LUBIN-TATE AND ALGEBRAIC GEOMETRY IN CHROMATIC HOMOTOPY THEORY

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Abstract. The algebro-geometric language of stacks invaded chromatic homotopy theory in the 1990's, largely due to Hopkins and his study of Lubin-Tate deformation spaces. This expository paper tells the basic stories behind each of these three things and describes the initial connections.

Contents

There are three intertwined stories to tell.

Chromatic homotopy theory begins with the universality of complex cobordism under Quillen's correspondence between complex-oriented cohomology theories and formal group laws [cf §1-3]. The Landweber exact functor theorem gives a partial converse to this correspondence, and it gives rise to the Morava E-theories via the universal Lubin-Tate rings [cf §8]. These theories $E(n)$ along with their counterparts the Morava K-theories $K(n)$ govern the behavior of the stable homotopy category, as illustrated by the thick subcategory, nilpotence, and chromatic convergence theorems [cf §9]. In particular, the picture they produce of the stable homotopy category leads to the core idea captured by tmf [cf §10].

The language of stacks enters chromatic homotopy theory via the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups [cf §5]. The geometry of $\mathcal{M}_{\mathrm{fg}}$ elucidates the nature of the Lubin-Tate rings from before by interpreting them as the universal deformations of the strata of $\mathcal{M}_{\mathrm{fg}}$ [cf §6]. Moreover, the Morava theories behave geometrically on $\mathcal{M}_{\mathrm{fg}}$: for example $K(n)$ stratifies $\mathcal{M}_{\mathrm{fg}}$ and the functor $L_{E(n)}$ behaves like restriction to $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ and $L_{K(n)}$ like completion along $\mathcal{M}_{\mathrm{fg}}^{n}$. These behaviors intuit the main results of chromatic homotopy theory [cf §9-10].

Lubin-Tate theory originates from an alternate approach to class field theory known as the Lubin-Tate approach, wherein certain formal $\mathcal{O}$-modules determine some torsion which generate the abelian extensions of any local field [cf §4]. This approach motivates the Lubin-Tate spaces, which are moduli spaces of deformations of formal $\mathcal{O}$-modules and are analogous to those in Lubin-Tate deformation theory [cf §6]. The cohomology of these Lubin-Tate spaces generalizes the Lubin-Tate approach to a nonabelian class field theory [cf §7].

References. The main sources for most of the content in this paper are Jacob Lurie's lecture notes [cf [1]] for his 292x course on chromatic homotopy theory taught at Harvard in 2010 and the underground lecture notes known as COCTALOS [cf [2]], short for *Complex-Oriented Cohomology Theories and the Language of Stacks*, which were compiled for the course taught by Mike Hopkins at MIT in 1999. As for the classical viewpoint, there is the so-called Orange book [cf [3]].

1. Cobordism and Thom's theorem

Thom's theorem reduces the geometric problem of classifying manifolds up to cobordism to the algebraic problem of computing stable homotopy groups of Thom spectra. Thom introduced all of these ideas in his celebrated 1954 paper *Quelques propriétés globales des variétés différentiables* for which he was awarded the Fields medal. To prove his results, in the same paper Thom invented the notion of transversality and what is known today as the Pontryagin-Thom construction, all in a time where spectra had not even been invented.

History. In fact this classification was not even the point of this 1954 paper. The original goal was to answer the following question posed by Steenrod: can any homology class of a finite polyhedron be represented as an image of the fundamental class of some manifold? Thom produced the following answer: yes for mod 2 homology, but in any dimension ≥ 7 there are integral homology classes not representable via compact differentiable manifolds.

Reference. An English translation of Thom's original paper is available at Novikov and Taimanov's *Topological Library; Part 1: Cobordisms and Their Applications* [cf [4]].

The simplest variant of cobordism is the unoriented case. Here two smooth compact manifolds are said to be *cobordant* if their disjoint union is the boundary of a manifold one dimension higher. For example, S^1 is cobordant to $S^1 \sqcup S^1$ via the pair of pants, and every manifold M is cobordant to itself because $\partial(M \times I) \cong M \sqcup M$. The set Ω_* of all smooth compact manifolds up to cobordism forms a ring graded by dimension where the addition is disjoint union and the multiplication is cartesian product. Thom's idea was to realize Ω_* as the stable homotopy groups of a spectrum MO.

The *Thom spectrum* MO is the spectrum whose qth space $(MO)_q = \mathrm{Th}(\gamma^q)$ is the Thom space of the qth universal vector bundle and whose structure maps are

$$\Sigma(MO)^q = \Sigma\mathrm{Th}(\gamma^q) \cong \mathrm{Th}(\mathbb{R} \oplus \gamma^q) \to \mathrm{Th}(\gamma^{q+1}) = (MO)_{q+1}.$$

Here we invoke functoriality of the Thom space construction and the fact that the pullback of γ^{q+1} along the classifying map $BO(q) \to BO(q+1)$ is $\mathbb{R} \oplus \gamma^q$. It turns out that the stable homotopy groups $\pi_*(MO)$ form a ring because MO is a ring spectrum.

The *Pontryagin-Thom construction* works as follows to give a map $\Omega_* \to \pi_*(MO)$. Let M be a smooth compact n-dimensional manifold, and embed it into $\mathbb{R}^{n+k}$ for sufficiently large k using the Whitney embedding theorem. Let T be a tubular neighborhood of M so that by the tubular neighborhood theorem it is diffeomorphic to the normal bundle ν. Then $\mathrm{Th}(\nu)$ is homeomorphic to $\overline{T}/\partial\overline{T}$. The Pontryagin-Thom collapse map $\mathbb{R}^{n+k} \to \mathrm{Th}(\nu)$ is defined to be the above identification on T and the constant map at the basepoint on the complement of T, and one easily checks that this is continuous. In summary, this yields a map

$$S^{n+k} \cong \mathbb{R}^{n+k} \cup \{\infty\} \to \overline{T}/\partial\overline{T} \cong \mathrm{Th}(\nu) \to \mathrm{Th}(\gamma^k) = (MO)_k$$

by Thomifying the classifying map of ν, and this represents an element of $\pi_n(MO)$.

Thom's theorem in this unoriented case asserts that the Pontryagin-Thom construction is a ring isomorphism $\Omega_* \to \pi_*(MO)$. In fact it is straightforward to construct an inverse. An arbitrary element of $\pi_n(MO)$ is a continuous map $S^{n+k} \to (MO)_k = \mathrm{Th}(\gamma^k)$, and this factors by compactness through the canonical bundle on $\mathrm{Gr}_k(\mathbb{R}^q)$ for large q. By the Whitney approximation theorem we may homotope this map to be transverse to the zero section and also to avoid ∞ so that now the preimage of the zero section gives a smooth compact n-dimensional manifold.

All of this generalizes to manifolds equipped with more structure, for instance with an orientation, spin, or almost-complex structure. To make this notion of extra structure precise, there is the below definition for arbitrary vector bundles. Specializing this definition to the normal bundles of manifolds yields the desired notions since as long as a manifold is embedded in a large enough Euclidean space, the structures on any two embeddings are in bijection.

Definition. A (B_n, f_n)-*structure* on an n-dimensional vector bundle over a base space X is a fibration f_n along with a lift over the classifying map as shown:

$$\begin{tikzcd} & B_n \arrow[d, "f_n"] \\ X \arrow[ur] \arrow[r] & BO(n). \end{tikzcd}$$

A (B, f)-*structure* is a sequence of compatible (B_n, f_n)-structures.

In particular, for a structure $\mathscr{B}$ there is the cobordism ring $\Omega_*^{\mathscr{B}}$ which demands that the cobordism respect $\mathscr{B}$. Along the same lines, there is the Thom spectrum $M\mathscr{B}$ defined by Thomifying the universal $\mathscr{B}$-vector bundles $f_n^*(\gamma^n)$, and it is a ring spectrum whenever $\mathscr{B}$ is multiplicative.

Example 1.1. The case $B_n = EO(n)$ and f_n the universal projection is called the framing structure, and the Thom spectrum is given by

$$(M\mathrm{Framed})_n = \mathrm{Th}(f_n^*(\gamma^n)) = \mathrm{Th}(\gamma^n) = S^n.$$

In other words $M\mathrm{Framed} = S$.

Let us tabulate some $\mathscr{B}$-structures, their geometric interpretations, and the notations for their Thom spectra.

Table 1.2.

B_n	f_n	structure	$M\mathscr{B}$
$BO(n)$	id	none	MO
$BSO(n) = EO(n)/SO(n)$	projection	orientation	MSO
$B\mathrm{Spin}(n) = EO(n)/\mathrm{Spin}(n)$	projection	spin	$M\mathrm{Spin}$
$B\mathrm{String}(n) = EO(n)/\mathrm{String}(n)$	projection	string	$M\mathrm{String}$
$EO(2n)/U(n)$	projection	almost complex	MU
$EO(n)$	projection	framing of ν	$M\mathrm{Framed}$

In fuller generality, Thom's theorem is then as follows.

Theorem (Thom)**.** *For a multiplicative structure $\mathscr{B}$, the Pontryagin-Thom construction gives a ring isomorphism $\Omega_*^{\mathscr{B}} \to \pi_*(M\mathscr{B})$.*

The utility of Thom's theorem is enormous because it leads to the classification of manifolds up to various types of cobordism. For a fantastic example, Example 1.1 gives that the manifolds up to framed cobordism are given by the stable homotopy groups of spheres. Certainly more tractable, one can use the Adams spectral sequence to compute the manifolds up to unoriented cobordism and complex cobordism.

Computation 1.3.
- $\pi_*(MO) = \mathbb{Z}/2[x_i \mid i \geq 1 \text{ and } i \neq 2^j - 1]$ where x_i has degree i
- $\pi_*(MU) = \mathbb{Z}[y_i \mid i \geq 1]$ where y_i has degree $2i$.

