APPLICATIONS OF GALOIS COHOMOLOGY

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Contents

This paper gives an overview of some applications of Galois cohomology and Poitou-Tate duality. Most of the content in this paper is based on material from Chapter IX of the book by Neukirch, Schmidt, and Wingberg (cf references), and all references in this paper refer to parts of that book.

The Hasse Principle

There are many Hasse principles, also known as local-global principles. Classically, the Hasse-Minkowski theorem says that a quadratic form has a solution in $\mathbb{Q}$ if and only if it has a solution at all completions $\mathbb{Q}_v$. Along the same lines, the Albert-Brauer-Hasse-Noether theorem says that a central simple algebra A over a number field K is a matrix algebra over K if and only if it splits over every completion K_v. More generally, the sequence

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_{\mathfrak{p}} \mathrm{Br}(k_{\mathfrak{p}}) \xrightarrow{\mathrm{inv}_k} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where inv_k is the sum of the local invariant maps (cf 7.1.4), is exact (cf 8.1.17). This is morally speaking a statement about cohomology groups via the relations

$$H^2(G(L|k), L^*) \cong B(L/k) \quad \text{and} \quad B(k) = \bigcup_{L \text{ fin cyc}} B(L/k).$$

In this section, we express the notion of a Hasse principle cohomologically, state some cases in which it holds, and then dualize.

Notation.

- k a global field
- S a set of primes of k
- T a subset of S
- $\mathcal{O}_{k,S}$ the ring of S-integers
- $\mathbb{N}(S) = \{n \in \mathbb{N} \mid n \in \mathcal{O}_{k,S}^{\times}\}$

- k_S the maximal unramified outside S extension of k
- $G_S = G(k_S|k)$
- A a G_S-module
- $A' = \mathrm{Hom}(A, \mathcal{O}_S^\times)$,

Here is the cohomological notion of a Hasse principle. Intuitively, we think of T as the set of primes where we have local information, and the Hasse principle holding means that this information is enough to recover the global information.

Definition. We say the Hasse principle holds (in dimension 1) for A if

$$Ш^1(k_S, T, A) = \ker\left(H^1(k_S|k, A) \to \prod_{\mathfrak{p}\in T} H^1(k_\mathfrak{p}, A)\right)$$

vanishes.

There is then an obvious question: how big does T need to be in order for the Hasse principle to hold? Due to the generality of the definition, the Hasse principle may not hold even in easy cases. For example, even if $T = S$, i.e. we have maximum local information, the Hasse principle fails when k is a number field, $A = \mathbb{Z}/p\mathbb{Z}$, and S is a finite set of primes containing p and the infinite places, since then $Ш^1(k_S, S, A)$ is the (dual of the) p-primary part of $\mathrm{Cl}_S(k)$, which may be nontrivial. Furthermore, even if $T = S =$ all primes, the Hasse principle may still fail (cf page 523).

Here are some answers to our question, i.e. some cases where the Hasse principle does hold. We will use these Hasse principles throughout this paper.

Notation.

- $\delta(T)$ the Dirichlet density of T
- $\mathrm{cs}(L|k)$ the primes that completely split in a finite separable extension L

Theorem (cf 9.1.9)**.** *Suppose A is finite. Then the Hasse principle holds in the following two situations:*

(*i*) *A is a trivial G_S-module and $\delta(T) > 1/p$, where p is the smallest prime divisor of $|A|$*

(*ii*) *$A = \mu_m$, where $m = p_1^{r_1}\cdots p_n^{r_n}$ with $p_i \in \mathbb{N}(S)$, and*

$$\delta(\mathrm{cs}(k(\mu_{p_i}^{r_i})|k) \cap T) > \frac{1}{p_i[k(\mu_{p_i}^{r_i}) : k]}$$

for all i, except in certain so-called (cf 9.1.7) special cases.

Consider now the dual question of whether there exists a global object, i.e. a cohomology class, which restricts to some prescribed local ones. In other words, we are interested in when the following cokernel vanishes:

$$H^i(k_S|k, A) \xrightarrow{\mathrm{res}^i} {\prod_T}' H^i(k_\mathfrak{p}, A) \longrightarrow \mathrm{coker}^i(k_S, T, A),$$

where the prime indicates a restricted product.

There is again then an obvious question: how small does T need to be in order for this cokernel to vanish? Here are three situations where it vanishes. The first two situations come from the Hasse principles that we have already seen, translated via local and global duality theorems. The third one we omit the proof of (cf 9.2.5).

