Problem F01.1. Let K be a compact set of real numbers and let f be a continuous function on K. Prove that there exists $x_0 \in K$ such that $f(x) \le f(x_0)$ for all $x \in K$.

Solution. We first prove that $f(K)$ is compact. Suppose $(U_\alpha)_{\alpha \in I}$ is an open cover of $f(K)$. For any, $x \in K$ we have $f(x) \in f(K)$. Then $f(x) \in U_\alpha$ for some α and so $x \in f^{-1}(U_\alpha)$. Hence $(f^{-1}(U_\alpha))_{\alpha \in I}$ is a cover for K. Also each of $f^{-1}(U_\alpha)$ is open since f is continuous so we have an open cover of K. Then by compactness of K, there is a finite subcover $(f^{-1}(U_{\alpha_i}))_{i=1}^N$. Take $y \in f(K)$. Then there is $x \in K$ such that $f(x) = y$. Then $x \in f^{-1}(U_{\alpha_i})$ for some $i = 1, \ldots, N$ so $y = f(x) \in f(f^{-1}(U_{\alpha_i})) = U_{\alpha_i}$. Then $(U_{\alpha_i})_{i=1}^N$ forms a cover of $f(K)$. Hence any cover of $f(K)$ has a finite subcover, so $f(K)$ is compact.

Since $f(K)$ is compact, it is bounded, and thus has a finite supremum $M \in \mathbb{R}$. Also $f(K)$ is closed so it contains its supremum. Hence $M \in f(K)$ and so there is $x_0 \in K$ such that $f(x) = M$. Since M is an upper bound for $f(K)$, we have that $y \le f(x_0)$ for all $y \in f(K)$, or put another way, $f(x) \le f(x_0)$ for all $x \in K$.

Problem F01.4. Let $\mathcal{S}$ be the set of all sequences $(x_1, x_2, \ldots)$ such that for all n,

$$x_n \in \{0, 1\}.$$

Prove that there is not a one-to-one mapping from $\mathbb{N}$ onto $\mathcal{S}$.

Solution. There *is* a one-to-one mapping; there *is not* an onto mapping. We prove this as such. For any mapping $f : \mathbb{N} \to \mathcal{S}$, list the images:

$$\begin{aligned} f(1) &= (x_{1,1}, \;\; x_{1,2}, \;\; x_{1,3}, \ldots) \\ f(2) &= (x_{2,1}, \;\; x_{2,2}, \;\; x_{2,3}, \ldots) \\ f(3) &= (x_{3,1}, \;\; x_{3,2}, \;\; x_{3,3}, \ldots) \\ &\;\;\vdots \end{aligned}$$

Construct the sequence $(y_1, y_2, y_3, \ldots)$ so that

$$y_1 \ne x_{1,1}, \quad y_2 \ne x_{2,2} \quad y_3 \ne x_{3,3}, \quad \ldots.$$

Then there is no $n \in N$ such that $f(n) = (y_1, y_2, y_3, \ldots)$ because the n^{th} component of $f(n)$ differs from y_n by construction.

Problem W02.6. State some reasonably general conditions on a function $f : \mathbb{R}^2 \to \mathbb{R}$ under which

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

and prove the formula under the conditions you give.

Solution. We claim that if the mixed partials are continuous, then the formula holds. Indeed, assume the mixed partials are both continuous. Define

$$F(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y) - \frac{\partial^2 f}{\partial y \partial x}(x, y), \quad (x, y) \in \mathbb{R}^2.$$

Then F is continuous and integrating this function over any rectangle $[a,b]\times[c,d]$ in $\mathbb{R}^2$, we see

$$I := \int_c^d \int_a^b F(x,y)dx\,dy = \int_c^d \int_a^b \frac{\partial^2 f}{\partial x \partial y}(x,y)dx\,dy - \int_c^d \int_a^b \frac{\partial^2 f}{\partial y \partial x}(x,y)dx\,dy.$$

Using Fubini's theorem on the latter integral, we get

$$\begin{aligned} I &= \int_c^d \int_a^b \frac{\partial^2 f}{\partial x \partial y}(x,y)dx\,dy - \int_a^b \int_c^d \frac{\partial^2 f}{\partial y \partial x}(x,y)dy\,dx \\ &= \int_c^d \left[\frac{\partial f}{\partial y}(b,y) - \frac{\partial f}{\partial y}(a,y)\right]dy - \int_a^b \left[\frac{\partial f}{\partial x}(x,d) - \frac{\partial f}{\partial x}(x,c)\right]dx \\ &= f(b,d) - f(b,c) - f(a,d) + f(a,c) - f(b,d) + f(a,d) + f(b,c) - f(a,c) = 0. \end{aligned}$$

Thus F integrates to zero over any rectangle in $\mathbb{R}^2$. Assume that F is not identically zero. Then there is $(x^*,y^*)\in\mathbb{R}^2$ such that (wlog) $F(x^*,y^*) = \varepsilon > 0$. By continuity, we can find $\delta > 0$ so that for

$$(x,y) \in [x^*-\delta, x^*+\delta]\times[y^*-\delta, y^*+\delta]$$

we have $F(x,y) > \varepsilon/2$. Then

$$\int_{y^*-\delta}^{y^*+\delta}\int_{x^*-\delta}^{x^*+\delta} F(x,y)dx\,dy > \int_{y^*-\delta}^{y^*+\delta}\int_{x^*-\delta}^{x^*+\delta} \frac{\varepsilon}{2}dx\,dy = 2\varepsilon\delta^2 > 0.$$

This contradicts that F integrates to zero over any rectangle, hence F is identically zero which implies that

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y), \quad (x,y)\in\mathbb{R}^2.$$

Problem W02.7. Suppose $F = (F_1,F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is everywhere differentiable and that the first derivative matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$$

is continuous and nonsingular everywhere. Suppose also that

$$||F(x,y)|| \ge 1 \quad \text{if} \quad ||(x,y)|| = 1 \quad \text{and} \quad F(0,0) = (0,0).$$

Put $U = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 < 1\}$. Prove that $U \subset F(U)$.

Solution. Once you have proven that $U\cap F(U)$, we know that $U\cap F(U) = \varnothing$ or $U \cap F(U) = U$ since U is connected. However, $U\cap F(U) \neq \varnothing$ because $(0,0)$ is in the intersection.

To prove that the set is open and closed, appeal to the Inverse Function Theorem. I can't be bothered to write down the proof.

Problem F03.2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an infinitely differentiable function and assume that for each $x\in[0,1]$, there is a positive integer m such that $f^{(m)}(x) \neq 0$.

Prove the following stronger statement: there is an integer M such that for each element $x \in [0,1]$ there is a positive integer $m \le M$ such that $f^{(m)}(x) \neq 0$.

Solution. Define

$$E_n = \{x \in [0,1] : \text{ there exists } k \in \mathbb{N},\ 0 < k \le n \text{ such that } f^{(k)}(x) \neq 0\}.$$

By assumption, each $x \in [0,1]$ is in E_m where m is as in the statement of the problem. Thus

$$[0,1] = \bigcup_{n=1}^{\infty} E_n.$$

Also,

$$E_n = \bigcup_{k=1}^{n} \left(f^{(k)}\right)^{-1} (\mathbb{R} - \{0\}).$$

Since f is C^∞, each of $f^{(k)}$ is continuous and thus $\left(f^{(k)}\right)^{-1} (\mathbb{R} - \{0\})$ is open for all $k \in \mathbb{N}$ since it is the pullback of an open set under a continuous function. Thus E_n is open as a union of open sets. Thus E_n, $n \in \mathbb{N}$ forms and open cover of $[0,1]$. By compactness, there is a finite subcover

$$[0,1] = \bigcup_{i=1}^{N} E_{n_i}.$$

Putting $M = n_N$ gives the result.

Problem S03.3. Find a subset S of $\mathbb{R}$ such that both of the following hold for S:

1. S is not a countable union of closed sets
2. S is not a countable intersection of open sets

Solution. Let S be the union of the positive rationals with the negative irrationals. Assume that S is a countable intersection of open sets $D_n, n = 1, 2, 3, \ldots$. Then $\mathbb{Q}^+$ is also a countable intersection of open sets since

$$\mathbb{Q}^+ = \bigcap_n D_n^+ \quad \text{where} \quad D_n^+ = D_n \cap (0, \infty).$$

Since $\mathbb{Q}^+$ is contained in each D_n^+, we know that each D_n^+ is dense in $\mathbb{R}^+$. Let (q_n) denumerate the positive rationals. Then the sets $A_n = \mathbb{R}^+ - \{q_n\}$ are also all open and dense in $\mathbb{R}^+$. Then by the Baire Category Theorem, the intersection of all the A_n and D_n^+ sets must be open and dense in $\mathbb{R}^+$. This is impossible because

$$\bigcap_n D_n^+ = \mathbb{Q}^+ \quad \text{whereas} \quad \bigcap_n A_n = \mathbb{R}^+ - \mathbb{Q}^+.$$

The contradiction means that (2) is satisfied.

Assume that S is a countable union of closed sets:

$$S = \bigcup_n C_n.$$

Then

$$\mathbb{R} - S = \bigcap_n (\mathbb{R} - C_n)$$

which is a countable intersection of open sets. This is impossible by the same reasoning as above since $\mathbb{R} - S$ contains the negative rationals.

Problem S03.4. Consider the following equation for a function $F(x, y)$ on $\mathbb{R}^2$:

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2}.$$

(a) Show that if a function F has the form $F(x, y) = f(x+y)+g(x-y)$ where $f, g : \mathbb{R} \to \mathbb{R}$ are twice differentiable, then F satisfies the equation.

(b) Show that if $F(x, y) = ax^2 + bxy + cy^2$ then $F(x, y) = f(x + y) + g(x - y)$ for some polynomials f, g in one variable.

Solution.

(a) By the chain rule

$$\frac{\partial F}{\partial x} = f'(x + y)\frac{d}{dx}(x + y) + g'(x - y)\frac{d}{dx}(x - y) = f'(x + y) + g'(x - y)$$

and

$$\frac{\partial^2 F}{\partial x^2} = f''(x + y)\frac{d}{dx}(x + y) + g''(x - y)\frac{d}{dx}(x - y) = f''(x + y) + g''(x - y).$$

Likewise

$$\frac{\partial F}{\partial y} = f'(x + y) - g'(x - y)$$

and

$$\frac{\partial^2 F}{\partial y^2} = f''(x + y) + g''(x - y).$$

Thus

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2}.$$

(b) The equation immediately yields $a = c$ so $F(x, y) = ax^2 + bxy + ay^2$. Then

$$\begin{aligned} F(x, y) &= a(x^2 + y^2) + bxy \\ &= a\left(\tfrac{1}{2}(x + y)^2 + \tfrac{1}{2}(x - y)^2\right) + b\left(\tfrac{1}{2}(x + y)^2 - \tfrac{1}{2}(x - y)^2\right) \\ &= \tfrac{1}{2}(a + b)(x + y)^2 + \tfrac{1}{2}(a - b)(x - y)^2. \end{aligned}$$

Thus taking $f(x) = \frac{1}{2}(a + b)x^2$ and $g(x) = \frac{1}{2}(a - b)x^2$ works.

Problem F04.3. Show that if $f_n \to f$ uniformly on the closed bounded interval $[a,b]$, then

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

Solution. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon/2$ when $n \geq N$. By definition of the Riemann integral, there are some piecewise constant functions g_n, h_n such that g_n majorizes f_n and h_n minorizes f_n for all n and

$$\int_a^b g_n(x)dx - \int_a^b f_n(x)dx < \varepsilon \quad \text{and} \quad \int_a^b f_n(x)dx - \int_a^b h_n(x)dx < \varepsilon.$$

Since f stays within $\varepsilon/2$ of $f_n(x)$ for n suffieciently large, it follows that for such n, $g_n + \varepsilon$ is piecewise constant and majorizes f and $h_n - \varepsilon$ is piecewise constant and minorizes f. Then

$$\overline{\int_a^b} f(x)dx \leq \int_a^b (g_n(x) + \varepsilon)dx \leq \int_a^b f_n(x)dx + \varepsilon + \varepsilon(b-a) = \int_a^b f(x)dx + (b-a+1)\varepsilon.$$

Likewise, we have

$$\underline{\int_a^b} f(x)dx \geq \int_a^b (h_n(x) - \varepsilon)dx \geq \int_a^b f_n(x)dx - \varepsilon - (b-a)\varepsilon = \int_a^b f_n(x)dx - (b-a+1)\varepsilon.$$

Since ε is arbitrarily small, this proves that $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

Problem F04.4. Suppose that (X,d) is a metric space, $x \in X$ and (x_n) is a sequence in X converging to x. Show that for every $y \in X$, $d(x_n,y) \to d(x,y)$.

Solution. From the triangle inequality, we have

$$d(x_n,y) \leq d(x,y) + d(x_n,x) \quad \Longrightarrow \quad d(x_n,y) - d(x,y) \leq d(x_n,x)$$

and

$$d(x,y) \leq d(x_n,y) + d(x_n,x) \quad \Longrightarrow \quad d(x,y) - d(x_n,y) \leq d(x_n,x)$$

which imply

$$|d(x_n,y) - d(x,y)| \leq d(x_n,x) \to 0 \quad \text{as} \quad n \to \infty.$$

Problem F04.5. Prove that the space $C[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ with the supremum norm $||f||_\infty = \sup\limits_{x\in[0,1]} |f(x)|$ is complete.

Solution. Let (f_n) be a Cauchy sequence in $C[0,1]$. Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that $m,n > N$ implies

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty < \varepsilon$$

for all $x \in [0,1]$. This means that for each $x \in [0,1]$, the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ and thuse converges to some $a_x \in \mathbb{R}$. Define $f : [0,1] \to \mathbb{R}$ so that $f(x) = a_x$ for every $x \in [0,1]$. It clear that $f_n \to f$ pointwise, and thus in the supremum norm. We need to prove that $f \in C[0,1]$. Let $\varepsilon > 0$. Then for any $x, y \in [0,1]$, there are $M, N \in \mathbb{N}$ and $\delta > 0$ so that

(1) $|f(x) - f_n(x)| < \varepsilon/3$ when $n \geq N$,

(2) $|f_n(x) - f_n(y)| < \varepsilon/3$ when $|x - y| < \delta$,

(3) $|f_n(y) - f(y)| < \varepsilon/3$ when $n \geq M$.

Setting $N^* = \max\{M, N\}$ and taking $n > N^*$, we have

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

which shows that f is continuous and so $C[0,1]$ is complete.

Problem F04.6. The Bolzano-Weierstrass Theorem in $\mathbb{R}^n$ states that if S is a closed, bounded subset in $\mathbb{R}^n$, then every sequence in S has a subsequence converging in S. Assume the theorem for $n = 1$ and prove the theorem for $n = 2$.

Solution. Let (x_n) be a sequence in S and let $M > 0$ be a bound on the norm of the elements of S. We see for each component $x_n^{(i)}$ of x_n,

$$\left|x_n^{(i)}\right| \leq ||x_n|| \leq M.$$

Thus, the sequence of first components $(x_n^{(1)})$ is a bounded sequence in $\mathbb{R}$, which, by assumption has a subsequence converging to an element of $\mathbb{R}$, say $x^{(1)}$. Call this subsequence $(x_{n_k}^{(1)})$. Then $(x_{n_k}^{(2)})$ is also a bounded sequence in $\mathbb{R}$ and thus has a convergent subsequence $(x_{n_{k_\ell}}^{(2)})$ which converges to $x^{(2)}$. Since any subsequence of a convergent sequence converges to the same limit, we know $(x_{n_{k_\ell}}^{(1)})$ still converges to $x^{(1)}$. Then the sequence (x_n) converges to the vector $x = (x^{(1)}, x^{(2)})$. Since S is closed, we know x must be in S.

Problem F04.7. Observe that the point $P = (1, 1, 1)$ belongs to the set S of points in $\mathbb{R}^3$ satisfying the equation

$$x^4y^2 + x^2z + yz^2 = 3.$$

Explain how the Implicit Function Theorem allows us to conclude that there is a differentiable function $f(x, y)$ such that $(x, y, f(x, y)) \in S$ for all (x, y) in a small open neighborhood of $(1, 1)$.

Solution. The implicit function theorem tells us that is $g(x, y, z)$ is continuously differentiable, $g(x_0, y_0, z_0) = a \in \mathbb{R}$ and $\frac{dg}{dz}(x_0, y_0, z_0) \neq 0$ then there is a neighborhood U of (x_0, y_0), a neighborhood V of z_0 and a continuously differentiable function $f : U \to V$ such that

$$\{(x, y, f(x, y)) : x, y \in U\} = \{(x, y, z) : g(x, y, z) = a\}.$$

Here, if we set

$$g(x, y, z) = x^4y^2 + x^2z + yz^2$$

and $(x_0, y_0, z_0) = (1, 1, 1)$ then g is smooth, $g(x_0, y_0, z_0) = 3$ and $\frac{dg}{dz}(x_0, y_0, z_0) = 3 \neq 0$. Then there are open neighborhoods U of (1,1), V of 1 and a continuously differentiable function $f : U \to V$, such that

$$S = \{(x, y, z) : g(x, y, z) = 3\} = \{(x, y, f(x, y)) : (x, y) \in U\}.$$

This is exacly the conclusion we wanted to draw.

Problem F04.8. Let $A \in M_n(\mathbb{R})$ be symmetric and let $Q(v) = v \cdot Av$ be the associated quadratic form defined for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

1. Show that $\nabla Q_v = 2Av$ where ∇Q_v is the gradient at v of the function Q.
2. Let M be the minimum value of $Q(v)$ on the unit sphere $S^n = \{v \in \mathbb{R}^n : ||v|| = 1\}$ and let $u \in S^n$ be a vector such that $Q(u) = M$. Prove using Lagrange multipliers that u is an eigenvector of A with eigenvalue M.

