Math 32B Final Review Problems

Problem 0. Basic computations. Credit to Paul’s online notes.

(i) Evaluate the double integral
\[ \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \cos(x^2 + y^2) \, dy \, dx. \]

Answer: \( \pi \sin(1)/2 \).

(ii) Find the volume of the region bounded below by \( z = x^2 + y^2 \) and above by \( z = 16 \). Answer: \( 128\pi \).

(iii) Evaluate the triple integral
\[ \iiint_E xz \, dV \]
where \( E \) is inside both \( x^2 + y^2 + z^2 = 4 \) and the cone (pointing upward, opening downward) that makes an angle of \( \pi/3 \) with the negative \( z \)-axis and has \( x \leq 0 \). Answer: \( 8\sqrt{3}/5 \).

(iv) Compute the surface area of the part of \( z = xy \) that lies in the cylinder \( x^2 + y^2 = 1 \). Answer: \( \frac{2\pi}{3} (2^{3/2} - 1) \).

(v) Evaluate the line integral
\[ \int_C xyz \, ds \]
where \( C \) is the helix given by \( r(t) = (\cos(t), \sin(t), 3t) \) for \( 0 \leq t \leq 4\pi \). Answer: \( -3\sqrt{10}\pi \).

(vi) Evaluate the line integral
\[ \int_C (\sin(\pi y) \, dy + yx^2 \, dx) \]
where \( C \) is the line segment from \((1, 4)\) to \((0, 2)\). Answer: \( -7/6 \).

(vii) Evaluate the line integral of \( F \) along \( C \) where \( F = (xz, 0, -yz) \) and \( C \) is the line segment from \((-1, 2, 0)\) to \((3, 0, 1)\). Answer: 3.

(viii) Evaluate the surface integral \( \iint_S z \, dS \) where \( S \) is the upper half of a sphere of radius \( 2 \). Answer: \( 8\pi \).

(ix) Evaluate
\[ \iint_S (0, y, -z) \cdot dS \]
where \( F = \) and \( S \) is the paraboloid \( y = x^2 + z^2 \) for \( 0 \leq y \leq 1 \) with normal vector pointing in the \(+y\) direction. Answer: \( \pi \).
Problem 1. Consider the following vector field $F$:

(i) Explain why the integral of $F$ along the piecewise linear paths $(-4, -4) \to (-4, 4) \to (4, 4)$ and $(-4, -4) \to (4, 4)$ and $(-4, -4) \to (4, -4) \to (4, 4)$ are the same.

(ii) Using (i), describe three closed curves based at $(-4, -4)$ along which the integral of $F$ is zero.

(iii) Line integrals of conservative vector fields along closed curves are zero. Does (ii) therefore imply that $F$ is conservative?

(iv) In fact $F(x, y) = (ax + 2y, bx + cy)$ for some integers $a, b, c$. Find $a, b, c$.

(v) Use (iv) to show that $F$ is conservative.

(vi) Use (v), what is curl($F$)? Using (iv), compute div($F$). Explain how your answers are reflected in the picture.

Problem 2. Compute

$$\iint_S \left< -2xe^z z, y\sin(z), e^{z^2} + \cos(z) \right> \cdot dS,$$

where $S$ is the upper half of the sphere of radius 3 centered at the origin.

Problem 3. An ellipse with width $2a$ and height $2b$ is given by the equation

$$\left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1.$$

(a) Find a coordinate transformation from the unit disk to the ellipse. Justify your answer, i.e. show that it is a one-to-one correspondence.

(b) Use (a) to compute the area of the ellipse.
(c) Consider the following region $D$:

The outer boundary is an ellipse with width 10 and height 4, and the inner boundaries are circles with radius 1. Suppose $F$ is a vector field such that $\text{curl}(F) = 2$ and such that the clockwise line integral around each inner circle is $5\pi$. Find the clockwise line integral around the ellipse.

**Problem 4.** For each of the following statements, write true or false and briefly justify, or write not-sure. If you write true or false, you get 3 points if your answer and justification are correct and 0 points if not. If you write not-sure, you get 2 points.

