Some structure theorems for $RO(G)$-graded cohomology

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$RO(G)$-graded cohomology

- CW-complex $X \rightsquigarrow$ cohomology $H^*(X)$
- Suspension $\Sigma^m X = S^m \wedge X = (S^m \times X)/(S^m \vee X)$
- Suspension isomorphism $\tilde{H}^n(X) \cong \tilde{H}^{n+m}(\Sigma^m X)$

$G$ - finite group

- $G$-CW complex
- $V$ - real representation of $G$
- $S^V = \hat{V}$ one-point compactification
- Suspension $\Sigma^V X = S^V \wedge X$
- Suspension isomorphism $\tilde{H}^G_\alpha(X) \cong \tilde{H}^G_{\alpha+V}(\Sigma^V X)$
$RO(C_2)$-graded cohomology

$G = C_2$

- Representations $V = \mathbb{R}^{p,q} = (\mathbb{R}_{\text{triv}})^{p-q} \oplus (\mathbb{R}_{\text{sgn}})^q$
- Representation spheres $S^V = S^{p,q}$

- Coefficients in the constant Mackey functor: $\underline{\mathbb{F}_2}$
- Write $H_G^\alpha(X; \underline{\mathbb{F}_2}) = H^{p,q}(X; \underline{\mathbb{F}_2}) = H^{p,q}(X)$
Cohomology of a point

\[ \mathcal{M}_2 = H^{*, *}(pt; \mathbb{F}_2) \]

\[ \tilde{H}^{*, *}(S^{p, q}) \cong \Sigma^{p, q} \mathcal{M}_2 \]
Examples

For any $X$, $H^{*,*}(X)$ is an $\mathbb{M}_2$-module via $X \to pt$

\[ \cdot = \mathbb{F}_2 \]

\[ \cdot \tau \]

\[ \cdot \rho \]

\[ H^{*,*}(C_2; \mathbb{F}_2) \quad H^{*,*}(S^n_a; \mathbb{F}_2) \quad H^{*,*}(S^n_a; \mathbb{F}_2) \]
Torus examples

Cohomologies of $C_2$-actions on a torus with $\mathbb{F}_2$-coefficients
Structure theorem

Theorem (M, 2018)

If $X$ is a finite $C_2$-CW complex then $H^{*,*}(X; \mathbb{F}_2)$ is a direct sum of shifted copies of $\mathbb{M}_2 = H^{*,*}(pt; \mathbb{F}_2)$ and $H^{*,*}(S^n_2; \mathbb{F}_2)$.
$RO(C_3)$-graded cohomology

\[ G = C_3 \]

- Representations $V = \mathbb{R}^{p,q} = (\mathbb{R}_{\text{triv}})^{p-q} \oplus (\mathbb{R}_{\text{rot}}^{2})^{q/2}$
- Representation spheres $S^V = S^{p,q}$

\[ S^{1,0} \quad S^{2,0} \quad S^{2,2} \]

- Coefficients in the constant Mackey functor: $\mathbb{F}_3$
- Write $H^\alpha_G(X; \mathbb{F}_3) = H^{p,q}(X; \mathbb{F}_3)$ for $q = \text{even}$
Cohomology of a point

\[ \tilde{H}^*,*(S^p,q) \cong \sum_{p,q} M_3 \]
Examples

For any $X$, $H^{*,*}(X; \mathbb{F}_3)$ is an $\mathbb{M}_3$-module via $X \to pt$

\[ \cdot = \mathbb{F}_3 \]
\[ \cdot x \]
\[ \cdot y \]
\[ \cdot z \]

\[ H^{*,*}(C_3; \mathbb{F}_3) \]
\[ H^{*,*}(S^1_{\text{free}}; \mathbb{F}_3) \]
\[ H^{*,*}(S^3_{\text{free}}; \mathbb{F}_3) \]
Egg-beater

Cofiber sequence $C_{3+} \rightarrow S^{0,0} \rightarrow EB$

For $G = C_2$ this cofiber sequence is $C_{2+} \rightarrow S^{0,0} \rightarrow S^{1,1}$
Structure theorem

“Theorem” (M, in progress)

If $X$ is a finite $C_3$-CW complex then $H^*,*(X; \mathbb{F}_3)$ is a direct sum of shifted copies of:

$$\mathbb{M}_3 = H^*,*(\text{pt}), \quad H^*,*(C_3), \quad H^*,*(S^{2n+1}_{\text{free}}), \quad \text{and} \quad \tilde{H}^*,*(EB).$$
Thank you!