Solution.

1. Let $\mathbf{a}_j$ be the columns of A. We see

$$Q(v) = v \cdot Av = v \cdot \left(\sum_{j=1}^{n} v_j \mathbf{a}_j \right) = \sum_{j=1}^{n} v_j (v \cdot \mathbf{a}_j) = \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{n} v_i a_{ij} \right)$$

Then

$$\frac{\partial Q}{\partial v_k}(v) = \sum_{i=1}^{n} v_i a_{ik} + \sum_{j=1}^{n} v_j a_{kj}.$$

But since the matrix is symmetric,

$$\frac{\partial Q}{\partial v_k}(v) = 2 \sum_{i=1}^{n} v_i a_{ik} = 2v \cdot \mathbf{a}_k$$

Then

$$\nabla Q(v) = 2Av.$$

2. The problem is to minimize $Q(u)$ subject to $g(u) = ||u||^2 - 1 = 0$. Any minimizer u must satisfy

$$\nabla Q(u) = \lambda \nabla g(u) \quad \Longleftrightarrow \quad 2Au = \lambda(2u) \quad \Longleftrightarrow \quad Au = \lambda u,$$

for some $\lambda \in \mathbb{R}$. Taking the inner product of the above equation with u, we see

$$Q(u) = \lambda(u, u) = \lambda.$$

But $Q(u) = M$ so $M = \lambda$ is an eigenvalue of A.

Problem S04.4. Are there infinite compact subsets of $\mathbb{Q}$? Prove your assertion.

Solution. Yes. The set $A = \{0\} \cup \left\{\frac{1}{n} : n = 1, 2, 3, \ldots\right\}$ is obviously infinite, bounded and contained in $\mathbb{Q}$. Recall that a subset of $\mathbb{R}$ is compact iff it is closed and bounded. A is closed because

$$\mathbb{R} - A = (-\infty, 0) \cup (1, \infty) \cup \left(\bigcup_{n=1}^{\infty} \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right) \right)$$

is a union of open sets and is thus open. This implies that A is infinite and compact.

Problem S04.5. Suppose that $G \subset \mathbb{R}^n$ is open, $f : G \to \mathbb{R}^m$ and that $x_0 \in G$.

(a) Carefully define what is meant by $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$.

(b) Suppose that I is a line segment in G such that $f'(x)$ is defined for $x \in I$. Show that if f is differentiable at all points of I, then there is a point $c \in I$ such that

$$||f(q) - f(p)|| \le ||f'(c)|| \, ||q - p||$$

where p, q are distinct points in I.

Solution.

(a) We say that $f : G \to \mathbb{R}^m$ is differentiable at a point $x_0 \in G$ if there is a linear map $T : G \to \mathbb{R}^m$ such that

$$\frac{||f(x_0 + h) - f(x_0) - T(h)||}{||h||} \to 0 \qquad \text{as} \qquad ||h|| \to 0.$$

If such a T exists, we define $f'(x_0) = T$. That is $f'(x_0)$ is a linear map.

(b) If $f(q) = f(p)$, the result is trivial. Otherwise, we see that

$$||f(q) - f(p)|| = (f(q) - f(p)) \cdot \frac{(f(q) - f(p))}{||f(q) - f(p)||} := (f(q) - f(p)) \cdot w$$

where w is a unit vector. Define $g : G \to \mathbb{R}$ by

$$g(p) = f(p) \cdot w.$$

Since f is differentiable on I, so is g. then for any $x \in I$, $g'(x)$ is a functional on G and

$$g'(x)(u) = (f'(x)(u)) \cdot w.$$

By the mean value theorem, there is $c \in I$ such that

$$g(q) - g(p) = g'(c)(p - q) = (f'(c)(p - q)) \cdot w.$$

But $g(q) - g(p) = ||f(q) - f(p)||$. Then using Cauchy-Schwarz,

$$||f(q) - f(p)|| \le ||(f'(c)(p - q))|| \, ||w|| = ||(f'(c)(p - q))|| \, .$$

But now we use the fact that $||T(v)|| \le ||T||\,||v||$ for any linear operator T and vector v. This yields

$$||f(q) - f(p)|| \le ||f'(c)||\,||q-p||\,,$$

which completes the proof.

Problem S04.6. Let $||\cdot||$ be any norm on $\mathbb{R}^n$.

(a) Prove that there is a constant d with $||x|| \le d\,||x||_2$ for all $x \in \mathbb{R}^n$, and use this to show that $N(x) = ||x||$ is continuous in the usual topology.

(b) Prove that there is a constant c with $||x|| \ge c\,||x||_2$. (Hint: use the fact that N is continuous on $\{x : ||x||_2 = 1\}$)

(c) Show that if L is an n-dimensional subspace of an arbitrary normed space V, then L is closed.

Solution.

(a) Take any $x \in \mathbb{R}^n$ and write $x = \sum_{i=1}^n x_i e_i$ where e_i are the standard basis vectors. Then

$$||x|| = \left|\left|\sum_{i=1}^n x_i e_i\right|\right| \le \sum_{i=1}^n |x_i|\,||e_i|| \le \sqrt{\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n ||e_i||^2\right)} = d\,||x||_2$$

where $d = \left(\sum_{i=1}^n ||e_i||^2\right)$. Taking $x, y \in \mathbb{R}^n$ such that $||x-y||_2 < \delta = \frac{\varepsilon}{d}$. Then by the reverse triangle inequality,

$$|||x|| - ||y||| \le ||x-y|| \le d\,||x-y||_2 \le d \cdot \frac{\varepsilon}{d} = \varepsilon,$$

and hence the norm is continuous.

(b) Since the norm is continuous on the unit ball (with respect to the twonorm), and since the unit ball is compact, the norm achieves a minumum there. Say $c = \min_{||x||_2=1} ||x||$. Then for any $y \in \mathbb{R}^n$,

$$||y|| = ||y||_2 \left|\left|\frac{y}{||y||_2}\right|\right| \ge c\,||y||_2\,.$$

(c) Suppose L is an n-dimensional subspace of a normed space V. We prove that L contains its limit points and is thus closed. Let x be a limit point of L. Then there is a sequence (x_n) in L converging to x. Since L is $n$dimensional, there is a linear bijection $T : L \to R^n$. Define the norm $||\cdot||$ on $\mathbb{R}^n$ by $||y|| = ||T^{-1}y||_V$, $y \in \mathbb{R}^n$. From (a), (b), there are constants c, d such that for all $y \in \mathbb{R}^n$

$$c\,||y||_2 \le ||y|| \le d\,||y||_2\,.$$

Then

$$c\,||Tx_m - Tx_n||_2 \le ||Tx_m - Tx_n|| = ||x_n - x_m||_V \le d\,||Tx_n - Tx_m||_2$$

for all $m, n \in \mathbb{N}$. Let $\varepsilon > 0$. Since (x_n) is convergent, in particular, it is Cauchy, so there are $m, n \in \mathbb{N}$ so that $||x_n - x_m|| < c\varepsilon$. Choosing these show that (Tx_n) is Cauchy in R^n with respect to the two norm and so has a limit point in $y \in \mathbb{R}^n$ since the two norm is complete. Using the other side of the above inequality, we can show $y = Tx$ and so $x = T^{-1}y \in L$. Thus L is closed.

Problem F05.1. A real number α is said to be algebraic if for some finite set of integers $a_0, \ldots, a_n$ (not all zero), we have

$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0.$$

Prove that the set of algebraic numbers is countable.

Solution. Let A be the set of algebraic numbers and $\mathbb{Z}[x]$ the set of polynomials over $\mathbb{Z}$. We see

$$A = \bigcup_{p \in \mathbb{Z}[x]} \{x \in \mathbb{R} : p(x) = 0\}.$$

Since $\mathbb{Z}$ is countable, so is $\mathbb{Z}[x]$. Thus the union above is a countable union of finite sets and hence countable so A is countable.

Problem F05.3.

(a) Prove that if $f_j : [0,1] \to \mathbb{R}$ is a sequence of continuous functions which converge uniformly to $F : [0,1] \to \mathbb{R}$, then

$$\lim_{j\to\infty} \int_0^1 f_j(x)^2 dx = \int_0^1 F(x)^2 dx.$$

(b) Give an example of a sequence $f_j : [0,1] \to \mathbb{R}$ which converge to a continuous function $F : [0,1] \to \mathbb{R}$ pointwise and for which

$$\lim_{j\to\infty} \int_0^1 f_j(x)^2 dx \text{ exists, but}$$

$$\lim_{j\to\infty} \int_0^1 f_j(x)^2 dx \neq \int_0^1 F(x)^2 dx.$$

Solution.

(a) Let $\varepsilon > 0$. Since $f_j \to F$ uniformly, we know that F is continuous and hence bounded on $[0,1]$ by some $M \in \mathbb{R}$. Then by uniform convergence, there is $N \in \mathbb{N}$ such that $j \geq N$ implies

$$|f_j(x) - F(x)| < \sqrt{\frac{\varepsilon}{2}} \quad \text{and} \quad |f_j(x) - F(x)| < \frac{\varepsilon}{4M}, \quad x \in [0,1].$$

Letting $j \geq N$, we see

$$\begin{aligned}\left|f_j(x)^2 - F(x)^2\right| &= \left|(f_j(x) - F(x))^2 + 2f_j(x)F(x) - 2F(x)^2\right| \\ &\leq |f_j(x) - F(x)|^2 + 2\,|F(x)|\,|f_j(x) - F(x)| \\ &< \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M} = \varepsilon, \quad x \in [0,1].\end{aligned}$$

Thus $f_j^2 \to F^2$ uniformly and the result follows by the previous problem.

(b) Define

$$f_j(x) = \begin{cases} 0, & x < \frac{1}{2j}, \\ \sqrt{4j^2\left(x - \frac{1}{2j}\right)}, & \frac{1}{2j} \leq x < \frac{1}{j}, \\ \sqrt{4j^2\left(\frac{3}{2j} - x\right)}, & \frac{1}{j} \leq x < \frac{3}{2j}, \\ 0, & x \geq \frac{3}{2j} \end{cases}$$

Then each f_j^2 is a triangular spike centered at $\frac{1}{j}$ with base length $\frac{1}{j}$ and height $2j$. Thus

$$\int_0^1 f_j(x)^2 dx = \frac{1}{2} \cdot \frac{1}{j} \cdot 2j = 1 \quad \text{so} \quad \lim_{j\to\infty} \int_0^1 f_j(x)^2 dx = 1.$$

However, for any $x \in [0,1]$, there is some $J_x \in \mathbb{N}$ such that $\frac{3}{2j} < x$ when $j \geq J_x$. For $j \geq J_x$, we have $f_j(x) = 0$. Thus

$$\lim_{j\to\infty} f_j(x)^2 = 0, \quad x \in [0,1].$$

Hence f_j^2 converge pointwise to the zero function so

$$\int_0^1 \lim_{j\to\infty} f_j(x)^2 dx = 0.$$

Problem F05.4. Suppose $F : [0,1] \to [0,1]$ is a C^2 function with $F(0) = 0, F(1) = 0$ and $F''(x) < 0$ for all $x \in [0,1]$ Prove that the arc length of the curve $\{(x, F(x)) : x \in [0,1]\}$ is less than 3.

Solution. The arc length is given by

$$L = \int_0^1 \sqrt{1 + F'(x)^2} dx.$$

However, $\sqrt{a^2 + b^2} \leq |a| + |b|$ for all $a, b \in \mathbb{R}$, so

$$L \leq \int_0^1 (1 + |F'(x)|) dx = 1 + \int_0^1 |F'(x)|\, dx.$$

Now $F''(x) < 0$ implies that F' is a decreasing function of x. Since $F(0) = F(1) = 0$, we know F must increase to a maximum and then decrease back to 0. Let the maximum of F occur at $x = x_M$. Then

$$L \leq 1 + \int_0^{x_M} F'(x)dx - \int_{x_M}^1 F'(x)dx = 1 + F(x_M) + F(x_M) = 1 + 2F(x_M).$$

Then since $F(x_M) \leq 1$, we have $L \leq 3$.

Problem F05.8. For a real $n \times n$ matrix A set $||A|| = \sup_{||x||=1} ||Ax||$.

(a) Prove that $||A + B|| \leq ||A|| + ||B||$.

(b) Use part (a) to check that the set M of all $n \times n$ matrices is a metric space if the distance dunction d is defined by

$$d(A, B) = ||B - A|| .$$

(c) Prove that (M, d) is complete.

Solution.

(a) For any particular $x \in \mathbb{R}^n$ with $||x|| = 1$, we know

$$||Ax|| \leq ||A|| ,$$

since the norm is the supremum over all such x. Taking $||x|| = 1$, we see

$$||(A + B)x|| = ||Ax + Bx|| \leq ||Ax|| + ||Bx|| \leq ||A|| + ||B|| .$$

Since the inequality holds for all such x, it holds for the supremum. Thus the inequality is proven.

(b) The non-negativity and symmetry is obvious. If $d(A, B) = 0$ then $||(B - A)x|| = 0$ for all $x \in \mathbb{R}^n, ||x|| = 1$. Taking $x = e_i$, for each $i = 1, \ldots, n$ shows that each column of $B - A$ is the zero vector so $B - A = 0$ and $A = B$. The triangle inequality follows from (a) since for any A, B, C, we have

$$d(A, B) = ||B - A|| = ||B - C + C - A|| \leq ||B - C|| + ||C - A|| = d(A, C) + d(B, C).$$

Thus (M, d) is a metric space.

(c) Let (A_n) be a Cauchy Sequence in M. We notice that

$$\begin{aligned} d(A_n, A_m) &= ||A_m - A_n|| \\ &\geq ||(A_m - A_n)e_i|| \\ &\geq \left|\left|a_i^{(m)} - a_i^{(n)}\right|\right| \quad \text{(the } i^{\text{th}} \text{ column of each matrix)} \\ &\geq \left|a_{ij}^{(m)} - a_{ij}^{(n)}\right| . \end{aligned}$$

This shows that each entry of each matrix forms a Cauchy sequence in $\mathbb{R}$. So the sequences of each of the entries converge and the matrices converge. Thus (M, d) is complete.

Problem S05.7. Let X, Y be topological spaces. We say that a continuous function $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact in X whenever K is compact in Y.

(a) Give an example of a function that is proper but not a homeomorphism.

(b) Give an example of a function which is continuous but not proper.

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is C^1 and

$$|f'(x)| \geq 1, \quad \text{for all } x \in \mathbb{R}.$$

Show that f is proper.

Solution.

(a) Let X be any compact metric space, and let $f : X \to \mathbb{R}$ be defined $f(x) = 0$ for all $x \in X$. Then clearly f is continuous. Also, if $K \subset \mathbb{R}$ is compact, then

$$f^{-1}(K) = \{x \in X : f(x) \in K\} = \begin{cases} \varnothing, & 0 \notin K, \\ X, & 0 \in K \end{cases}$$

is compact. However, the function is not a bijection since it is not a surjection and thus it is not a homeomorphism.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = 0$ for all $x \in \mathbb{R}$. Then clearly f is continuous, but for any compact $K \subset \mathbb{R}$ such that $0 \in K$, we have $f^{-1}(K) = \mathbb{R}$ which is not compact. Thus f is not proper.

(c) Suppose $K \subset \mathbb{R}$ is compact. Let $(U_\alpha)_{\alpha \in I}$ be an open cover of $f^{-1}(K)$. Then $(f(U_\alpha))_{\alpha \in I}$ is an open cover of $f(f^{-1}(K)) = K$. Since K is compact, there is finite subcover $(f(U_i))_{i=1}^N$ for some $N \in \mathbb{N}$.

Let $x \in f^{-1}(K)$. Then $f(x) \in f(U_i)$ for some i. But this implies there is $y \in U_i$ such that $f(y) = f(x)$. If we assume that $x \neq y$ (wlog $x < y$), then by the mean value theorem, there is $c \in (x, y)$ such that $f(x) - f(y) = f'(c)(x - y)$. But since $f(x) = f(y)$, this leads to

$$0 = |f'(c)|\,|x - y| \geq |x - y|,$$

which is a contradiction. Thus $x = y$, so $x \in U_i$ for some i. Hence $(U_i)_{i=1}^N$ is an finite subcover for $f^{-1}(K)$ so $f^{-1}(K)$ is compact and so f is proper.

Problem S05.8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1. Show that

$$\lim_{n \to \infty} \sum_{j=1}^{n} \left| f\left(\tfrac{j-1}{n}\right) - f\left(\tfrac{j}{n}\right) \right|$$

is equal to

$$\int_0^1 |f'(t)|\, dt.$$

Solution. Let $\varepsilon > 0$. Since f' is continuous on $[0,1]$ so is $|f'|$ and since $[0,1]$ is compact, we know that $|f'|$ is uniformly continuous there. Let $N \in \mathbb{N}$ be large enough that for all $x, y \in [0,1]$,

$$|x-y| < \frac{1}{N} \quad \implies \quad \Big| \, |f'(x)| - |f'(y)| \, \Big| < \varepsilon.$$

Then for $n \geq N$, by the mean value theorem, there is $x_n \in ((j-1)/n, j/n)$ such that $\left|f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right)\right| = \frac{1}{n}|f'(x_n)|$. Also, there is $y_n \in ((j-1)/n, j/n)$ which maximizes $|f'|$ on that interval. Then for $n > N$,

$$\begin{aligned}
\left| \int_0^1 |f'(t)|\,dt - \sum_{j=1}^n \left|f\left(\tfrac{j-1}{n}\right) - f\left(\tfrac{j}{n}\right)\right| \right| &= \left| \sum_{j=1}^n \int_{(j-1)/n}^{j/n} |f'(t)|\,dt - \left|f\left(\tfrac{j-1}{n}\right) - f\left(\tfrac{j}{n}\right)\right| \right| \\
&\leq \left| \sum_{j=1}^n \frac{1}{n}|f'(y_n)| - \frac{1}{n}|f'(x_n)| \right| \\
&\leq \frac{1}{n}\sum_{j=1}^n \Big| \, |f'(y_n)| - |f'(x_n)| \, \Big| < \frac{1}{n} n\varepsilon = \varepsilon.
\end{aligned}$$

Thus $\lim_{n\to\infty} \sum_{j=1}^n \left|f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right)\right| = \int_0^1 |f'(t)|\,dt.$

Problem S05.10. Let K be the set of $f : [0,1] \to \mathbb{R}$ that obey

(i) $|f(x) - f(y)| \leq |x-y|$,

(ii) $\displaystyle\int_0^1 f(x)dx = 1.$

Prove that K is a compact subset of $C[0,1]$.