Spelling it out for the unoriented case, this computation of $\pi_*(MO)$ says that there are i-dimensional manifolds x_i for i not of the form $2^j - 1$ such that every manifold up to unoriented cobordism is a disjoint union of cartesian products of the x_i. In fact, explicit representatives are known. For even i, the real projective space $\mathbb{R}\mathrm{P}^i$ is a valid representative for x_i because it is not nullbordant. For odd i not of the form $2^j - 1$, write $i = 2^k(2\ell + 1) - 1 = 2^{k+1}\ell + 2k - 1$ for some positive k and ℓ. Now for $m < n$, let $H_{m,n}$ be the $(m + n - 1)$-dimensional hypersurface in $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^n$ defined by $x_0y_0 + \cdots + x_my_m = 0$. These are the *Dold manifolds,* and it turns out that $H_{2^{k+1}\ell, 2^k}$ is a representative for x_i.

Reference. *Concise* exposits the unoriented case [cf Ch 25 of [5]]. Kochman's *Bordism, Stable Homotopy, and Adams Spectral Sequences* proves the generalized version of Thom's theorem and computes $\pi_*(MO)$ and $\pi_*(MU)$ [cf Ch 1-3 of [6], specifically Thms 1.5.10, 3.7.6, 3.7.7]. The generalized version was initially due to Richard Lashof in his paper *Poincaré Duality and Cobordism* [cf [7]].

2. Formal group (laws), the Lazard ring, and height

To understand Quillen's correspondence between complex-oriented cohomology theories and formal group laws, we first need to know what formal group laws are.

Definition 2.1. A *formal group law* over a commutative ring A is a power series $F \in A[\![X, Y]\!]$ satisfying
- $F(X, Y) = F(Y, X) = X + Y +$ higher order terms
- $F(X, F(Y, Z)) = F(F(X, Y), Z) \in A[\![X, Y, Z]\!]$.

A *morphism $f\colon F \to G$ of formal group laws* is a power series $f \in TA[\![T]\!]$ such that $f(F(X, Y)) = G(f(X), f(Y))$. The category of formal group laws over a ring A is denoted $\mathrm{fgl}(A)$.

There are three main examples of formal group laws. The first two are the simplest one could imagine.

Example 2.2. There is the *additive formal group law* given by $\mathbb{G}_a(X, Y) = X + Y$ and the *multiplicative formal group law* given by $\mathbb{G}_m(X, Y) = X + Y + XY$. It is straightforward to check that these satisfy the requirements of Definition 2.1.

If we are working over a ring which is a $\mathbb{Q}$-algebra, then we can define $\ln(T + 1)$ and $\exp(T) - 1$ via their usual power series so that

$$\mathbb{G}_m \underset{\ln(T)+1}{\overset{\exp(T)-1}{\rightleftarrows}} \mathbb{G}_a$$

are inverse isomorphisms. In fact in this case every formal group law is isomorphic to the additive law $\mathbb{G}_a$, but this is certainly not true if the ring has positive characteristic for instance.

The third main example is the *universal formal group law* F over the Lazard ring L, which determines a natural bijection $\mathrm{Hom}(L, A) \to \mathrm{fgl}(A)$ for any ring A. One implication of the existence of $F \in \mathrm{fgl}(L)$ is then that the functor $\mathrm{fgl}\colon \mathrm{Ring} \to \mathrm{Set}$ given by substitution of coefficients is corepresentable by L. The construction of these things is amusing.

Construction 2.3. Observe that any formal group law $F \in \mathrm{fgl}(A)$ may be written in the form $F(X, Y) = \sum_{i,j} c_{i,j} X^i Y^j$ for $c_{i,j} \in A$. The trick is to view these coefficients $c_{i,j}$ as indeterminates. It then turns out that the defining properties of a formal group law translate to polynomial relations in the ring $\mathbb{Z}[c_{i,j}]$, for example the first property in Definition 2.1 translates to $c_{i,j} = c_{j,i}$ and $c_{0,0} = 0$ and $c_{1,0} = c_{0,1} = 1$. The *Lazard ring* L is the polynomial ring $\mathbb{Z}[c_{i,j}]$ modulo these relations, and by construction the power series F is a formal group law over L and is universal.

Though the Lazard ring L seems like an intractable object, it in fact has a simple structure as a graded ring. Topologists endow $\mathbb{Z}[c_{i,j}]$ with the grading where $c_{i,j}$ has degree $2(i + j - 1)$, with the point that this grading descends to L and makes it a nonnegatively graded ring.

Theorem 2.4 (Lazard's theorem)**.** *The Lazard ring L is isomorphic to the graded polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ where x_i has degree $2i$.*

Algebraic geometry makes its first appearance here, with the following innocuous observation. In analogy with the functor fgl, let isofgl assign to a ring the isomorphisms of formal group laws over A, which are tracked by the coordinate transformations $W \cong L[b_0^{\pm 1}, b_1, b_2, \dots]$.

Observation 2.5. Via their functors of points $\mathrm{fgl} \cong \mathrm{Spec}(L)$ and $\mathrm{isofgl} \cong \mathrm{Spec}(W)$.

Formal groups provide a coordinate-free framework in which to think about formal group laws. They are just functors R-Alg $\to$ Ab. Writing $X +_F Y = F(X, Y)$ for a formal group law $F \in \mathrm{fgl}(A)$ endows $TA[\![T]\!]$ with an abelian group structure, and in general there is an *associated formal group* $\mathcal{G}_F\colon R\text{-Alg} \to \mathrm{Ab}$ by endowing the set $\mathcal{G}_F(A)$ of nilpotent elements in A with the group operation $+_F$. Note $F(a, b)$ makes sense when a and b are nilpotent because then only finitely many terms are nonzero. A *coordinatizable formal group* is one that arises this way for some F.

Notions such as height for formal group laws extend to formal groups via coordinatizability. For example, we say that a formal group $F\colon R\text{-Alg} \to \mathrm{Ab}$ has height

$\geq n$ if $F|_{A\text{-Alg}}$ has height $\geq n$ whenever A is an R-algebra such that $F|_{A\text{-Alg}}$ is coordinatizable. Having gotten ahead of ourselves, let us explain what the height of a formal group law is.

Height is an invariant that classifies formal group laws over fields of positive characteristic p. Of particular importance in the definition of height are the coefficients $v_0, \ldots, v_n$ which live in the field and which will repeatedly come up in future sections. To formulate the definition, let F be a formal group law over a ring of characteristic p, then recursively define $F^{(n)}(T) = F(F^{(n-1)}(T), T)$ with the base case $F^{(0)}(T) = 0$.

Definition 2.6. Let v_i denote the coefficient of T^{p^i}. One can show that if $F^{(p)}(T)$ is nonzero, then its first nonzero term is in degree p^n for some n. We say F has *height at least* n if the first nonzero term is $v_n T^{p^n}$, and if in addition v_n is invertible, then we say f has *height exactly* n. If $F^{(p)}(T)$ vanishes, then F has *infinite height.*

For example, consider the additive and multiplicative laws from Example 2.2. When the coefficient ring is a $\mathbb{Q}$-algebra, a logarithm like the one in the same example exhibits an isomorphism from any formal group law to $\mathbb{G}_a$. On the other hand, over a ring with characteristic $p > 0$ it turns out that $\mathbb{G}_m \ncong \mathbb{G}_a$. Indeed the multiplicative law $\mathbb{G}_m(X, Y) = X + Y + XY$ has nth iterate $\mathbb{G}_m^{(n)}(T) = (1+T)^n - 1$, so it follows that it has height exactly one. In contrast, the additive law $\mathbb{G}_a(X, Y) = X + Y$ has nth iterate $\mathbb{G}_a^{(n)}(T) = nT$, so it has infinite height. It turns out that in general there is exactly one formal group law for every height. The following theorem summarizes this discussion.

Theorem 2.7. *Over a field of characteristic zero, there is exactly one formal group law up to isomorphism. Over a separably closed field of characteristic* $p > 0$, *for each height there is exactly one formal group law up to isomorphism.*

3. Quillen's theorem on MU and Landweber exactness

Daniel Quillen in his landmark 1969 paper [cf [8]] outlined how a complex-oriented cohomology theory determines a formal group law and then proved that MU determines the universal group law over L.

History and Reference. This paper is strange. Quillen went on to give a convenient construction of the Brown-Petersen spectrum BP which Brown and Petersen had constructed in 1966 to be direct summands of a splitting of the localization $MU_{(p)}$, and in doing so Quillen proved that BP is in fact a ring spectrum. Furthermore, Quillen fully described the cohomology operations BP_*BP. All told, this paper started the connection between homotopy theory, algebraic geometry, and number theory.

The issue was that the paper was only six pages long and included only one proof! It was published as a Bulletin announcement. It was not until five years later that Frank Adams had figured out and published the details in his book *Stable Homotopy and Generalized Homology* [cf [9]], the second part of which has since become the canonical reference to the paper.

Let us describe this connection between complex-oriented cohomology theories and formal group laws.

Definition. A multiplicative cohomology theory E is *complex-orientable* if it comes with a natural and multiplicative choice of Thom class for every complex vector bundle. That is, $\Phi_{f^*\xi} = f^*\Phi_\xi$ and $\Phi_{\xi\oplus\zeta} = \Phi_\xi\Phi_\zeta$.

