RO(G)-graded cohomology

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RO($G$)-graded cohomology

$G$ - finite group

- $G$-CW complex: attach orbit cells $G/K \times D^n$, for $K \leq G$
- Bredon cohomology $H^*_G(-)$ determined by values $H^*_G(G/K)$

$V$ - real representation of $G$

$\Sigma^V X = S^V \wedge X$

- For $W \in RO(G)$ any virtual representation, $M$ a Mackey functor, get $H^W_G(X; M)$
- Suspension isomorphism $\tilde{H}^W_G(X; M) \cong \tilde{H}^{W+V}_G(\Sigma^V X; M)$
$RO(C_2)$-graded cohomology

$G = C_2$

- Two orbits: $pt = C_2/C_2$ and $C_2 = C_2/e$
- Representations $V = \mathbb{R}^{p,q} = \mathbb{R}^{triv} \oplus \mathbb{R}^{sgn}$
- Representation spheres $S^V = S^{p,q}$

$S^{1,0}$ $S^{1,1}$ $S^{2,0}$ $S^{2,1}$ $S^{2,2}$

- Coefficients in the constant $\mathbb{Z}/2$ Mackey functor: $\mathbb{Z}/2$
- We write $H^V_G(X; \mathbb{Z}/2) = H^{p,q}(X; \mathbb{Z}/2) = H^{p,q}(X)$
Examples

\[ M_2 = H^{*,*}(pt) \]
Examples

For any $X$, $H^{*,,*}(X)$ is an $\mathbb{M}_2$-module via $X \to pt$

- $\bullet = \mathbb{Z}/2$
- $q \cdot \tau$
- $p \cdot \rho$

$H^{*,,*}(C_2)$  \quad  $H^{*,,*}(S^n_a)$  \quad  $H^{*,,*}(S^n_a)$
Some $\mathbb{M}_2$-modules

\[ \text{ker } d \quad \text{cok } d \]

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Torus examples

Cohomologies of $C_2$-actions on a torus
Main result

Theorem (M, in progress)

If $X$ is a finite $C_2$-CW complex then $H^\ast,\ast(X)$ is a direct sum of (shifted) copies of $H^\ast,\ast(pt)$ and $H^\ast,\ast(S_a^n)$. 
Ingredients for the proof

\[ \mathbb{M}_2 = H^*,*(pt) \]

\[ H^*,*(S^n) \]

\[ \mathbb{M}_2[\rho^{-1}] \]

- If \( x \in H^*,*(X) \) and \( \theta x \neq 0 \) then \( \mathbb{M}_2 \langle x \rangle \hookrightarrow H^*,*(X) \).
- \( \mathbb{M}_2 \) is injective
- For a finite \( C_2\)-CW complex

\[ H^*,*(X)[\rho^{-1}] \cong H^*_{\text{sing}}(X^{C_2}; \mathbb{Z}/2) \otimes \mathbb{M}_2[\rho^{-1}] \]

- \( \langle \tau, \theta, \rho \rangle = 1 \)
Toda bracket

Use $\langle \tau, \theta, \rho \rangle = 1$ to exclude many $\mathbb{M}_2$-modules.

Lemma

If $x \in H^{*,*}(X)$ and $\tau x = 0$ then $x = \rho y$ for some $y \in H^{*,*}(X)$.

Follows from $x = x \cdot \langle \tau, \theta, \rho \rangle = \langle x, \tau, \theta \rangle \cdot \rho$

Use several similar results to describe $H^{*,*}(X)$.
Application of theorem to $\mathbb{R}P^2_{tw}$

- Consider $\mathbb{R}P^2_{tw}$
- Cofiber sequence $S^{1,0} \hookrightarrow \mathbb{R}P^2_{tw} \rightarrow S^{2,2}$
- Long exact sequence in $\tilde{H}^*,*(-)$
- Extension problem $0 \rightarrow \text{cok } d \rightarrow \tilde{H}^*,*(\mathbb{R}P^2_{tw}) \rightarrow \ker d \rightarrow 0$
Freeness Theorems

- $G$-CW complex: attach orbit cells $G/K \times D^n$
- $\text{Rep}(G)$-complex: attach representation cells $D(V)$
  
  e.g. Grassmannian $Gr_k(\mathbb{R}^{p,q})$

**Theorem (Kronholm, 2010)**

*If $X$ is a finite $\text{Rep}(C_2)$-complex, $H^\ast,\ast(X)$ is a free $\mathbb{M}_2$-module.*

**Theorem (Ferland, 1999)**

*If $X$ is a finite $\text{Rep}(C_p)$-complex for $p$ odd and $X$ has only even dimensional cells, then $H^\ast_G(X)$ is a free $H^\ast_G(\text{pt})$-module (with coefficients in $\mathcal{A}$ or $\mathbb{Z}$).*

- Future work: Show main theorem implies Kronholm’s freeness theorem.
Thank you!