3 = 0$. Thus let us compute $\det A - 2I_3$ in terms of a.

$$\det A - 2I_3 = \det \begin{bmatrix} 2 & 0 & 2 \\ 2 & a-2 & 3 \\ 0 & a^2 & -1 \end{bmatrix} = 2\det \begin{bmatrix} a-2 & 3 \\ a^2 & -1 \end{bmatrix} - 0 + 2\det \begin{bmatrix} 2 & a-2 \\ 0 & a^2 \end{bmatrix}$$

$$= (-2a + 4 - 6a^2) + 4a^2 = -2a^2 - 2a + 4$$

This polynomial has roots of 1 and -2 by factoring or the quadratic equation. so the only values of a for which A has an eigenvalue of 2 are $a = 1$ and $a = -2$.

Exercise 3. Let $A = \begin{bmatrix} 19 & -12 \\ 30 & -19 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) Find bases for the eigenspaces of A.

(c) Using part (b), diagonalize A.

(d) Use diagonalization to find A^{100}.

(a) We compute that the characteristic polynomial of A is

$$(\lambda - 19)(\lambda + 19) + 12 * 30 = \lambda^2 - 361 + 360 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

Therefore, A has eigenvalues 1 and -1.

(b) Since A has eigenvalues 1 and -1, we must find bases for $\ker I_2 - A$ and $\ker -I_2 - A$.

Eigenvalue of 1: $\ker I_2 - A = \ker \begin{bmatrix} -18 & 12 \\ -30 & 20 \end{bmatrix}$. Row reducing, we find that $\ker I_2 - A = \text{span}\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix} \rangle$. So $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a basis for the subspace of eigenvectors of A with eigenvalue 1.

Eigenvalue of 2: $\ker -I_2 - A = \ker \begin{bmatrix} -20 & 12 \\ -30 & 18 \end{bmatrix}$. Row reducing, we find that $\ker -I_2 - A = \text{span}\langle \begin{bmatrix} 3 \\ 5 \end{bmatrix} \rangle$. So $v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is a basis for the subspace of eigenvectors of A with eigenvalue -1.

(c) Therefore, letting $B = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and recalling that v_1 has eigenvalue 1 and v_2 has eigenvalue -1, we have:

$$A = B \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} B^{-1}$$

We can explicitly compute B^{-1}, and it's not too bad since $\det B = 1$:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

(d) We have:

$$A^{100} = (BDB^{-1})^{100} = \overbrace{(BDB^{-1})(BDB^{-1})\dots(BDB^{-1})}^{100} = BD^{100}B^{-1}$$

We have

$$D^{100} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{100} = \begin{bmatrix} -1^{100} & 0 \\ 0 & 1^{100} \end{bmatrix} = I_2$$

Therefore,

$$A^{100} = BD^{100}B^{-1} = BI_2B^{-1} = BB^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 4. True or false:

(a) If V and W are dimension m and k subspaces of $\mathbb{R}^n$, then $V \cap W$ is dimension $m - k$.

(b) If A is invertible, then $\det(A^{-1}) = 1/(\det A)$.

(c) If A is an $m \times n$ matrix with $n > m$, there is a non-zero vector v such that $A \cdot v = \vec{0}$.

(d) The matrix $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is invertible.

(e) If A is an $m \times n$ matrix with $n < m$, there is always a solution to $Ax = \vec{b}$ for any $\vec{b} \in \mathbb{R}^n$.

(f) The rank of a matrix A is the number of leading ones in RREF.

(g) The dimension of the kernel of a matrix A is the number of columns without a leading one in RREF.

(h) If $A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for 2 by 2 matrices A and B, then either $A = 0$ or $B = 0$.

(i) If three vectors $v_1, v_2, v_3 \in \mathbb{R}^n$ are linearly independent, then $\dim \operatorname{span}\langle v_1, v_2, v_3 \rangle = 3$.

(j) There is an orthonormal basis of $\ker A$, where $A = \begin{bmatrix} 3 & 0 & 1 \\ -6 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) False

(b) True, since $\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I_n) = 1$.

(c) True, the RREF has at least one column without a leading 1 or by rank nullity since $\dim \mathrm{Im} A \leq m$, we have $\dim \ker A = n - \dim \mathrm{Im} A > 0$.

