Math 115A - Spring 2019
Exam 2

Full Name: ________________________________

UID: ________________________________

Instructions:

• Read each problem carefully.

• Show all work clearly and circle or box your final answer where appropriate.

• Justify your answers. A correct final answer without valid reasoning will not receive credit.

• All work including proofs should be well organized and clearly written using complete sentences.

• You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.

• Calculators are not allowed but you may have a 3 × 5 inch notecard.

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1. (10 points) True or False: Prove or disprove the following statements.

(a) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space over a field $\mathbb{F}$. Let $v$ and $w$ be two eigenvectors of $T$ with eigenvalue $\lambda \in \mathbb{F}$. Then any nonzero linear combination of $v$ and $w$ is also an eigenvector of $T$.

(b) Let $S, T : V \to V$ be linear operators on a finite-dimensional vector space. Assume that $S$ and $T$ commute, i.e. that $ST = TS$. If $T$ is injective then $S$ is injective.
2. (10 points) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation given by reflection about the line $y = 2x$.

(a) Give a basis for $\mathbb{R}^2$ consisting of eigenvectors for $T$ and find their corresponding eigenvalues.

(b) Is there a basis $\gamma$ for $\mathbb{R}^2$ such that $[T]_\gamma$ is the following matrix?

$$[T]_\gamma = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

If so, find the basis $\gamma$. If not, justify why no such basis exists.
3. (10 points) Let $A, B \in M_{n \times n}(F)$ and let $\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$ be the trace of $A$.

(a) Show that if $A$ and $B$ are similar then $\text{tr}(A) = \text{tr}(B)$.

(b) Show that if $A^k = 0$ for some $k \geq 1$ then the determinant $\det(A) = 0$. 

4. (10 points) Let \( V = M_{2 \times 2}(\mathbb{R}) \) and \( W = P_3(\mathbb{R}) \). Let
\[
\beta = \left\{ w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, w_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},
\]
and
\[
\gamma = \{1, x, x^2, x^3\}
\]
be the standard bases. Consider the linear map \( T : V \rightarrow W \) defined by
\[
T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (a - c)x^3 + (a + c - 2b + 2d)x^2 + 3(c + d)x + 2(c + d).
\]

(a) Find \([T]_{\beta}^{\gamma}\).

(b) Prove that although \( V \cong W \), the map \( T \) is not an isomorphism. (\textit{Hint:} The proof that \( V \cong W \) should be one line.)