Math 32B Final Review Problems

Problem 0. *Basic computations.* Credit to Paul's online notes.

(*i*) Evaluate the double integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \cos(x^2+y^2)\,dy\,dx.$$

Answer: $\pi\sin(1)/2$.

(*ii*) Find the volume of the region bounded below by $z = x^2 + y^2$ and above by $z = 16$. *Answer:* 128π.

(*iii*) Evaluate the triple integral

$$\iiint_E xz\,dV$$

where E is inside both $x^2 + y^2 + z^2 = 4$ and the cone (pointing upward, opening downward) that makes an angle of $\pi/3$ with the negative z-axis and has $x \le 0$. *Answer:* $8\sqrt{3}/5$.

(*iv*) Compute the surface area of the part of $z = xy$ that lies in the cylinder $x^2 + y^2 = 1$. *Answer:* $\frac{2\pi}{3}(2^{\frac{3}{2}} - 1)$.

(*v*) Evaluate the line integral

$$\int_C xyz\,ds$$

where C is the helix given by $r(t) = \langle \cos(t), \sin(t), 3t\rangle$ for $0 \le t \le 4\pi$. *Answer:* $-3\sqrt{10}\pi$.

(*vi*) Evaluate the line integral

$$\int_C \big(\sin(\pi y)\,dy + yx^2\,dx\big)$$

where C is the line segment from $(1, 4)$ to $(0, 2)$. *Answer:* $-7/6$.

(*vii*) Evaluate the line integral of F along C where $F = \langle xz, 0, -yz\rangle$ and C is the line segment from $(-1, 2, 0)$ to $(3, 0, 1)$. *Answer:* 3.

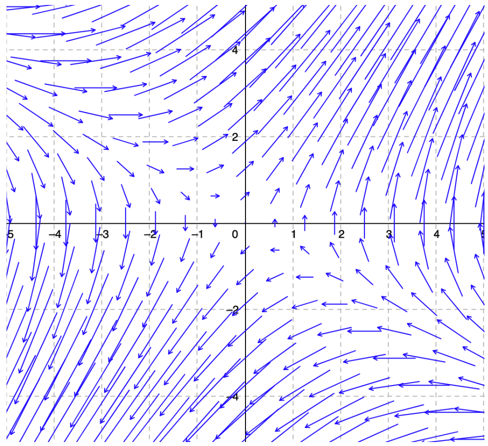
(*viii*) Evaluate the surface integral $\iint_S z\,dS$ where S is the upper half of a sphere of radius 2. *Answer:* 8π.

(*ix*) Evaluate

$$\iint_S \langle 0, y, -z\rangle \cdot dS$$

where $F =$ and S is the parabaloid $y = x^2 + z^2$ for $0 \le y \le 1$ with normal vector pointing in the $+y$ direction. *Answer:* π.

Problem 1. Consider the following vector field F:



(i) Explain why the integral of F along the piecewise linear paths $(-4,-4) \to (-4,4) \to (4,4)$ and $(-4,-4) \to (4,4)$ and $(-4,-4) \to (4,-4) \to (4,4)$ are the same.

(ii) Using (i), describe three closed curves based at $(-4,-4)$ along which the integral of F is zero.

(iii) Line integrals of conservative vector fields along closed curves are zero. Does (ii) therefore imply that F is conservative?

(iv) In fact $F(x,y) = \langle ax + 2y, bx + cy \rangle$ for some integers a, b, c. Find a, b, c.

(v) Use (iv) to show that F is conservative.

(vi) Using (v), what is curl(F)? Using (iv), compute div(F). Explain how your answers are reflected in the picture.

Problem 2. Compute

$$\iint_S \left\langle -2xe^{z^2} z, y\sin(z), e^{z^2} + \cos(z) \right\rangle \cdot dS,$$

where S is the upper half of the sphere of radius 3 centered at the origin.

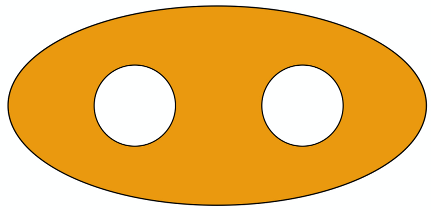
Problem 3. An ellipse with width $2a$ and height $2b$ is given by the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

(a) Find a coordinate transformation from the unit disk to the ellipse. Justify your answer, i.e. show that it is a one-to-one correspondence.

(b) Use (a) to compute the area of the ellipse.

(c) Consider the following region $\mathcal{D}$:



The outer boundary is an ellipse with width 10 and height 4, and the inner boundaries are circles with radius 1. Suppose F is a vector field such that $\mathrm{curl}(F) = 2$ and such that the clockwise line integral around each inner circle is 5π. Find the clockwise line integral around the ellipse.