Theorem. *Suppose T is finite, and if k is a number field, then suppose S contains S_∞. Let A be a finite G_S-module with $|A| \in \mathbb{N}(S)$. Then* $\operatorname{coker}^1(k_S, T, A) = 0$ *in the following three situations:*

(*i*) *A' is a trivial G_S-module, and $\delta(S) > 1/p$, where p is the smallest prime divisor of $|A|$*

(*ii*) *$A = \mathbb{Z}/m\mathbb{Z}$, where $m = p_1^{r_1} \cdots p_n^{r_n}$ with $p_i \in \mathbb{N}(S)$, and*

$$\delta(\operatorname{cs}(k(\mu_{p_i}^{r_i})|k) \cap S) > \frac{1}{p_i[k(\mu_{p_i}^{r_i}) : k]}$$

for all i, except in certain so-called (cf 9.2.3 or next section) special cases

(*iii*) *k is a global field with characteristic $p > 0$, the set T is a proper subset of S, and the G_S-module A is p-primary*

Proof (of situations 1 and 2). Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & 0 & & \operatorname{coker}^1(k_S, T, A)^\vee & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \prod_T H^1(k_\mathfrak{p}, A') & \xrightarrow{\Xi^1}_{\sim} & \prod_T H^1(k_\mathfrak{p}, A)^\vee & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ш^1(k_S, A') & \longrightarrow & H^1(k_S|k, A') & \to & \prod'_S H^1(k_\mathfrak{p}, A') & \longrightarrow & H^1(k_S|k, A)^\vee \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & Ш^1(k_S, S \setminus T, A') & \to & H^1(k_S|k, A') & \to & \prod'_{S \setminus T} H^1(k_\mathfrak{p}, A') & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Here the map Ξ^1 is an isomorphism (cf 8.5.2) since A is finite, and the third row comes from dualizing the long exact sequence of Poitou-Tate (cf 8.6.10). Now a simple diagram chase produces the following short exact sequence:

$$0 \longrightarrow Ш^1(k_S, A') \longrightarrow Ш^1(k_S, S \setminus T, A') \longrightarrow \operatorname{coker}^1(k_S, T, A)^\vee \longrightarrow 0.$$

Dualizing gives

$$\operatorname{coker}^1(k_S, T, A) \subset Ш^1(k_S, S \setminus T, A')^\vee,$$

so now the Hasse principles easily translate over, in particular since T is finite and so $\delta(S \setminus T) = \delta(S)$, etc. □

mth Powers and Grunwald-Wang

As applications of these Hasse principles, we derive the local-global principle for mth powers and the Grunwald-Wang theorem.

The local-global principle for mth powers is exactly what is sounds like.

Application (Local-global principle for mth powers)**.** Let $m \in \mathbb{N}$ be odd and prime to $\mathrm{char}(k)$, and assume $\delta(T) = 1$. Then, except in certain special cases, $\alpha \in k$ is an mth power if and only if $\alpha \in k_{\mathfrak{p}}$ is an mth power for every $\mathfrak{p} \in T$.

Proof. The special cases here correspond to the special cases in the above theorem, so we ignore them. Take S to be the set of all primes so that $k_S = \overline{k}$ and

$$H^1(k, \mu_m) = H^1(G(\overline{k}|k), \mu_m) = k^\times / k^{\times m}$$

by Kummer theory. Now consider the following diagram:

$$\begin{array}{ccccc} & & \dfrac{k^\times}{k^{\times m}} & \longrightarrow & \displaystyle\prod_{\mathfrak{p}\in T} \frac{k_{\mathfrak{p}}^\times}{k_{\mathfrak{p}}^{\times m}} \\ & & \Big| \sim & & \Big| \sim \\ Ш^1(k, T, \mu_m) & \longrightarrow & H^1(k, \mu_m) & \longrightarrow & \displaystyle\prod_{\mathfrak{p}\in T} H^1(k_{\mathfrak{p}}, \mu_m). \end{array}$$

It suffices to show that the upper map is injective, i.e. that the Hasse principle holds for μ_m. But we are in the situation (ii) in the above theorem because

$$\delta(\mathrm{cs}(k(\mu_{p_i}^{r_i})|k) \cap T) = \frac{\delta_{k(\mu_{p_i}^{r_i})}(T)}{[k(\mu_{p_i}^{r_i}) : k]} = \frac{1}{[k(\mu_{p_i}^{r_i}) : k]} > \frac{1}{p_i[k(\mu_{p_i}^{r_i}) : k]}. \qquad \square$$

Let us now discuss these so-called special cases.

