\textbf{\textit{\texttt{sl}}}_2 \textit{ AND } \textit{SL}_2 \textit{ OVER AN ACF OF CHARACTERISTIC } p

COLIN NI

\textbf{Contents}
\begin{itemize}
\item The $p$-center and $p$-reductions \hfill 1
\item Representations of $\textit{\texttt{sl}}_2(\mathbb{F})$ \hfill 2
\item Representations of $G$ versus $g$ \hfill 4
\item Weight decomposition \hfill 5
\item Representations of $\textit{SL}_2(\mathbb{F})$ \hfill 6
\item Steinberg decomposition \hfill 7
\item References \hfill 8
\end{itemize}

\textbf{Notation.}
\begin{itemize}
\item $\mathbb{F}$ an algebraically closed field of characteristic $p$, e.g. $\overline{\mathbb{F}}_p$
\item rep = representation
\item irrep = irreducible representation
\end{itemize}

In this paper we describe the reps of $\textit{\texttt{sl}}_2(\mathbb{F})$ and $\textit{SL}_2(\mathbb{F})$. All of this is based on Ivan Losev’s Math 7313 course at Yale from fall 2015, especially lecture 4.

\textbf{The $p$-center and $p$-reductions}

For a Lie algebra $g \subset \textit{gl}_n(\mathbb{F})$, in addition to the characteristic zero center in $U(g)$, there is also a new large central subalgebra.

\textbf{Notation.}
\begin{itemize}
\item $g \subset \textit{gl}_n(\mathbb{F})$ a Lie algebra
\item $x^p \in U(g)$ the $p$th power of $x \in U(\textit{gl}_n(\mathbb{F}))$
\item $x^{[p]} \in U(g)$ the $p$th power of $x \in \textit{gl}_n = \textit{M}_n(\mathbb{R})$
\end{itemize}

\textbf{Definition.} The central embedding is
\[
\iota: \begin{array}{c}
g \longrightarrow Z(U(g)) \\
x \longmapsto x^p - x^{[p]},
\end{array}
\]
and the $p$-center is the subalgebra generated by $\iota(g)$.

This is well-defined by the following lemma, and it deserves its name by the following proposition.

\textbf{Lemma.} $x^{[p]} \in g$, and $x^p - x^{[p]} \in Z(U(g))$.  

Proof. For the first statement, see Jantzen page 148. To see this by hand for \( \mathfrak{sl}_n(\mathbb{F}) \), note that if \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( x \), then \( \lambda_1^p, \ldots, \lambda_n^p \) are the eigenvalues of \( x^p \), so \( x^p \in \mathfrak{sl}_n(\mathbb{F}) \) since

\[
\lambda_1^p + \cdots + \lambda_n^p = (\lambda_1 + \cdots + \lambda_n)^p = 0.
\]

For the second statement, observe that if \( A \) is an \( \mathbb{F} \)-algebra and \( x, y \in A \), then

\[
\text{ad}(x)^p y = \sum_{i=0}^{p} \binom{p}{i} (-1)^i x^i y x^{p-i} = x^p y - y x^p = [x^p, y]
\]

since the nontrivial binomial coefficients vanish in characteristic \( p \). Now \( x^p - x^p \) is central because

\[
[x^p, y] = \text{ad}(x)^p y = [x^p, y]
\]

for any \( y \in \mathfrak{g} \). In the first equality we apply the observation with \( A = U(\mathfrak{g}) \) and the second with \( A = M_n(\mathbb{F}) \), and the definition of \( U(\mathfrak{g}) \) ensures the adjoint operations agree. \( \square \)

**Proposition.** The central embedding is additive, semi-linear in the sense that \( \iota(\lambda x) = \lambda^p x \) for \( \lambda \in \mathbb{F} \), \( G \)-equivariant, and takes a basis of \( \mathfrak{g} \) to a set of algebraically independent elements.

**Proof.** It is straightforward to see that \( \iota \) is semilinear and \( G \)-equivariant, and the last statement is just PBW.