- \( \int_b^a \int_c^d f \, dx \, dy = \int_c^d \int_a^b f \, dy \, dx \) for any $a, b, c, d \in \mathbb{R}$ and any continuous function $f$.
- \( \int_0^1 \int_0^1 e^{x^2+y^2} \, dx \, dy = \left( \int_0^1 e^{x^2} \, dx \right) \left( \int_0^1 e^{y^2} \, dy \right) \).
- $\mathbb{R}^2$ minus the origin is simply connected.
- $\mathbb{R}^3$ minus the origin is simply connected.
- If a vector field $F$ equals $\nabla f$ for some $f$, then $F$ is conservative.
- A vector field on a domain that is not simply connected is not conservative.
- A vector field on a domain that is simply connected is conservative.
- If $\text{curl}(F) = 0$, then $F$ is conservative.
- If $F$ is conservative, then $\text{curl}(F) = 0$.
- If $\text{div}(F) = 0$, then $F$ is conservative.
- If $F$ is conservative, then $\text{div}(F) = 0$.
- If $F = \nabla f$ is conservative and $C$ is a curve from $P$ to $Q$, then $\int_C F \cdot dr = f(Q) - f(P)$.
- If $F = \nabla f$ is conservative and $C$ is a loop, then $\int_C F \cdot dr = 0$.
- If the line integral of $F$ along all circles is zero, then $F$ is conservative.
• If the line integral of $F$ along all loops is zero, then $F$ is conservative.
• If $\int_C F \cdot dr = 0$, then $C$ is a closed curve.
• If $\int_C F \cdot dr \neq 0$, then $C$ is not a closed curve.

**Problem 5.** Let

$$F = \left\langle \frac{-y, x}{x^2 + y^2} \right\rangle$$

be the vortex field.

(a) Show that $F$ is conservative on the right half plane $x > 0$. Similarly show that $F$ is conservative on the left, lower, and upper half planes. *Hint: The potential function is the $\theta$ coordinate, roughly speaking.*

(b) Use (a) to determine the line integral of $F$ along each of the following paths:

(c) Show that $\text{curl}(F) = 0$.

(d) Why is this not enough to show that $F$ is conservative? Using (b), explain why $F$ is not conservative.

**Problem 6.** For each of the following regions, set up an integral to compute its volume.

(i) The lower half of the ball of radius 2 centered at the origin

(ii) The intersection of the cylinder $y^2 + z^2 \leq 9$ with the cone $z \geq \sqrt{x^2 + y^2}$

(iii) The region in the unit ball centered at the origin bounded below by the paraboloid $z = x^2 + y^2 - 1$.

(iv) A ball of radius 5 centered at the origin with the cylinder $r \leq 3$ drilled out.

**Problem 7.** For each of the following statements, write true or false and briefly justify, or write not-sure. If you write true or false, you get 3 points if your answer and justification are correct and 0 points if not. If you write not-sure, you get 2 points.

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• $G(u, v) = (u, v^2)$ is a coordinate transformation from $[-1, 1] \times [0, 2]$ to $[-1, 1] \times [0, 4]$.

• As a follow-up to above, $G$ is a coordinate transformation from $[-1, 1] \times [-2, 2]$ to $[-1, 1] \times [0, 4]$.

• The Jacobian of a coordinate transformation does not vanish.

• If $G = F^{-1}$ and $\text{Jac}(F)$ is nowhere zero, then $\text{Jac}(G) = (\text{Jac}(F))^{-1}$.

• A surface integral depends on the choice of orientation.

• Flux depends on the choice of orientation.

• Let $M$ denote the Möbius strip. Even though $M$ is a nonorientable surface, the surface integral $\int_M f \, dA$ still makes sense.

• As a follow-up to above, the flux $\int_M F \cdot n \, dA$ still makes sense.

• In Stokes’ Theorem $\int_{\partial S} F \cdot dr = \int_S \text{curl}(F) \cdot dS$, the orientation of $\partial S$ is such that the region stays on the left.

