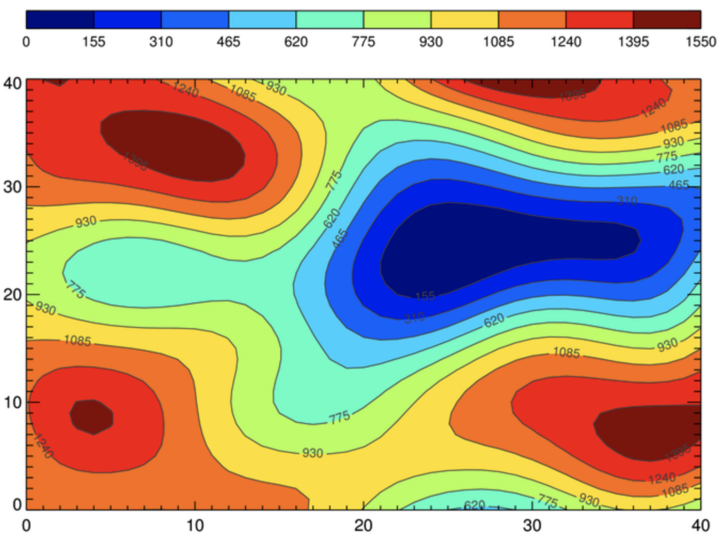
Math 32A Final Review Problems

Problem 1. Consider the contour plot for $f(x, y)$ below:



- Find and classify the critical points of f (estimating as necessary).
- Draw $\nabla f(20, 20)$ and $\nabla f(30, 30)$.
- Find P such that $\nabla f(P)$ is in the direction $(-1, 1)$. Describe how the sign of the directional derivative at P changes as we vary the direction.
- Where is $\|\nabla f\|$ maximal?
- Assuming that $D(4, 9)$ (determinant of Hessian) is nonzero, determine whether the following numbers are positive, negative, or zero: $f_x(4, 9)$, $f_{xx}(4, 9)$, $f_y(4, 9)$, $f_{yy}(4, 9)$, and $D(4, 9)$.
- Classify the critical points of f subject to the constraint $x + y = 40$.
- Classify the critical points of f subject to the constraint

$$(x - 20)^2 + (y - 20)^2 = 100.$$

Problem 2. The *Laplace equation*

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

has many solutions. Which of the following is not a solution?

- $w(x,y) = Ax + By + C$
- $w(x,y) = A(x^2 + y^2) - Bxy$
- $w(x,y) = \frac{Ax+By}{x^2+y^2} + C$

Here A, B, and C are constants.

Problem 3. Show that

$$\lim_{(x,y)\to(0,0)} \frac{y^2(1-\cos(2x))}{x^4+y^2}$$

exists. Now demonstrate that the following limit agrees on all constant speed lines (t, mt) (use the identity $\cos(2x) = \cos^2(x) - \sin^2(x)$):

$$\lim_{(x,y)\to(0,0)} \frac{y^2+(1-\cos(2x))^2}{x^4+y^2}.$$

Despite this, show that this limit does not exist.

Problem 4. Draw a large detailed picture of the surface

$$4x^2 + y^2 - z^2 = 1.$$

Find a trace which is a hyperbola, and draw it onto the surface. Find a point P on the hyperbola (i.e. write it down explicitly), label it on your picture, and draw a normal vector to the surface at P. Now compute the normal vector, and check that it agrees with your picture.

Problem 5. For each statement, pick exactly one of the following: true, false, or not-sure. You get 2 points for correctly selecting true or false, and 1 point for selecting not-sure.

(i) A product of two continuous functions is continuous.

(ii) If f and g are continuous functions and g is never zero, then f/g is continuous.

(iii) A composition of continuous functions is continuous.

(iv) Continuous functions are differentiable.

(v) There exists a continuous function f such that $\lim_{(x,y)\to(a,b)} f(x,y) \neq f(a,b)$.

(vi) If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then $f(x,y)$ is continuous at (a,b).

(vii) If the limit of $f(x,y)$ at a point (a,b) exists along every line, then $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

(viii) Bounded sets are closed.

(ix) The set $\{(x,y) \mid 1 \le x+y \le 2\}$ is bounded.

(x) The set $\{(x,y) \mid 1 \le x^2+y^2 \le 2\}$ is bounded.

(xi) Closed sets are bounded.

(xii) The set $\{(x,y) \mid 1 \le x+y \le 2\}$ is closed.

(xiii) The set $\{(x,y) \mid 1 \le x^2+y^2 \le 2\}$ is closed.