Here is a sketch of how to show that \( \iota \) is additive. Let

\[
\Delta: U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})
\]

be the coproduct which takes \( x \mapsto x \otimes 1 + 1 \otimes x \) for every \( x \in \mathfrak{g} \). This is well defined since \( [\Delta(a), \Delta(b)] = \Delta([a, b]) \) by rote computation. An element \( u \in U(\mathfrak{g}) \) is called primitive if \( \Delta(u) = u \otimes 1 + 1 \otimes u \). For example the monomial \( x^p \) of degree \( p \) is primitive because

\[
\Delta(x^p) = \sum_{i=0}^{p} \binom{p}{i} (x \otimes 1)^i (1 \otimes x)^{p-i} = x^p \otimes 1 + 1 \otimes x^p.
\]

However, a monomial in \( U(\mathfrak{g}) \) of degree \( < p \) if primitive if and only if it has degree 1, i.e. it is in \( \mathfrak{g} \). Now \( \iota \) is additive because

\[
\iota(x + y) = (x + y)^p - (x + y)^{[p]} = x^p - x^p + y^p - y^p = \iota(x) + \iota(y)
\]

where \( x + y \) has degree \( < p \) and is thus primitive. \( \square \)

Irreps of \( \mathfrak{g} \) factor through a so-called \( p \)-reduction of \( U(\mathfrak{g}) \). Namely, if \( U(\mathfrak{g}) \rightarrow \mathfrak{g}(V) \) is a finite-dimensional irrep of \( \mathfrak{g} \), then since elements of the \( p \)-center act as scalars, the irrep induces a unique so-called \( p \)-character \( \alpha: \mathfrak{g} \rightarrow \mathbb{F} \).

**Definition.** For a \( p \)-character \( \alpha: \mathfrak{g} \rightarrow \mathbb{F} \), the \( p \)-reduction of \( U(\mathfrak{g}) \) with respect to \( \alpha \) is

\[
U_{\alpha}(\mathfrak{g}) = \frac{U(\mathfrak{g})}{\langle \iota(x) - \alpha(x) \mid x \in \mathfrak{g} \rangle}.
\]

The following fact will not be used in this paper.
Proposition. The $p$-reduction $U_\alpha(g)$ has $\mathbb{F}$-dimension $p^{\dim g}$, with basis given by ordered monomials
\[ x_1^{m_1} \cdots x_n^{m_n} \quad \text{with} \quad 0 \leq m_i \leq p - 1, \]
where $x_1, \ldots, x_n$ is a basis of $g$.

Representations of $\mathfrak{sl}_2(\mathbb{F})$

Let $g = \mathfrak{sl}_2(\mathbb{F})$ and $G = \text{SL}_2(\mathbb{F})$. Recalling that $p > 2$ and
\[ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
the $p$-center of $U(g)$ is thus generated by
\[ e^p - e^{[p]} = e^p \]
\[ f^p - f^{[p]} = f^p \]
\[ h^p - h^{[p]} = h^p - h. \]
Note it thus does not contain the Casimir element $C = ef + fe + \frac{1}{2}h^2 \in U(\mathfrak{sl}_2(\mathbb{F}))$ which spans the center in characteristic zero, i.e. the $p$-center is a distinct feature of being in characteristic $p$.

Since $p > 2$, the trace form $\langle x, y \rangle = \text{tr}(xy)$ is non-degenerate and $G$-invariant, so $g \cong g^*$ via $x \mapsto (y \mapsto \text{tr}(xy))$. Up to $G$-conjugacy, i.e. Jordan canonical form, the $p$-reductions and the actions of $e, f, h$, i.e. the value of $\iota(x) - \alpha(x)$, are are then as follows:
\[
\begin{array}{ccc}
\alpha & e^p & f^p & h^p - h \\
(0 & 0 & 0 & 0) & 0 & 0 & 0 \\
(0 & 1 & 0 & 0) & 0 & 1 & 0 \\
(0 & 0 & -\frac{2}{2} & \frac{2}{2}) & 0 & 0 & \lambda \\
\end{array}
\]
As in the characteristic zero case, there is an $h$-eigenvector with eigenvalue $z$ killed by $e$. In the first two cases $z^p = z$, i.e. $z \in \mathbb{F}_p$, and in the third case $z^p - z = \lambda$.

