110AH Final Review Problems

Colin Ni

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Star means highly recommended.

**Problem 1***. Let \( n \geq 3 \). Construct an injection \( D_{2n} \rightarrow S_n \). Prove or disprove: \( S_n \) is the smallest symmetric group into which \( D_{2n} \) embeds.

**Problem 2***. Let \( A \) and \( B \) be abelian groups. Denote by \( \text{Hom}(A, B) \) the set of group homomorphisms \( A \rightarrow B \).

(a) Explain how \( \text{Hom}(A, B) \) is naturally an abelian group.

(b) Describe \( \text{Hom}(\mathbb{Z}, B) \) and \( \text{Hom}(C_n, B) \).

(c) In particular, for \( A \) and \( B \) cyclic, compute \( \text{Hom}(A, B) \).

**Problem 3***. A theorem of Gauss says that \( (\mathbb{Z}/n\mathbb{Z})^\times \), where \( n \geq 1 \), is cyclic if and only if \( n \) is 1, 2, 4, or \( p^k \) or \( 2p^k \) for some odd prime \( p \) and \( k > 0 \). Use this to help fill out the following table of information about \( (\mathbb{Z}/n\mathbb{Z})^\times \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>cyclic</th>
<th>order</th>
<th>structure</th>
<th>gens</th>
<th># gens</th>
<th>min size gen set</th>
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<td>yes</td>
<td>6</td>
<td>( C_6 )</td>
<td>1, 5</td>
<td>2</td>
<td>1 (e.g. {5})</td>
</tr>
<tr>
<td>4</td>
<td>no</td>
<td>4</td>
<td>( C_2 \times C_2 )</td>
<td>0</td>
<td>none</td>
<td>2 (e.g. {3, 5})</td>
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<tr>
<td>5</td>
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<tr>
<td>7</td>
<td>yes</td>
<td>6</td>
<td>( C_6 )</td>
<td>1, 5</td>
<td>2</td>
<td>1 (e.g. {5})</td>
</tr>
<tr>
<td>8</td>
<td>no</td>
<td>4</td>
<td>( C_2 \times C_2 )</td>
<td>0</td>
<td>none</td>
<td>2 (e.g. {3, 5})</td>
</tr>
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</tbody>
</table>

**Problem 4***. Find the smallest \( n \geq 1 \) where \( S_n \) has an element of order \( 5n \).
Problem 5. Let $p$ be an odd prime. Show that the only groups of order $2p$ are $C_{2p}$ and $D_{2p}$.

Problem 6. Is the following $4 \times 4$ sliding tile puzzle solvable?:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 15 & 14
\end{array}
\]

Problem 7*.

(a) Show that every dihedral group has an index 2 subgroup, and generalize this to exhibit an infinite nonabelian group that has an index 2 subgroup.

(b) Denote by $S_\infty$ the group of permutations of $\mathbb{N}$, where $S_n \hookrightarrow S_\infty$ in the natural way. A theorem of Schreier-Ulam says that the only proper nontrivial normal subgroups of $S_\infty$ are $\bigcup_{n \geq 1} S_n$ and $\bigcup_{n \geq 1} A_n$. Use this to show that $S_\infty$ does not have an index 2 subgroup.

(c) (Optional) Show that the only groups whose proper nontrivial subgroups all have index 2 are the simple cyclic groups, $C_4$, and $C_2 \times C_2$.

Problem 8. Prove, or disprove and find a minimal counterexample:

- If $G$ is a finite group and $d \mid |G|$, then $G$ has an element of order $d$.
- If $G$ is a finite group and $d \mid |G|$, then $G$ has a subgroup of order $d$.

You may use that the list of non-abelian groups in increasing order starts with $D_6, D_8, Q_8, D_{10}, D_{12}, A_4, \ldots$.

Problem 9*.

(a) Show that if $S \subset G$ is a normal subset of a group, i.e. $gSg^{-1} \subset S$ for all $g \in G$, then $\langle S \rangle$ is normal.

(b) Show that $A_{3,5,2,19}$ is generated by the permutations of the form

\[(a_1 a_2 a_3)(b_1 b_2 b_3 b_4 b_5)(c_1 c_2 c_3 c_4 c_5)(d_1 d_2 \cdots d_{18} d_{19})\]

where the $a_i, b_i, c_i, d_i$ are pairwise distinct.

(c) Show that a nontrivial simple group is generated by its elements of order $p$ if and only if contains an element of order $p$.