Example 3.1. Being complex-orientable is equivalent by the splitting principle to asking for the map $E^2(\mathbb{C}\mathrm{P}^\infty) \to E^2(S^2)$ induced by $S^2 \cong \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{C}\mathrm{P}^\infty$ to be surjective. Hence:

- Singular cohomology is complex-orientable since $H^2(\mathbb{C}\mathrm{P}^\infty) \to H^2(S^2)$ is an isomorphism
- Complex K-theory is complex-orientable because $\mathcal{O}(-1)-1$ restricts under the map $\widetilde{KU}^2(\mathbb{C}\mathrm{P}^\infty) \to \widetilde{KU}^2(S^2)$ to the Bott element $h-1$ which generates $\widetilde{KU}^2(S^2)$
- Real K-theory is not complex-orientable since $\widetilde{KO}^2(\mathbb{C}\mathrm{P}^\infty) \to \widetilde{KO}^2(S^2)$ is multiplication by two on the integers
- Complex cobordism is complex-orientable with an orientation in $\widetilde{MU}^2(\mathbb{C}\mathrm{P}^\infty)$ represented by the identity on $\mathbb{C}\mathrm{P}^\infty \cong (MU)_1$.

The main reason for considering complex-orientable theories E is that they beget the key computation $E^*(\mathbb{C}\mathrm{P}^\infty) \cong (\pi_*E)[\![T]\!]$ [cf Prop 4.3.2 in [6]], which is analogous to the result for singular cohomology. In short, this is because the Atiyah-Hirzebruch spectral sequence degenerates. Explicitly, for $\mathbb{C}\mathrm{P}^n$ the AHSS reads

$$E_2^{p,q} = H^p(\mathbb{C}\mathrm{P}^n; E^q(*)) \Rightarrow E^{p+q}(\mathbb{C}\mathrm{P}^n),$$

and by universal coefficients the E_2 page is isomorphic to $(\pi_*E)[T]/(T^{n+1})$. Since E is complex-orientable, the element $T \in E_2^{2,0}$ arises via restriction from $E^2(\mathbb{C}\mathrm{P}^\infty)$ and hence from $E^2(\mathbb{C}\mathrm{P}^n)$, so it follows that the AHSS collapses at page two. The extension problem is trivial, hence

$$E^*(\mathbb{C}\mathrm{P}^n) \cong (\pi_*E)[T]/(T^{n+1}).$$

Finally, the maps $E^*(\mathbb{C}\mathrm{P}^{n+1}) \to E^*(\mathbb{C}\mathrm{P}^n)$ are surjective and compatible and so satisfy the Mittag-Leffler condition, thus the $\lim^1$ term in the Milnor sequence vanishes and

$$E^*(\mathbb{C}\mathrm{P}^\infty) = E^*\left(\varinjlim \mathbb{C}\mathrm{P}^n\right) = \varprojlim (\pi_*E)[T]/(T^{n+1}) = (\pi_*E)[\![T]\!].$$

With this computation in hand, the construction of a formal group law from a complex-orientable theory E is straightforward, assuming one believes that similarly $E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) = (\pi_*E)[\![X,Y]\!]$. Note that since $\mathbb{C}\mathrm{P}^\infty$ is a $K(Z,2)$, it is a topological abelian group [cf §16.5 in [5]], so in particular it has a multiplication $\mathrm{mult}\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$.

Correspondence. The formal group law associated to a complex-orientable cohomology theory E is the image of T under the map

$$(\pi_*E)[\![T]\!] \cong E^*(\mathbb{C}\mathrm{P}^\infty) \xrightarrow{\mathrm{mult}^*} E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) = (\pi_*E)[\![X,Y]\!]$$

induced by the multiplication on $\mathbb{C}\mathrm{P}^\infty$. It is straightforward to show that this indeed defines a formal group law [cf Prop 4.4.3 in [6]].

One way to interpret this formal group law is in terms of Chern classes. Recall that for singular cohomology $c_1(\xi \otimes \zeta) = c_1(\xi) + c_1(\zeta)$. However, for generalized cohomology theories this is not in general true, and the associated formal group law measures the discrepancy in the universal case in the sense that the indeterminates in $E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) = (\pi_* E)[\![X, Y]\!]$ are pullbacks of $\mathcal{O}(1)$ along the two projections $\pi_1, \pi_2 \colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$. For example, for singular cohomology the associated formal group law is just the additive law $X+Y$, and for complex K-theory it is the multiplicative law $X + Y + XY$ because the multiplication on $\mathbb{C}\mathrm{P}^\infty$ classifies the tensor product of line bundles and so [cf Ex 3.1]

$$\begin{aligned} \mathrm{mult}^*(\mathcal{O}(-1) - 1) &= \pi_1^*\mathcal{O}(-1) \otimes \pi_2^*\mathcal{O}(-1) - 1 \\ &= \pi_1^*(\mathcal{O}(-1) - 1) \otimes \pi_2^*(\mathcal{O}(-1) - 1) \\ &\qquad + \pi_1^*(\mathcal{O}(-1) - 1) + \pi_2^*(\mathcal{O}(-1) - 1). \end{aligned}$$

Quillen's theorem answers the natural question of whether there is a theory whose associated law is the universal formal group law over the Lazard ring L [cf Constr 2.3]. The obvious candidate is complex cobordism because its formal group law lives over the ring $\pi_* MU$ which is isomorphic to L [cf Comp 1.3 and Thm 2.4]. In particular, this formal group law is classified by a ring map $L \to \pi_* MU$, and Quillen's theorem gives everything one could wish for, namely that this map $L \to \pi_* MU$ is an isomorphism. We can formulate this as follows.

Theorem (Quillen's theorem on MU)**.** *The formal group law associated to complex cobordism MU is the universal group law over the Lazard ring $L \cong \pi_* MU$.*

There is an indicator pointing to the truth of this theorem. For a structure $\mathscr{B} = (B, f)$ a choice of compatible E-Thom classes for the universal $\mathscr{B}$-vector bundles $f_n^*(\gamma^n)$ is the same data as a map of ring spectra $M\mathscr{B} \to E$. Since we are interested in complex-orientable theories, we specialize to the statement that complex-orientations on a ring spectrum E are in bijection with maps $MU \to E$ [cf Lemma 4.1.13 in [10]]. In this way MU arises as the universal complex-orientable theory, so one might expect the law associated to MU to also be universal.

The Landweber exact functor theorem is a partial converse to Quillen's correspondence. Note that the coefficient ring $\pi_*(E)$ of any complex-oriented cohomology theory E is an algebra over the Lazard ring L via the classifying map $L \to \pi_*(E)$ of the formal group law associated to E. Thus one wonders given a formal group law $F \colon L \to R$ whether there is a cohomology theory E with coefficient ring $\pi_*(E) = R$ that determines F.

One attempt to construct such an E in the setting of homology is to define

$$E_*(-) = MU_*(-) \otimes_L R,$$

where R is endowed with an L-module structure via F. Certainly this functor satisfies $\pi_*(E) = R$, produces F, and is defined on the homotopy category of CW complexes, in particular because tensor products commute with direct limits. However, tensoring is not in general exact. Even worse, in this case $- \otimes_L R$ is almost never exact intuitively because L is so large.

The key observation is that $MU_*(X)$ contains the extra structure of a (L, W)-comodule, where we recall $W \cong L[b_0^{\pm 1}, b_1, b_2, \dots]$ tracks the isomorphisms of formal group laws [cf Obs 2.5], which suggests paying attention to when $- \otimes_L R$ is exact on the category of (L, W)-comodules. Such R are called *Landweber exact*. This is a weaker condition because it suffices for $\mathrm{Tor}_L(R, -)$ to vanish on just (L, W)-comodules rather than on all L-modules.

The Landweber exact functor theorem provides an algebraic condition for Landweber exactness, which is easy to check in practice. To formulate it, set $I_n =$

$v_0R + \cdots + v_{n-1}R$ for $i, n \geq 0$, where $v_i \in L$ is defined as in the definition of height [cf Def 2.6] for the universal law over L.

Theorem 3.2. *For a formal group law $L \to R$, the assignment*

$$E_*(X) = MU_*(X) \otimes_L R$$

defines a homology theory if and only if $v_0 = p, v_1, v_2, \ldots \in L$ is a regular sequence for R for all primes p.

4. The Lubin-Tate approach to class field theory

In classical number theory, the Kronecker-Weber theorem asserts that every finite abelian extension of $\mathbb{Q}$ is contained in some cyclotomic field. It follows that $\mathbb{Q}^{\mathrm{ab}}$ can be constructed by adjoining all nth roots of unity to $\mathbb{Q}$. One may think of this as adjoining all n-torsion points of $\mathbb{C}^\times$, or in other words the torsion of $\mathbb{G}_m(\mathbb{C})$ where $\mathbb{G}_m$ is the multiplicative group scheme.

Similarly in the theory of elliptic curves, there is the following construction of K^{ab} for a quadratic imaginary field K. Initially, if E is an elliptic curve with complex multiplication by $\mathcal{O}_K$, then the extension $K(j(E), E_{\mathrm{tors}})$ obtained by adjoining its j-invariant and torsion points is abelian over $K(j(E))$. The hindrance to being over K is that there is no guarantee that the torsion points of E are even algebraic since E is only defined over $\mathbb{C}$. To remedy this, one defines the Weber function $h\colon E \to \mathbb{P}^1$ associated to E which has the desired property that $h(P)$ is algebraic for any torsion point $P \in E$. Then $K(j(E), h(E_{\mathrm{tors}}))$ realizes the maximal abelian extension K^{ab} [cf Cor 5.7 in [11]].

The Lubin-Tate approach to class field theory produces an analogous and streamlined construction of the maximal abelian extension K^{ab}. The following table summarizes the analogy.

	Kronecker-Weber	Complex multiplication	Lubin-Tate
Type of field K	$\mathbb{Q}$	quadratic imaginary	local
K^{ab}	$K(\mathbb{G}_m(\mathbb{C})_{\mathrm{tors}})$	$K(j(E), h(E_{\mathrm{tors}}))$	$K(\mathfrak{m}_{\mathrm{tors}})K^{\mathrm{ur}}$

History. Seeking an extension of his theorem, Kronecker once described the complex multiplication case as his *liebster Jugendtraum* or "dearest dream of his youth." As a result, Hilbert's twelfth problem, which asks for the above table to be extended to any number field K, is often known as Kronecker's Jugendtraum.