Detail. One case where the local-global principle for mth powers does not hold is

$$k = \mathbb{Q}, \quad m = 8, \quad S = T = \text{odd primes}.$$

Indeed, 16 is certainly not an 8th root in $\mathbb{Q}$, but it is an 8th root in $\mathbb{R}$ and moreover in every $\mathbb{Q}_p$ with $p \in S$ by the following argument. Let $p \in S$, and note that at least one of $-1, 2, -2$ is a square mod p by multiplicativity of the Legendre symbol. Thus by Hensel's lemma $\mathbb{Q}_p$ contains at least one of $\sqrt{-1}$, $\sqrt{2}$, or $\sqrt{2}i$, so $\mathbb{Q}_p$ contains at least one of the roots

$$\pm\sqrt{2}, \quad \pm\sqrt{2}i, \quad \pm 1 \pm i$$

of $X^8 - 16$.

In fact, even if we maximize T, the local-global principle still fails in the case

$$k = \mathbb{Q}(\sqrt{7}), \quad m = 8, \quad S = T = \text{odd primes}.$$

Indeed $\sqrt{2}, \sqrt{2}i \notin \mathbb{Q}(\sqrt{7})$, and 16 is still an 8th root at the archimedean primes and in $\mathbb{Q}_p(\sqrt{7})$ for p odd. But 16 is now also an 8th root in $\mathbb{Q}_2(\sqrt{7})$ since this field contains i.

In general, the local-global principle for mth powers holds except in the special cases where the following hold:

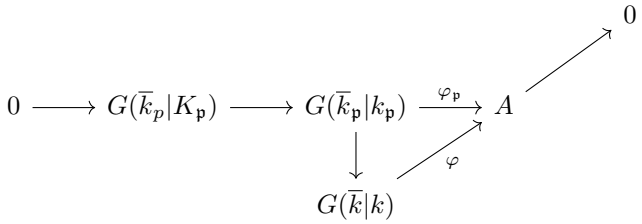
- $r \geq 3$ where $m = 2^r m'$ with m' odd
- $k(\mu_{2^r})|k$ is not cyclic
- all primes $\in T$ over 2 decompose in $k(\mu_{2^r})|k$.

These special cases correspond to the special cases in situation (ii) where the Hasse principle does not hold (cf. Remark 3 on page 528).

The Grunwald-Wang theorem produces a global field with, roughly speaking, prescribed completions and Galois groups at a finite number of primes.

Application (The Grunwald-Wang Theorem)**.** Suppose S is finite, and let A be a finite abelian group. For each $\mathfrak{p} \in S$, let $K_\mathfrak{p}|k_\mathfrak{p}$ be a finite abelian extension such that $G(K_\mathfrak{p}|k_\mathfrak{p})$ embeds into A. Then, except for some special cases, there exists an abelian extension $K|k$ whose Galois group is A and whose completions at every $\mathfrak{p} \in S$ is $K_\mathfrak{p}$.

Proof. Consider the following diagram:



Here the given extensions $K_\mathfrak{p}|k_\mathfrak{p}$ correspond to the continuous homomorphisms $\varphi_\mathfrak{p}$ in the top row. It suffices to find φ such that the diagram commutes for all $\mathfrak{p}$, since then $\ker \varphi = G(\overline{k}|K)$ produces the desired global field K.

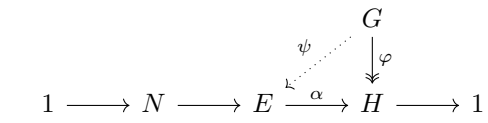
In other words, it suffices to find a surjective φ such that

$$\begin{array}{ccc} H^1(k, A) & \longrightarrow & \bigoplus_S H^1(k_\mathfrak{p}, A) \\ \| & & \| \\ \mathrm{Hom}(G(\overline{k}|k), A) & \longrightarrow & \bigoplus_S \mathrm{Hom}(G(\overline{k}_\mathfrak{p}|k_\mathfrak{p}), A) \\ \varphi & \longmapsto & (\varphi_\mathfrak{p})_{\mathfrak{p}\in S} \end{array}$$

Since finite direct sums, roughly speaking, come out, by decomposing A into primary parts using the structure theorem for finitely generated abelian groups, situation (*iii*) applies, so these maps are surjective. This produces φ. To guarantee that φ is surjective, we add into S a finite number of extra primes so that the images of $\varphi_\mathfrak{p}$ as $\mathfrak{p}$ runs through S generate A. □

Embedding Problems

Definition. A solution to an embedding problem



is a map $\psi\colon G \to E$ making the diagram commute, and it is proper if ψ is surjective.