Solution. Arzela-Ascoli says that a subset of $C[0,1]$ is compact if and only if it is uniformly bounded and equicontinuous. It is clear that $K \subset C[0,1]$ because the (i) implies continuity. Let $f \in K$. If there was $x \in [0,1]$ such that $f(x) \geq 3$, then $|f(y) - f(x)| \leq |y - x| \leq 1$ implies that $f(y) \geq 2$ for all $y \in [0,1]$ which means that f couldn't satisfy (ii). Thus K is uniformly bounded. Further, (i) implies equicontinuity directly. Thus K is compact.

Problem S05.11. Make $M_n(\mathbb{C})$ into a metric space in the following fashion

$$d(A,B) = \sqrt{\sum_{i,j} |A_{ij} - B_{ij}|^2}.$$

(a) Suppose $F : \mathbb{R} \to M_n(\mathbb{C})$ is continuous. Show that the set

$$\{x : F(x) \text{ is invertible}\}$$

is open in the usual topology on $\mathbb{R}$.

(b) Show that on the set given above, $x \mapsto [F(x)]^{-1}$ is continuous.

Solution.

(a) We see
$$\{x : F(x) \text{ is invertible}\} = \{x : \det F(x) \neq 0\}.$$
The latter is open because it is the pullback of an open set under the composition function $\det F(x)$ which is continuous since both det and F are continuous.

(b) The maps is continuous because F is continuous and $A \mapsto A^{-1}$ is continuous since the entries of the inverse are polynomials in the entries of A.

Problem S05.12. Let (X, d) be a metric space. Prove that the following are equivalent:

(a) There is a countable dense set in X.

(b) There is a countable basis for the topology on X.

Solution. If there is a countable basis $\{B_n\}_{n\in\mathbb{N}}$ for the topology, then taking $x_n \in B_n$ for each $n \in \mathbb{N}$ gives a countable dense set $\{x_1, x_2, x_3, \ldots\}$.

Conversely, assume that $(x_n)_{n\in\mathbb{N}}$ is a countable dense set. Consider the set
$$\beta = \{B(x_n, 1/m) : n \geq 1, m \geq 1\}$$
Let U be open in X with $x \in U$. The ball $B(x, 2/k) \subset U$ for sufficiently large k. Also there is some n, such that $x \in B(x_n, 1/k)$. By the triangle inequality, We have $B(x_n, 1/k) \subset U$. Thus for every $x \in U$, there is $B_x \in \beta$ such that $x \in B_x \subset U$. Then
$$U = \bigcup_{x\in U} B_x.$$
Ergo, we can build any open set in X from sets in β so β is a basis for the topology on X and clearly β is countable since it can be put in bijection with $\mathbb{N} \times \mathbb{N}$.

Problem W06.1. Show that for each $\varepsilon > 0$, there exists a sequence of intervals (I_n) with the properties
$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} |I_n| < \varepsilon.$$

Solution. Let $(q_n)_{n\in\mathbb{N}}$ be a denumeration of the rationals. Define $I_n = (q_n - \varepsilon/2^{n+2}, q_n + \varepsilon/2^{n+2})$. Then clearly
$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$$
and
$$\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} 2 \cdot \frac{\varepsilon}{2^{n+2}} = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} < \varepsilon.$$

Problem W06.3. Consider a function $f : [a, b] \to \mathbb{R}$ which is twice continuously differentiable. Let $a = x_0 < x_1 < \ldots < x_n = b$ be the uniform partition of $[a, b]$. Show that there exists M such that for all $n \geq 1$,

$$\left| \frac{b-a}{n} \left(\frac{1}{2} f(x_0) + f(x_1) + \ldots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right) - \int_a^b f(x)dx \right| \leq \frac{M}{n^2}.$$

Solution. Call the expression on the left hand side E. Then we rewrite E:

$$\begin{aligned} E &= \left| \left(\sum_{i=0}^{n-1} \frac{b-a}{2n} (f(x_{i+1}) - f(x_i)) \right) - \int_a^b f(x)dx \right| \\ &= \left| \sum_{i=0}^{n-1} \left(\frac{b-a}{2n} (f(x_{i+1}) - f(x_i)) - \int_{x_i}^{x_{i+1}} f(x)dx \right) \right| \\ &\leq \sum_{i=0}^{n-1} \left| \frac{b-a}{2n} (f(x_{i+1}) - f(x_i)) - \int_{x_i}^{x_{i+1}} f(x)dx \right|. \end{aligned}$$

Integrating by parts, we see

$$\int_{x_i}^{x_{i+1}} f(x)dx = \left[(x-C)f(x) \Big|_{x_i}^{x_{i+1}} \right] - \int_{x_i}^{x_{i+1}} (x-C)f'(x)dx,$$

where C is a parameter we can choose. Letting $C = (x_{i+1} + x_i)/2$, we see

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{b-a}{2n} (f(x_{i+1}) - f(x_i)) - \int_{x_i}^{x_{i+1}} (x-C)f'(x)dx.$$

Then if E_i is the i^{th} member of the sum above, we have

$$E_i = \left| \int_{x_i}^{x_{i+1}} \left(x - \frac{x_{i+1} + x_i}{2} \right) f'(x)dx \right|.$$

Integrating by parts again, we get

$$E_i = \left| \left[\frac{1}{2} \left(\left(x - \frac{x_{i+1} + x_i}{2} \right)^2 - B \right) f'(x) \Big|_{x_i}^{x_{i+1}} \right] - \int_{x_i}^{x_{i+1}} \frac{1}{2} \left(\left(x - \frac{x_{i+1} + x_i}{2} \right)^2 - B \right) f''(x)dx \right|.$$

We choose $B = \left(\frac{b-a}{2n}\right)^2$ so that the boundary term goes to zero. Then

$$\begin{aligned} E_i &= \left| \int_{x_i}^{x_{i+1}} \frac{1}{2} \left(\left(x - \frac{x_{i+1} + x_i}{2} \right)^2 - \left(\frac{b-a}{2n} \right)^2 \right) f''(x)dx \right| \\ &\leq \int_{x_i}^{x_{i+1}} \frac{1}{2} \left| \left(x - \frac{x_{i+1} + x_i}{2} \right)^2 - \left(\frac{b-a}{2n} \right)^2 \right| |f''(x)| \, dx. \end{aligned}$$

Now f'' is continuous on $[a,b]$ and thus bounded by some $K \in \mathbb{R}$. Hence

$$\begin{aligned} E_i &\le \frac{K}{2} \int_{x_i}^{x_{i+1}} \left(\frac{b-a}{2n}\right)^2 - \frac{1}{2}\left(x - \frac{x_{i+1}+x_i}{2}\right)^2 dx \\ &= \frac{K(b-a)^3}{12n^3}. \end{aligned}$$

Then summing over all i, we see

$$E \le \frac{K(b-a)^3}{12n^2}.$$

Problem W06.5. Consider a function $f(x,y)$ which is twice continuously differentiable. Suppose that f has a unique minimum at $(x,y) = (0,0)$. Prove that at $(0,0)$,

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} \ge \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

[You may use without proof that the mixed partials are equal for C^2 functions.]

Solution. If f has a minimum at a point (x,y) then $\nabla f(x,y) = \vec{0}$. By Taylor's Theorem in a neighborhood of (0,0),

$$\begin{aligned} f(x,y) &= f(0,0) + \nabla f(0,0)\cdot(x,y) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}x^2 + 2\frac{\partial^2 f}{\partial x \partial y}xy + \frac{\partial^2 f}{\partial y^2}y^2\right) + E(x,y) \\ &= f(0,0) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}x^2 + 2\frac{\partial^2 f}{\partial x \partial y}xy + \frac{\partial^2 f}{\partial y^2}y^2\right) + E(x,y) \end{aligned}$$

where $E(x,y) \to 0$ quickly as $(x,y) \to (0,0)$ and the second derivatives are evaluated at $(0,0)$. We know $f(x,y) - f(0,0) \ge 0$ since $f(0,0)$ is a minimum. This gives

$$-E(x,y) \le f(x,y) - f(0,0) - E(x,y) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}x^2 + 2\frac{\partial^2 f}{\partial x \partial y}xy + \frac{\partial^2 f}{\partial y^2}y^2\right).$$

But since $E(x,y) \to 0$, it follows that, in some neighborhood of $(0,0)$,

$$\left(\frac{\partial^2 f}{\partial x^2}x^2 + 2\frac{\partial^2 f}{\partial x \partial y}xy + \frac{\partial^2 f}{\partial y^2}y^2\right) \ge 0.$$

Consider the matrix

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix};$$

again, with the functions evaluated at $(0,0)$. The above inequality tells us that this matrix is non-negative definite. Hence its eigenvalues are non-negative and so its determinant (which is the product of the eigenvalues) is non-negative. Thus at $(0,0)$,

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} \ge \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

Problem S06.3. Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{5/2}}$$

converges for all $x \in \mathbb{R}$ and that $f(x)$ is continuous on $\mathbb{R}$ with a continuous derivative.

Solution. For any $x \in \mathbb{R}$, we have

$$|f(x)| = \left| \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{5/2}} \right| \le \sum_{n=1}^{\infty} \frac{|\sin(nx)|}{n^{5/2}} \le \sum_{n=1}^{\infty} \frac{1}{n^{5/2}},$$

which converges. Hence the sum converges for each $x \in \mathbb{R}$; in fact, this shows that the convergence is uniform.

Let (f_N) be defined by $f_N(x) = \sum_{n=1}^{N} \frac{\sin(nx)}{n^{5/2}}$. Then each f_N is continuous as a finite sum of continuous functions. Also, for any $x \in \mathbb{R}$,

$$|f(x) - f_N(x)| = \left| \sum_{n=N+1}^{\infty} \frac{\sin(nx)}{n^{5/2}} \right| \le \sum_{n=N+1}^{\infty} \frac{1}{n^{5/2}}$$

which goes to zero as $N \to \infty$. Thus $f_N \to f$ uniformly. Since the uniform limit of continuous functions is continuous, we know that f is continuous.

Further, each f_N is actually differentiable, with derivative

$$f_N'(x) = \sum_{n=1}^{N} \frac{\cos(nx)}{n^{3/2}}.$$

These also converge uniformly by the same reasoning as above, so we know that f is also differentiable with derivative

$$f'(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{3/2}}.$$

Problem S06.5.

(A) Define uniform continuity for a function f on a metric space X with the distance function $\rho(x, y)$.

(B) Prove that if $0 < \alpha < 1$, then $F(x) = x^{\alpha}$ is uniformly continuous on $[0, \infty)$.

Solution.

(A) We say a function $f : (X, \rho) \to (Y, \sigma)$ is uniformly continuous if for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in X$, $\rho(x, y) < \delta \implies \sigma(f(x), f(y)) < \varepsilon$.

(B) We prove that $F(x)$ is uniformly continuous on $[0, 2]$ and on $[1, \infty)$. Since $F(x)$ is continuous on $[0, 2]$ and $[0, 2]$ is compact, $F(x)$ is uniformly continuous there. Let $\varepsilon > 0$ and take $\delta = \varepsilon/\alpha$. Then for $x, y \in [1, \infty)$, by the mean value theorem, there is $z \in (x, y)$ such that

$$|F(x) - F(y)| = |F'(z)| \, |x - y| = \alpha z^{1-\alpha} \, |x - y| \le \alpha \, |x - y| < \varepsilon,$$

when $0 < |x - y| < \delta$. Thus F is uniformly continuous on $[1, \infty)$.

Now, for $\varepsilon > 0$, there is $\delta_1 > 0$ and $\delta_2 > 0$ such that for $x, y \in [0, 2]$,

$$|x - y| < \delta_1 \quad \implies \quad |F(x) - F(y)| < \varepsilon$$

and for $z, w \in [1, \infty)$,

$$|z - w| < \delta_2 \quad \implies \quad |F(z) - F(w)| < \varepsilon.$$

Take $\delta = \min\left\{\delta_1, \delta_2, \frac{1}{2}\right\}$. Then for $x, y \in [0, \infty)$ with $|x - y| < \delta$ we have that $x, y \in [0, 2]$ or $x, y \in [1, \infty)$. Then since δ is less than both δ_1 and δ_2, we have that $|F(x) - F(y)| < \varepsilon$. Thus F is uniformly continuous.

Problem S06.6. Let W be the subset of $C[0, 1]$ satisfying

$$|f(x) - f(y)| < |x - y| \quad \text{and} \quad \int_0^1 f(x)^2 dx = 1.$$

(A) Prove that W is uniformly bounded.

(B) Prove that W is a compact subset of $C[0, 1]$.

Solution.

(A) Suppose that for some $f \in W$, there is an $x \in [0, 1]$ such that $f(x) > 3$. Then

$$|f(x) - f(y)| < |x - y| \le 1$$

which means that $f(y) > 2$ for all $y \in [0, 1]$. Then $\int_0^1 f(x)^2 dx > \int_0^1 4dx > 4$, a contradiction. Hence $f(x) < 3$ for all $x \in [0, 1]$. Likewise, assume there is $x \in [0, 1]$ such that $f(x) < -3$. Then

$$|f(x) - f(y)| < |x - y| \le 1$$

implies that $f(y) < -2$ for all $y \in [0, 1]$ and thus $f(y)^2 > 4, y \in [0, 1]$. Then $\int_0^1 f(x)^2 dx > 4$, a contradiction. Thus $f(x) \ge -3, x \in [0, 1]$. Hence W is uniformly bounded.

(B) By the Arzela-Ascoli theorem, W is compact if and only if it is equicontinuous and uniformly bounded. We proved the uniform bound in part (A). The first condition directly implies equicontinuity of W. Thus W is compact.

Problem F07.2. Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable in $(a, b) - \{c\}$. If $\lim_{x \to c} f'(x) = d \in \mathbb{R}$, show that f is differentiable at c and $f'(c) = d$.

Solution. It suffices to show that $|f(x) - (f(c) + d(x - c))|$ gets arbitrarily small if x is near c; this will prove that $f'(c)$ exists and equals d.

Let $\varepsilon > 0$. Then since $\lim_{x \to c} f'(x) = d$, there is $\delta_1 > 0$, such that $|f'(x) - d| < \varepsilon/4$ when $|x - c| < \delta_1$.

Also, since f is continuous at c, there is $\delta_2 > 0$ such that $|f(x) - f(c)| < \varepsilon/4$ when $|x - c| < \delta_2$.

Take $y \in (a, b), 0 < |y - c| < \min\{\delta_1, \delta_2\}$. Then f is differentiable at y and so there is $\delta_3 > 0$ such that $|x - y| < \delta_3$ implies that $|f(x) - (f(y) + f'(y)(x - y))| < \varepsilon/4$.

Now taking $|x - c|$ small enough, will ensure that $|x - y| < \delta_4$, and so

$$\begin{aligned} |f(x) - (f(c) + d(x - c))| &\leq |f(x) - (f(y) + f'(y)(x - y))| + \\ &\qquad |(f(y) + f'(y)(x - y)) - (f(c) + d(x - c))| \\ &< \varepsilon/4 + |f(y) - f(c)| + |f'(y)(x - y) - d(x - c)| \\ &< \varepsilon/4 + \varepsilon/4 + |f'(y)(x - y) - d(x - y) + d(x - y) - d(x - c)| \\ &\leq \varepsilon/2 + |f'(y) - d|\,|x - y| + |d|\,|y - c| \\ &< \varepsilon/2 + \varepsilon\delta_3/4 + |d| \min\{\delta_1, \delta_2\}. \end{aligned}$$

Decreasing each δ (if necessary) will ensure that this is less than ε. Thus $f'(c) = d$.

Problem F07.4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice-differentiable and its second derivative satisfies $|f''(x)| \leq B$.

a) Prove that

$$\left| 2Af(0) - \int_{-A}^{A} f(x)dx \right| \leq \frac{A^3 B}{3}.$$

b) Use the result from part a) to justify the estimate:

$$\left| \int_a^b f(x)dx - \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{2k-1}{2n}(b-a) \right) \right| \leq Cn^{-2}$$

where C is a constant which does not depend on n.

Solution.

a) Notice that $2Af(0) = \int_{-A}^{A} f(0)dx$.