Let us now describe the Lubin-Tate approach. Fix the following objects: K a local field, $\mathcal{O}$ its ring of integers, $\pi \in K$ a uniformizer, and K^{s} a separable algebraic closure of K. The idea is to consider the family

$$\mathscr{F}_\pi = \{f \in \mathcal{O}[\![T]\!] \mid f(T) = \pi T + (\text{higher order terms}) \text{ and } f(T) \equiv T^q \pmod{\pi}\}$$

of laws which are analogous to Eisenstein polynomials with respect to π, for instance $\pi X + X^q \in \mathscr{F}_\pi$. Such a law f admits the following two constructions.

Construction 4.1. By successively approximating them with polynomials to satisfy their defining properties, one obtains:

- the *Lubin-Tate formal group law* F_f associated to f, uniquely determined to admit $f \in \mathscr{F}_\pi$ as an endomorphism
- for any $a \in \mathcal{O}$, *the canonical endomorphism* $[a]_f \in \mathrm{End}(F_f)$ associated to a, characterized by $[a]_f \equiv aX + (\text{higher order terms})$. For example $[\pi]_f = f$.

Together, these endow the maximal ideal $\Lambda_f = \mathfrak{m}^{\mathrm{s}}$ in the maximal separable extension K^{s} with an $\mathcal{O}$-module structure via $+_{F_f}$ and $a\alpha = [a]_f(\alpha)$.

Construction 4.2. The π^n torsion adjoined to K generates an abelian extension $K_{\pi,n}$ with Galois group $\mathcal{O}/(\pi^n)^\times$, and adjoining all such torsion $K_\pi = \bigcup_n K_{\pi,n}$ and taking the composite with K^{ur} yields the maximal abelian extension K^{ab}. Here K^{ur} is the maximal unramified extension of K, which is just the separable closure since K is local. To construct the Artin map, first note that $K_\pi \cap K^{\mathrm{ur}} = K$ since K_π is totally ramified, then for any $f \in \mathscr{F}_\pi$ define

$$\begin{aligned} K^\times \cong \mathcal{O} \times \pi^{\mathbb{Z}} &\xrightarrow{\phi_\pi} \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(K_\pi K^{\mathrm{ur}}/K) \\ u\pi^m &\longmapsto \begin{cases} [u^{-1}]_f \text{ on } K_\pi \\ \mathrm{Frob}^m \text{ on } K^{\mathrm{ur}}. \end{cases} \end{aligned}$$

The hard work for all this goes into the first sentence of the following summary.

Theorem. *Everything here is independent of the choice of uniformizer π. The constructions $K^{\mathrm{ab}} = K_\pi K^{\mathrm{ur}}$ and $\phi = \phi_\pi$ realize respectively the maximal abelian extension of K and the Artin map from local class field theory.*

As for the choice of $f \in \mathscr{F}_\pi$, note the map $a \mapsto [a]_f$ is an injection $\mathcal{O} \hookrightarrow \mathrm{End}(F_f)$ because the linear term aX recovers a. This suggests the key notion of a formal $\mathcal{O}$-module, reported below. The point is that as formal $\mathcal{O}$-modules over $\mathcal{O}$, the Lubin-Tate law F_f is independent of f up to isomorphism, so everything is independent of the choice of $f \in \mathscr{F}_\pi$ as well.

Definition 4.3. A *formal $\mathcal{O}$-module over an $\mathcal{O}$-algebra A* is a formal group law $F \in \mathrm{fgl}(A)$ together with a ring homomorphism $\mathcal{O} \hookrightarrow \mathrm{End}_A(F)$ satisfying the above defining property of a canonical endomorphism for a Lubin-Tate formal group law.

Let us work out the cyclotomic case.

Example. Consider the case $K = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$, and $\pi = p$, where by the theorem it suffices to fix $f(T) = (1+T)^p - 1$ in the family $\mathscr{F}_p$. We make the following computations. The Lubin-Tate law $F_f(X,Y) = X + Y + XY$ is just the multiplicative law because

$$f(F_f(X,Y)) = (1+X)^p(1+Y)^p - 1 = F_f(f(X), f(Y)).$$

The canonical endomorphism is given by $[a]_f(T) = (1+T)^a - 1$ because this satisfies its defining properties, where for $a \in \mathbb{Z}_p$ it is valid to define $(1+X)^a = \sum_{k=0}^\infty \binom{a}{n} X^k$. The p^n-torsion is simply $\{\zeta - 1 \mid \zeta^{p^n} = 1\}$ because $[p^n]_f(T) = (1+T)^{p^n} - 1$. Observe that $K_{p,n}$ is just the p^n-cyclotomic field, which indeed has Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times = U/U_n = \mathbb{Z}_p^\times/(1+p^n\mathbb{Z}_p)$. Moreover, just as in Kronecker-Weber, taking the composite $\bigcup_n K_{p,n}$ yields the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$. The Lubin-Tate approach gives the Artin map explicitly for an element $a = p^t u \in \mathbb{Q}_p^\times$ on the maximal unramified extension as Frob^t and on the totally ramified extension as $\zeta_{p^n} \mapsto \zeta_{p^n}^{u^{-1}}$.

5. The moduli stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups

The language of stacks pervades chromatic homotopy theory. Stacks originated in the work of Grothendieck in 1959 who made the following observation, which nLab calls a widespread slogan.

Slogan. A type of object that has nontrivial automorphisms cannot have a fine moduli space.

One example of this obstruction arises in attempting to construct $\mathbb{C}$ to be the fine moduli space of elliptic curves via the j-invariant, which we recall classifies elliptic curves up to isomorphism. Elliptic curves have nontrivial automorphisms, for example the involution.

To be explicit, consider the family of elliptic curves $y^2 = x^3 - t$ parametrized by t as it runs over the affine line. Every curve in this family has j-invariant zero and hence is isomorphic to a fixed elliptic curve E, but this family is not the trivial family. Indeed, this family is rational because it is affine:

$$k[x, y, t]/(x^3 - y^2 - t) \cong k[x, y].$$

However, the trivial family $E \times \mathbb{C}^\times$ is not rational because the composition $\mathbb{P}^2 \to \mathbb{C}^\times \times E \to E$ of a rational map with the projection would make E unirational, hence rational by Lüroth's theorem and in particular genus zero, contrary to elliptic curves having genus one.

The solution to this general problem is to use moduli stacks. In the above example, consider the category $\mathscr{M}$ of families $X \to B$ of elliptic curves where the morphisms are pullback squares that are isomorphisms. It is fibered $\mathscr{M} \to \mathrm{Sch}$ over schemes in groupoids, where the fiber $\mathscr{M}(B)$ is just the groupoid of families over B. Such families glue over surjective families of open immersions, so in the Zariski topology on Sch descent datum is effective. In this language a family $X \to B$ is just a functor $\mathrm{Sch}/B \to \mathscr{M}$ from the slice category. The category $\mathscr{E}$ of families equipped with non-vanishing sections together with the forgetful functor $\mathscr{E} \to \mathscr{S}$ then form a universal family of elliptic curves in the sense that any family $X \to B$ is just the pullback of $\mathscr{E} \to \mathscr{S}$ along the corresponding functor $\mathrm{Sch}/B \to \mathscr{M}$.

Reference. Gabriele Edidin spells this example out in his friendly column entry *What is a Stack?* [cf [12]] in the Notices of the AMS.

In general, there is the following notion of a stack. Note that a scheme X is certainly a stack via its functor of points $\mathrm{Hom}(-, X)\colon \mathrm{Sch} \to \mathrm{Set}$.

Definition. A *prestack* over a site $\mathscr{C}$ is a fibered category $p\colon \mathscr{M} \to \mathscr{C}$ fibered in groupoids such that the presheaf $\mathrm{Isom}(x, y)$ is a sheaf on the site $\mathscr{C}/U$ for every $U \in \mathscr{C}$ and $x, y \in \mathscr{M}(U)$. It is a *stack* if in addition any descent datum for any covering in $\mathscr{C}$ is effective.

Usually stacks arise in constructing a moduli space, where one builds a larger parametrizing space and then quotients by a group action that accounts for the automorphisms. In the setting of schemes this in general does not work well, and in fact this failure leads to the subject of Mumford's geometric invariant theory. In the setting of stacks, however, if G is a smooth affine group scheme acting on a scheme X, then there is a quotient stack X/G taking a scheme T to the groupoid of objects $X \leftarrow E \to T$ consisting of a G-torsor over T with a G-equivariant map to X. Moreover X/G is algebraic because the morphism $X \to X/G$ that $X \leftarrow E \times X \to X$ determines is étale and surjective.

Stacks $\mathcal{M}_{(A,\Gamma)}$ also arise through Hopf algebroids (A, Γ) and in such a nice way that $\mathrm{QCoh}\left(\mathcal{M}_{(A,\Gamma)}\right) \cong (A, \Gamma)\text{-Comod}$ [cf Rem 2.39 in Goerss], which is visibly analogous to the equivalence $\mathrm{QCoh}(\mathrm{Spec}(R)) \cong R\text{-Mod}$ from algebraic geometry.

Briefly, given a Hopf algebroid (A,Γ) and a ring R, one forms the groupoid $G_\Gamma(R)$ consisting of objects $\mathrm{Hom}(A,R)$ and morphisms $\mathrm{Hom}(\Gamma,R)$. This functor G_Γ is a sheaf on the category of affine schemes equipped with any Grothendieck topology in which the representable presheaves are sheaves, but it is almost never a stack. Thus, one constructs the *stack* $\mathcal{M}_{(A,\Gamma)}$ *associated to* (A,Γ) to be its stackification.