Notation.

- $\mathfrak{c}$ a full class of groups
- $F_\omega(\mathfrak{c})$ a free pro-$\mathfrak{c}$-group of rank ω (i.e. countable rank)

For which G does every embedding problem with E a finite $\mathfrak{c}$-group have a proper solution? Certainly taking $G = F_\omega(\mathfrak{c})$ works, as follows.

Proof. Pick a basis X of $F_\omega(\mathfrak{c})$, and note that $X \setminus \ker\varphi$ is finite whereas $X \cap \ker\varphi$ is infinite. For each $x \in X \setminus \ker\varphi$, use surjectivity of α to pick a lift to E. Now pick a surjection of sets $X \cap \ker\varphi \to N$ that sends all but finitely many elements to 1. By the universal property of $F_\omega(\mathfrak{c})$ (cf 3.5.14) and by construction, this determines a surjective map ψ. □

According to Iwasawa, this is the only such one that works.

Proposition (Isawasa, 3.5.19, 3.5.20)**.** A pro-$\mathfrak{c}$-group G with rank ω is free if and only if every embedding problem with E a finite $\mathfrak{c}$-group has a proper solution.

Embedding problems come from the following motivating situation. Suppose $L|K$ is Galois and that $G(L|K)$ is a quotient of some group E. Then the problem of extending L to a field M so that $G(\overline{K}|M) = E$ is the same problem as finding a surjective ψ making the following diagram commute:

$$\begin{array}{ccccccccc} & & & & & & G(\overline{K}|L) & \hookleftarrow & G(\overline{K}|M) \\ & & & & & & \downarrow & \swarrow & \\ & & & & & & \mathrm{Gal}(\overline{K}|K) & & \\ & & & & & \overset{\psi}{\swarrow} & \downarrow \varphi & & \\ 1 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{\alpha} & G(L|K) & \longrightarrow & 1. \end{array}$$

That is, a proper solution to the embedding problem gives the desired field extension.

The Inverse Galois Problem and Iwasawa's Theorem

The following elementary problem has been open since the 1800's. It is a so-called sink problem, as opposed to a source problem, in the sense that its resolution would have no significant implications.

Problem (Inverse Galois problem)**.** Every finite group is a Galois group over $\mathbb{Q}$.

Class field theory of course realizes the finite abelian groups. Beyond this, there is much evidence that it is doable: most finite simple groups have been realized as Galois groups over $\mathbb{Q}^{\mathrm{ab}}$.

The inverse Galois problem is trivially equivalent to the embedding problem for the short exact sequence

$$1 \to G \to G \to 1 \to 1,$$

and by the proposition from the previous section, this is equivalent to $G(\overline{\mathbb{Q}}|\mathbb{Q}^{\mathrm{ab}}) = F_\omega(\mathrm{fin})$. More generally, there is the following conjecture which specializes to this statement in the case $k = \mathbb{Q}$, using that $\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\mu)$ by Kronecker-Weber.

Conjecture (Šaferevič)**.** $G(\overline{k}|k(\mu)) \cong F_\omega(\mathrm{fin})$ for a global field k.

The solvable case was proved by Iwasawa.

Theorem (Iwasawa, cf 9.5.4(ii))**.** *The maximal pro-solvable quotient of $G(\overline{k}|k(\mu))$ is $F_\omega(solv)$ for a global field k.*

In other words, the Galois group of the maximal solvable extension of k is free pro-solvable of rank ω.

Šafarevič's Theorem

Perhaps the biggest partial result of the inverse Galois problem is the following celebrated theorem of Šafarevič, which settles the problem for solvable groups.

Theorem (Šafarevič, cf 9.6.1)**.** *Every solvable group is a Galois group over k.*

Šafarevič's original proof, which uses the technique of shrinking obstructions, is the only known proof of this theorem. In the remainder of this section we will roughly describe the method.