By Taylor's theorem, for any $x \in [-A, A]$ there is $\xi_x \in [-A, A]$ such that $f(x) = f(0) + f'(0)x + \frac{f''(\xi_x)}{2}x^2$. Then for each x,

$$|f(x) - f(0) - xf'(0)| = \left| \frac{f''(\xi_x)}{2} x^2 \right| \leq \frac{Bx^2}{2}.$$

From this, we see

$$\begin{aligned}
\left|2Af(0) - \int_{-A}^{A} f(x)dx\right| &= \left|\int_{-A}^{A} f(x) - f(0)dx\right| \\
&= \left|\int_{-A}^{A} f(x) - f(0) - xf'(0) + xf'(0)dx\right| \\
&\le \left|\int_{-A}^{A} f(x) - f(0) - xf'(0)dx\right| + \left|\int_{-A}^{A} f'(0)xdx\right| \\
&\le \int_{-A}^{A} \left|f(x) - f(0) - xf'(0)\right| dx \\
&\le \int_{-A}^{A} \frac{Bx^2}{2} dx = \frac{A^3B}{3}.
\end{aligned}$$

b) Let $a = x_0 < x_1 < \cdots < x_n = b$ be the uniform partition of $[a, b]$. Put $A_k = a + \frac{2k-1}{2n}(b-a)$. Then A_k is the midpoint of $[x_{k-1}, x_k]$ for each k. Put $f_k(x) = f(A_k - x)$ for each k. Then

$$f\left(a + \frac{2k-1}{2n}(b-a)\right) = f(A_k) = f_k(0)$$

and

$$\begin{aligned}
\int_{x_{k-1}}^{x_k} f(x)dx &= \int_{x_{k-1}-A_k}^{x_k-A_k} f(x - A_k)dx \\
&= \int_{-(b-a)/2n}^{(b-a)/2n} f(x - A_k)dx \\
&= \int_{-(b-a)/2n}^{(b-a)/2n} f(A_k - x)dx \\
&= \int_{-(b-a)/2n}^{(b-a)/2n} f_k(x)dx.
\end{aligned}$$

Finally, but $h = \frac{b-a}{2n}$. Then,

$$\begin{aligned}
\left|\int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{2k-1}{2n}(b-a)\right)\right| &= \left|\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} f(x)dx - 2h\sum_{k=1}^{n} f_k(0)\right| \\
&= \left|\sum_{k=1}^{n}\int_{-h}^{h} f_k(x)dx - 2h\sum_{k=1}^{n} f_k(0)\right| \\
&\le \sum_{k=1}^{n}\left|\int_{-h}^{h} f_k(x)dx - 2hf_k(0)\right| \\
&\le \sum_{k=1}^{n} \frac{h^3B}{3} = \frac{nh^3B}{3} = \frac{B(b-a)^3}{12n^2},
\end{aligned}$$

which proves the claim.

Problem F07.5.

a) Show that, given a continuous function $f : [0,1] \to \mathbb{R}$ such that $f(1) = 0$, there is a sequence of polynomials which vanish at 1 and converge uniformly to f on $[0,1]$.

b) If f is continuous on $[0,1]$ and

$$\int_0^1 f(x)(x-1)^k dx = 0, \text{ for all } k = 1, 2, 3, \ldots,$$

show that $f(x) \equiv 0$.

Solution.

a) Let $\varepsilon > 0$.

By the Stone-Weierstrass theorem, there is a polynomial $p : [0,1] \to \mathbb{R}$ such that $||f - p||_\infty < \varepsilon/3$. We show that the sequence $\{p(x)(1 - x^n)\}$ converges uniformly to f on $[0,1]$. Let p_n be the n^{th} member of the sequence. It is clear that each p_n is a polynomial which vanishes at 1.

Since $f(1) = 0$, we know $|p(1)| < \varepsilon/3$. Hence by continuity, there is $\delta > 0$ (and wlog, $\delta < 1$), such that $|P(x)| < \varepsilon/2$ whenever $x \in [1-\delta, 1]$.

Since p is continuous on $[0,1]$ it is bounded; say $|p(x)| \le B$ for all $x \in [0,1]$. Let $N \in \mathbb{N}$ be such that $(1-\delta)^n < \varepsilon/3B$, whenever $n \ge N$.

Take $n \ge N$. Now for any $x \in [0,1]$, we either have $x \in [0, 1-\delta]$ or $x \in (1-\delta, 1]$. In the former case, $|x^n p(x)| \le |1-\delta|^n |p(x)| < B \cdot \varepsilon/3B = \varepsilon/3$. In the latter case, $|x^n p(x)| \le |p(x)| < \varepsilon/2$. Hence for all $x \in [0,1]$, $|x^n p(x)| < \varepsilon/2$ and so $||x^n p||_\infty \le \varepsilon/2$.

For such n, we see

$$||f - p_n||_\infty = ||f - p(1-x^n)||_\infty \le ||f - p||_\infty + ||px^n||_\infty \le \varepsilon/3 + \varepsilon/2 < \varepsilon.$$

Hence, $p_n \to f$ uniformly.

b) Suppose p_n is a sequence of polynomials vanishing at 1 which converge uniformly to f. Then each p_n can be written as a finite number of terms of the form $(x-1)^k$ so

$$\int_0^1 f(x)(x-1)^k dx = 0, \text{ for all } k \quad \Longrightarrow \quad \int_0^1 f(x) p_n(x) dx = 0, \text{ for all } n.$$

By uniform convergence, we can switch limits and integrals, so

$$\int_0^1 f(x)^2 dx = \int_0^1 \left(\lim_{n\to\infty} p_n(x) \right) f(x) dx = \lim_{n\to\infty} \int_0^1 p_n(x) f(x) dx = \lim_{n\to\infty} 0 = 0.$$

Hence $||f||_2 = 0$ and so $f \equiv 0$.

Problem F07.9. Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded and suppose for each $n \in \mathbb{N}$, $u_n : \mathbb{R} \to \mathbb{R}$ is differentiable and solves

$$u_n'(x) = F(u_n(x), x), \quad x \in \mathbb{R}.$$

a) Assume that $u_n \to u$ uniformly. Show that u is differentiable and solves
$$u'(x) = F(u(x), x).$$

b) Suppose that
$$u'(x) = F(u(x), x), \quad u(x_0) = y_0,$$
has a unique solution $u : \mathbb{R} \to \mathbb{R}$ and that $u_n(x_0) \to y_0$ as $n \to \infty$. Show that $u_n \to u$ uniformly.

Solution.

a) In particular, all u_n are continuous and thus u is continuous as the uniform limit of continuous functions. For fixed $x_0 \in \mathbb{R}$, if we can prove that
$$u(x) = u(x_0) + \int_{x_0}^{x} F(u(t), t)dt,$$
by continuity of u and F, it will actually be the case that u is differentiable and by the fundamental theorem of calculus, $u'(x) = F(u(x), x)$. It is clearly true that
$$u_n(x) = u_n(x_0) + \int_{x_0}^{x} F(u_n(t), t)dt.$$
Let $\varepsilon > 0$. Then for $x \neq x_0$, by continuity of F, there is $\delta > 0$ such that
$$|F(u,t) - F(v,t)| < \frac{\varepsilon}{3|x - x_0|}$$
when $||u - v||_\infty < \delta$. Without loss of generality, we can take $\delta < \varepsilon/3$. Further, by uniform convergence of $u_n \to u$, there is $N \in \mathbb{N}$ such that $||u - u_n||_\infty < \varepsilon/3$ for all $n \geq N$. For such n,
$$\begin{aligned}\left|u(x) - u(x_0) - \int_{x_0}^{x} F(u(t), t)\right| &= \left|u(x) - u_n(x) + u_n(x_0) - u(x_0) + \int_{x_0}^{x} F(u_n, t) - F(u, t)dt\right| \\ &\leq |u(x) - u_n(x)| + |u_n(x_0) - u(x_0)| + \int_{x_0}^{x} |F(u_n, t) - F(u, t)|\, dt \\ &= \varepsilon/3 + \varepsilon/3 + \int_{x_0}^{x} \frac{\varepsilon}{3|x - x_0|} dt = \varepsilon.\end{aligned}$$
Since $\varepsilon > 0$ was arbitrary, this implies that
$$u(x) = u(x_0) + \int_{x_0}^{x} F(u(t), t)dt,$$
which completes the proof.

b) We simply need to show that u_n converge uniformly to *some* function u_* because then by the above work, u_* will satisfy

$$u_*'(x) = y_0 + \int_{x_0}^{x} F(u_*(t), t)dt$$

and by uniqueness, we will have $u \equiv u_*$.

Consider, since F is bounded (by, say $L > 0$), we see

$$|u_n(x) - u_n(y)| = \left| \int_y^x F(u_n(t), t)dt \right| \le \left| \int_y^x L dt \right| = L\,|x - y|\,.$$

Hence the sequence (u_n) is equicontinuous.

Further, since $u_n(x_0) \to y_0$, there is $N \in \mathbb{N}$ such that $|u_n(x_0) - y_0| \le 1, n \ge N$. Then for any $n \ge N$,

$$\begin{aligned} |u_n(x)| &= \left| u_n(x_0) + \int_{x_0}^{x} F(u_n(t), t)dt \right| \\ &\le 1 + |y_0| + L\,|x - x_0| \end{aligned}$$

which shows that (u_n) is pointwise bounded. The set is also closed by the work from part *a*).

Thus by Arzela-Ascoli, there is a subsequence of (u_n) which converges uniformly on any compact subset of $\mathbb{R}$. The same reasoning would work for any subsequence of (u_n). Thus (u_n) converges uniformly on any compact subset of $\mathbb{R}$.

Problem F07.11. Let f be a bounded real function on $[0, 1]$. Show that f is Riemann integrable if and only if f^3 is Riemann integrable.

Solution. Let $M \in \mathbb{R}$ be the bound for f on $[0, 1]$. Suppose that f is Riemann integrable. Let $\varepsilon > 0$. Then there is a piecewise constant function g which majorizes f (and without loss of generality, we can take g to be bounded by M as well) for which $\int_0^1 (g(x) - f(x))dx < \varepsilon/3M^2$. Since x^3 is non decreasing on $\mathbb{R}$, it follows that g^3 majorizes f^3. Further

$$\begin{aligned} \int_0^1 (g(x)^3 - f(x)^3)dx &= \int_0^1 (g(x) - f(x))(g(x)^2 + g(x)f(x) + f(x)^2)dx \\ &\le \int_0^1 (g(x) - f(x))(M^2 + M \cdot M + M^2)dx \\ &= 3M^2 \int_0^1 (g(x) - f(x))dx < \varepsilon. \end{aligned}$$

Similarly, if we take a piecewise constant h which minorizes f and such that $\int_0^1 (f(x) - h(x))dx < \varepsilon/3M^2$, we find that h^3 minorizes f^3 and

$$\int_0^1 (f(x)^3 - h(x)^3)dx < \varepsilon.$$

Since ε was arbitrary, this implies that f^3 is Riemann integrable.

I can't figure out an elementary proof in the other direction.

[Note: the easiest soluton is as follows. The functions f and f^3 have the same points of continuity (since $x \mapsto x^3$ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$). A function is Riemann integrable if and only if it is continuous allmost everywhere. Hence f is Riemann integrable iff f is continuous almost everywhere iff f^3 is continuous almost everywhere iff f^3 is Riemann integrable.]

Problem S07.6. Consider the integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))ds \tag{$*$}$$

where $f(t, y)$ is continuous on $[0, T] \times \mathbb{R}$ and is Lipschitz in y with Lipschitz constant K. Assume you have shown that the iterates defined by

$$y_0(t) \equiv y_0, \quad y_n(t) = y_0 + \int_0^t f(s, y_{n-1}(s))ds, \quad t \in [0, T], n \in \mathbb{N}$$

converge uniformly to a solution y of $(*)$. Show that if $Y(t)$ is a solution of $(*)$ and satisfies $|Y(t) - y_0| \le C$ for some constant C and all $t \in [0, T]$, then $Y(t) \equiv y(t)$ on $[0, T]$.

Solution. We prove by induction that $|Y(t) - y_n(t)| \le \frac{C(tK)^n}{n!}$, $t \in [0, T]$. The base case:

$$|Y(t) - y_0| \le \frac{C(tK)^0}{0!} = C, \ \ t \in [0, T]$$

is given by assumption. Assume that

$$|Y(t) - y_n(t)| \le \frac{C(tK)^n}{n!}, \ \ t \in [0, T].$$

Then

$$\begin{aligned}
|Y(t) - y_{n+1}(t)| &= \left| y_0 + \int_0^t f(s, Y(s))ds - y_0 - \int_0^t f(s, y_n(s))ds \right| \\
&= \left| \int_0^t f(s, Y(s)) - f(s, y_n(s))ds \right| \\
&\le \int_0^t |f(s, Y(s)) - f(s, y_n(s))| \, ds \\
&\le K \int_0^t |Y(s) - y_n(s)| \, ds \\
&\le K \int_0^t \frac{C(sK)^n}{n!} ds = \frac{C(tK)^{n+1}}{(n+1)!}, \ \ t \in [0, T].
\end{aligned}$$

However, $\frac{C(tK)^n}{n!} \le \frac{C(TK)^n}{n!} \to 0$ as $n \to \infty$. Thus y_n converge uniformly to Y on $[0, T]$. Hence, since limits are unique, $Y(t) = y(t)$ on $[0, T]$.

Problem S07.8. Suppose the functions f_n are twice continuously differentiable on $[0,1]$ and satisfy

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \text{for all } x \in [0,1] \quad \text{and}$$

$$|f_n'(x)| \le 1, \;\; |f_n''(x)| \le 1, \quad \text{for all } x \in [0,1], \;\; n \ge 1.$$

Prove that $f(x)$ is continuously differentiable on $[0,1]$.

Solution. Since each $f_n(x)$ is differentiable with derivative bounded by 1, we know that for any $x, y \in [0,1]$ by the mean value theorem, there is $c \in (x,y)$ such that

$$|f_n(x) - f_n(y)| = |f'(c)|\,|x-y| \le |x-y|$$

Note, this holds for *all* n, so this shows that (f_n) is equicontinuous. Also, since $(f_n(0))$ converges as a sequence in $\mathbb{R}$, in particular, the sequence is bounded. Say $|f_n(0)| \le M$, $n \in \mathbb{N}$. Then for all $x \in [0,1]$, by the triangle inequality,

$$\begin{aligned} |f_n(x)| \le |f_n(x) - f_n(0)| + |f_n(0)| &= \left| \int_0^x f_n'(t)dt \right| + M \\ &\le \int_0^x |f_n'(x)|\,dx + M \le \int_0^x 1 \; dx + M = x + M \le 1 + M. \end{aligned}$$

Since this holds for all $x \in [0,1], n \in \mathbb{N}$, we see that (f_n) is uniformly bounded. Hence by Arzela-Ascoli, there is a subsequence of (f_n) which converges uniformly (*uniformly* since $[0,1]$ is compact). Since the same reasoning could be applied to any subsequence of (f_n), we see that (f_n) converges uniformly. Since the pointwise limit of (f_n) is f, it must be the case that the uniform limit of (f_n) is f. Since (f_n) is a sequence of continuous function which converge uniformly, f must be continuous.

The same reasoning as above could be applied to (f_n') to show that they converge uniformly to some continuous function g on $[0,1]$.

Then since (f_n) converges pointwise (in particular) and (f_n') converges uniformly, we know that f is differentiable and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Problem S07.10. Suppose the functions (f_n) on $\mathbb{R}$ satisfy

(i) $0 \le f_n(x) \le 1$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$,

(ii) $f_n(x)$ is an increasing function of x for all $n \in \mathbb{N}$,

(iii) $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for $x \in \mathbb{R}$ where $f(x)$ is continuous on $\mathbb{R}$,

(iv) $\lim\limits_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$.

Show that $f_n \to f$ uniformly in $\mathbb{R}$.

Solution. Let $\varepsilon > 0$.

By (i) and (ii), $f(x)$ is increasing and contained in $[0,1]$. By (iv), there are $m, M \in \mathbb{R}$ such that

$$x \le m \implies f(x) < \varepsilon/2$$

and

$$x \ge M \implies f(x) > 1 - \varepsilon.$$

We first prove that $f_n \to f$ uniformly on $[m, M]$.

Since $[m, M]$ is compact, f is uniformly continuous on $[m, M]$. Thus there is $\delta > 0$ such that for any $x, y \in [m, M]$,

$$0 < |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon/5.$$

Cover $[m, M]$ with intervals $[x_0, x_1], \ldots, [x_{n-1}, x_n]$ such that $x_0 = m, x_n = M$ and each interval has length less than δ. Since $f_n \to f$ pointwise, for each x_i, there is $N_i \in \mathbb{N}$ such that $n \ge N_i \implies |f(x_i) - f_n(x_i)| < \varepsilon/5$. Take $N = \max\limits_{0 \le i \le n} N_i$. Then for any $x \in [m, M]$, there is i such that $x \in [x_i, x_{i+1}]$, taking $n \ge N$, we have

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) - f_n(x_i) + f_n(x_i) - f(x_i) + f(x_i) - f(x)| \\ &= |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &< (f_n(x) - f_n(x_i)) + \varepsilon/5 + \varepsilon/5 \\ &< f_n(x_{i+1}) - f_n(x_i) + 2\varepsilon/5. \end{aligned}$$

But $f_n(x_{i+1})$ is at most $f(x_{i+1}) + \varepsilon/5$ and $f_n(x_i)$ is at least $f(x_i) - \varepsilon/5$. Then

$$\begin{aligned} |f_n(x) - f(x)| &< f(x_{i+1}) + \varepsilon/5 - (f(x_i) - \varepsilon/5) + 2\varepsilon/5 \\ &= (f(x_{i+1}) - f(x_i)) + 4\varepsilon/5 \\ &< \varepsilon/5 + 4\varepsilon/5 = \varepsilon. \end{aligned}$$

Since δ, N don't depend on x, this implies that $f_n \to f$ uniformly on $[m, M]$.

For $x \le m$ and $n \ge N$, we have

$$|f_n(x) - f(x)| \le f_n(m) - 0 \le f(m) - \varepsilon/5 < \varepsilon/2 + \varepsilon/5 < \varepsilon$$

and for $x \ge M$ and $n \ge N$, we have

$$|f_n(x) - f(x)| \le 1 - f(M) < 1 - (1 - \varepsilon) < \varepsilon$$

so $f_n \to f$ uniformly on all of $\mathbb{R}$.

Problem S07.11.