It turns out that the prestack of formal group laws is not a stack because it does not satisfy effective descent, so instead we turn our attention to formal groups, which do form a moduli stack $\mathcal{M}_{\mathrm{fg}}$.

For now, let us construct $\mathcal{M}_{\mathrm{fg}}$ and describe some immediate consequences, namely how previous results such as the Landweber exact functor theorem and the classification of formal group laws by height translate to geometric statements about $\mathcal{M}_{\mathrm{fg}}$. Recall that $\mathrm{fgl} \cong \mathrm{Spec}(L)$ and $\mathrm{isofgl} \cong \mathrm{Spec}(W)$ [cf Obs 2.5].

Construction. There are several ways to construct the *moduli stack of formal groups* $\mathcal{M}_{\mathrm{fg}}$, each with its own upside. The first construction is concrete; the second makes it clear that $\mathcal{M}_{\mathrm{fg}}$ is an algebraic stack; and the third highlights the equivalence $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \cong (L,W)$-Comod.

- DIRECTLY: Let $\mathcal{M}_{\mathrm{fg}}(R)$ be the groupoid of formal groups over R and their isomorphisms so that $\mathcal{M}_{\mathrm{fg}}$ becomes a category fibered in groupoids over schemes.
- QUOTIENT STACK: The group scheme $\Lambda = \mathrm{Spec}(\mathbb{Z}[b_0^{\pm}, b_1, \dots])$ of coordinate transformations is such that $\Lambda(R) = TR[\![T]\!]^\times$, and Λ acts on fgl via $(\phi F)(X,Y) = \phi^{-1}(F(\phi(X),\phi(Y))$. Take the quotient stack $\mathcal{M}_{\mathrm{fg}} = \mathrm{fgl}/\Lambda$.
- HOPF ALGEBROIDS: The Hopf algebroid (L,W) arises via the group action $\mathrm{fgl} \times \Lambda \to \mathrm{fgl}$ or equivalently via complex cobordism: $(L,W) = (MU_*, MU_*MU)$. Either way, its associated stack is $\mathcal{M}_{\mathrm{fg}}$.

Reference. Paul Goerss details these constructions in his paper *Quasi-coherent sheaves on the Moduli Stack of Formal Groups* [cf intro in [13]].

The Landweber exact functor theorem [cf Thm 3.2] translates to a statement about flatness in the language of stacks. A quasi-coherent sheaf M on the stack $\mathcal{M}_{\mathrm{fg}}$ is flat if the R-module $M(\eta)$ is flat over R for every R-point η. Now if $q\colon \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$ is such an R-point, then there is a right adjoint q_* to the pullback

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) & \underset{q^*}{\overset{q_*}{\leftrightarrows}} & R\text{-Mod} \\ M & \longmapsto & M(\eta), \end{array}$$

so now we say that q is flat over $\mathcal{M}_{\mathrm{fg}}$ if $q_*(R)$ is flat.

Theorem 5.1. *Landweber exactness of R is equivalent to the map* $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$ *being flat.*

The classification of formal group laws [cf Thm 2.7] by height translates to a stratification of $\mathcal{M}_{\mathrm{fg}}$. Here we are working at a prime p, but we still write $\mathcal{M}_{\mathrm{fg}}$ for the moduli stack $\mathcal{M}_{\mathrm{fg}} \times \mathbb{Z}_{(p)}$ of formal groups over p-local rings. Taking inspiration from the classification, consider the elements $v_0, \dots, v_{n-1}$ [cf Def 2.6] that detect the height of a formal group law. By working in the universal case these elements are elements of $L_{(p)}$, the Lazard ring L localized at the prime p. The model of $\mathcal{M}_{\mathrm{fg}}$ as the stack quotient $\mathrm{Spec}(L)/\Lambda$ then suggests the following constructions.

Constructions 5.2. Let
$$\mathcal{M}_{\mathrm{fg}}^{\geq n} = \mathrm{Spec}(L_{(p)}/(v_0, \dots, v_{n-1}))/\Lambda$$
be the closed substack of $\mathcal{M}_{\mathrm{fg}}$ consisting of height $\geq n$ formal groups, where the ideal $(v_0, \dots, v_{n-1})$ is Λ^+-invariant because height is invariant up to isomorphism. Similarly, let
$$\mathcal{M}_{\mathrm{fg}}^{n} = \mathcal{M}_{\mathrm{fg}}^{\geq n} - \mathcal{M}_{\mathrm{fg}}^{\geq n+1} = \mathrm{Spec}(L_{(p)}[v_n^{-1}/(v_0, \dots, v_{n-1}))/\Lambda$$
be the locally closed substack consisting of formal groups of height exactly n.

This gives a stratification of the moduli stack $\mathcal{M}_{\mathrm{fg}}$ by closed substacks. The $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ are all closed and reduced, but they are also canonical in the sense of the following uniqueness theorem [cf Thm 5.14 in [13]].

Theorem 5.3. *Any closed reduced substack of $\mathcal{M}_{\mathrm{fg}}$ is one of $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ for some n.*

6. Lubin-Tate deformation theory

It is easy to paint a picture of each stratum $\mathcal{M}_{\mathrm{fg}}^{n}$. Working in the universal case, let $\overline{\mathbb{F}}_p$ denote the algebraic closure of $\mathbb{F}_p$ and let $\Gamma_n \in \mathrm{fgl}(\overline{\mathbb{F}}_p)$ be the Lubin-Tate formal group law of height n [cf Constr 4.1] arising from $\mathbb{F}_{p^n}$ and $f(x) = px + x^{p^n}$. By the classification of formal groups by height, $\mathcal{M}_{\mathrm{fg}}^{n}$ looks like

$$\begin{tikzcd} \Gamma_n \arrow[loop right, "\mathrm{Aut}_{\mathbb{F}_p}(\Gamma_n)"] \end{tikzcd}$$

where $\mathrm{Aut}_{\mathbb{F}_p}(\Gamma_n)$ consists of the automorphisms fixing $\mathbb{F}_p$ and which fits in the exact sequence
$$0 \to \mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_n) \to \mathrm{Aut}_{\mathbb{F}_p}(\Gamma_n) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 0.$$

From this picture we can motivate Lubin-Tate deformation theory in a couple of different ways.

The overarching motivation is that we would like to apply the Landweber exact functor theorem to this map $\mathrm{Spec}(\mathbb{F}_{p^n}) \to \mathcal{M}_{\mathrm{fg}}$. However, as intuition from algebraic geometry should suggest, the issue is that the inclusion of a geometric point is rarely ever flats, so what Lubin-Tate deformation theory gives is an infinitesimal neighborhood of this geometric point whose inclusion is easily checked to be flat. Landweber then yields what is called the Morava E-theories, which play a central role in chromatic homotopy theory.

A more elementary motivation is to understand the global structure of $\mathcal{M}_{\mathrm{fg}}$, having already understood the structure of each stratum. Figuring out how the strata fit together requires figuring out what small neighborhoods of points in $\mathcal{M}_{\mathrm{fg}}^{n}$ look like. Thus we restrict attention to formal group laws, that is, geometric points that correspond to a coordinatizable formal group. Let us use this motivation to describe the theory.

Notation 6.1. Fix a perfect field k of characteristic $p > 0$ and a formal group law $\Gamma \in \mathrm{fgl}(k)$ of height n.

We are interested in certain factorizations
$$\mathrm{Spec} k \to \mathrm{Spec} B \to \mathcal{M}_{\mathrm{fg}}$$

where the first morphism is a so-called *infinitesimal thickening* of k. The formal group laws $G \in \mathrm{fgl}(B)$ that are mapped to Γ under the infinitesimal thickening are called *deformations*. To keep things simple in the following definition, assume B is a complete local noetherian ring, and denote its quotient map by $q\colon B \to B/\mathfrak{m}$.

$$\begin{tikzcd} & \mathrm{fgl}(B) \arrow[d, "q_*"] \\ \mathrm{fgl}(k) \arrow[r, "i_*"] & \mathrm{fgl}(B/\mathfrak{m}) \end{tikzcd}$$

Definition 6.2. A *deformation* of $\Gamma \in \mathrm{fgl}(k)$ to the ring B is a triple (G, i, f) consisting of a formal group law $G \in \mathrm{fgl}(B)$, a ring homomorphism $i\colon k \to B/\mathfrak{m}$, and an isomorphism $f\colon q_*G \to i_*\Gamma$. The groupoid consisting such deformations and isomorphisms is denoted $\mathrm{Def}_B(\Gamma)$.

These groupoids $\mathrm{Def}_\Gamma(B)$ have a distinct structure: $\pi_1(\mathrm{Def}_\Gamma(B), D) = 0$. In other words there are no nontrivial automorphisms of deformations, hence $\mathrm{Def}_\Gamma(B)$ is equivalent to $\pi_0(\mathrm{Def}_\Gamma(B))$, the category of isomorphism classes with only identity morphisms. One way to interpret this is that although formal groups in general admit automorphisms, this stackiness is not visible within small neighborhoods of geometric points.

The main result which kicks off Lubin-Tate theory is that there is a universal deformation for Γ. Recall first the *ring of Witt vectors* $\mathbb{W}(k)$ for the field k.

Recall. For any such k [cf Notn 6.1] there exists a unique complete valuation ring $\mathbb{W}(k)$ that has p as a uniformizer and k as its residue field, and for B as in Definition 8.2 this ring satisfies the following universal property:

$$\begin{tikzcd} \mathbb{W}(k) \arrow[r, dotted, "\exists!"] \arrow[d, "\text{residue}"] & B \arrow[d, "q"] \\ k \arrow[r] & B/\mathfrak{m} \end{tikzcd}$$

The example which motivated the idea and the construction is the p-adic integers from $\mathbb{F}_p$, where we have $\mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$. This is all proved in Serre's book on class field theory [cf Thm 2 and Prop 10 in Ch 2 of [14]].