Recall. In the characteristic zero case $g = \mathfrak{sl}_2(\mathbb{C})$, recall the classification of irreps via the Verma modules. Define the following:
- $b \subset g$ the subalgebra generated by $h, e$
- $\mathbb{C}_z$ the one-dimensional $b$-module with $e$ acting trivially and $h$ by $z \in \mathbb{C}$
- $v$ a basis of $\mathbb{C}_z$
- the Verma module $\Delta(z) = \text{Ind}_b^g \mathbb{C}_z = U(g) \otimes_{U(b)} \mathbb{C}_z$.

Tensoring over $U(b)$ means $hv = zv$ and $ev = 0$, so $\Delta(z)$ has basis $f^kv$ for $k \in \mathbb{Z}$. More generally, easy computations show that $e, f, h$, roughly speaking, raise, fix, and lower the basis elements respectively:
\[ f : f^kv = f^{k+1}v, \quad hf^kv = (z - 2k)f^kv, \quad ef^kv = (z - k + 1)kf^{k-1}v. \]
Due to the \((z-k+1)k\) term, \(\Delta(z)\) is reducible if and only if \(z \in \mathbb{Z}_{\geq 0}\). In this case, letting \(n = z\), the module \(\Delta(n)\) has a unique proper submodule spanned by \(f^{n+1}v, f^{n+2}v, \ldots\) which is isomorphic to \(\Delta(-2-n)\), so
\[
0 \to \Delta(-2-n) \to \Delta(n) \to L(n) \to 0
\]
for some \(L(n)\).

Now let \(V\) be a finite-dimensional irrep. Since
\[
\text{Hom}_{U(\mathfrak{g})}(\Delta(z), V) = \text{Hom}_{U(\mathfrak{b})}(\mathbb{C}z, V) = \{v \in V \mid hv = zv, ev = 0\}
\]
and \(V\) is irreducible, the latter set is nonzero, so there is a nonzero \(\Delta(z) \to V\), i.e. \(V\) is a proper quotient of \(\Delta(z)\). Thus \(z \in \mathbb{Z}_{\geq 0}\), and \(V = L(n)\).

In the characteristic \(p\) case we proceed in the same way. Define the following:
- \(\mathfrak{b} \subset \mathfrak{sl}_2(\mathbb{F})\) the subalgebra generated by \(h, e,\)
- \(\mathbb{F}z\) the one-dimensional \(U(\mathfrak{b})\)-module with \(e\) acting trivially and \(h\) by \(z\)
- \(v\) a basis of \(\mathbb{F}z\)
- the \textit{baby Verma module} \(\Delta_{\alpha}(z) = \text{Ind}_\mathfrak{b}^\mathfrak{g} \mathbb{F}z = U_{\alpha}(\mathfrak{g}) \otimes U_{\alpha}(\mathfrak{b}) \mathbb{F}z\).

Once again \(hv = zv\) and \(ev = 0\), but now \(\Delta_{\alpha}(z)\) has basis \(f^kv\) for \(k \in [0,p)\) since \(U_{\alpha}(\mathfrak{b})\) has basis \(h^i e^j\) with \(i, j \in [0,p)\). We still have
\[
ef^k v = k(z-k+1)f^{k-1}v.
\]

**Theorem.** The irreducible \(U_{\alpha}(\mathfrak{g})\)-modules are as follows.

- \(\text{If } \alpha = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right), \text{ then there are } p \text{ of them, one for each } z \text{ and of dimension } z + 1.\)
- \(\text{If } \alpha = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), \text{ then there are } (p+1)/2 \text{ of them, given by the } \Delta_{\alpha}(z), \text{ where } \Delta_{\alpha}(z) \cong \Delta_{\alpha}(z') \text{ if and only if } z + z' = -2.\)
- \(\text{If } \alpha = \left(\begin{array}{cc} \lambda & 0 \\ 0 & -\lambda \end{array}\right), \text{ then there are } p \text{ of them, given by the } \Delta_{\alpha}(z).\)

**Proof.** Recall that for the first two cases the number \(z\) is in \(\mathbb{F}_p\), whereas for the third case it is not. To see that the described modules are pairwise non-isomorphic, just note that they kill vectors of different weights.