Problem 10. A group $G$ is said to be $k$-abelian if $(ab)^k = a^kb^k$ for every $a, b \in G$. Show that if a group $G$ is $k$-, $(k+1)$-, and $(k+2)$-abelian for some $k \in \mathbb{Z}$, then $G$ is abelian.
Problem 11. Let $p$ be an odd prime. The Legendre symbol $(\frac{a}{p}) : (\mathbb{Z}/p\mathbb{Z})^* \to \{\pm 1\}$ is defined as

$$\left( \frac{a}{p} \right) = \begin{cases} 
+1 & a \text{ is a square in } (\mathbb{Z}/p\mathbb{Z})^* \\
-1 & a \text{ is not a square in } (\mathbb{Z}/p\mathbb{Z})^*. 
\end{cases}$$

Prove that $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$ for any $a, b \in (\mathbb{Z}/p\mathbb{Z})^*$.

Problem 12*. Let $G \leq \mathbb{C}^*$ the group of $p$-power roots of unity, where $p$ is a fixed prime. Show that there exists a nontrivial $N \triangleleft G$ such that $G \cong G/N$.

Problem 13. For which $n, m$ can $S_n$ be embedded into $A_m$?

Problem 14*. A group $G$ is finitely generated if there exists a finite set $S \subset G$ such that $G = \langle S \rangle$. Obviously finite groups are finitely generated, so let us examine infinite groups.

(a) Show that $\mathbb{Z}^n$ is finitely generated.

(b) Show that $\mathbb{Q}$ is not finitely generated because its finitely generated subgroups are cyclic.

(c) Show that $\mathbb{R}$ is not finitely generated but that it has finitely generated subgroups that are not cyclic.

(d) Show that the finitely generated group $\langle \begin{pmatrix} 1 & 1 \\
0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\
0 & 1 \end{pmatrix} \rangle \leq \text{GL}_2(\mathbb{Q})$ has a subgroup that is not finitely generated, namely the one consisting of the matrices in the group with ones on the diagonal.

Remark. A [theorem of Higman, Neumann, and Neumann] says that every countable group can be embedded into a group generated by two elements.

Problem 15. Given a set of symbols $S$ and a set of relations $R$ which are words in these symbols, the group $\langle S \mid R \rangle$ is the quotient of the free group generated by $S$ by the normal subgroup generated by $R$. Find a presentation of the groups $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z}$.

Problem 16. Denote by

$$Q_8 = \left\langle -1, i, j, k \bigg| i^2 = j^2 = k^2 = (ij)^2 = 1, -1 \text{ is central} \right\rangle$$

the quaternion group. For $G \in \{Q_8, D_8\}$ do the following:

(a) Show that $|G| = 8$, and write down the multiplication table of $G$.

(b) Determine the subgroup lattice of $G$, and optionally for each subgroup determine its normalizer.
(c) Find all 2-element subsets \( S \subset G \) such that \( \langle S \rangle = G \).

(d) For each \( N \trianglelefteq G \), compute the isomorphism class of \( G/N \).

(e) Determine the conjugacy classes of \( G \).

**Problem 17*.** (Do Problem 16 first, or look at the answers to it in Solutions.)

Let \( G \) be a finite group. Prove or disprove:

(a) If all subgroups of \( G \) are normal, then \( G \) is abelian.

(b) There exists \( H, K \leq G \), one normal, such that \( G = HK \) and \( H \cap K = 1 \).

(c) There exists an injection \( G \hookrightarrow S_{|G|} \).

(d) If \( H \leq G \), then there exists \( N \leq G \) such that \( G/N \cong H \).

(e) If \( N \leq G \), then there exists \( H \leq G \) such that \( G/N \cong H \).

(f) If \( H, K \leq G \) and \( G/H \cong G/K \), then \( H \cong K \).

(g) If \( H, K \leq G \) and \( H \cong K \), then \( G/H \cong G/K \).

**Remark.** Cayley’s theorem exhibits an injection \( G \hookrightarrow S_{|G|} \) for any finite group \( G \), so part (c) is asking whether this \( |G| \) is sharp.

**Problem 18*.** Show that a transitive group action is the same thing as left-multiplication on a coset space. More precisely, show that if \( G \) acts transitively on a set \( X \), then \( X \cong G/G_x \) as \( G \)-sets for any \( x \in X \).