• In the Divergence Theorem $\iint_{\partial W} F \cdot dS = \iiint_W \text{div}(F) \, dV$, the normal vectors point outside $W$.

**Problem 8.** Consider the following vector field $F$:

(i) Is $F$ conservative?

(ii) For any number $w > 0$, find a loop based at the origin along which the line integral of $F$ is $w$.

(iii) Do (ii) but now for $w < 0$.

(iv) For any number $\ell > 0$, find a loop with length $\ell$ along which the line integral of $F$ is zero.
(v) In fact \( F(x, y) = (ax + by, cx + dy) \) for some integers \( a, b, c, d \). Determine the values of \( a, b, c, d \).

(vi) Using (v), what are \( \text{curl}(F) \) and \( \text{div}(F) \)? Explain how your answers are reflected in the picture.

**Problem 9.** Let \( \mathcal{D} \) be the region bounded by the simple polygon with vertices \((0, 0), (5, 1), (6, 1), (6, 4), (1, 3)\). Compute

\[
\iint_{\mathcal{D}} \frac{x - 5y + 1}{y - 3x + 1} \, dx \, dy
\]

by splitting \( \mathcal{D} \) into two regions and using a coordinate change on one of them.

**Problem 10.** Compute

\[
\iiint_{\mathcal{V}} (2e^z + e^{x^3 \sin(y)}(2z - 3)) \, dV,
\]

where \( \mathcal{V} \) is the volume bounded by the cylinder defined by \( r = 3 \) and \( z \in [0, 3] \).

*Hint: Be clever with the \( 2e^z \) term. What else can \( 2e^z \) come from?*

**Problem 11.** Let

\[ F = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}. \]

(a) Show that \( F \) is conservative.

(b) Consider the following two paths from \((3, 3)\) to \((-1, -1)\):

![Diagram](image)

Use (a) to find the line integral of \( F \) along the two paths.

**Problem 12.** Let

\[ F = \langle 5x^4y^2z^2, 2x^5yz^2 + 2yz^3, 2x^5y^2z + 3y^2z^2 \rangle. \]

and

\[ G = \langle 5x^4y^2z^2, 2x^5yz^2 + 2yz^3, 2x^5y^2z + 2y^2z^2 \rangle. \]
(i) Show that the sum of two conservative vector fields is conservative. Show that a scalar multiple of a conservative vector field is conservative.

(ii) Only one of $F$ and $G$ is conservative. Determine which one is conservative, and use (i) to show that the other one is not.

(iii) Find the line integral of the conservative one along the path

$$r(t) = \langle t \cos(2\pi t), t \sin(2\pi t), t^2 \rangle$$

where $0 \leq t \leq 10$.

**Problem 13.** (Everything here is smooth. Parts (ii) and (iii) are hard.)

(i) On $\mathbb{R}^2$ show that the composition of any two consecutive maps equals 0 in the following sequence:

$$0 \xrightarrow{\text{functions}} 1 \xrightarrow{\text{vector fields}} 2 \xrightarrow{\text{vector fields}} 3 \xrightarrow{\text{functions}} 0$$

(ii) On $\mathbb{R}^2$, show that conversely if something goes to 0, then it comes from something.

(iii) In general, if this property holds for a domain $X$ at one of the spots $i = 0, 1, 2$, then we say that

$$H^i(X) = 0,$$

i.e. the $i$th cohomology vanishes. If it does not hold, then we say $H^i(X) \neq 0$. Intuitively, $H^i(X)$ detects whether $X$ has an $i$-dimensional hole, and in general it counts how many there are. Part (ii) shows $H^i(\mathbb{R}^2) = 0$ for all $i$. Show that

$$H^1(\mathbb{R}^2 - 0) \neq 0 \quad \text{and} \quad H^2(\mathbb{R}^2 - 0) = 0.$$ 

Finally, show that

$$H^1(\mathbb{R}^3 - 0) = 0 \quad \text{and} \quad H^2(\mathbb{R}^3 - 0) \neq 0.$$