The first case works exactly like in characteristic \(p\): the dimension \(z + 1\) irreducible \(U_{\alpha}(\mathfrak{g})\)-module \(L_{\alpha}(z)\) is a simple quotient of \(\Delta_{\alpha}(z)\). Moreover \(\Delta_{\alpha}(p-1)\) is already irreducible, and for \(z \neq p-1\) there is an exact sequence
\[
0 \to L_{\alpha}(-2-z) \to \Delta_{\alpha}(z) \to L_{\alpha}(z) \to 0.
\]

In the second case, since \(f^p = 1\), the basis vectors wrap around. Now if \(z = p-1\), then there is only one vector killed by \(e\) since \(z-k+1 > 0\), whereas otherwise there are two, one with weight \(z\) and another with weight \(-2-z\). In the latter case, by flipping, we get a homomorphism \(\Delta_{\alpha}(z) \to \Delta_{\alpha}(-2-z)\) which is automatically an isomorphism.

In the third case, \(z-k+1 \neq 0\) since \(z \notin \mathbb{F}_p\), so every \(\Delta_{\alpha}(z)\) is irreducible and has a vector with weight \(z\) killed by \(e\). \(\square\)
Representations of $G$ versus $\mathfrak{g}$

Notation.
- $G$ a linear algebraic group, i.e. a subgroup of $\text{GL}_n(\mathbb{F})$
- $\mathfrak{g}$ its Lie algebra
- $\Phi: G \rightarrow G'$ an algebraic group homomorphism
- $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ a Lie algebra homomorphism

In this section we discuss the correspondence between such $\Phi$ and $\varphi$: $\{\Phi: G \rightarrow G'\}$ \xrightarrow{\text{tangent map}} $\{\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'\}$

Trivially, the tangent map $\varphi$ of $\Phi$ is always a Lie algebra homomorphism. However, nontrivial $G$-modules can induce trivial $\mathfrak{g}$-modules, as the following example demonstrates.

Example. Take $G = \text{GL}_1(\mathbb{F}) = \mathbb{F}^\times$ and $\mathfrak{g} = \mathbb{F}$. Reps $\Phi: G \rightarrow \mathbb{F}^\times$ of $G$ are polynomials on $\mathbb{F}^\times$, which if $\mathbb{F}$ has infinite cardinality must be $x^n$ for some $n$. The corresponding rep $\varphi: \mathfrak{g} \rightarrow \mathbb{F}$ of $\mathfrak{g}$ is just multiplication by $n$, which is trivial.

What about the converse, i.e. given $\varphi$, when does there exist $\Phi$ whose tangent map is $\varphi'$? In characteristic zero, such a $\Phi$ exists when $G$ is connected and simply connected and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ (cf Proposition 1.4, Lecture 3), and moreover when such a $\Phi$ exists, it is unique. However, in characteristic $p$ it is no longer necessarily unique, as the following example demonstrates.

Example. Let $\text{Fr}: G \rightarrow G$ be the Frobenius automorphism, which is just the usual Frobenius acting element-wise. For a rational rep $\rho$ of $G$, the pullback $\text{Fr}^* \rho$ is also a rational rep of $G$. But the tangent map of $\text{Fr}$ is zero since $px^{p-1} = 0$, so the tangent map of $\text{Fr}^* \rho$ is as well. Thus there can be many reps of $G$ whose pullbacks are nonisomorphic yet all have zero tangent maps.

Weight Decomposition

In characteristic zero, the root system of a Lie algebra and the weights of its reps describe all of its rep theory.

Recall. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a field of characteristic zero, and let $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a rep. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal subalgebra that is abelian and has $\text{ad}(\mathfrak{h})$ consisting of semisimple operators, i.e. diagonalizable matrices; these are then simultaneously diagonalizable. Thus $\mathfrak{g}$ decomposes as $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_{\lambda}$ where $\mathfrak{g}_{\lambda} = \{x \in \mathfrak{g} \mid \text{ad}(h)x = \lambda(h)x \text{ for all } h \in \mathfrak{h}\}$, and $\Phi = \{\lambda \in \mathfrak{h}^* - 0 \mid \mathfrak{g}_{\lambda} \neq 0\}$ is called the root system of $\mathfrak{g}$. Similarly $\rho$ decomposes as $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda}$ where $V_{\lambda} = \{v \in V \mid \rho(h)(v) = \lambda(h)(v) \text{ for all } h \in \mathfrak{h}\}$, and the $\lambda$ for which $V_{\lambda} \neq 0$ are called the weights of $\rho$. By the theorem of the highest weight, any finite-dimensional irrep $\rho$ has a highest weight with respect to a partial ordering on $\mathfrak{h}^*$ which is so-called dominant integral, and given a root $\lambda \in \Phi$ which is dominant integral, there exists a finite-dimensional irrep with highest weight $\lambda$. 
The same thing works in characteristic $p$ for $SL_2(\mathbb{F})$, by the same arguments.