Problem 4. For each of the following statements, write true or false and briefly justify, or write not-sure. If you write true or false, you get 3 points if your answer and justification are correct and 0 points if not. If you write not-sure, you get 2 points.

- $\int_a^b \int_c^d f\,dx\,dy = \int_c^d \int_a^b f\,dy\,dx$ for any $a, b, c, d \in \mathbb{R}$ and any continuous function f.
- $\int_0^1 \int_0^1 e^{x^2+y^2}\,d, dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_0^1 e^{y^2}\,dy\right)$.
- $\mathbb{R}^2$ minus the origin is simply connected.
- $\mathbb{R}^3$ minus the origin is simply connected.
- If a vector field F equals ∇f for some f, then F is conservative.
- A vector field on a domain that is not simply connected is not conservative.
- A vector field on a domain that is simply connected is conservative.
- If $\mathrm{curl}(F) = 0$, then F is conservative.
- If F is conservative, then $\mathrm{curl}(F) = 0$.
- If $\mathrm{div}(F) = 0$, then F is conservative.
- If F is conservative, then $\mathrm{div}(F) = 0$.
- If $F = \nabla f$ is conservative and $\mathcal{C}$ is a curve from P to Q, then $\int_{\mathcal{C}} F \cdot dr = f(Q) - f(P)$.
- If $F = \nabla f$ is conservative and $\mathcal{C}$ is a loop, then $\int_{\mathcal{C}} F \cdot dr = 0$.
- If the line integral of F along all circles is zero, then F is conservative.

- If the line integral of F along all loops is zero, then F is conservative.
- If $\int_{\mathcal{C}} F \cdot dr = 0$, then $\mathcal{C}$ is a closed curve.
- If $\int_{\mathcal{C}} F \cdot dr \neq 0$, then $\mathcal{C}$ is not a closed curve.

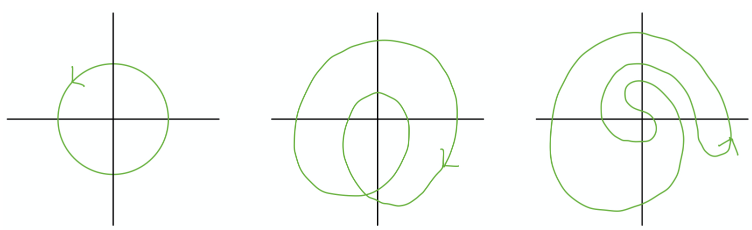
Problem 5. Let

$$F = \frac{\langle -y, x \rangle}{x^2 + y^2}$$

be the vortex field.

(a) Show that F is conservative on the right half plane $x > 0$. Similarly show that F is conservative on the left, lower, and upper half planes. *Hint: The potential function is the θ coordinate, roughly speaking.*

(b) Use (a) to determine the line integral of F along each of the following paths:



(c) Show that $\mathrm{curl}(F) = 0$.

(d) Why is this not enough to show that F is conservative? Using (b), explain why F is not conservative.

Problem 6. For each of the following regions, set up an integral to compute its volume.

(i) The lower half of the ball of radius 2 centered at the origin

(ii) The intersection of the cylinder $y^2 + z^2 \leq 9$ with the cone $z \geq \sqrt{x^2 + y^2}$

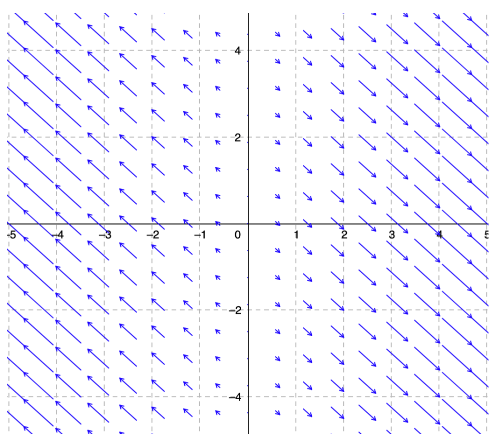
(iii) The region in the unit ball centered at the origin bounded below by the paraboloid $z = x^2 + y^2 - 1$.

(iv) A ball of radius 5 centered at the origin with the cylinder $r \leq 3$ drilled out.

Problem 7. For each of the following statements, write true or false and briefly justify, or write not-sure. If you write true or false, you get 3 points if your answer and justification are correct and 0 points if not. If you write not-sure, you get 2 points.