(a) Consider the equations

$$u^3 + xv - y = 0, \quad v^3 + yu - x = 0.$$

Can these equations be solved uniquely for u, v in terms of x, y in a neighborhood of $x = 0, y = 1, u = 1, v = -1$?

(b) Give an example where the conclusion of the implicit function theorem is true but the conditions are not met.

Solution.

(a) Let $f = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$ be defined

$$f(u,v,x,y) = \begin{bmatrix} f_1(u,v,x,y) \\ f_2(u,v,x,y) \end{bmatrix} = \begin{bmatrix} u^3 + xv - y \\ v^3 + yu - x \end{bmatrix}, \quad (u,v,x,y) \in \mathbb{R}^4.$$

We see $f(1,-1,0,1) = (0,0)$. Also at $(1,-1,0,1)$, we have

$$\det\left(\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\right) = -2 \neq 0.$$

Thus the matrix is invertible. By the implicit function theorem, there is a open neighborhood U of $(1,-1)$, an open neighborhood V of $(0,1)$ and a continuously differentiable map $g = (g_1, g_2) : U \to V$ such that

$$\{(u, v, g_1(u,v), g_2(u,v)) : (u,v) \in U\} = \{(u,v,x,y) : f(u,v,x,y) = (0,0)\}.$$

That is, we can uniquely for x, y in terms of u, v in a neighborhood of $(1,-1,0,1)$. (I did it backwards, but do that same thing but using the u, v derivatives to form the matrix to see the answer is yes.)

(b) Let $F(x,y) = x^3 - y$ and consider solving $F(x,y) = 0$ in a neighborhood of $(0,0)$. We know this has unique solution $x = y^{1/3}$. However, $\frac{\partial F}{\partial x} = x^2 = 0$ at $x = 0$ which is not invertible, so the conditions of the implicit function theorem are not met.

Problem S07.12. Let c_0 be the normed space of real sequences $x = (x_1, x_2, \ldots)$ such that $\lim\limits_{k\to\infty} x_k = 0$ with the norm $||x|| = \sup\limits_k |x_k|$.

(a) Show that c_0 is complete.

(b) Is the unit ball $\{x \in c_0 : ||x|| \leq 1\}$ compact?

(c) Is the set $E = \left\{ x \in c_0 : \sum\limits_{k=1}^{\infty} k\,|x_k| \leq 1 \right\}$ compact?

Solution.

(a) Let $(x^{(m)})_{m\in\mathbb{N}}$ be a Cauchy sequence in c_0. Let $\varepsilon > 0$, then for each $\ell \in \mathbb{N}$,

$$\left| x_\ell^{(m)} - x_\ell^{(n)} \right| \leq \left|\left| x^{(m)} - x^{(n)} \right|\right| < \varepsilon$$

for sufficiently large m, n. This shows that for each $\ell \in \mathbb{N}$, the sequence $(x_\ell^m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and thus coverges to some $x_\ell \in \mathbb{R}$. Then $(x^{(m)})$ converges term-by-term to $x = (x_1, x_2, \ldots)$ and so converges in norm. We need only check that $x \in c_0$.

For this, consider, for $\varepsilon > 0$, since $\lim_{\ell\to\infty} x_\ell^{(m)} = 0$ for each m and $\lim_{m\to\infty} x^{(m)} = x$, there is $N \in \mathbb{N}$ so that $\ell, m \geq N$ implies that

$$\left|x_\ell - x_\ell^{(m)}\right| < \varepsilon \quad \text{and} \quad \left|x_\ell^{(m)}\right| < \varepsilon.$$

For such m, ℓ, we have

$$|x_\ell| \leq \left|x_\ell - x_\ell^{(m)}\right| + \left|x_\ell^{(m)}\right| < 2\varepsilon$$

which implies that $\lim_{\ell\to\infty} x_\ell = 0$ and so $x \in c_0$.

(b) The unit ball is not compact. Consider a sequence $(x^{(m)})$ where $x_k^{(m)} = 1, k \leq m$ and $x_k^{(m)} = 0, k \geq m$. Each member of the sequence is eventually constant at *zero* and so has limit zero. Also each has $\left|\left|x^{(m)}\right|\right| = 1$ so each member of the sequence is in the unit ball. However, the sequence has no limit in c_0 since the limiting sequence is one that does not go to zero.

(c) Put $c_0^* = \{x \in c_0 : \sum_k k\,|x_k| \leq 1\}$. We prove that c_0^* is sequentially compact and thus compact. Suppose that $\left(x^{(n)}\right)_{n\in\mathbb{N}}$ be a sequence in c_0^*. Then since

$$\sum_{k=1}^{\infty} k\left|x_k^{(n)}\right| \leq 1, \quad \text{for all } n \in \mathbb{N},$$

we see that in particular, that $\left|x_k^{(n)}\right| \leq \frac{1}{k}$ for all $n, k \in \mathbb{N}$. Thus, by the Bolzano-Weierstrass Theorem, for all $k \in \mathbb{N}$, the sequence $\left(x_k^{(n)}\right)_{n\in\mathbb{N}}$. has a convergent subsequence. Let $\left(x_1^{(n,1)}\right)_{n\in\mathbb{N}}$ be a convergent subsequence of $\left(x_1^{(n)}\right)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} x_1^{(n,1)} = x_1 \in \mathbb{R}.$$

Then $\left(x_2^{(n,1)}\right)_{n\in\mathbb{N}}$ is a bounded sequence and so has a convergent subsequence $\left(x_2^{(n,2)}\right)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} x_2^{(n,2)} = x_2 \in \mathbb{R}.$$

Then we still have

$$\lim_{n\to\infty} x_1^{(n,2)} = x_1 \in \mathbb{R}.$$

Continuing this process inductively, for each $m \in \mathbb{N}$, we get subsequence $\left(x^{(n,m)}\right)_{n\in\mathbb{N}}$ such that $\left(x^{(n,m)}\right)_{n\in\mathbb{N}}$ is a subsequence of $\left(x^{(n,m-1)}\right)_{n\in\mathbb{N}}$ and for all $k \leq m$, we get

$$\lim_{n\to\infty} x_k^{(n,m)} = x_k.$$

Put $y^{(n)} = x^{(n,n)}, n \in \mathbb{N}$ and $x = (x_1, x_2, \ldots)$. Then $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is a subsequence of $\left(x^{(n)}\right)_{n\in\mathbb{N}}$. We prove that $y^{(n)} \to x$ and that $x \in c_0^*$.

First, it is clear that

$$\lim_{n\to\infty} y_k^{(n)} = x_k$$

or all $k \in \mathbb{N}$. Since for each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{\infty} k \left| y_k^{(n)} \right| \le 1.$$

Thus

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} k \left| y_k^{(n)} \right| \le 1.$$

We can switch the sum and the limit since all terms in the sum are positive, so

$$\sum_{k=1}^{\infty} k \lim_{n\to\infty} \left| y_k^{(n)} \right| \le 1 \quad \Longrightarrow \quad \sum_{k=1}^{\infty} k \left| x_k \right| \le 1 \quad \Longrightarrow \quad x \in c_0^*.$$

Finally, since $x \in c_0^*$, we have $|x_k| \le \frac{1}{k}$ for all $k \in \mathbb{N}$ (and similary $\left| y_k^{(n)} \right| \le \frac{1}{k}$ for all $k \in \mathbb{N}$). Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $\frac{1}{N} < \frac{\varepsilon}{4}$. Then for each $k = 1, \ldots, N$ there is $M_k \in \mathbb{N}$ such that for $n \ge M_k$, we have

$$\left| y_k^{(n)} - x_k \right| < \frac{\varepsilon}{2}.$$

put $M = \max\{M_1, \ldots, M_N\}$.

Take $n \ge M$. Then for $k \le N$, we have

$$\left| y_k^{(n)} - x_k \right| < \varepsilon/2$$

while for $k > N$, we have

$$\left| y_k^{(n)} - x_k \right| \le \left| y_k^{(n)} \right| + |x_k| \le \frac{1}{k} + \frac{1}{k} < \frac{1}{N} + \frac{1}{N} < \frac{\varepsilon}{2}.$$

Thus $\left| y_k^{(n)} - x_k \right| < \varepsilon/2$ for all $k \in \mathbb{N}$ and so $\sup_k \left| y_k^{(n)} - x_k \right| \le \varepsilon/2 < \varepsilon$. Since this holds for all $n \ge M$, we see that $y^{(n)} \to x$ in norm.

Hence c_0^* is sequentially compact and thus compact.

[Note: for those interested, switching the sum and the limit when all terms are positive is an application of the Monotone Convergence Theorem when integration is taken with respect to the counting measure on $\mathbb{N}$ and the sequences are viewed as functions $x : \mathbb{N} \to \mathbb{R}$.]

Problem F08.1. For which $a = 0, 1, 2$ is $f(t) = t^a$ uniformly continuous on $[0, \infty)$?

Solution. The function is uniformly continuous when $a = 0, 1$ but not when $a = 2$.

For $a = 0$, this is trivial, since $f(t) - f(s) = 0$ for all $t, s \in \mathbb{R}$.

For $a = 1$, let $\varepsilon > 0$ and set $\delta = \varepsilon$ which does not depend on $s, t \in \mathbb{R}$. Then

$$0 < |s - t| < \delta \implies |s - t| < \varepsilon \implies |f(s) - f(t)| < \varepsilon,$$

which implies that $f(t) = t$ is uniformly continuous.

For $a = 2$, take $\varepsilon = 1$. Then for any $\delta > 0$, we can take $n \geq 1/\delta$ and take $s = n, t = n + \delta/2$. Then

$$|f(s) - f(t)| = \left|s^2 - t^2\right| = |s + t|\,|s - t| = (2n + \delta/2)(\delta/2) = n\delta + \delta^2/4 > n\delta > 1.$$

Thus $f(t) = t^2$ is <u>not</u> uniformly continuous.

Problem F08.2. Suppose A is a non-empty connected subset of $\mathbb{R}^2$.

(a) Prove that if A is open then A is path-connected.

(b) Is the same true if A is closed?

Solution.

(a) Since A is non-empty, choose $x \in A$ and let $B \subseteq A$ be the set of all elements of A which can be connected to x by a path lying entirely in A.

Clearly B is non-empty since $x \in B$. Take $y \in B$. Then $y \in A$ and since A is open, there is an $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq A$. But for any $z \in B(y, \varepsilon)$, we can connect z by a path to y which remains in A and then continue to x remaining in A. Thus $z \in B$ and so $B(y, \varepsilon) \subseteq B$ so B is open.

Now assume that $A - B$ is non-empty. Taking $y \in A - B$,there is an $r > 0$ such that the ball $B(y, r) \subseteq A - B$ because otherwise, there would be $z \in B(y, r)$ which could be connected to x by a path lying in A which could then be extended to a path connectin x to y. But $y \in B - A$ procludes the existence of such path. Thus $B(y, r) \subset A - B$ and $A - B$ is open.

But now we see $A = B \cup (A - B)$ and clearly $B \cap (A - B) = \varnothing$ and we have proven that $B, A - B$ are open. This implies that A is disconnected, a contradiction. Thus our assumption that $A - B$ is non-empty must be incorrect. Hence $A - B = \varnothing$.

Thus $A = B$, so all points in A can be connected to x and so to each other. This means that A is path-connected.

(b) No. It is well known that the set

$$A = \left\{\left(x, \sin\tfrac{1}{x}\right) : x \in \left(0, \tfrac{1}{\pi}\right]\right\} \cup \{(0, x) : x \in [-1, 1]\} := B \cup C$$

is connected. If A was path-connected, then there is a continuous function $f : [0, 1] \to A$ such that $f(0) = (\frac{1}{\pi}, 0) \in B$ and $f(1) = (0, 0) \in C$. Define

$$s = \inf_{t \in [0,1]} \{t : f(t) \in C\}.$$

Then $f([0,s])$ can contain only one point in C, otherwise the minimality of s is violated. However, $f([0,s])$ will contain all points in B and thus $\overline{f([0,s])}$ contains all of C. Thus $f([0,s])$ is not closed (since it doesn't equal its closure) and not compact. But $[0,s]$ is compact and continuous functions map compact sets to compact sets so we have a contradiction. Thus A is not path-connected.

Problem F08.4. Suppose that K and F are closed subsets of $\mathbb{R}^2$ with $K \cap F = \varnothing$.

(a) Prove that if K is bounded, then $d(K,F) > 0$ where

$$d(K,F) = \inf\left\{d(x,y) : x \in K, y \in F\right\}.$$

(b) Is (a) necessarily true if K is not bounded?

Solution.

(a) Assume to the contrary that $d(K,F) = 0$. Then for any $\varepsilon > 0$, there are $x \in K, y \in F$ such that $d(x,y) < \varepsilon$. In particular, for any $n \in \mathbb{N}$, there are $x_n \in K, y_n \in F$ such that $d(x_n, y_n) < \frac{1}{n}$. Since K is bounded, by Bolzano-Weierstrass, the sequence (x_n) has a subsequence (x_{n_m}) which converges to some $x \in \mathbb{R}^2$ and since K is closed, $x \in K$. We show that (y_{n_m}) also converges to x. Let $\varepsilon > 0$. Then there is $M \in \mathbb{N}$, such that $d(x_{n_m}, x) < \varepsilon/2$. Taking m large enough so that $n_m > \max\{M, 2/\varepsilon\}$. We see

$$d(x, y_{n_m}) \le d(x, x_{n_m}) + d(x_{n_m}, y_{n_m}) < \varepsilon/2 + \tfrac{1}{n_m} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus (y_{n_m}) converges to x as well and since F is closed, $x \in F$. But then $K \cap F \neq \varnothing$ as was assumed. The contradiction implies that $d(K,F) > 0$.

(b) No. Let $F = \{(x,0) : x \in \mathbb{R}\}$, $K = \left\{\left(x, \frac{1}{x}\right) : x > 0\right\}$. Then it is clear that K, F are closed and for any $\varepsilon > 0$, taking $x > 1/\varepsilon$ gives existence of points $p \in K, q \in F$ such that $d(p,q) < \varepsilon$. Letting $\varepsilon \to 0$ shows that $d(K,F) = 0$.

Problem F08.10. Given $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we let

$$||v|| = \sqrt{\sum_{i=1}^n v_i^2}.$$

If $f = (f_1, \ldots, f_n) : [a,b] \to \mathbb{R}^n$, we define

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, \ldots, \int_a^b f_n(t)dt\right).$$

Prove that

$$\left|\left|\int_a^b f(t)dt\right|\right| \le \int_a^b ||f(t)||\, dt.$$

Solution. If $\left|\left|\int_a^b f(t)dt\right|\right| = 0$ then the inequality is trivial and we are done. Assume that $\left|\left|\int_a^b f(t)dt\right|\right| > 0$.

For convenience, put $x_i = \int_a^b f_i(t)dt$ and $x = (x_1, \dots, x_n)$ so that $x = \int_a^b f(t)dt$. We see

$$\begin{aligned} ||x||^2 = \left|\left|\int_a^b f(t)dt\right|\right|^2 &= \sum_{i=1}^n \left(\int_a^b f_i(t)dt\right)^2 \\ &= \sum_{i=1}^n x_i \int_a^b f_i(t)dt \\ &= \int_a^b \left(\sum_{i=1}^n x_i f_i(t)\right) dt \\ &\le \int_a^b ||x|| \, ||f(t)|| \, dt \\ &= ||x|| \int_a^b ||f(t)|| \, dt \\ &= \left|\left|\int_a^b f(t)dt\right|\right| \int_a^b ||f(t)|| \, dt, \end{aligned}$$

where the inequality incurred follows from Cauchy-Schwarz. Dividing by $\left|\left|\int_a^b f(t)dt\right|\right|$ gives

$$\left|\left|\int_a^b f(t)dt\right|\right| \le \int_a^b ||f(t)|| \, dt.$$

Problem S08.2. Let (f_n) be a sequence on functions on $[0,1]$ such that $f_n(x) \ge 0$ for all $n \in \mathbb{N}, x \in [0,1]$ and for each $x \in [0,1]$,

$$\lim_{n\to\infty} f_n(x) = 0.$$

Prove or give a counterexample to the assertion that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

Solution. We give a counterexample. For each $n \in \mathbb{N}$, $n \ge 3$, define

$$f_n(x) = \begin{cases} 0, & x < \frac{1}{n}, \\ n^2\left(x - \frac{1}{n}\right), & \frac{1}{n} \le x < \frac{2}{n}, \\ n^2\left(\frac{3}{n} - x\right), & \frac{2}{n} \le x < \frac{3}{n}, \\ 0, & \frac{3}{n} \le x. \end{cases}$$

Now $f_n(0) = 0$, for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} f_n(0) = 0$ and for any $x \in (0,1]$, there is $N_x \in \mathbb{N}$, such that $3/N_x < x$. Hence for all $n \ge N_x$, $f_n(x) = 0$ and so $\lim_{n\to\infty} f_n(x) = 0$. Thus $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

However, each f_n is a triangular spike of height n and base $2/n$. This triangle has area 1. Thus for each $n \in \mathbb{N}$, we have

$$\int_0^1 f_n(x)dx = 1 \quad \text{and so} \quad \lim_{n\to\infty} \int_0^1 f_n(x)dx = 1.$$

Problem S08.3. Assuming $f \in C^4[a,b]$ is real, derive a formula for the error of approximation $E(h)$ when the second derivative is replaced by the finite difference formula

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

where h is the mesh size. (It is alright to assume that $x-h, x, x+h \in (a,b)$).