**Notation.**
- $T$ the subgroup of diagonal matrices in $SL_2(\mathbb{F})$
- $V$ a rational rep of $SL_2(\mathbb{F})$

**Lemma.** A rational rep of $\mathbb{F}^\times$ is a direct sum of one-dimensional rational reps.

Thus since reps of $\mathbb{F}^\times$ are just $x^n$, as we have seen, $V$ decomposes as

$$V = \bigoplus_{n \in \mathbb{Z}} V_n \quad \text{where} \quad V_n = \left\{ v \in V \mid \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} v = z^n v \right\}.$$ 

In the next section, we will show that there is a unique irrep $L(n)$ of $SL_2(\mathbb{F})$ with highest weight $n$ for each $n \geq 0$.

**Representations of $SL_2(\mathbb{F})$**

**Notation.**
- $G = SL_2(\mathbb{F})$

In characteristic zero we used the universality property

$$\text{Hom}_{U(\mathbb{F})}(\Delta(z), V) = \text{Hom}_{U(\mathbb{F})}(\mathbb{C}z, V) = \{ v \in V \mid hv = zv, ev = 0 \}$$

in order to show that any irrep $V$ is a quotient of $\Delta(z)$. In characteristic $p$ we do an analogous thing, but there are some obstacles to overcome, namely that our reps here are rational and that we do not have an explicit description of the $SL_2(\mathbb{F})$-action as we did for $sl_2(\mathbb{F})$.

**Definition.**
- $W^\vee(n) = S^n(\mathbb{F}^2)$ the dual Weyl module
- $W(n) = W^\vee(n)^*$ the Weyl module
- $x, y \in \mathbb{F}^2$ a basis so that $W^\vee(n)$ is spanned by $x^n, x^{n-1}y, \ldots, y^n$
- $B$ the group of upper triangular matrices
- $\mathbb{F}_n$ the one-dimensional rep of $B$ where $b = \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix}$ acts by $z^n$

**Theorem.** For each $n \geq 0$, there is a unique irrep $L(n)$ of $G$ with highest weight $n$ which is the unique irreducible quotient of $W(n)$.

**Lemma.**

$$\text{Hom}_G(W(n), V) = \text{Hom}_B(\mathbb{F}_n, V) = \{ v \in V \mid bv = z^n v \text{ for all } b \in B \}$$

for rational reps $V$ of $G$.

**Idea of proof.** The main idea is to view $W^\vee(n)$ as the global sections of $\mathcal{O}(n)$ on the homogeneous space $\mathbb{P}^1 = G/B$. Then $W^\vee(n) = \text{Ind}_{B}^{G}(\mathbb{F}_n)$, and $G \times B \mathbb{F}_{-n} = \mathcal{O}(n)$.

In more detail, note that by taking duals, it suffices to show that

$$\text{Hom}_G(V, W^\vee(n)) \cong \text{Hom}_B(V, \mathbb{F}_{-n}).$$
There are then the following identifications:

\[ \text{Hom}_G(V, W^\vee(n)) = \{ \varphi: G \to V^* \mid \varphi(gb) = z^n \varphi(g) \}^G \]
\[ = \{ \varphi: G \to V^* \mid \varphi(gb) = gz^n \varphi(1) \} \]
\[ = (V^* \otimes \mathbb{F}_{-n})^B \]
\[ = \text{Hom}_B(V, \mathbb{F}_{-n}). \]

The second and fourth identifications are just unraveling definitions. For the first identification, observe that

\[ \varphi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \varphi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \varphi \left( \begin{pmatrix} a & a + b \\ c & c + d \end{pmatrix} \right) \]

and \( \varphi(gb) = z^n \varphi(g) \) mean \( \varphi \) is a homogeneous polynomial of degree \( n \) in the entries \( a, c \), i.e. global sections of \( \mathcal{O}(n) \). Here the \( a, c \) entries correspond to the projective coordinates of \( \mathbb{P}^1 \). For the third identification, the map \( \varphi \mapsto \varphi(1) \) is an isomorphism. \( \square \)

**Proof of theorem.** Let \( V \) be a finite-dimensional irrep with highest weight \( n \), so \( V_n \neq 0 \). By the above lemma, it suffices to show there exists \( v \in V \) such that \( bv = z^n v \) for all \( b \in B \) since then, like the proof in characteristic zero, \( V \) is an irreducible quotient of \( W(n) \) and is thus \( L(n) \). We will show that any nonzero \( v \in V_n \) works.

