Homework 7 :

Problem 1)

To show that Gauss - Jacobi, and Gauss - Seidel converge, we need to check if matrix is strictly diagonally dominant :

row 1 : $4 > 2$
2 $4 > 3$
3 $4 > 2$
4 $4 > 2$
5 $4 > 3$
6 $4 > 2$

o.k.

Homework 7, bonus problems

Corollary 7.20, part 2:

If $\|T\| < 1$ for any natural norm, then the following error bounds hold for sequence $\{x^{k}\}$, defined by $x^{(k)} = Tx^{(k-1)} + c$:

i) $\|x - x^{(k)}\| \le \|T\|^{k} \, \|x^{(0)} - x\|$

ii) $\|x - x^{(k)}\| \le \dfrac{\|T\|^{k}}{1 - \|T\|} \, \|x^{(1)} - x^{(0)}\|$

Proof:

$$\cancel{\|(Tx^{(k-1)} + c) - (Tx + c)\|}$$

i) $$\|x^{(k)} - x\| = \|Tx^{(k-1)} - Tx\|$$

$$= \|T(x^{(k-1)} - x)\|$$

$$\le \|T\| \, \|x^{(k-1)} - x\|$$

Corollary 7.10 ~~repeat argument~~

$$\cdots$$

$$\le \|T\|^{k} \, \|x^{(0)} - x\|$$

ii) $$\|x^{(k+1)} - x^{(k)}\| = \|Tx^{(k)} - Tx^{(k-1)}\|$$

$$= \|T(x^{(k)} - x^{(k-1)}\|$$

$$\le \|T\| \, \|x^{(k)} - x^{(k-1)}\|$$

$$\cdots$$

$$\le \|T\|^{k} \, \|x^{(1)} - x^{(0)}\|$$

Now, for $m > k > 1$:

$$\| x^{(m)} - x^{(k)} \| = \| x^{(m)} - x^{(m-1)} + x^{(m-1)} - x^{(m-2)} + x^{(m-2)} - \dots - x^{(k)} \|$$

$$\leq \| x^{(m)} - x^{(m-1)} \| + \| x^{(m-1)} - x^{(m-2)} \| + \dots + \| x^{(k+1)} - x^{(k)} \|$$

$$\leq \|T\|^{m-1} \| x^{(1)} - x^{(0)} \| + \|T\|^{m-2} \| x^{(1)} - x^{(0)} \| + \dots + \|T\|^{k} \| x^{(1)} - x^{(0)} \|$$

$$= \left(\|T\|^{m-1} + \|T\|^{m-2} + \dots + \|T\|^{k} \right) \cdot \| x^{(1)} - x^{(0)} \|$$

for $m \to \infty$: $x^{(m)} \to X$

$$\Rightarrow \| X - x^{(k)} \| \leq \|T\|^{k} \cdot \underbrace{\sum_{i=0}^{\infty} \|T\|^{i}}_{\substack{\text{for } \|T\| < 1: \\ \text{geometric series,} \\ \frac{1}{1 - \|T\|}}} \cdot \| x^{(1)} - x^{(0)} \|$$

$$\| X - x^{(k)} \| \leq \frac{\|T\|^{k}}{1 - \|T\|} \| x^{(1)} - x^{(0)} \|$$