Since Γ is height n, we adjoin $n-1$ variables $v_1, \ldots, v_{n-1}$ to the Witt ring to track the vanishing of the $n-1$ coefficients of the same name [cf Def 2.6]. The resulting polynomial ring is the universal ring which is called the *Lubin-Tate ring*. Now in order to construct the universal deformation it suffices to produce a morphism $L_{(p)} \to \mathbb{W}(K)[\![v_1, \ldots, v_{n-1}]\!]$. In the following diagram the top right corner is exact, and the classification map takes t_{p^i-1} to 0 for $i = 1, \ldots, n-1$:

$$\begin{tikzcd} (p, v_1, \ldots, v_{n-1}) \arrow[r, hook] & \mathbb{W}(k)[\![v_1, \ldots, v_{n-1}]\!] \arrow[d] \\ L_{(p)} = \mathbb{Z}_{(p)}[t_1, t_2, \ldots] \arrow[ur, dotted] \arrow[r, "\text{classify } \Gamma"'] & k \end{tikzcd}$$

Picking any lift yields the universal deformation. The universality of $\mathbb{W}(k)$ and the efficient use of the vanishing variables serve as evidence for the following result.

Theorem 6.3. *The functor* $\pi_0(\mathrm{Def}_\Gamma(-))$ *is representable by the Lubin-Tate ring* $\mathbb{W}(k)[\![v_1,\ldots,v_{n-1}]\!]$.

In particular, for any infinitesimal thickening $B \to k$ the universal deformation induces a natural bijection

$$\mathrm{Hom}_{\mathrm{Ring}/k}(\mathbb{W}(k)[\![v_1,\ldots,v_{n-1}]\!], B) \to \mathrm{Def}_\Gamma(B).$$

7. Lubin-Tate spaces and nonabelian class field theory

The Lubin-Tate approach to local class field theory and the deformation theory from the previous sections are just the beginning of a much more general project, namely the local Langlands program, which can be thought of as a nonabelian class field theory. We will only discuss a tiny sliver of this enormous program.

Reference. Jared Weinstein's lecture notes *The Geometry of Lubin-Tate Spaces* are the source for the content in this section.

The entire Lubin-Tate approach is encapsulated in the zeroth *Lubin-Tate space* which is constructed using the Lubin-Tate deformation ring from before. This approach only generates abelian extensions because it only considers the torsion of formal group laws [cf Constr 4.2], which are commutative. The idea, then, is to consider certain moduli spaces of deformations of formal $\mathcal{O}$-modules [cf Def 4.3] which are known are the *higher Lubin-Tate spaces*. These fit together in the *Lubin-Tate tower*, and the Galois action on the cohomology of this tower is what generates the nonabelian extensions that are studied in this nonabelian class field theory.

To construct the zeroth Lubin-Tate deformation space $\mathcal{M}_0$, recall that the functor $\pi_0(\mathrm{Def}_B(-))$ classifies deformations (G, i, f) and is represented by the adic ring $\mathbb{W}(k)[\![v_1,\ldots,v_{h-1}]\!]$ [cf Thm 6.3]. Here we are adopting Notation 6.1 but writing h for the height of $\Gamma \in \mathrm{fgl}(k)$. Setting $M_0 = \pi_0(\mathrm{Def}_B(-))$, another way of saying this is that

$$M_0 \cong \mathrm{Spf}(\mathbb{W}(k)[\![v_1,\ldots,v_{h-1}]\!]).$$

The rigid generic fiber, in other words the p-adic rigid open ball of radius one, is the zeroth Lubin-Tate space $\mathcal{M}_0$.

Along the same lines, Drinfeld constructed the higher Lubin-Tate spaces $\mathcal{M}_n$. These are rigid spaces $\mathcal{M}_n$ that classify triples $(\mathscr{G}, i, \varphi)$, where now $\mathscr{G}$ is a formal group over $\mathcal{O}_K$ and φ is a Drinfeld level structure, an isomorphism $(\mathbb{Z}/p^n\mathbb{Z})^{\oplus h} \to \mathscr{G}[p^n]$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules with some additional requirements.

The inverse limit $\mathcal{M} = \varprojlim_n \mathcal{M}_n$ of these spaces is the Lubin-Tate tower. The Lubin-Tate tower is quite symmetric, admitting group actions by three different groups:

- $G = \mathrm{GL}_h(\mathbb{Q}_p)$
- the units J in the division algebra $\mathrm{End}(G_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of dimension h
- the Weil group $W_{\mathbb{Q}_p}$, the preimage of $\mathbb{Z}$ under $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

For example, when $h = 1$ the tower $\mathcal{M}$ is just the group of units $\mathbb{Q}_p(1)^\times$, and the action of $W_{\mathbb{Q}_p}$ is through the Artin reciprocity map $W^{\mathrm{ab}}_{\mathbb{Q}_p} \to \mathbb{Q}_p^\times$.

Let us finally describe the correspondence in this nonabelian class field theory. By studying the vanishing cycle cohomology of the spectrum of the deformation ring of formal $\mathcal{O}_K$-modules of height n with level structure, one obtains for $\ell \neq p$

the ℓ-adic étale cohomology

$$H_c^i(\mathcal{M}, \mathbb{Q}_\ell) = \varinjlim_n H_c^i(\mathcal{M}_n, \mathbb{Q}_\ell).$$

Just as the Artin map $F^\times \to W_F^{\mathrm{ab}}$ relates the one-dimensional representations of W_F with the irreducible representations of $F^\times$, this space $H_c^i(\mathcal{M}, \mathbb{Q}_\ell)$ relates the representations of all three groups G, J, and $W_{\mathbb{Q}_p}$. To make this precise, there is the following theorem by Harris and Taylor [cf [15]]. Recall that Jacquet-Langlands gives a correspondence JL between, roughly, irreducible representations of G and representations of J [cf Thm 2 and following in [16]].

Theorem. *There exists a bijection between irreducible supercuspidal representations π of G and irreducible h-dimensional representations of $W_{\mathbb{Q}_p}$ having the property that for all irreducible supercuspidal representations π with $\mathbb{Q}_\ell$-coefficients we have (up to a sign)*

$$\mathrm{Hom}_G(\pi, H_c^*(\mathcal{M}, \mathbb{Q}_\ell)) = \mathrm{JL}(\pi) \otimes \mathrm{rec}(\pi)$$

as (virtual) representations of $J \times W_{\mathbb{Q}_\ell}$.

8. Morava E- and K-theories

To be upfront, there is a Morava E-theory $E(n)$ and a Morava K-theory $K(n)$ for each prime number and every $n \geq 0$. This notation suppresses the prime, so in this section we will fix a prime p.

Slogan. The Morava E- and K-theories govern the behavior of the stable homotopy category $\mathrm{SH}_{(p)}^{\mathrm{fin}}$ of finite p-local spectra.

In the next two sections, we will discuss two remarkable theorems from chromatic homotopy theory that can attest to this slogan: the thick subcategory theorem and the chromatic convergence theorem. The former concerns the kernels of Morava K-theories, and the latter involves the Bousfield localization for Morava E-theories.

Let us first discuss the constructions of these things. The Morava E-theories arise from the Landweber exact functor theorem applied to the Lubin-Tate deformation ring $\mathbb{W}(k)[\![v_1, \ldots, v_{n-1}]\!]$. The sequence $v_0 = p, v_1, \ldots, v_{n-1}$ is trivially regular, and v_n is invertible because Γ by assumption has height exactly n. Hence $E(n)$ has coefficient ring

$$\pi_*(E(n)) = \mathbb{W}(k)[\![v_1, \ldots, v_{n-1}]\!][\beta^{\pm 1}],$$

where β has degree two. Unfortunately, the Morava K-theories are not Landweber exact, and their construction is somewhat involved [cf [17]]. Lurie's notes offer the following construction [cf Lect 22 of [1]]: with the notation $\pi_* MU_{(p)} \cong \mathbb{Z}_{(p)}[t_1, t_2, \ldots]$, let $M(k) = \mathrm{cofib}(t_k \colon \Sigma^{2k} MU_{(p)} \to MU_{(p)})$, then set

$$K(n) = MU_{(p)}[v_n^{-1}] \bigwedge_{i \neq p^n - 1} M(i),$$

where we recall that n and p are fixed. From this we can immediately compute

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}]/(t_i \mid i \geq 0, i \neq p^n - 1) \cong \mathbb{F}_p[v_n^{\pm 1}].$$

Despite their difficult upbringing, the Morava K-theories behave quite nicely. As ring spectra they are complex-oriented, and actually they are the fields in the category of ring spectra in the sense that every module spectrum over $K(n)$ is a wedge of suspensions of $K(n)$.

Example 8.1. Up to Bousfield equivalence $\langle E(n)\rangle = \langle K(0) \vee \cdots \vee K(n)\rangle$, where we recall that two spectra are said to be Bousfield equivalent if their homology theories have the same kernels.

As theories the Morava K-theories are even better. The formal group law associated to the cohomology theory $K(n)$ has height exactly n. In fact $K(n)$ can be characterized as the unique complex-oriented ring spectra that has this property and that has coefficient ring $\mathbb{F}_p[v_n^{\pm 1}]$ where v_n has degree $2(p^n - 1)$. Moreover, as cohomology theories they detect p-torsion, becoming more sensitive as n increases:

$$H\mathbb{Q} = K(0), K(1), \ldots, K(\infty) = H\mathbb{Z}/p,$$

where in particular $K(1)$ is one of the $p-1$ isomorphic summands of mod p complex K-theory. As homology theories the Morava K-theories satisfy the following.