First, note that the subgroup

\[ \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \]

fixes \( v \), by the following argument. The action is a polynomial, i.e.

\[ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v = f(x)v \]

for some polynomial \( f(x) \), but

\[ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & z^2x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \]

implies that \( f(x) = f(z^2x) \) for every \( z \in \mathbb{F} \). Thus \( f \) is a constant, and

\[ f(0)v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v = v \]

shows \( f = 1 \).

Now the computation

\[ \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} v = \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & z^{-1}y \\ 0 & 1 \end{pmatrix} v = \begin{pmatrix} z & y + x \\ 0 & z^{-1} \end{pmatrix} v, \]

shows that \( bv \) depends only on \( z \). Since \( T \subset B \) and since the action of \( T \) on \( v \) is just multiplication by \( z^n \), so too is the action of \( B \). \( \square \)
STEINBERG DECOMPOSITION

There is the following nice description of the structure of $L(n)$.

**Notation.**

- $(\text{Fr}^*)^n$ the $n$th iterated pullback along Frobenius

**Theorem.** Write $n = a_0 + a_1 p + \cdots + a_n p^n$ in base $p$. Then

$$L(n) \cong W(a_0) \otimes \text{Fr}^* W(a_1) \otimes \cdots \otimes (\text{Fr}^*)^{(n)} W(a_n).$$

**Proof.** For the base case $n < p$, note $S^n(V)^* = S^n(V)$ for finite-dimensional $V$ since $n < p$, so $W^\vee(n) = W(n)$. Moreover $L(n) \cong L(n)^*$ since both are irreps with highest weight $n$, so

$$W(n) = W^\vee(n) \hookrightarrow L(n)^* = L(n)$$

by taking a dual of the quotient $W(n) \twoheadrightarrow L(n)$. This shows $L(n) = W(n)$.

For the inductive step, it suffices to show that

$$L(q p + r) = L(r) \otimes \text{Fr}^* L(q) \quad \text{for } r < p \text{ and } q \geq 0.$$ 

Since both sides have highest weight $q p + r$, it suffices to show that the RHS is irreducible. Thus, let $U$ be a $G$-submodule of $L(r) \otimes \text{Fr}^* L(q)$.

In general, for $A$ an $\mathbb{F}$-algebra, $V$ a finite-dimensional irreducible $A$-module, and $M$ a finite-dimensional $\mathbb{F}$-vector space with $A$ acting trivially, we have

$$U = V \otimes \text{Hom}_A(V, U)$$

for every $A$-submodule $U \subset V \otimes M$. Indeed

$$\text{Hom}_A(V, U) \hookrightarrow \text{Hom}_A(V, V \otimes M) = M,$$

where the identification is by Schur’s lemma using that $V$ is irreducible and that $A$ only acts on the first factor of $A \otimes M$.

Thus since have seen that Fr$^* L(q)$ is trivial as a $g$-module, we have

$$U = L(r) \otimes \text{Hom}_g(L(r), U).$$

Now consider the following inclusion of $G$-modules:

$$\text{Hom}_g(L(r), U) \hookrightarrow \text{Hom}_g(L(r), L(r) \otimes \text{Fr}^* L(q)) \twoheadrightarrow \text{Fr}^* L(q)$$

Here the identification is as in the above paragraph, and we use that the Hom$_g$ of two $G$-modules is a $G$-module. But Fr$^* L(d)$ is irreducible because the pullback of an irreducible module along a group homomorphism is irreducible, so

$$\text{Hom}_g(L(r), U) = \text{Fr}^* L(q).$$

**References**

Ivan Losev. Yale Math 7313 (Fall 2015). [https://gauss.math.yale.edu/~il282/RT/RTMI.html](https://gauss.math.yale.edu/~il282/RT/RTMI.html)