Notation.

– G a finite group

Unsurprisingly, we begin by translating the statement of the theorem into a statement about finding proper solutions to certain embedding problems, namely the following one (cf 9.6.6):

$$\begin{array}{ccccccccc} & & & & & & G(\overline{k}|k) & & \\ & & & & & & \downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & H \rtimes G & \underset{\longleftarrow}{\longrightarrow} & G & \longrightarrow & 1 \end{array}$$

every embedding problem for $G(\overline{k}|k)$ with a split short exact sequence of finite groups and nilpotent H has a proper solution. This suffices because the Fitting subgroup $F(G)$ of G has a solvable partial complement U, so there exists a surjection $F(G) \rtimes U \to G$, where by induction on the order of G the complement U is a Galois group.

We further reduce to the case where $H = \mathcal{F}(n)/\mathcal{F}(n)^{(v)}$ is a so-called generic kernel. The following details are not important to the main idea.

Detail.

- If F_d is a pro-p-group of rank d, then
$$\mathcal{F}(d) = \underset{G}{\ast} F_d$$
is a free pro-p-G operator group of rank d.
- For a pro-p-group P, such as $\mathcal{F}(d)$, there is the descending p-central series P^i and the descending central series P_i. Then
$$P^v = (P^i \cap P_j)P^{i+1}$$
are normal characteristic subgroups and form a refinement of the descending p-central series. Here $v = (i, j)$ is a pair with $1 \leq i \leq j$ and ordered lexicographically so that $v + 1$ is defined in the obvious way, reading from left to right, bottom to top.
- A finite nilpotent group is the direct product of its p-Sylow subgroups, and every finite G-operator p-group is a quotient of $\mathcal{F}(n)/\mathcal{F}(n)^{(v)}$ for some n and v.

Here is the key technical technique of shrinking obstructions, which Neukirch deems remarkable and highly instructive. Roughly speaking, the technique allows

one to shrink a module in such a way that kills a prescribed collection of elements. For a vector $a \in \mathbb{F}_p^r$, denote

$$\begin{aligned} \varphi_a \colon \qquad & M^{\oplus r} \longrightarrow M \\ & x \longmapsto a \cdot x. \end{aligned}$$

Lemma. Let M and N be finitely generated $\mathbb{F}_p[G]$-modules, and let $s, t \in \mathbb{N}$. Then for $r \in \mathbb{N}$ sufficiently large, the following holds: for any

$$z_1, \dots, z_t \in \left(M^{\oplus r}\right)^{\otimes s} \otimes N,$$

there exists a nonzero vector $a \in \mathbb{F}_p^r$ such that

$$\begin{aligned} (\varphi_a^{\otimes s}) \otimes \mathrm{id} \colon \qquad & \left(M^{\oplus r}\right)^{\otimes s} \otimes N \longrightarrow M^{\otimes s} \otimes N \\ & z_1, \dots, z_t \longmapsto 0. \end{aligned}$$

Proof. Take

$$r > st \cdot \dim_{\mathbb{F}_p}(M^{\otimes s} \otimes N).$$

Consider the set of vectors $a \in \mathbb{F}_p^r$ such that $(\varphi_a^{\otimes s}) \otimes \mathrm{id}$ takes $z_1, \dots, z_t$ to zero. It is the set of common zeros of n polynomials of degree s, and it contains the zero vector, so by the Chevalley-Warning theorem it contains a nonzero vector. □

Now let us give an extremely rough outline of the proof. It proceeds by induction on ν, letting n be arbitrary. The case base $\nu = 1$ is trivial, and the inductive step is summarized in the following core result.

Theorem (cf 9.6.7)**.** *Let e, n be natural numbers, p a prime, and ν a tuple. The split embedding problem*

$$\begin{array}{ccccccccc} & & & & & & G(\overline{k}|k) & & \\ & & & & & & \downarrow \varphi & & \\ 1 & \longrightarrow & \dfrac{\mathcal{F}(n)}{\mathcal{F}(n)^{(\nu)}} & \longrightarrow & \dfrac{\mathcal{F}(n)}{\mathcal{F}(n)^{(\nu)}} \rtimes G & \rightleftarrows & G & \longrightarrow & 1, \end{array}$$

has a proper solution which corresponds to an extension $N_\nu^n|k$, and if $p \neq \mathrm{char}(k)$, then we can choose the solution to satisfy the following conditions:

- *if $\mathfrak{p}$ is in S_p or S_∞ or ramifies, then it completely decomposes in $N_\nu^n|K$*
- *if $\mathfrak{p}$ is ramified in $N_\nu^n|K$, then $\mathfrak{p}$ splits completely in $K(\mu_{p^e})|k$, and $N_{\nu,\mathfrak{p}}^n|k_{\mathfrak{p}}$ is a cyclic totally ramified extension of local fields*

The reason for these conditions is that the embedding problems in the inductive steps are

$$\begin{array}{ccccccccc} & & & & & & G(\overline{k}|k) & & \\ & & & & & \overset{\varphi_{n,\nu+1}}{\swarrow} & \downarrow \varphi_{n,\nu} & & \\ 1 & \longrightarrow & \dfrac{\mathcal{F}(n)^{(\nu)}}{\mathcal{F}(n)^{(\nu+1)}} & \longrightarrow & \dfrac{\mathcal{F}(n)}{\mathcal{F}(n)^{(\nu+1)}} \rtimes G & \longrightarrow & \dfrac{\mathcal{F}(n)}{\mathcal{F}(n)^{(\nu)}} \rtimes G & \longrightarrow & 1, \end{array}$$

where $\varphi_{n,\nu+1}$ denotes the desired solution, and these embedding problems are not actually solvable. The conditions thus provide a nice enough starting point to allow the embedding problem to be solved at each level n, and they are also reasonable

enough so that the constructed solutions can be changed to satisfy them, allowing for the induction.

Roughly speaking, at each inductive step we change the given $\varphi_{n,\nu}$ and then locally prove the existence of a solution $\varphi_{n,\nu+1}$ (cf first step, page 586). Then we use a Hasse principle to show that this produces a global one (cf second step, page 588). It then remains to check that the two conditions hold for the solution, which they do only after changing the solution once again (cf third and fourth steps, page 590).

To change the given solution $\varphi_{n,\nu}$, we use the shrinking technique. Given ν, n, and a finitely generated $\mathbb{F}_p[G]$-module T, shrinking produces a surjective pro-p-G operator

$$\psi\colon \mathcal{F}(m) \to \mathcal{F}(n)$$

which for m sufficiently large induces a map

$$\hat{H}^k(G, \mathcal{F}(m)^{(\nu)}/\mathcal{F}(m)^{(\nu+1)} \otimes T) \to \hat{H}^k(G, \mathcal{F}(n)^{(\nu)}/\mathcal{F}(n)^{(\nu+1)} \otimes T)$$

which can kill a number of cohomology classes. Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & & & G(\overline{k}|k) & & \\
 & & & & & & \downarrow{\scriptstyle \varphi_{m,\nu}} & & \\
1 & \longrightarrow & \dfrac{\mathcal{F}(m)^{(\nu)}}{\mathcal{F}(m)^{(\nu+1)}} & \longrightarrow & \dfrac{\mathcal{F}(m)}{\mathcal{F}(m)^{(\nu+1)}} \rtimes G & \longrightarrow & \dfrac{\mathcal{F}(m)}{\mathcal{F}(m)^{(\nu)}} \rtimes G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \overline{\psi}} & & \downarrow{\scriptstyle \psi_{\nu+1}} & & \downarrow{\scriptstyle \psi_\nu} & & \\
1 & \longrightarrow & \dfrac{\mathcal{F}(n)^{(\nu)}}{\mathcal{F}(n)^{(\nu+1)}} & \longrightarrow & \dfrac{\mathcal{F}(n)}{\mathcal{F}(n)^{(\nu+1)}} \rtimes G & \longrightarrow & \dfrac{\mathcal{F}(n)}{\mathcal{F}(n)^{(\nu)}} \rtimes G & \longrightarrow & 1.
\end{array}$$

Let α_m and α_n respectively denote the 2-classes corresponding to the upper and lower rows of group extensions. Then the embedding problem on level n is solvable if and only if $\varphi_{n,\nu}^*(\alpha_n) = 0$ (cf 3.5.9), but since $\psi_\nu^*(\alpha_n) = \overline{\psi}_*(\alpha_m)$ (cf 1.5 exercise 4), this is equivalent to ψ killing the class $\varphi_{m,\nu}^*(\alpha_m)$, which we arrange by choosing m large enough.

References

J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of Number Fields.*