Solution. By Taylor's Theorem,

$$f(x \pm h) = f(x) \pm hf'(x) + h^2 \frac{f''(x)}{2} \pm h^3 \frac{f'''(x)}{6} + h^2 \frac{f^{(4)}(\xi)}{24}$$

for some ξ between x and $x \pm h$. Let $K = \sup_{x \in [a,b]} \left| f^{(4)}(x) \right|$. The supremum is finite (and actually it is achieved for some $x_0 \in [a,b]$) since $f^{(4)}$ is continuous and $[a,b]$ is compact. Then

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + h^4 \frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_2)}{24} \leq 2f(x) + h^2 f''(x) + h^4 \frac{K}{12},$$

for some $\xi_1, \xi_2 \in (x-h, x+h)$. Then

$$\left| \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) \right| \leq h^2 \frac{K}{12} = \frac{\sup_{x \in [a,b]} \left| f^{(4)}(x) \right|}{12} h^2.$$

Thus the error $E(h)$ is $\mathcal{O}(h^2)$ [the exact formula can be taken to be the one above, though this is actually a bound on the error].

Problem S08.6. Let Y be a complete and countable metric space. Prove there is $y \in Y$ such that $\{y\}$ is open.

Solution. Assume to the contrary that $\{y\}$ is not open for all $y \in Y$. In any metric space (and more generally, any Hausdorff space), singleton sets are closed, so $\{y\} = \overline{\{y\}}$ and since the only subsets thereof are $\varnothing$ and $\{y\}$ we conclude that the interior of $\{y\} = \overline{\{y\}}$, which is the largest open subset, is $\varnothing$. That is, if no singleton is open, then for each $y \in Y$, the set $\overline{\{y\}}$ has empty interior and thus $\{y\}$ is nowhere dense. Then

$$Y = \bigcup_{y \in Y} \{y\}$$

is a countable union of closed no-where dense sets and since Y is complete, the Baire Category Theorem (metric space version) implies that Y is nowhere dense. But then since Y is both open and closed we have $Y = \text{int}(\overline{Y}) = \varnothing$ which is an impossibility. Thus some $\{y\}$ is open.

Problem S08.7. Let $a(x)$ be a function on $\mathbb{R}$ such that

(i) $a(x) \geq 0$ for all $x \in \mathbb{R}$, and

(ii) there exists $M < \infty$ such that for all *finite* $F \subset \mathbb{R}$,

$$\sum_{x \in F} a(x) \le M.$$

Prove that $A = \{x : a(x) > 0\}$ is countable.

Solution. Without loss of generality, take $M \in \mathbb{N}$. Define

$$A_n = \left\{x : a(x) > \tfrac{1}{n}\right\}, n = 1, 2, 3, \ldots$$

so that

$$A = \bigcup_n A_n.$$

Suppose that A_n has more that Mn elements. Then there is a finite subset $F \subset A_n$ with at least $Mn + 1$ elements. This gives

$$\sum_{x \in F} a(x) \ge \sum_{x \in F} \frac{1}{n} \ge \frac{Mn+1}{n} = M + \frac{1}{n}, \text{ which contradicts (ii).}$$

Thus each A_n has less then Mn elements. Then A is countable since it is a countable union of finite sets.

Problem F09.1.

(i) For each $n \in \mathbb{N}$, let $f_n : \mathbb{N} \to \mathbb{R}$ be a function with $|f_n(m)| \le 1$ for all $m, n \in \mathbb{N}$. Prove that there is an infinite subsequence of positive integers (n_i) such that for each $m \in \mathbb{N}$, $(f_{n_i}(m))_{i=1}^{\infty}$ converges.

(ii) For n_i as in (i), assume that in addition $\lim_{m \to \infty} \lim_{i \to \infty} f_{n_i}(m) = 0$. Prove or disprove that $\lim_{i \to \infty} \lim_{m \to \infty} f_{n_i}(m) = 0$.

Solution.

(i) Let $X = \{f : \mathbb{N} \to [-1, 1]\}$ and equip X with the metric

$$d(f, g) = \sup_{m \in N} |f(m) - g(m)|, \; f, g \in X.$$

Then $d(f, g) \le 2$ for all $f, g \in X$ so X is totally bounded. Suppose (f_k) is a Cauchy sequence in X. Then for each $m \in \mathbb{N}$, $(f_k(m))$ is a Cauchy sequence in $[-1, 1]$ and thus converges to some value $a_m \in [-1, 1]$. Define $f : \mathbb{N} \to [-1, 1]$ by $f(m) = a_m$. Now for any $\varepsilon > 0$, $m \in \mathbb{N}$, there is $K \in \mathbb{N}$ such that

$$|f_k(m) - f(m)| < \varepsilon/2 \quad \text{for all} \quad k \ge K.$$

Taking the supremum over all m gives $d(f_k, f) \le \varepsilon/2 < \varepsilon$. Thus (f_k) converges to f. Hence X is complete. Since X is complete and totally bounded, it is compact and thus sequentially compact.

Let (f_n) be a sequence in X. Then by sequential compactness, there is a subsequence (f_{n_i}) which converges in the metric. It follows that $(f_{n_i}(m))$ converges for each $m \in \mathbb{N}$.

(ii) No. Define $f_{n_i}(m) = 1, m > n_i$ and $f_{n_i}(m) = 0, m \le n_i$. It's clear that (f_{n_i}) converges to a function f which maps all natural numbers to zero, so $f \in X$. Then

$$\lim_{m\to\infty} \lim_{i\to\infty} f_{n_i}(m) = 0 \quad \text{whereas} \quad \lim_{i\to\infty} \lim_{m\to\infty} f_{n_i}(m) = 1.$$

Problem F09.2.

(i) Let X be a complete metric space with respect to the distance function d. We say that a map $T : X \to X$ is a contraction if for some $\lambda \in (0, 1)$ and all $x, y \in X$:

$$d\left(T(x), T(y)\right) \le \lambda d(x, y).$$

Prove that any contraction mapping on X has a unique fixed point.

(ii) Using (i), show that given a differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose first derivative satisfies $f'(x) = e^{-x^2} - e^{-x^4}, x \in \mathbb{R}$, there exists $\alpha \in \mathbb{R}$ such that $f(\alpha) = \alpha$.

Solution.

(i) Let $\varepsilon > 0$.

Let $x_0 \in X$ and define a sequence (x_n) by $x_n = Tx_{n-1}, n = 1, 2, 3, \ldots$. We prove that (x_n) is Cauchy. Indeed $d(x_2, x_1) = d(Tx_1, Tx_0) \le \lambda d(x_1, x_0)$. Further, $d(x_3, x_2) = d(Tx_2, Tx_1) \le \lambda d(x_2, x_1) \le \lambda^2 d(x_1, x_0)$. Indeed, by induction $d(x_{n+1}, x_n) \le \lambda^n d(x_1, x_0)$. Let $m, n \in \mathbb{N}$, $n > m$. Then

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\le (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m) d(x_1, x_0) \\ &\le d(x_1, x_0) \sum_{j=m}^{\infty} \lambda^j \\ &= d(x_1, x_0) \frac{\lambda^m}{1 - \lambda}. \end{aligned}$$

But $0 < \lambda < 1$ means that $\lambda^m \to 0$ as $m \to \infty$. Take $N \in \mathbb{N}$ such that $\frac{d(x_1,x_0)\lambda^m}{1-\lambda} < \varepsilon$ when $m \ge N$. Then for $n, m \ge N$, we have $d(x_n, x_m) < \varepsilon$. Thus (x_n) is Cauchy and since X is complete, (x_n) converges to some $x \in X$. Then by continuity of T (which follows from the contraction property), we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\left(\lim_{n\to\infty} x_{n-1}\right) = Tx.$$

To prove uniqueness, we consider $x, y \in X$ such that $Tx = x, Ty = y$. Then

$$d(x, y) = d(Tx, Ty) \le \lambda d(x, y) \quad \Longrightarrow \quad (1 - \lambda) d(x, y) \le 0 \quad \Longrightarrow d(x, y) \le 0$$

which implies that $x = y$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ have derivative $f'(x) = e^{-x^2} - e^{-x^4}$. Then for $x \in [0,1]$, we have

$$|f'(x)| = \left| e^{-x^2} - e^{-x^4} \right| = e^{-x^4} - e^{-x^2} \le e^{-x^4} - e^{-1} \le e^0 - e^{-1} = 1 - \tfrac{1}{e} < 1.$$

Also, for $x \in [1, \infty)$,

$$|f'(x)| = \left| e^{-x^2} - e^{-x^4} \right| = e^{-x^2} - e^{-x^4} \le e^{-x^2} \le \tfrac{1}{e} \le 1 - \tfrac{1}{e}.$$

Thus for any $x, y \in \mathbb{R} x \neq y$, by the mean value theorem, there is c between x and y such that

$$|f(x) - f(y)| = |f'(c)| \, |x - y| \le \left(1 - \tfrac{1}{e}\right) |x - y| \, .$$

Thus f is a contraction on $\mathbb{R}$. Since $\mathbb{R}$ is complete with respect to the absolute value, by the above proof, f has a unique fixed point.

Problem F09.3. The purpose ofthis problem is to give a multi-variable calculus proof of the geometric mean-arithmetic mean inequality.

(i) Let $\mathbb{R}_n^+ \subset \mathbb{R}^n$ be the subset of vectors all of whom components are positive and let $f : \mathbb{R}_n^+ \to \mathbb{R}$ be defined by

$$f(x_1, \ldots, x_n) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n}$$

Explain why f attains a global (not necessarily unique) minimum as some $p \in \mathbb{R}_n^+$.

(ii) Find p.

(iii) Deduce that if all $x_i \in \mathbb{R}$ are positive and $\prod x_i = 1$ then $\sum x_i \ge n$.

Solution.

(i) We see that

$$\lim_{x_i \to 0} f(x_1, \ldots, x_n) = \lim_{x_i \to \infty} f(x_1, \ldots, x_n) = \infty,$$

for any $i = 1, \ldots, n$. Further f is continuous so it will attain a minimum on any set of the form $[1/L, L]^n$, $L > 1$ since the set is compact. Fix $M > 0$ and let $L > 1$ be such that

$$f(x_1, \ldots, x_n) > M$$

outside $[1/L, L]$. Then f attains a minimum (say m) inside $[1/L, L]$ and is larger than M outside $[1/L, L]$. Thus $\min\{m, M\}$ will be a global minimum for f.

(ii) Setting $\frac{\partial f}{\partial x_j} = 0$ for all i simultaneously, we see that

$$1 - \frac{1}{x_1 \cdots x_j^2 \cdots x_n} = 0 \quad \implies \quad x_j = \frac{1}{\prod_{j=1}^n x_j}.$$

Then

$$\prod_{j=1}^n x_j = \frac{1}{\left(\prod_{j=1}^n x_j\right)^n} \quad \implies \quad \left(\prod_{j=1}^n x_j\right)^{n+1} = 1.$$

But since all x_j are positive, we conclude that the minimum must occur along the hypersurface

$$\prod_{j=1}^{n} x_j = 1.$$

But then $x_j = \frac{1}{\prod_{j=1}^{n} x_j} = 1$. Thus the minimum occurs at $(1, \ldots, 1)$.

(iii) We see that for all $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ such that $\prod_{j=1}^{n} x_j = 1$,

$$1 + \sum_{j=1}^{n} x_j = f(x_1, \ldots, x_n) \geq f(1, \ldots, 1) = n + 1$$

which implies the desired inequality.

Problem F09.8. For a matrix $A \in M_n(\mathbb{R})$, define $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$. Let $v_0 \in \mathbb{R}^n$. Prove that the function $v : \mathbb{R} \to \mathbb{R}^n$ given by $v(t) = e^{At} v_0$ solves the differential equation

$$\begin{aligned} v'(t) &= Av(t) \\ v(0) &= v_0. \end{aligned}$$

Solution. It is clear that $v(0) = v_0$. Note that the matrices At and Ah commute for any $t, h \in \mathbb{R}$. Thus $e^{A(t+h)} = e^{At} e^{Ah}$. Then

$$\begin{aligned} v'(t) &= \lim_{h \to 0} \frac{e^{A(t+h)} v_0 - e^{At} v_0}{h} \\ &= \lim_{h \to 0} \frac{e^{Ah} - I}{h} e^{At} v_0. \end{aligned}$$

Further

$$e^{Ah} - I = hA + \frac{h^2 A^2}{2} + \frac{h^3 A^3}{6} + \cdots .$$

So

$$\frac{e^{Ah} - I}{h} = A + o(h).$$

Hence

$$v'(t) = \lim_{h \to 0} \frac{e^{Ah} - I}{h} e^{At} v_0 = A e^{At} v_0 = Av(t).$$

Problem F09.10.

(a) If $f : I \to \mathbb{R}$ is a continuous function such that $\int_0^2 f(x)dx = 36$, prove that there is an $x \in [0, 2]$ such that $f(x) = 18$.

(b) Let $g : [0, 2] \times [0, 2] \to \mathbb{R}$ be a continuous function such that $\int_0^2 \int_0^2 g(x, y) dx\, dy = 36$. Show that there is an $(x, y) \in I^2$ such that $g(x, y) = 9$.

Solution. We can answer both these in one go. Suppose that there is no $x \in [0,2]$ and no $(x,y) \in [0,2] \times [0,2]$ such that $f(x) = 18$ or $g(x,y) = 9$. Then by continuity (and the intermediate value theorem), we have (wlog) $f(x) > 18$ for all $x \in [0,2]$ and $g(x,y) > 9$ for all $x \in [0,2] \times [0,2]$. But then $\int_0^2 f(x)dx > \int_0^2 18\,dx = 36$ and $\int_0^2 \int_0^2 g(x,y)dx\,dy > \int_0^2 \int_0^2 9\,dx\,dy = 36$. Both of these are contradictions with the assumptions, thus we do have such x and (x,y).

Problem S09.4. Let (X,d) be a metric space.

(a) Give a definition of compactness of X using open covers.

(b) Define completeness of X.

(c) Define connectedness of X.

(d) Is $\mathbb{Q}$ connected with the usual metric on $\mathbb{R}$?

(e) Suppose X is complete. Prove that X is compact if and only if for any $r > 0$, X can be covered by finitely many balls of radius r.

Solution.

(a) X is compact if every open cover of X admits and finite subcover.

(b) X is complete if every Cauchy sequence in X converges to a limit in X.

(c) X is connected if there is no decomposition $X = A \cup B$ with non-empty, open $A, B \subseteq X$ and $A \cap B = \varnothing$.

(d) No, $\mathbb{Q}$ is not connected. We find just such a decomposition. Let $A = \mathbb{Q} \cap (-\infty, \pi)$ and $B = \mathbb{Q} \cap (\pi, \infty)$. Then clearly A, B are nonempty, $\mathbb{Q} = A \cup B$ and $A \cap B = \varnothing$. Further, for $x \in A$, the rationals in the interval $(x - \pi, \pi)$ is an open neighborhood of x contained in A so A is open. B is also open by an identical arguement. Thus A, B form a decomposition of $\mathbb{Q}$ which proves that $\mathbb{Q}$ is disconnected.

(e) Suppose X is complete and compact. Let $r > 0$. Then $\{B(x,r)\}_{x \in X}$ forms an open cover of X. By compactness, there is a finite subcover, $\{B(x_1,r), \ldots, B(x_k,r)\}$. Then X is covered by finitely many balls of radius r.

Suppose X is complete and for any $r > 0$, X can be covered by finitely many balls of radius r. We prove that X is sequentially compact which implies that X is compact in the sense of covers. Let (x_n) be a sequence in X. Cover X with finitely many balls of radius $1/2$. Then one of the balls must contain infinitely many members of the sequence (x_n). Let $(x_{2,n})$ be a subsequence of (x_n) which are all contained in a ball of radius $1/2$. Cover this ball with finitely many balls of radius $1/3$. Then infinitely many members of $(x_{2,n})$ will be contained in a single ball of radius $1/3$. Cover this ball with finitely many balls of radius $1/4$ and repeat the process. By induction, we have sequences $(x_{k,n})_{n=1}^\infty$ such that each $(x_{k,n})$ is a subsequence of $(x_{k-1,n})$ and each $(x_{k,n})$ is contained in a ball of radius $1/k$. Letting $y_n = x_{n,n}, n \in \mathbb{N}$, we have a sequence (y_n).

Then $(y_n)_{n=1}^{\infty}$ is a subsequence of (x_n) and $(y_n)_{n=k}^{\infty}$ is a subsequence of $(x_{k,n})$ for each $k \in \mathbb{N}$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ so that $2/k < \varepsilon$. Then by construction of $(x_{k,n})$, for any $m, n \geq k$,

$$d(y_m, y_n) < \tfrac{2}{k} < \varepsilon.$$

Thus (y_n) is a Cauchy sequence and thus converges in X since X is complete. Thus every sequence (x_n) has a subsequence that converges and so X is sequentially compact.

Problem S09.6. Show that a continuous function $f : [0, 1) \to \mathbb{R}$ is uniformly continuous if and only if there is a continuous function $g : [0, 1] \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in [0, 1)$.

Solution. If such a g exists, then it is uniformly continuous, since it is continuous on a compact set. Thus $f = g|_{[0,1)}$ will be uniformly continuous as well.

Conversely, suppose f is uniformly continuous on $[0, 1)$. Define $x_n = 1 - \frac{1}{n}$, $n \in \mathbb{N}$. Then

$$|x_n - x_m| = \left| \frac{1}{m} - \frac{1}{n} \right| \leq \max \left\{ \frac{1}{m}, \frac{1}{n} \right\} \to 0$$

as $m, n \to \infty$. Thus (x_n) is a Cauchy sequence. Let $\varepsilon > 0$. Then by uniform continuity, there is $\delta > 0$ such that

$$0 < |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Since (x_n) is Cauchy, there is $N \in \mathbb{N}$ such that $m, n \geq N$ will give $|x_n - x_m| < \delta$. For such m, n, we have

$$|x_n - x_m| < \delta \quad \Longrightarrow \quad |f(x_n) - f(x_m)| < \varepsilon.$$

Thus $(f(x_n))$ is a Cauchy sequence and has a limit in $\mathbb{R}$. Say $\lim\limits_{n\to\infty} f(x_n) = f_1 \in \mathbb{R}$.