**Problem 19*.** Show that a finite group is not the union of the conjugates of one of its proper subgroups.

**Problem 20*.** Let \( G \) be a finite group, and let \( d \in \mathbb{N} \). Prove and generalize, or disprove:

(a) If \( d \mid |G| \), then \( G \) acts transitively on a set with \( d \) elements.

(b) If \( |G| = 144 \), then \( G \) acts transitively on a set with 9 elements.

**Problem 21.** A group \( G \) is **solvable** if there exist subgroups

\[
1 = N_1 \leq N_2 \leq \cdots \leq N_{r-1} \leq N_r = G
\]

such that \( N_{i+1}/N_i \) is abelian for \( i = 1, \ldots, r - 1 \). Prove the following using the isomorphism theorems:

(a) A subgroup of a solvable group is solvable.

(b) The homomorphic image of a solvable group is solvable.

(c) Show that if \( N \trianglelefteq G \) and \( G/N \) are solvable, then \( G \) is solvable.
Problem 22. Show that any $p$-group or any group $G$ with order $pq$, $p^2q$, $p^2q^2$, or $pqr$ where $p, q, r$ are primes is solvable.

Problem 23. Recall that

$$S_n \cong \left\langle x_1, \ldots, x_{n-1} \left| \begin{array}{c} x_i^2 \text{ for } i = 1, \ldots, n - 1 \\ (x_i x_{i+1})^3 \text{ for } i = 1, \ldots, n - 2 \\ (x_i x_j)^2 \text{ for } i < j \text{ and } |j - i| > 1 \end{array} \right. \right\rangle$$

via the isomorphism $\tau_i = (i \ i + 1) \in [x_i]$.

(a) Two triple transpositions in $S_6$ share 0, 1, 2, or 3 transpositions. In each case, what is the cycle type of their product?

(b) Find an automorphism $S_6 \to S_6$ that takes transpositions to triple transpositions, and hence is not an inner automorphism.

Problem 24*. Let $G$ be a group. The commutator of $x, y \in G$ is defined to be $[x, y] = xyx^{-1}y^{-1}$, and the commutator subgroup $G' \leq G$ is the subgroup generated by all commutators.

(a) Show that $G$ is a abelian if and only if $G' = 1$.

(b) Show that $G'$ is the smallest normal subgroup with abelian quotient, i.e. if $N \trianglelefteq G$ and $G/N$ is abelian, then $G' \leq N$.

(c) Show that any subgroup containing $G'$ is normal.

Problem 25. Show that a proper subgroup of a $p$-group is properly contained in its normalizer.

Problem 26*. Compute the order of the normalizer $N_{Sp}(C)$ where $C \leq Sp$ is a cyclic subgroup of order $p$.

Problem 27. Let $G$ be a finite group and $X$ a finite $G$-set. Prove Burnside’s lemma:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Deduce that a finite group acting transitively on a non-singleton set has a fixed-point-free element.

Problem 28*. (Optional) Prove the following extension of Bézout’s identity: For $a, b \in \mathbb{N}$ coprime and $c \geq (a - 1)(b - 1)$, there exists $x, y \geq 0$ such that $ax + by = c$.

(b) Let $G$ be a finite group of order 35. Determine the set of the sizes of the finite $G$-sets with no fixed points. Optionally, generalize.

Problem 29*. Let $H$ be a nontrivial $p$-group for some prime $p$. 

5
(a) Show that the center of $H$ is nontrivial, using that the size of a conjugacy class in a finite group divides the order of the group.

(b) Write $|H| = p^n$ for some $n \geq 1$. Show that $H$ has a subgroup of order $p^k$ for every $0 \leq k \leq n$.

(c) Suppose $H$ injects into a finite group $G$ with coprime order. Prove and generalize, or disprove and fix: $H$ contains all elements in $G$ that have order $p$.

Problem 30. Suppose $G$ is a finite simple group that has a proper subgroup of index $n$. Recall that $|G| | n!$. Show that in fact $|G| \mid \frac{1}{2}n!$.

Problem 31. (Optional) The homophonic group $H$ is the group generated by the 26 letters of the English alphabet modulo homophones, i.e. two English words with the same pronunciation are equal in $H$. Show that $H$ is trivial.

Problem 32. Let $G$ be a group, and let $S, T \leq G$ be subgroups.

(a) Show that $ST = TS$ if and only if $ST \leq G$ if and only if $TS \leq G$.

(b) Show that if $S$ or $T$ is normal, then equivalent statements in part (a) hold.