Problem 6. Show that the planes

$$2x - y + z = 9 \qquad \text{and} \qquad x + 4y - 4z = -1$$

intersect in a line. Find a parametrization for L.

Problem 7. Find and classify the critical points of

$$f(x,y) = 8x - x\sqrt{y-1} + x^3 + \frac{1}{2}y - 12x^2.$$

Problem 8. Is there a number c that makes the following function continuous?

$$f(x,y) = \begin{cases} \dfrac{10x^2 + 11xy + y^2}{10x^2 - 39xy - 4y^2} & (x,y) \neq (-1,10) \\ c & (x,y) = (-1,10) \end{cases}$$

Hint: Shift coordinates.

Problem 9. Let

$$f(x,y,z) = x^2 - 10z.$$

Classify the critical points of f. Now classify them with the constraint $x^2+y^2+z^2 = 36$. Redo this with $f(x,y,z) = xyz$ and the constraint $x^2+2y^2+4z^2 = 24$.

Problem 10. Find the tangent plane to

$$f(x,y) = e^{3xy} + x^2 e^{3-x}$$

at $(1,0)$.

Problem 11. For each statement, pick exactly one of the following: true, false, or not-sure. You get 2 points for correctly selecting true or false, and 1 point for selecting not-sure.

(i) The gradient is the unique direction that maximizes the rate of change.

(ii) If $\nabla f(-1,1) = (1,1)$, then there is only one direction vector u such that $D_u f(-1,1) = 1$.

(iii) If $f(x,y)$ is a differentiable function and P is the tangent plane of f at (a,b), then there exists a neighborhood around (a,b) where $f = P$.

(iv) There exists a function $f(x, y)$ whose gradient at a criticial point does not vanish.

(v) The number of critical points of a function $f(x, y)$ subjected to a constraint is at most the number of critical points of f without the constraint.

(vi) There exists a function $f(x, y)$ where $D(a, b)$ (determinant of Hessian) vanishes at a critical point (a, b) and has a local minimum at (a, b).

(vii) For any point (a, b), if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local maximum of f.

(viii) A function can have only finitely many critical points.

(ix) For every differentiable function $f(x, y)$, there is a point p and direction u such that the directional derivative $D_u(p)$ of f vanishes.

(x) If all partials of $f(x, y)$ exist, then f is differentiable.

(xi) If all second partials of $f(x, y)$ exist, then $f_{xy} = f_{yx}$.

(xii) If f is smooth, then $f_{xy} = f_{yx}$.

Problem 12. The acceleration of a particle is

$$a(t) = \left(6t, \cos(t), \frac{1}{t^2}\right).$$

Its velocity at $t = \pi$ is $(3\pi^2, -3, -1/\pi)$, and its position at $t = \pi$ is $(0, 0, 0)$. Determine its position as a function of t.

Problem 13. Let

$$r(u, v) = u^2 v - 3 \qquad \text{and} \qquad s(u, v) = \sin(uv),$$

and set $z(r, s) = s^2 e^r$. Compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Problem 14. Suppose u, v, w span a parallelpiped of volume 57. Determine whether there exists such u, v, w under each of the following requirements:

- they are pairwise orthogonal
- they are pairwise acute
- they are pairwise obtuse
- they all have length at most 3
- they all have length at least 5
- the parallelpiped enters the interior of 3 octants
- the parallelpiped enters the interior of 4 octants

Problem 15. Let $r(t) = (t^{-2}, 1, t^2)$. At $t = 2$, find the following things:

- the velocity $v(t)$, acceleration $a(t)$, unit tangent $T(t)$, and unit normal $N(t)$ vectors
- the projection of $v(t)$ onto $N(t)$ and vice versa
- the projection of $a(t)$ onto $N(t)$ and vice versa

In general, of the 16 possible projections between $v(t), a(t), T(t), N(t)$, how many are always zero?

Problem 16. The *heat equation*

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0$$

has many solutions. Which of the following is not a solution?

- $w(x,t) = A(x^2 + 2t) + B$
- $w(x,t) = A\exp(\mu^2 t \pm \mu x) + B$
- $w(x,t) = A\exp(-\mu x)\cos(\mu t - 2\mu^2 x + B) + C$
- $w(x,t) = A\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right) + B$
- $w(x,t) = A\exp(-\mu^2 t)\cos(\mu x + B) + C$s

Here A, B, C, and μ are constants.

Problem 17. Which points on the surface $f(x,y) = y\sin(x)$ have tangent plane normal to $(\sqrt{2}, \sqrt{2}, 2)$? What about $(\sqrt{2}, \sqrt{2}, 1)$?