Define $g : [0, 1] \to \mathbb{R}$ so that $g(x) = f(x), x \in [0, 1)$ and $g(1) = f_1$. We prove that g is continuous. It is clear that g is continuous on $[0, 1)$ since it is equal to a continuous function there. Let (y_n) be any sequence in $[0, 1)$ converging to 1. Then

$$|g(y_n) - f_1| \leq |g(y_n) - f(x_n)| + |f(x_n) - f_1|,$$

where x_n is as defined above. However, since $y_n \in [0, 1)$, $g(y_n) = f(y_n)$ and since x_n and y_n go to the same limit, they get arbitrarily close. That is, there is $N \in \mathbb{N}$ so that $n \geq N$ gives $|y_n - x_n| < \delta$ which then leads to $|f(y_n) - f(x_n)| < \varepsilon$. Also there is also $N \in \mathbb{N}$ such that $n \geq N$ gives $|f(x_n) - f_1| < \varepsilon$. Taking the max of each N, we see

$$|g(x_n) - f_1| < 2\varepsilon$$

which proves that $g(x_n) \to f_1$. Hence by the sequential criterion theorem, $\lim_{x\to 1^-} g(x) = f_1 = g(1)$ so g is continuous.

Problem S09.10.

(a) Rigorously justify the limit:

$$\int_0^1 \frac{1}{1 + x^2} dx = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-1)^n}{2n + 1}.$$

(b) Deduce the value of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

Solution.

(a) Consider, for $x \in [0, 1)$,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Moreover, the convergence is uniform on any interval $[0, t]$, $t < 1$. Thus

$$\int_0^t \frac{1}{1+x^2} dx = \int_0^t \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx = \sum_{n=0}^{\infty} (-1)^n \int_0^t x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{2n+1}$$

By the alternating series test, the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

converges and thus by Abel's Theorem,

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &= \lim_{t \to 1^-} \int_0^t \frac{1}{1+x^2} dx \\ &= \lim_{t \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{2n+1} \\ &= \sum_{n=0}^{\infty} \lim_{t \to 1^-} \frac{(-1)^n t^{2n+1}}{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \end{aligned}$$

(b) Evaluating the integral, we see

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \int_0^1 \frac{1}{1+x^2} dx = \arctan(x)\Big|_0^1 = \arctan(1) = \frac{\pi}{4},$$

which was the desired deduction.

Problem S09.12. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Show that

$$\text{div}(F) = \rho$$

for all $(x, y, z) \in \mathbb{R}^3$ if and only if

$$\iint_{\partial\Omega} F \cdot dS = \iiint_{\Omega} \rho \, dx \, dy \, dz$$

for all open balls $\Omega \subset \mathbb{R}^3$.

[You may use without proof the various standard theorems of vector calculus.]

Solution. The Divergence theorem tells us that if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is smooth and D is a simple solid region with smooth boundary ∂D, then

$$\iint_{\partial D} F \cdot dS = \iiint_D \text{div} F \, dx \, dy \, dz.$$

Assume $\rho = \text{div}F$. Then the divergence theorem immediately implies that

$$\iint_{\partial \Omega} F \cdot dS = \iiint_\Omega \rho \, dx \, dy \, dz$$

for any open ball $\Omega \subset \mathbb{R}^3$.

Conversely, assume that

$$\iint_{\partial \Omega} F \cdot dS = \iiint_\Omega \rho \, dx \, dy \, dz$$

for any open ball $\Omega \subset \mathbb{R}^3$. Then

$$\iiint_\Omega \text{div} F \, dx \, dy \, dz = \iiint_\Omega \rho \, dx \, dy \, dz \quad \implies \quad \iiint_\Omega G \, dx \, dy \, dz = 0$$

for any open ball $\Omega \subset \mathbb{R}^3$ where $G \coloneqq \text{div}F - \rho$. Assume that $G \not\equiv 0$ on $\mathbb{R}^3$. Then there is a point $x \in \mathbb{R}^3$ such that (wlog) $G(x) = \varepsilon > 0$. By countinuity (since G is smooth), there is $\delta > 0$ such that

$$G(v) > \varepsilon/2 \quad \text{whenever} \quad ||x - v|| < \delta.$$

Now $B(x, \delta) \subset \mathbb{R}^3$ is certainly an open ball, but

$$\iiint_{B(x,\delta)} G \, dx \, dy \, dz > \iiint_{B(x,\delta)} \frac{\varepsilon}{2} \, dx \, dy \, dz = \frac{2\pi\varepsilon\delta^3}{3} > 0.$$

This contradicts that G integrates to zero over any open ball in $\mathbb{R}^3$. Thus $G \equiv 0$ so $\rho = \text{div}F$ on all of $\mathbb{R}^3$.

Problem F09.6. Consider the function $f(x, y) = \sin^3(xy) + y^2 |x|$ defined on $S \subset \mathbb{R}^2$ given by

$$S = \{(x, y) \in \mathbb{R}^2 : x^{2010} + y^{2010} \leq 1\}.$$

Define what it means for f to be uniformly continuous on S and prove that f is uniformly continuous on S.

Solution. f is uniformly continuous on S if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any $u, v \in S$,

$$||u - v|| < \delta \quad \implies \quad |f(u) - f(v)| < \varepsilon.$$

Here $||\cdot||$ can be any norm on $\mathbb{R}^2$ restricted to S but is usually taken to be the Euclidean norm.

It is clear that f is continuous on S since f is composed of functions which are continuous everywhere. We recall that continuous functions on compact spaces are uniforly continuous. Hence it is enough to show that S is compact. A subset of $\mathbb{R}^2$ is compact if and only if it is closed and bounded. S is bounded, since for example, $|x|, |y| \le 1$. We need only show that S is closed. Let (x, y) be a limit point of S. Then there is a sequence (x_n, y_n) in S which converges to (x, y). Then

$$0 \le x_n^{2010} + y_n^{2010} \le 1$$

for all $n \in \mathbb{N}$. Then $x_n^{2010} + y_n^{2010} \to x^{2010} + y^{2010}$ (this follows since $F(x, y) = x^{2010} + y^{2010}$ is continuous). But $[0, 1]$ is compact, so it contains its limit points which means that $x^{2010} + y^{2010} \in [0, 1]$ and hence $(x, y) \in S$. Thus S is closed and so f is uniformly continuous on S.

Problem F10.3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ have continuous derivatives up to order 3.

(a) State Taylor's Theorem with remainder for f and g.

(b) Assuming the theorem for f, prove the theorem for g.

Solution.

(a) Let $x, a \in \mathbb{R}$. Taylor's Theorem for f says that there is ξ in between x and a such that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(\xi)}{3!}(x - a)^3.$$

Let $\mathbf{x}, \mathbf{a} \in \mathbb{R}^2$. Define

$$H_g(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2}(\mathbf{x}) & \frac{\partial^2 g}{\partial x \partial y}(\mathbf{x}) \\ \frac{\partial^2 g}{\partial y \partial x}(\mathbf{x}) & \frac{\partial^2 g}{\partial y^2}(\mathbf{x}) \end{bmatrix}$$

where x, y are the coordinates of $\mathbf{x}$. Then Taylor's Theorem for g says that

$$g(\mathbf{x}) = g(\mathbf{a}) + \nabla g(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \mathbf{a} \cdot H_g(\mathbf{x})\mathbf{a} + E(\mathbf{x})$$

where $E(\mathbf{x}) \to 0$ faster than $||\mathbf{x} - \mathbf{a}||_2^2$ as $\mathbf{x} \to \mathbf{a}$. If $\mathbf{x} = (x, y)$ and $\mathbf{a} = (a, b)$ it is sometimes convenient to write this in the form

$$\begin{aligned} g(x, y) = g(a, b) &+ \frac{\partial g}{\partial x}(a, b)(x - a) + \frac{\partial g}{\partial y}(a, b)(y - b) \\ &+ \frac{1}{2}\left(\frac{\partial^2 g}{\partial x^2}(a, b)(x - a)^2 + 2\frac{\partial^2 g}{\partial x \partial y}(a, b)(x - a)(y - b) + \frac{\partial^2 g}{\partial y^2}(a, b)(y - b)^2\right) \\ &+ E(x, y). \end{aligned}$$

We note that this is equivalent because mixed partials of g are equal since g is three times continuous differentiable.

(b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be three times continuously differentiable. For fixed $(x, y), (a, b) \in \mathbb{R}^2$, define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(t) = g(a + t(x - a), b + t(y - b)).$$
Then f is also three times continuously differentiable, so
$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(\xi)}{3!}t^3 \tag{1}$$
for some ξ between 0 and t. Now $f(0) = g(a, b)$. Using the chain rule, we see
$$f'(t) = \frac{\partial g}{\partial x}(a + t(x - a), b + t(y - b))(x - a) + \frac{\partial g}{\partial y}(a + t(x - a), b + t(y - b))(y - b)$$
so
$$f'(t) = \frac{\partial g}{\partial x}(a, b)(x - a) + \frac{\partial g}{\partial y}(a, b)(y - b).$$
Further (dropping the arguments), we see
$$f''(t) = \frac{\partial^2 g}{\partial x^2} \cdot (x - a)^2 + 2\frac{\partial^2 g}{\partial x \partial y} \cdot (x - a)(y - b) + \frac{\partial^2 g}{\partial y^2} \cdot (y - b)^2,$$
so
$$f''(0) = \frac{\partial^2 g}{\partial x^2}(a, b)(x - a)^2 + 2\frac{\partial^2 g}{\partial x \partial y}(a, b)(x - a)(y - b) + \frac{\partial^2 g}{\partial y^2}(a, b)(y - b)^2.$$
Finally, it is clear that $f'''(t)$ will contain terms multiplied by $(x - a)^3, (x - a)^2(y - b), (x - a)(y - b)^2$ and $(y - b)^3$. Thus $f''' \to 0$ faster than $(x - a)^2 + (y - b)^2$ as $(x, y) \to (a, b)$. Evaluting (1) at $t = 1$, and noting that $f(1) = g(x, y)$, we see
$$\begin{aligned} g(x, y) = g(a, b) &+ \frac{\partial g}{\partial x}(a, b)(x - a) + \frac{\partial g}{\partial y}(a, b)(y - b) \\ &+ \frac{1}{2}\left(\frac{\partial^2 g}{\partial x^2}(a, b)(x - a)^2 + 2\frac{\partial^2 g}{\partial x \partial y}(a, b)(x - a)(y - b) + \frac{\partial^2 g}{\partial y^2}(a, b)(y - b)^2\right) \\ &+ E(x, y), \end{aligned}$$
where $E(x, y) \to 0$ fast as $(x, y) \to (a, b)$. This proves Taylor's Theorem for g.

Problem F10.4.

(a) Show that given a real-valued continuous function f on $[0, 1] \times [0, 1]$ and $\varepsilon > 0$, there exist real valued continuous functions $g_1, \dots, g_n, h_1, \dots, h_n$ on $[0, 1]$ for some finite n such that
$$\left| f(x, y) - \sum_{i=1}^{n} g_i(x)h_i(y) \right| \le \varepsilon, \;\; 0 \le x, y \le 1.$$

(b) If $f(x, y) = f(y, x)$ for all $x, y \in [0, 1]$, can this be done with $g_i = h_i$ for all i?

Solution.

(a) Let $\varepsilon > 0$. Since f is continuous on a compact set it is uniformly continuous. Take $N \in \mathbb{N}$ so that $|(x,y)-(z,w)| < \frac{1}{N}$ gives $|f(x,y)-f(z,w)| < \varepsilon$. Define a grid $x_i = i/N, i = 0, 1, \dots, N$ and let

$$g_i(x) = \begin{cases} 0, & x \le x_{i-1}, \\ N(x - x_{i-1}), & x_{i-1} \le x \le x_i, \\ N(x_{i+1} - x), & x_i \le x \le x_{i+1}, \\ 0, & x \ge x_{i+1}. \end{cases}$$

for $i = 0, \dots, N$. Then $|g_i(x)| \le 1$ for all i, x and

$$\left| f(x_i, y) - \sum_{i=0}^{N} g_i(x_i) f(x_i, y) \right| = 0 \text{ for all } x = x_i, i = 0, \dots, N, y \in [0,1].$$

If $x \ne x_i$ for any i, then $x \in (x_j, x_{j+1})$ for some j. In this case, all $g_i(x)$ are zero except when $i = j, j+1$ and we have $g_j(x) + g_{j+1}(x) = 1$. Thus

$$\begin{aligned} \left| f(x,y) - \sum_{i=0}^{N} g_i(x) f(x_i, y) \right| &= |f(x,y) - g_j(x) f(x_j, y) - g_{j+1} f(x_j, y)| \\ &= |(g_j(x) + g_{j+1}(x)) f(x,y) - g_j(x) f(x_j, y) - g_{j+1} f(x_{j+1}, y)| \\ &\le g_j(x) \,|f(x,y) - f(x_j, y)| + g_{j+1}(x)\,|f(x,y) - f(x_{j+1}, y)| \\ &< g_j(x)\varepsilon + g_{j+1}(x)\varepsilon = \varepsilon. \end{aligned}$$

Hence the difference is small for all $x \in [0,1]$.

By the Stone-Weierstrass theorem, there are polynomials $h_0, \cdots, h_N$ such that for each x_i,

$$|f(x_i, y) - h_i(y)| < \varepsilon/(N+1)$$

for all $y \in [0,1]$. Then

$$\begin{aligned} \left| f(x,y) - \sum_{i=0}^{N} g_i(x) h_i(y) \right| &\le \left| f(x,y) - \sum_{i=0}^{N} g_i(x) f(x_i, y) \right| + \left| \sum_{i=0}^{N} g_i(x) f(x_i, y) - \sum_{i=0}^{N} g_i(x) h_i(y) \right| \\ &\le \varepsilon + \sum_{i=0}^{N} |g_i(x)|\,|f(x_i, y) - h_i(y)| \\ &\le \varepsilon + \sum_{i=0}^{N} \varepsilon/(N+1) = 2\varepsilon. \end{aligned}$$

Rescaling the epsilons and re-indexing the sums gives the solution. (For those interested, the functions g_i defined above are the functions used in the linear spline Galerkin finite element method which is an important numerical method for solving PDE.)

(b) No. If f is a negative function then taking $g_i = h_i$, we will need to satisfy

$$\left| f(x,x) - \sum_{i=1}^{n} g_i(x)^2 \right| < \varepsilon$$

which isn't possible for all ε since f is negative but the sum is nonnegative

Problem F10.10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous. For $k \in \mathbb{N}$, define $x_k : [0,1] \to \mathbb{R}$ by $x_k(0) = 0$ and

$$x_k(t) = x_k(n2^{-k}) + (t - n2^{-k})f(x_k(n2^{-k}))$$

for

$$n2^{-k} < t \leq (n+1)2^{-k}, \quad n \in \mathbb{N}.$$

Explain why x_k uniformly converges to a solution $x : [0,1] \to \mathbb{R}$ of the ODE

$$x'(t) = f(x(t)), x(0) = 0,$$

as $k \to \infty$.

Solution. I'm not sure this problem is worth the time or effort required to complete it. The function x_k are the approximations to the solution of $x' = f(x)$ using Euler's Method with step size 2^{-k}. It is well know that these will converge uniformly to x.

Problem F10.12. Define $D(t) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r(t)^2\}$ where $r(t) : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function. For a given smooth nonnegative function $u(x,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, express the following in terms of a surface integral:

$$\frac{d}{dt}\left(\int_{D(t)} u(x,t)dx\right) - \int_{D(t)} \frac{\partial u}{\partial t}(x,t)dx.$$

Solution. Write the first integral above in polar coordinates:

$$\frac{d}{dt}\left(\int_{D(t)} u(x,t)dx\right) = \frac{d}{dt}\int_0^{2\pi}\int_0^{r(t)} u(\theta,\rho,t)\rho\, d\rho\, d\theta.$$

By Liebniz rule,

$$\frac{d}{dt}\int_0^{r(t)} u(\theta,\rho,t)\rho\, d\rho = u(\theta,r(t),t)r(t)r'(t) + \int_0^{r(t)} \frac{\partial u}{\partial t}(\theta,\rho,t)\rho\, d\rho.$$

Thus

$$\frac{d}{dt}\int_0^{2\pi}\int_0^{r(t)} u(\theta,\rho,t)\rho\, d\rho\, d\theta - \int_0^{2\pi}\int_0^{r(t)} \frac{\partial u}{\partial t}(\theta,\rho,t)\rho\, d\rho\, d\theta = \int_0^{2\pi} u(\theta,r(t),t)r(t)r'(t)\, d\theta.$$

Hence

$$\begin{aligned}\frac{d}{dt}\left(\int_{D(t)} u(x,t)dx\right) &- \int_{D(t)} \frac{\partial u}{\partial t}(x,t)dx \\ &= \int_0^{2\pi} u(\theta,r(t),t)r(t)r'(t)\, d\theta = r(t)r'(t)\int_{\partial D(t)} u(x,t)dx\end{aligned}$$

is the desired representation as a surface integral.

Problem S10.9. Assume that $f(x,y,z)$ is a real valued, continuously differentiable function such that $f(x_0,y_0,z_0)=0$. If $\nabla f(x_0,y_0,z_0)\neq\vec{0}$, show that there is a differentiable surface given parametrically by $(x(s,t),y(s,t),z(s,t))$ with $(x(0,0),y(0,0),z(0,0))=(x_0,y_0,z_0)$ on which $f=0$.