Problem 33*. Let $G$ be a group with $N \triangleleft G$ and $H \leq G$. Show that the following definitions for $G$ being the inner semidirect product of $N$ and $H$ are equivalent:

(i) $G = NH$ and $N \cap H = 1$

(i)' $G = HN$ and $H \cap N = 1$

(ii) for every $g \in G$, there exists unique $n \in N$ and $h \in H$ such that $g = nh$

(ii)' for every $g \in G$, there exists unique $h \in H$ and $n \in N$ such that $g = hn$

(iii) $H \hookrightarrow G \twoheadrightarrow G/N$ is an isomorphism

Problem 34*. Show that $D_{2n}$, where $n \geq 3$, is a nontrivial semidirect product but that neither $C_4$ nor $Q_8$ is.

Problem 35. Let $p$ be a prime, set $X = \{1, \ldots, p\}$, and let $G \leq S_p$ be transitive.

(a) Show that $G$ acts on $X$ transitively if and only if $G$ has a Sylow $p$-subgroup.
(b) Define $n_G$ and $r_G$ for a Sylow $p$-subgroup $P \leq G$ as follows:

$$
\begin{array}{ccc}
&S_p & \\
G & \downarrow & N_{S_p}(P) \\
& n_G & \downarrow N_G(P) \\
& & r_G \\
& & p \\
& & 1
\end{array}
$$

Show that $n_G$ and $r_G$ are independent of the Sylow $p$-subgroup $P \leq G$. Note that $|G| = n_Gr_GP$ and that $r_G \mid (p - 1)$ by Problem 26.

(c) Show that if $r_G = 1$, then $G \cong C_p$.

(d) Suppose $|G| = nrp$ where $r < p$ is also prime, $n > 1$, and $n \equiv 1 \mod p$.
Show that $r = r_G$ and $n = n_G$. Moreover, show that any nontrivial $N \trianglelefteq G$ is transitive and that $n_N = n$ and $r_N = r$. Deduce that $G$ is simple.

Problem 36. A Steiner system $S(\ell, m, n)$ for positive integers $\ell < m < n$ is a collection of distinct size-$m$ subsets of $\{1, \ldots, n\}$ called blocks such that every size-$\ell$ subset of $\{1, \ldots, n\}$ is contained in exactly one block. The automorphism group $\text{Aut}(S(\ell, m, n))$ is the subgroup of $S_n$ taking blocks to blocks.

(a) Explain how the following picture depicts a $S(2, 3, 7)$:

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(b) Suppose there exists a $S(\ell, m, n)$ for some $\ell \geq 2$. Show that there exists a $S(\ell - 1, m - 1, n - 1)$ such that its automorphism group is a stabilizer subgroup of the action of $S(\ell, m, n)$ on $\{1, \ldots, n\}$. Moreover, show that if $\text{Aut}(S(\ell, m, n))$ is $k$-transitive, then $\text{Aut}(S(\ell - 1, m - 1, n - 1))$ is $(k - 1)$-transitive.

(c) There exists a unique $S(5, 6, 12)$ and a unique $S(5, 8, 24)$. Denote by $M_{24}$ and $M_{12}$ their automorphism groups which are both 5-transitive and which
are called \textit{Mathieu groups}. Spam part \((b)\) to fill out or make sense of the first three columns of the following table:

<table>
<thead>
<tr>
<th>group</th>
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<th>transitivity</th>
<th>simple</th>
<th>sporadic</th>
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<tr>
<td>(M_{23})</td>
<td></td>
<td></td>
<td>yes</td>
<td></td>
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<tr>
<td>(M_{22})</td>
<td></td>
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<tr>
<td>(M_{21})</td>
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</tr>
<tr>
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\((d)\) Show that \(M_{24}, M_{23}, M_{22}, M_{12}, \) and \(M_{11}\) are simple, using that \(M_{21}\) is simple (but not sporadic), Problem 35, and the following simplicity criterion, which is Theorem 9.25 in Rotman’s \textit{Introduction to the Theory of Groups}. Let \(X\) be a faithful \(k\)-transitive \(G\)-set for some \(k \geq 2\), and assume \(G\) has a simple stabilizer subgroup. Then the following are true:

- If \(k \geq 4\), then \(G\) is simple.
- If \(k \geq 3\) and \(|X|\) is not a power of 2, then \(G \cong S_3\) or \(G\) is simple.
- If \(k \geq 2\) and \(|X|\) is not a prime power, then \(G\) is simple.