(d) True find the determinant

(e) No, zero matrix with any non-zero vector

(f) True

(g) True (also, (f) + (g) proves rank nullity!)

(h) No, $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(i) True

(j) True. By Gram Schmidt, we can always find an orthonormal basis for *any* subspace of $\mathbb{R}^n$.

Exercise 4. Diagonalize the following matrices or show that they cannot be diagonalized by showing that the geometric multiplicity of an eigenvalue is less than its algebraic multiplicity:

(a) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix}$ for $a \neq b \in \mathbb{R}$.

(a) Characteristic polynomial is $P(x) = x^2 - 5x + 6 = (x-3)(x-2)$.
Eigenvalue of 2:
$A - 2I_2 = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$, which has RREF $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and thus kernel of the form $\ker(A - 2I_2) = \left\{ \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \, x_2 \in \mathbb{R} \right\} = \text{span}\langle \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rangle$ so $\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \}$ is a basis for the eigenspace of A with eigenvalue 2.
Eigenvalue of 3:
$A - 3I_2 = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and thus has basis of the kernel given by $\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \}$.
Thus, we have $A = SDS^{-1}$ for $S = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. We can compute $S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(b) Characteristic polynomial is $P(x) = x^2 - 2x + 1 = (x-1)^2$.
Eigenvalue of 1:
$A - I_2 = \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ and thus has basis of the kernel given by $\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \}$.
Thus, the geometric multiplicity of $(\lambda = 1)$ is 1 since the eigenspace has a basis with a single vector and is thus one dimensional, but the algebraic multiplicity of $(\lambda = 1)$ is 2 since $P(x) = (x-1)^2$. Thus, A is not diagonlizable.

(c) *(Note: if $a = b$, this matrix is not diagonalizable since the algebraic multiplicity of $(\lambda = a = b)$ is 2 but the geometric multiplicity is 1.*
The characteristic polynomial is $P(x) = (x-a)(x-b)$. Since $a \neq b$, P has two eigenvalues.
Eigenvalue of $\lambda = a$:
$A - aI_2 = \begin{bmatrix} 0 & 1 \\ 0 & b-a \end{bmatrix}$ which has RREF $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for $\ker A - aI_2$ is given by $\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$.
Eigenvalue of $\lambda = b$:
$A - bI_2 = \begin{bmatrix} a-b & 1 \\ 0 & 0 \end{bmatrix}$. Since $a \neq b$, $a - b \neq 0$, so this matrix has RREF $\begin{bmatrix} 1 & \frac{1}{a-b} \\ 0 & 0 \end{bmatrix}$ and thus has basis for $\ker A - bI_2$ given by $\{ \begin{bmatrix} 1 \\ b-a \end{bmatrix} \}$.

Thus, we have

$$S = \begin{bmatrix} 1 & 1 \\ 0 & b-a \end{bmatrix} \qquad D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

and $S^{-1} = \begin{bmatrix} 1 & -\frac{1}{b-a} \\ 0 & \frac{1}{b-a} \end{bmatrix}$.

Exercise 5. Find a basis of the kernel and image of the matrix $A = \begin{bmatrix} 0 & -2 & 1 & 4 \\ 1 & 0 & 1 & 3 \\ -1 & -4 & 1 & 5 \end{bmatrix}$.

Proof. Row reducing we have

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1/2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since only the first two columns have leading ones, the first two columns form a basis of A, so $\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} \}$ is a basis for ImA. Using RREF to express the solutions to $Ax = 0$, we find that $\{ \begin{bmatrix} -1 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \}$ form a basis for $\ker A$. □

Exercise 6. Find the QR factorization of the following invertible matrix:

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}$$

Performing Gram Schmidt on the columns of A (in the order they appear), we have:

$$v_1^\perp = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$v_2^\perp = v_2 - \frac{v_2 \cdot v_1^\perp}{||v_1^\perp||^2} v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_3^\perp = v_3 - \frac{v_3 \cdot v_1^\perp}{||v_1^\perp||^2} - \frac{v_3 \cdot v_2^\perp}{||v_2^\perp||^2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Normalizing, we have

$$u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad u_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \qquad u_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Thus, we have

$$Q = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

We now want to find R such that $A = QR$. Since Q is orthogonal, $Q^T = Q^{-1}$ and Q is invertible. Therefore, $R = Q^{-1}A = Q^TA$. Therefore,

$$R = Q^TA = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$