Properties 8.2.

- Künneth: $K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$
- Reflects duality: $\mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*(Y)) \cong K(n)_*(DX \otimes Y)$
- Decreasing kernels: If $K(n)_*(X) = 0$, then $K(n-1)_*(X) = 0$.

9. Thick subcategories, nilpotence, and chromatic convergence

To state the thick subcategory theorem, let $\mathscr{C}_{p,n} = \ker(K_{p,n}) \subset \mathrm{SH}^{\mathrm{fin}}_{(p)}$ be the full subcategory of the $K_{p,n}$-acylic spectra. These are called the type n spectra, and are analogous to the closed reduced substacks $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ [cf Constr 5.2]. A basic property of the Morava K-theories is that $\mathscr{C}_{p,n-1} \subset \mathscr{C}_{p,n}$, [cf Properties 8.3] and a highly nontrivial result which is part of the nilpotence theorem is that these are actually proper inclusions.

In contrast it is easier to see that if any two out of the three spectra in a cofiber sequence are type n, then so is the third by the long exact sequence for $K_{p,n}$. Moreover, certainly retracts of type n spectra are type n, and of course this subcategory contains zero. In general we call such a subcategory *thick*. In analogy with the uniqueness of the substacks $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ [cf Thm 5.3], the thick subcategory theorem asserts that in fact these are the only thick subcategories.

Theorem 9.1 (Thick subcategory theorem)**.** *The $\mathscr{C}_{p,n}$ are precisely the thick subcategories of $\mathrm{SH}^{\mathrm{fin}}_{(p)}$.*

The thick subcategory theorem is typically seen to be a consequence of the nilpotence theorem, which was proved by Devinatz, Hopkins, and Smith in 1980.

Theorem (Nilpotence theorem, ring spectrum form)**.** *The kernel of the Hurewicz map $\pi_*(R) \to MU_*(R)$ consists of nilpotent elements for any ring spectrum R.*

The proof of the nilpotence theorem is incredibly difficult, but the hard work pays off. As a first demonstration of the power of this theorem, let us examine the stable homotopy group $\pi_n(S)$. By Serre it is torsion, so its image under the Hurewicz map $\pi_*(S) \to MU_*(S)$ is as well. But since $MU_*(S) \cong L$ is non-torsion by Lazard's theorem, $\pi_n(S)$ must be in the kernel and hence by the nilpotence theorem consists of nilpotent elements. This proves the following classical theorem.

Corollary (Nishida 1973)**.** *Every element of $\pi_n(S)$ is nilpotent.*

Now let us show how nilpotence proves the thick subcategory theorem [cf §5.3 in [3]]. First we state an implication of (a reformation) of it [cf Cor 5.1.5 in [3]].

Corollary. *Let $f\colon X \to Y$ be a map of p-local finite spectra. Then $X \otimes f^{(k)}$ is nullhomotopic for large k if $K(n)_*(X \otimes f) = 0$ for all $n \geq 0$.*

The strategy is to take a thick subcategory $\mathscr{T}$ of $\mathrm{SH}^{\mathrm{fin}}_{(p)}$ and consider the minimal type among all spectra in $\mathscr{T}$. Call this number n. Certainly $\mathscr{T} \subset \mathscr{C}_{p,n}$, so it suffices to show that any X with type $\geq n$ is in $\mathscr{T}$.

Let $S \to X \otimes DX$ be the adjoint to the identity, and let $f\colon F \to S$ be its fiber. Here we recall that the Spanier-Whitehead dual DX of a finite spectrum X is analogous to the dual of a vector space, for example $[X, Y]_* \cong \pi_*(DX \otimes Y)$. Importantly, this is reflected in Morava K-theory [cf Properties 8.3]:

$$\mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*(Y)) \cong K(n)_*(DX \otimes Y).$$

The first step is to show that $X \otimes \mathrm{cofib}(f^{\otimes k}) \in \mathscr{T}$ for all k. Since $\mathscr{T}$ is thick, the smash of X with any finite spectrum is in $\mathscr{T}$, so in particular $\mathrm{cofib}(f) \otimes X \in \mathscr{T}$. Now in the following diagram start with the upper-left square and invoke the standard result that the cofibers in the bottom-right corner agree:

$$\begin{tikzcd}
F^{\otimes k} \arrow[r, "f\otimes 1^{\otimes(k-1)}"] \arrow[d, "f^{\otimes k}"] & F^{\otimes(k-1)} \arrow[r] \arrow[d, "f^{\otimes(k-1)}"] & \mathrm{cofib}(f \otimes 1^{\otimes(k-1)}) \arrow[d] \\
S \arrow[r, equal] \arrow[d] & S \arrow[r] \arrow[d] & * \arrow[d] \\
\mathrm{cofib}(f^{\otimes k}) \arrow[r] & \mathrm{cofib}(f^{\otimes(k-1)}) \arrow[r] & \Sigma F^{\otimes(k-1)} \otimes \mathrm{cofib}(f).
\end{tikzcd}$$

Tensoring the cofiber sequence along the bottom with X gives the inductive step.

The adjoint $S \to X \otimes DX$ induces an injection $K(m)_*(S) \hookrightarrow K(m)_*(X \otimes DX)$ for $m \geq n$, so $K(m)_*(f) = 0$ for $m \geq n$. On the other hand, $K(m)_*(f) = 0$ for $m \leq n$ because Y having type $\geq n$ means $K(m)_*(X) = 0$ [cf Properties 8.3] and so by above

$$K(m)_*(DX \otimes X) = \mathrm{Hom}_{K(m)_*}(K(m)_*(X), K(m)_*(X)) = 0.$$

Hence by the nilpotence theorem, or rather the above corollary, $X \otimes f^{\otimes k}$ is nullhomotopic for some large k, so the cofiber is

$$X \otimes \mathrm{cofib}(f^{\otimes k}) \cong X \vee \Sigma(X \otimes F^{\otimes k}).$$

This is in $\mathscr{T}$ by above and has X as a retract, so by thickness $X \in \mathscr{T}$, as desired.

Chromatic convergence is a more geometrically intuitive theorem. The behaviors on $\mathcal{M}_{\mathrm{fg}}$ of the Bousfield localization functors for the Morava theories are geometric: $L_{E(n)}$ behaves like restriction to the open substack $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ and $L_{K(n)}$ like completion along the locally closed substack $\mathcal{M}_{\mathrm{fg}}^{n}$ [cf Constr 5.2].

Just like how a sheaf on a variety is determined on a nested sequence of open subvarieties, the chromatic convergence theorem asserts that X can be understood just by examining the open substacks $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ [cf Thm 1 in Lect 32 of [1]]. In the following, the relation $\langle E(n) \rangle = \langle K(0) \vee \cdots \vee K(n) \rangle$ [cf Example 8.2] gives the transformations $L_{E(n+1)} \to L_{E(n)}$.

Theorem (Chromatic convergence theorem)**.** *For any finite p-local spectrum X, its $E(n)$-localizations*

$$\cdots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X$$

recover X as the homotopy limit $\lim_n L_{E(n)}X$.

Along the same lines, just like how a sheaf is determined via a Mayer-Vietoris principle by its restriction to a formal neighborhood of a closed subvariety and its open complement along with gluing data, there is the following chromatic fracture square [cf Lect 23 of [1]].

Theorem (Chromatic fracture square)**.** *The following square is a pullback:*

$$\begin{tikzcd}
L_{E(n)} \arrow[r] \arrow[d] & L_{K(n)} \arrow[d] \\
L_{E(n-1)} \arrow[r] & L_{E(n-1)}L_{K(n)}.
\end{tikzcd}$$

Of particular interest are the fibrations $M_n = \mathrm{fib}(L_{E(n)} \to L_{E(n-1)})$ which for a spectrum X give its *monochromatic layers* M_nX. Since $L_{E(n)}L_{K(n)} \cong L_{K(n)}$, the map $M_n \to M_nL_{K(n)}$ sits above the fracture square, so now the square being cartesian implies $M_n \cong M_nL_{K(n)}$. Conversely $L_{K(n)}M_n = L_{K(n)}$ since applying $L_{K(n)}$ to the fiber sequence $M_n \to L_{E(n)} \to L_{E(n-1)}$ yields $L_{K(n)}M_n \to L_{K(n)} \to *$.

In fact $L_{K(n)}$ and M_n define an equivalence of categories $L_{K(n)}\mathrm{SH}^{\mathrm{fin}}_{(p)} \cong M_n\mathrm{SH}^{\mathrm{fin}}_{(p)}$ [cf Prop 12 in Lect 34 of [1]], so in this sense the chromatic layers M_n determine the same data as the $K(n)$-localizations. It is these monochromatic layers that give chromatic homotopy theory its name.

10. The chromatic picture and a glimpse into tmf

Here is an elegant picture of the stable homotopy category of finite spectra:

$$\begin{tikzcd}
\mathrm{Spc}\left(\mathrm{SH}^{\mathrm{fin}}\right) = \arrow[ddddddd, "\rho"'] & \mathscr{P}_{2,\infty} \arrow[d, dash] & \mathscr{P}_{3,\infty} \arrow[d, dash] & \cdots & \mathscr{P}_{p,\infty} \arrow[d, dash] & \cdots \\
& \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
& \mathscr{P}_{2,n} \arrow[d, dash] & \mathscr{P}_{3,n} \arrow[d, dash] & \cdots & \mathscr{P}_{p,n} \arrow[d, dash] & \cdots \\
& \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
& \mathscr{P}_{2,2} \arrow[d, dash] & \mathscr{P}_{3,2} \arrow[d, dash] & \cdots & \mathscr{P}_{p,2} \arrow[d, dash] & \cdots \\
& \mathscr{P}_{2,1} \arrow[drrrr, dash] & \mathscr{P}_{3,1} \arrow[drrr, dash] & \cdots & \mathscr{P}_{p,1} \arrow[dr, dash] & \cdots \\
& & & & & \mathrm{SH}^{\mathrm{fin}}_{\mathrm{tors}} \\
\mathrm{Spec}(\mathbb{Z}) = & (2) \arrow[drrrr, dash] & (3) \arrow[drrr, dash] & \cdots & (p) \arrow[dr, dash] & \cdots \\
& & & & & (0)
\end{tikzcd}$$

As suggested visually, everything is mapped straight down. This picture in fact contains the statement of the thick subcategory theorem [cf Thm 9.1], and it leads to a big picture idea of tmf.