- $G(u, v) = (u, v^2)$ is a coordinate transformation from $[-1, 1] \times [0, 2]$ to $[-1, 1] \times [0, 4]$.
- As a follow-up to above, G is a coordinate transformation from $[-1, 1] \times [-2, 2]$ to $[-1, 1] \times [0, 4]$.
- The Jacobian of a coordinate transformation does not vanish.
- If $G = F^{-1}$ and $\mathrm{Jac}(F)$ is nowhere zero, then $\mathrm{Jac}(G) = \mathrm{Jac}(F)^{-1}$.
- A surface integral depends on the choice of orientation
- Flux depends on the choice of orientation
- Let M denote the Möbius strip. Even though M is a nonorientable surface, the surface integral $\int_M f \, dA$ still makes sense.
- As a follow-up to above, the flux $\int_M F \cdot n \, dA$ still makes sense.
- In Stokes' Theorem $\oint_{\partial \mathcal{S}} F \cdot dr = \iint_{\mathcal{S}} \mathrm{curl}(F) \cdot dS$, the orientation of $\partial \mathcal{S}$ is such that the region stays on the left.
- In the Divergence Theorem $\iint_{\partial \mathcal{W}} F \cdot dS = \iiint_{\mathcal{W}} \mathrm{div}(F) \, dV$, the normal vectors point outside $\mathcal{W}$.

Problem 8. Consider the following vector field F:



(i) Is F conservative?

(ii) For any number $w > 0$, find a loop based at the origin along which the line integral of F is w.

(iii) Do (ii) but now for $w < 0$.

(iv) For any number $\ell > 0$, find a loop with length ℓ along which the line integral of F is zero.

(v) In fact $F(x,y) = \langle ax+by, cx+dy \rangle$ for some integers a, b, c, d. Determine the values of a, b, c, d.

(vi) Using (v), what are $\text{curl}(F)$ and $\text{div}(F)$? Explain how your answers are reflected in the picture.

Problem 9. Let $\mathcal{D}$ be the region bounded by the simple polygon with vertices $(0,0), (5,1), (6,1), (6,4), (1,3)$. Compute

$$\iint_{\mathcal{D}} \frac{x-5y+1}{y-3x+1}\, dx\, dy$$

by splitting $\mathcal{D}$ into two regions and using a coordinate change on one of them.

Problem 10. Compute

$$\iiint_{\mathcal{V}} (2e^z + e^{x^3 \sin(y)}(2z-3))\, dV,$$

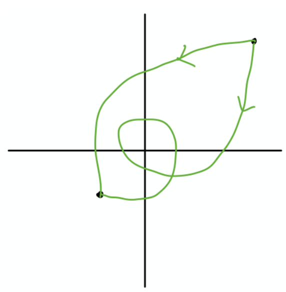
where $\mathcal{V}$ is the volume bounded by the cylinder defined by $r = 3$ and $z \in [0,3]$. *Hint: Be clever with the $2e^z$ term. What else can $2e^z$ come from?*

Problem 11. Let

$$F = \frac{\langle x, y \rangle}{\sqrt{x^2+y^2}}.$$

(a) Show that F is conservative.

(b) Consider the following two paths from $(3,3)$ to $(-1,-1)$:



Use (a) to find the line integral of F along the two paths.

Problem 12. Let

$$F = \langle 5x^4y^2z^2, 2x^5yz^2 + 2yz^3, 2x^5y^2z + 3y^2z^2 \rangle.$$

and

$$G = \langle 5x^4y^2z^2, 2x^5yz^2 + 2yz^3, 2x^5y^2z + 2y^2z^2 \rangle.$$

(*i*) Show that the sum of two conservative vector fields is conservative. Show that a scalar multiple of a conservative vector field is conservative.

(*ii*) Only one of F and G is conservative. Determine which one is conservative, and use (*i*) to show that the other one is not.

(*iii*) Find the line integral of the conservative one along the path

$$r(t) = \left\langle t\cos(2\pi t), t\sin(2\pi t), t^2 \right\rangle$$

where $0 \le t \le 10$.

Problem 13. (Everything here is smooth. Parts (*ii*) and (*iii*) are hard.)

(*i*) On $\mathbb{R}^2$ show that the composition of any two consecutive maps equals 0 in the following sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{functions} & \xrightarrow{\text{grad}} & \text{vector fields} & \xrightarrow{\text{curl}} & \text{vector fields} & \xrightarrow{\text{div}} & \text{functions} \\ & & 0 & & 1 & & 2 & & 3 \end{array}$$

(*ii*) On $\mathbb{R}^2$, show that conversely if something goes to 0, then it comes from something.

(*iii*) In general, if this property holds for a domain X at one of the spots $i = 0, 1, 2$, then we say that

$$H^i(X) = 0,$$

i.e. the *ith cohomology vanishes*. If it does not hold, then we say $H^i(X) \neq 0$. Intuitively, $H^i(X)$ detects whether X has an i-dimensional hole, and in general it counts how many there are. Part (*ii*) shows $H^i(\mathbb{R}^2) = 0$ for all i. Show that

$$H^1(\mathbb{R}^2 - 0) \neq 0 \quad \text{and} \quad H^2(\mathbb{R}^2 - 0) = 0.$$

Finally, show that

$$H^1(\mathbb{R}^3 - 0) = 0 \quad \text{and} \quad H^2(\mathbb{R}^3 - 0) \neq 0.$$