Problem 18. Use linear approximation to approximate

$$\frac{\sqrt{10 - (2.001)^2(1.001)}}{(5.002)^2}.$$

Check your answer using a calculator.

Problem 19. Explain how a partial derivative is a directional derivative.

Problem 20. For each statement, pick exactly one of the following: true, false, or not-sure. You get 2 points for correctly selecting true or false, and 1 point for selecting not-sure.

(i) The dot product of parallel vectors is zero.

(ii) The cross product of parallel vectors is zero.

(iii) Two planes orthogonal to a line are parallel.

(iv) Two planes parallel to a line are parallel.

(v) Two lines parallel to a plane are parallel.

(vi) For any three vectors u, v, and w, we have $u \times (v \times w) = -(u \times v) \times w$.

(vii) For any two vectors u and v, we have $u \times v = v \times u$.

(viii) For any two vectors u and v, we have $\|u + v\| \geq \|u\| + \|v\|$.

(ix) For any two vectors u and v, we have $u \cdot v \geq \|u\|\|v\|$.

(x) For any two vectors u and v, we have $u \cdot v = -v \cdot u$.

Problem 21. Let u have length 2 and v have length 3.

- If $\|u \times v\| = 3$, what is the angle that u and v form?
- If u and v form a 60 degree angle, what is $\|3u - v\|$?
- If $u \cdot v = 3\sqrt{2}$, what is the angle that u and v form?
- If $|u \cdot v| \leq 3\sqrt{3}$, what are the possible angles that u and v can form?

Problem 22. Is

$$f(x, y) = \begin{cases} \dfrac{x^2 + 4x - y^2 - 2y + 3}{x^2 + 4x + y^2 + 2y + 5} & (x, y) \neq (-2, -1) \\ 0 & (x, y) = (-2, -1) \end{cases}$$

continuous?

Problem 23. Let

$$f(x, y) = e^{xy} \sin(y).$$

Compute the directional derivative of f at $P = (0, \frac{\pi}{3})$ in the direction of $(3, 4)$. Explain why at P there is a unique direction in which f has the maximal rate of change, and compute this rate r. Find all directions u at P in which f has rate of change $-r/2$.

Problem 24. Given

$$f(x, y, z) = e^{-z} \cos(4y) \ln(2x),$$

find f_{xyyzxz}. Explain your answer carefully. *Hint*: Clairaut.

Problem 25. Draw a large detailed picture of the surface

$$x^2 + 9y^2 = z^2.$$

Find a trace which consists only of straight lines, and draw it onto the surface. Find a point $P \neq (0, 0, 0)$ on the trace (i.e. write it down explicitly), label it on your picture, and draw a normal vector to the surface at P. Now compute the normal vector, and check that it agrees with your picture.

Problem 26. Suppose

$$x^3 \sin(y+z) = z - ye^{x+y+z}.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Problem 27. Determine which of the following limits exists, and explain why the other does not:

$$\lim_{(x,y)\to(0,0)} \frac{xy + x^2y^2 + x^4y^4}{x^2+y^2} \qquad \text{versus} \qquad \lim_{(x,y)\to(1,0)} \frac{(x-1)^2(\ln x)^3}{(x-1)^2+y^2}$$

Problem 28. For each statement, pick exactly one of the following: true, false, or not-sure. You get 2 points for correctly selecting true or false, and 1 point for selecting not-sure.

(i) For a vector-valued function $r(t)$, the acceleration vector $r''(t)$ is always orthogonal to the unit normal vector $N(t)$.

(ii) For a vector-valued function $r(t)$, the unit normal vector $N(t)$ is always orthogonal to the unit tangent vector $T(t)$.

(iii) For a vector-valued function $r(t)$, the unit tangent vector $T(t)$ is always orthogonal to the acceleration vector $r''(t)$.

(iv) The curvature of a vector-valued function $r(t)$ is invariant up to t.

(v) The curvature of a vector-valued function $r(t)$ is invariant up to rotation of $r(t)$.

(vi) The curvature of a vector-valued function $r(t)$ is invariant up to scaling of $r(t)$.

(vii) If all of the contours of $f(x,y)$ are lines, then the graph of $f(x,y)$ is a plane.

(viii) If all of the contours of $f(x,y)$ are parallel lines, then the graph of $f(x,y)$ is a plane.

Problem 29. For $r(t) = (3\sin(t), 3\cos(t), 4t)$ and starting at $t = 8\pi$, we travel a distance of 10π. How far are we from the starting point?

Problem 30. Consider the line

$$\ell(t) = (8, -8, 0) + (1, 0, 1)t.$$

Explain why there is a unique point on ℓ that is closest to the origin, and determine this point.