Reference. The section titled *You could've invented* tmf of Aaron Mazel-Gee's notes on tmf [cf [18]] is where the content of this section comes from.

Let us first get a working understanding of this picture. For a tensor triangulated category $\mathscr{K}$, the *triangular spectrum* $\operatorname{Spc}(\mathscr{K})$ is the space of prime thick triangulated $\otimes$-ideals with topology given by the basis $\{\mathscr{P} \in \operatorname{Spc}(\mathscr{K}) \mid a \in \mathscr{P}\}$ for varying $a \in \mathscr{K}$. Certainly this resembles the spectrum of a ring, so one is led to look for a connection to the spectrum of the most obvious ring in $\mathscr{K}$, namely the endomorphism ring $R_{\mathscr{K}} = \operatorname{End}_{\mathscr{K}}(1)$ of its unit.

Theorem.

- *There is a natural continuous map* $\rho\colon \operatorname{Spc}(\mathscr{K}) \to \operatorname{Spec}(R_{\mathscr{K}})$ *given by*
$$\rho(\mathscr{P}) = \{f \in R_{\mathscr{K}} \mid \operatorname{cone}(f) \notin \mathscr{P}\}$$
- *If* $S \subset R_{\mathscr{K}}$ *is a multiplicative set and* $q\colon \mathscr{K} \to S^{-1}\mathscr{K}$ *is localization, then*
$$\operatorname{Spc}(S^{-1}\mathscr{K}) = \{\mathscr{P} \in \operatorname{Spc}(\mathscr{K}) \mid \rho(\mathscr{P}) \text{ does not meet } S\}.$$

Reference. Paul Balmer constructed this map ρ and wrote down this reformulation of nilpotence in his paper *Spectra, spectra, spectra – Tensor triangular spectra versus Zariski spectra of endomorphism rings* [cf [19]].

In the present case, $\mathrm{SH}^{\mathrm{fin}}$ is a tensor triangulated category where $R_{\mathrm{SH}^{\mathrm{fin}}} = \mathbb{Z}$, and letting $S = \mathbb{Z} - (p)$ we get the p-local category $\mathrm{SH}^{\mathrm{fin}}_{(p)}$ where $R_{\mathrm{SH}^{\mathrm{fin}}_{(p)}} = \mathbb{Z}_{(p)}$. In this language, the Morava theories $K_{p,n}$ are $\otimes$-functors from $\mathrm{SH}^{\mathrm{fin}}_{(p)}$ to a suitable category of free graded modules because they satisfy Künneth formulas [cf Properties 8.3], so the $\mathscr{C}_{p,n} = \ker(K_{p,n})$ are thick triangulated $\otimes$-ideals of $\mathrm{SH}^{\mathrm{fin}}_{(p)}$. In the picture we set $\mathscr{P}_{p,n} = q^{-1}(\mathscr{C}_{p,n})$.

The behavior of ρ is described by
$$\begin{array}{ccc} \operatorname{Spc}\left(\mathrm{SH}^{\mathrm{fin}}_{(p)}\right) & \xrightarrow{\rho} & \operatorname{Spec}\left(\mathbb{Z}_{(p)}\right) \\ \mathscr{C}_{p,0} & \longmapsto & 0 \\ \mathscr{C}_{p,n} & \longmapsto & p\mathbb{Z}_{(p)}. \end{array}$$

Let us see why. The first part is because $\mathscr{C}_{p,0} = \ker(K_{p,0}) = \ker(H\mathbb{Q})$ detects torsion. For the second part, note it suffices to consider just the $n = 1$ case because ρ is inclusion-reversing and because $p\mathbb{Z}_{(p)}$ is maximal. But $K_{p,1}$ is a direct summand of mod p complex K-theory, so we have $K_{p,1}(p\colon S^0 \to S^0) = 0$. Hence by definition $p\mathbb{Z}_{(p)} \subset \rho(\mathscr{C}_{p,1})$ since $K_{p,1}(\operatorname{cone}(p))$ is nonzero.

It remains to explain why $\mathscr{P}_{p,1}, \mathscr{P}_{p,2}, \dots, \mathscr{P}_{p,\infty}$ is the fiber above (p). The second bullet point in the theorem says it should be the image of $\rho^{-1}(p\mathbb{Z}_{(p)})$ under $\operatorname{Spc}\big(\mathrm{SH}^{\mathrm{fin}}_{(p)}\big) \to \operatorname{Spc}(\mathrm{SH}^{\mathrm{fin}})$. But $\rho^{-1}(p\mathbb{Z}_{(p)})$ contains the $\mathscr{C}_{p,n}$, and it containing nothing else is just the statement of the thick subcategory theorem [cf Thm 9.1].

To get a big picture idea of tmf, consider the two directions of $\operatorname{Spc}(\mathrm{SH}^{\mathrm{fin}})$, namely the vertical *chromatic direction* and the horizontal *arithmetic direction*. The Morava K-theories determine the points $\mathscr{P}_{p,n} = q^{-1}(\ker(K_{p,n}))$, and the Morava E-theories globalize in the chromatic direction via the chromatic convergence theorem and the chromatic fracture square from the previous section. In particular, $E_{n,p}$ recovers the points $\mathscr{P}_{0,p}, \dots, \mathscr{P}_{n,p}$.

This raises the question of how to globalize in the arithmetic direction, that is, how to recover $E_{n,2}, E_{n,3}, E_{n,5}, \dots$ for some number n that is called the *chromatic level*. For example $H\mathbb{Q}$ recovers the zeroth chromatic level because $H\mathbb{Q} = E_{0,p}$ for all p, and KU recovers up to the first chromatic level because $KU_{(p)} = E_{1,p}$. It turns out that what comes after ordinary cohomology and topological K-theory is in some sense tmf.

There are many reasons why people care about tmf. Let us describe two of them.

The mod 2 cohomology of the 2-completion $\mathrm{tmf}_{(2)}$ approximates that of the sphere:

$$\begin{aligned} H\mathbb{F}_2^* H\mathbb{F}_2 &= \mathcal{A} \\ H\mathbb{F}_2^* H\mathbb{Z} &= \mathcal{A} /\!\!/ \mathcal{A}(0) \\ H\mathbb{F}_2^* ko &= \mathcal{A} /\!\!/ \mathcal{A}(1) \\ H\mathbb{F}_2^* \mathrm{tmf}_{(2)} &= \mathcal{A} /\!\!/ \mathcal{A}(2) \\ &\vdots \\ H\mathbb{F}_2^* S &= \mathcal{A} /\!\!/ \mathcal{A}(\infty). \end{aligned}$$

Here $\mathcal{A}(n) = \langle \mathrm{Sq}^{2^i} \mid 0 \leq i \leq n \rangle$. One might wonder whether the list $H\mathbb{F}_2$, $H\mathbb{Z}$, ko, $\mathrm{tmf}_{(2)}$ of approximating spectra can be extended, but the Hopf invariant one problem obstructs this: there is no spectrum whose cohomology is $\mathcal{A} /\!\!/ \mathcal{A}(3)$.

History. There is a funny story behind this [cf intro of [20]]. The spectrum $\mathrm{tmf}_{(2)}$ was constructed by Mahowald and Hopkins around 1989, but earlier in 1982 Mahowald had published a paper with Davis describing an intricate spectral sequence argument which implied that there cannot even be a spectrum whose cohomology is $\mathcal{A} /\!\!/ \mathcal{A}_{(2)}$. Thus Mahowald's construction of $\mathrm{tmf}_{(2)}$ was at odds with his previous result! This conundrum was resolved only much later when Mahowald found a missing differential in his earlier paper around the 55th stem of the Adams spectral sequence for the sphere.

There are deep connections to physics. Just as the appearance of $H\mathbb{F}_2^* ko$ as a summand in $H\mathbb{F}_2^* M\mathrm{Spin}$ suggests the existence of the Atiyah-Bott-Shapiro orientation $M\mathrm{Spin} \to ko$, the appearance of $H\mathbb{F}_2^* \mathrm{tmf}_{(2)}$ as a summand in $H\mathbb{F}_2^* M\mathrm{String}$ suggests the existence of a string orientation. Indeed in 1987 Edward Witten defined a genus $M\mathrm{Spin} \to \mathbb{Z}[\![q]\!]$ which lands in the ring of modular forms ,provided the characteristic class $\frac{p_1}{2}$ vanishes, and in 2010 Ando-Hopkins-Rezk showed that it can be lifted to the ring spectrum map $\sigma \colon M\mathrm{String} \to \mathrm{tmf}$. This is the string orientation, which in particular induces a ring homomorphism

$$\sigma_* \colon \Omega_*^{\mathrm{String}} \to \{\text{topological modular forms}\}$$

from the string cobordism ring. Visually there is a tower [cf Ex 1.1 and Table 1.2]

$$\begin{tikzcd} S = M\mathrm{Framed} \arrow[r] & \cdots \arrow[r] & M\mathrm{String} \arrow[r] \arrow[d] & M\mathrm{Spin} \arrow[r] \arrow[d] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\ & & \mathrm{tmf} & ko & H\mathbb{Z} & H\mathbb{F}_2. \end{tikzcd}$$

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