Solution. Since $\nabla f(x_0,y_0,z_0)\neq\vec{0}$, assume without loss of generality that $\frac{\partial f}{\partial x}(x_0,y_0,z_0)\neq 0$. Then the matrix $[\frac{\partial f}{\partial x}(x_0,y_0,z_0)]$ is invertible so by the implicit function theorem there is a neighborhood U of (y_0,z_0), a neighborhood V of x_0 and a unique continuously differentiable function $g:U\to V$ with $g(y_0,z_0)=x_0$ such that

$$\{(g(y,z),y,z):y,z\in U\}=\{(x,y,z):f(x,y,z)=0\}.$$

Putting $y(s,t)=y_0+s$, $z(s,t)=z_0+t$, $x(s,t)=g(y(s,t),z(s,t))=g(y_0+s,z_0+t)$ for s,t sufficiently small, we see that

$$x(0,0)=x_0,\quad y(0,0)=y_0,\quad z(0,0)=z_0$$

and $f=0$ on the surface $(x(s,t),y(s,t),z(s,t))$.

Problem S10.10. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined

$$f(x,y)=\frac{xy}{\sqrt{x^2+y^2}}$$

when $(x,y)\neq(0,0)$ and $f(0,0)=(0)$.

(a) Compute the directional derivatives of f at $(0,0)$ in all directions where they exist.

(b) Is f differentiable at $(0,0)$?

Solution.

(a) Let v be a (unit) direction vector. The directional derivative of f at $(0,0)$ in the direction of v is

$$\lim_{h\to 0}\frac{f(hv_1,hv_2)-f(0,0)}{h}=\lim_{h\to 0}\frac{1}{h}\frac{h^2v_1v_2}{||hv||}=\lim_{h\to 0}\frac{hv_1v_2}{|h|}.$$

Since the limit of $h/|h|$ as $h\to 0$ doesn't exist, we need $v_1=0$ or $v_2=0$. Thus the directional derivatives only exist in along the x or y axis and it is zero in those directions.

(b) No. If f was differentiable at $(0,0)$, the derivative would have to be the zero map since the directional derivatives along the x,y axes are zero there. Then

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-0}{||(x,y)-(0,0)||}=\lim_{(x,y)\to(0,0)}\frac{|f(x,y)|}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{|xy|}{x^2+y^2}$$

would need to be zero regardless of how we approach the point $(0,0)$. However, approaching the origin along the line $y=x$ (from either direction), we have

$$\lim_{(x,y)\to(0,0)}\frac{|xy|}{x^2+y^2}=\lim_{x\to 0}\frac{x^2}{x^2+x^2}=\frac{1}{2}.$$

Hence f is not differentiable at $(0,0)$.

Problem S10.12. Assume that (f_n) is a sequence of nonnegative continuous function on $[0,1]$ such that $\lim_{n\to\infty}\int_0^1 f_n(x)dx = 0$. Is it necessarily true that

(a) there is $B \in \mathbb{R}$ such that $f_n(x) \le B$ for $x \in [0,1]$ for all $n \in \mathbb{N}$?

(b) there are points x_0 in $[0,1]$ such that $\lim_{n\to\infty} f_n(x_0) = 0$?

Solution.

(a) No. For $n \in \mathbb{N}$, $n \ge 2$, define

$$f_n(x) = \begin{cases} n^3\left(x - \left(\frac{1}{2} - \frac{1}{n^2}\right)\right), & \frac{1}{2} - \frac{1}{n^2} \le x < \frac{1}{2}, \\ n^3\left(\left(\frac{1}{2} + \frac{1}{n^2}\right) - x\right), & \frac{1}{2} \le x < \frac{1}{2} + \frac{1}{n^2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then each f_n is a triangular spike on $\left(\frac{1}{2} - \frac{1}{n^2}, \frac{1}{2} + \frac{1}{n^2}\right)$ of height n so clearly there is no uniform bound B. However,

$$\int_0^1 f_n(x)dx = \frac{2}{n^2}\cdot n = \frac{2}{n} \to 0 \quad \text{as } n \to \infty.$$

(b) No. Let (q_n) be a denumeration of rationals in $[0,1]$. Then define

$$f_n(x) = \begin{cases} n\left(x - \left(q_n - \frac{1}{n}\right)\right), & q_n - \frac{1}{n} \le x < q_n, \\ n\left(\left(q_n + \frac{1}{n}\right) - x\right), & q_n \le x < q_n + \frac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Then f_n is a spike of height 1 and base $2/n$ around the n^{th} member of the sequence. For any $x \in [0,1]$, there are infinitely many rationals in any neighborhood of x. Thus for infinitely many $n \in N$, $f_n(x) > 1/2$ (or any other number in $(0,1)$. Hence $f_n(x) \not\to 0$ for any $x \in [0,1]$.

Solution 2 (supplied by a classmate). Let χ_A denote the characteristic function of the set A; that is,

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Then the sequence

$$\chi_{[0,1/2]}, \chi_{[1/2,1]}, \chi_{[0,1/3]}, \chi_{[1/3,2/3]}, \chi_{[2/3,1]}, \chi_{[0,1/4]}, \chi_{[1/4,2/4]}, \chi_{[2/4,3/4]}, \chi_{[3/4,1]}, \cdots$$

satisfies the property since the measure of the sets go to zero but each $x \in [0,1]$ is contained in infinitely many sets of the form $[k/n,(k+1)/n]$. (Of course, these functions aren't continuous but you could mollify them).

Problem F11.3. Prove that the set of real numbers can be written as the union of uncountably many pairwise disjoint sets, each of which is uncountable.

Solution. Since $\mathbb{R} \times \mathbb{R}$ has the same cardinality of $\mathbb{R}$, there is a bijection $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Now

$$\mathbb{R} \times \mathbb{R} = \bigcup_{x \in \mathbb{R}} (\{x\} \times \mathbb{R}).$$

Then

$$\mathbb{R} = F(\mathbb{R} \times \mathbb{R}) = \bigcup_{x \in \mathbb{R}} F(\{x\} \times \mathbb{R}) := \bigcup_{x \in \mathbb{R}} F_x.$$

Each F_x is uncountable because it is an injective image of an uncountable set. Also they are pairwise disjoint since the sets $\{x\} \times \mathbb{R}$ are disjoint and F is injective. Thus we have achieved the appropriate representation.

Problem F11.4. If you rearrange the order of terms in a sum $\sum a_n$, sometime you can change the value of the sum. Find all resulting limiting values of the following series and prove your assertions.

1. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

2. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

Solution.

1. The sum can converge to any value $x \in \mathbb{R}$. Indeed put $a_n = (-1)^{n-1}/n$, $b_n = 1/(2n-1)$ and $c_n = -1/(2n), n = 1, 2, 3, \ldots$. Then clearly at least one of

$$\sum_{n \in \mathbb{N}} b_n \quad \text{or} \quad \sum_{n \in \mathbb{N}} c_n$$

must diverge because if both converged, then

$$\sum_{n \in \mathbb{N}} \frac{1}{n} = \sum_{n \in \mathbb{N}} b_n - \sum_{n \in \mathbb{N}} c_n$$

would converge as well. However,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n \in \mathbb{N}} b_n + \sum_{n \in \mathbb{N}} c_n$$

converges by the alternating series test, which would be impossible if one sum diverged while the other converged. Thus both sums diverge. We have that

$$\sum_{n \in \mathbb{N}} b_n = +\infty \quad \text{and} \quad \sum_{n \in \mathbb{N}} c_n = -\infty.$$

Let $x > 0$; the case that $x < 0$ is similar. Define $B_N = \sum_{n=1}^N b_n$ and $C_n = \sum_{n=1}^N c_n$. Then since $B_N \to \infty$, there is N_1 such that $B_{N_1} > x$ and the error will be no more than $1/N_1$. Then there must be N_2 so that $B_{N_1} + C_{N_2} < x$ and the error will not exceed $\max\{1/N_1, 1/N_2\}$. Then there must be N_3 so that $B_{N_3} + C_{N_2} > x$ with error no more than $\max\{1/N_3, 1/N_2\}$. Repeating this process, we get a sequences so that

$$B_{N_{k-1}} + C_{N_k} < x < B_{N_{k+1}} + C_{N_k}$$

where k is an even integer and the error from either side is at most

$$\max\{1/N_i : i = k-1, k, k+1\}.$$

By construction, the error will go to zero and so $\lim_{k\to\infty} B_{N_{k-1}} + C_{N_k} = x$. Define a rearrangement σ so that

$$a_{\sigma(1)} = b_1, \ldots, a_{\sigma(N_1)} = b_{N_1}, a_{\sigma(N_1+1)} = c_1, \ldots, a_{\sigma(N_1+N_2)} = c_{N_2}, \ldots.$$

Then $\displaystyle\sum_{n=1}^\infty a_{\sigma(n)} = \sum_{n=1}^\infty \frac{(-1)^{\sigma(n)-1}}{\sigma(n)} = x.$

2. The sum converges absolutely, so any rearrangement of the sum will have the same value. We prove this here in a couple steps. Assume that $(a_n)_{n\in\mathbb{N}}$ is a sequence such that

$$\sum_{n=1}^\infty |a_n|$$

converges. We first prove that

$$\sum_{n=1}^\infty a_n$$

converges. Indeed, we see that

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty (|a_n| + a_n) - \sum_{n=1}^\infty |a_n|.$$

Further

$$0 \le a_n + |a_n| \le 2|a_n|.$$

Then

$$s_N = \sum_{n=1}^N a_n + |a_n|$$

is a monotone sequence of terms which is bounded by $2\sum_{n=1}^\infty |a_n|$ and thus converges. Hence we have expressed

$$\sum_{n=1}^\infty a_n$$

as a difference of convergent sequence and thus $\sum_{n=1}^\infty a_n$ converges. Put

$$A = \sum_{n=1}^\infty a_n.$$

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be any bijection of $\mathbb{N}$ to itself. Let $\varepsilon > 0$. Then there is $N \geq 0$ such that for $k \geq N$, first

$$\left| A - \sum_{n=1}^k a_n \right| < \varepsilon/2$$

and second, for $k \geq N$,

$$\left| \sum_{n=k}^\infty a_n \right| < \varepsilon/5.$$

Since σ is a bijection, there is a finite $N^* \in \mathbb{N}$ so that $\{1, \ldots, N\} \subseteq \{\sigma(1), \ldots, \sigma(N^*)\}$. Let $\mathcal{N} = \{\sigma(1), \ldots, \sigma(N^*)\} \setminus \{1, \ldots N\}$. Then $\mathcal{N} \subseteq \{k+1, k+2, \ldots\}$. Now we see, for any $\ell \geq N^*$,

$$\begin{aligned}
\left| A - \sum_{n=1}^\ell a_{\sigma(n)} \right| &\leq \left| A - \sum_{n=1}^\ell a_n \right| + \left| \sum_{n=1}^\ell a_n - \sum_{n=1}^\ell a_{\sigma(n)} \right| \\
&\leq \frac{\varepsilon}{2} + \left| \sum_{n=N+1}^\ell a_n - \sum_{n \in \mathcal{N}} a_{\sigma(n)} \right| \\
&\leq \frac{\varepsilon}{2} + \left| \sum_{n=N+1}^\ell a_n \right| + \left| \sum_{n \in \mathcal{N}} a_{\sigma(n)} \right| \\
&\leq \frac{\varepsilon}{2} + \left| \sum_{n=N+1}^\infty a_n \right| + \left| \sum_{n=N+1}^\infty a_n \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} < \varepsilon.
\end{aligned}$$

Hence by definition,

$$\sum_{n=1}^\infty a_{\sigma(n)} = A.$$

Now we have established that any rearrangement of the sum will yield the same value. We simply need to find the value. Starting from

$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

we see

$$\sum_{n=1}^\infty \frac{1}{(2n)^2} = \frac{\pi^2}{24}.$$

Then

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} = \sum_{n=1}^\infty \frac{1}{n^2} - 2\sum_{n=1}^\infty \frac{1}{(2n)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{12} = \frac{\pi^2}{12}.$$

Problem S11.7. Prove there is a real number x such that

$$x^5 - 3x + 1 = 0.$$

Solution. Let $p(x) = x^5 - 3x + 1$. Then $p(1) = -1 < 0$ and $p(2) = 27 > 0$. Since polynomials are continuous, by the intermediate value theorem, we can conclude there is an $x \in [1, 2]$ such that $p(x) = 0$.

Problem S11.8. Give examples of:

1. A function $f(x)$ on $[0, 1]$ which is not Riemann integrable for which $|f(x)|$ is Riemann integrable.

2. Continuous functions f_n and f on $[0, 1]$ such that $f_n(t) \to f(t)$ for all $t \in [0, 1]$ but $\int_0^1 f_n(t)dt$ does not converge to $\int_0^1 f(t)dt$.

Solution.

1. Define
$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1], \\ -1, & x \in (\mathbb{R} - \mathbb{Q}) \cap [0, 1] \end{cases}$$
Then since both the rational and irrationals are dense, any piecewise constant majorizing function of f must remain greater than or equal to 1 and any piecewise constant minorizing function of f must be less than or equal to -1. Hence
$$\int_0^1 (g(x) - h(x))dx \geq 2$$
for any piecewise constant majorizer g of f and any piecewise constant minorizer h of f. Since this value cannot become arbitrarily small, f is not Riemann integrable. However, $|f(x)| = 1, x \in [0, 1]$ which is clearly Riemann integrable since it is a constant function.

2. Use triangles of height n and width $1/n$ (see **F05.3**, **S08.2**).

Problem S11.9. Prove that if f is a continuous function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx = 0$$

implies $f(x) = 0$ for all $x \in [a, b]$.

Solution. Suppose to the contrary,that $f \not\equiv 0$. Then there is $t \in [a, b]$ such that $f(t) \neq 0$. Without loss of generality, let $f(t) = \varepsilon > 0$. By continuity, there is $\delta > 0$ such that $f(x) > \varepsilon/2$ when $x \in (t-\delta, t+\delta)$. Since f is non-negative, we know that $\int_a^{t-\delta} f(x)dx, \int_{t+\delta}^b f(x)dx \geq 0$. Then

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^{t-\delta} f(x)dx + \int_{t-\delta}^{t+\delta} f(x)dx + \int_{t+\delta}^b f(x)dx \\ &\geq \int_{t-\delta}^{t+\delta} f(x)dx \geq \int_{t-\delta}^{t+\delta} \frac{\varepsilon}{2}\, dx = \delta\varepsilon > 0. \end{aligned}$$

This contradicts that $\int_a^b f(x)dx = 0$. Hence we conclude $f(x) = 0$ for all $x \in [a, b]$.

A small note: we implicitly assumed that $t \in (a, b)$. However, if $t = a$, we can bound f away from zero when $x \in [a, a + \delta)$ and if $t = b$, we do the same when $x \in (b - \delta, b]$ so the proof still works.

Problem S11.11.

1. Show that a connected subset $A \subset R$ is path connected.
2. Give an example of a subset of $\mathbb{R}^2$ which is connected but not path connected.

Solution.

1. First we claim that a connected subset of $\mathbb{R}$ is an interval. Indeed, if $A \subset \mathbb{R}$ is not an interval, then there is $x \notin A$ such that for some $y, z \in A$, $y < x < z$. Then $A = (A \cap (-\infty, x)) \cup (A \cap (x, \infty))$ shows that A is disconnected.

   Thus A, being connected, is an interval: $A = [a, b]$ (the interval could just as well be open, half-open or infinite; the proof is the same in all cases). Then for $x, y \in A$, the line $f(t) = x + t(y - x)$ will stay entirely inside A and connect x to y. Thus A is path connected.
2. The topologist's sine curve works (see **F08.1**).

Problem S11.12. Give a metric space (M, d) and a constant $0 < r < 1$, a continuous function $T : M \to M$ is called and r-contraction if

$$d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in M$. A well know fixed point theorem states that if M is complete and T is an r-contraction on M, then T has a unique fixed point (don't prove this). This result is often used to prove existence and uniqueness of solutions to differential equations.

1. Illustrate this technique for the equation

   $$f'(t) = f(t), \quad f(0) = 1 \tag{DE}$$

   by letting $M = C[0, c]$ for sufficiently small c with the uniform distance

   $$d(f, g) = \sup_{0 \le t \le c} |f(t) - g(t)|, \quad f, g \in M$$

   and defining

   $$(Tf)(t) = 1 + \int_0^t f(s)ds. \tag{2}$$
2. What approximations do you obtain from the sequence $T(0), T^2(0), T^3(0), \ldots$?

Solution.

1. By the fundamental theorem of calculus, we see that (DE) implies
$$f(t) = 1 + \int_0^t f(s)ds.$$
Further, if f is continuous, the step can be reversed. Thus for continuous f, we have that f satisfies (DE) if and only if f satisfies the above integral equation. Define T as in (2). Then f satisfies the integral equation if and only if f is a fixed point of T. That is, if we can prove that T has a unique fixed point, then (DE) has a unique solution. Consider for continuous x, y and $t \in [0, c]$,
$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \left| \int^t (x(s) - y(s))ds \right| \\ &\le \int_0^t |x(s) - y(s)|\, ds \qquad \le \|x - y\|_\infty \int_0^t ds \le c\, \|x - y\|_\infty . \end{aligned}$$
Taking the supremum over all t on the left hand side gives
$$\|Tx - Ty\|_\infty \le c\,\|x - y\|_\infty .$$
Thus if we take $0 < c < 1$, T is a contraction and thus has a unique fixed point. Hence (DE) has a unique solution on $[0, c]$.

2. Let $f_0(t) = 0$ and define $f_n = Tf_{n-1}$, $n = 1, 2, 3, \ldots$. Then
$$\begin{aligned} f_1(t) &= 1 + \int_0^t 0ds = 1, \\ f_2(t) &= 1 + \int_0^t 1ds = 1 + t, \\ f_3(t) &= 1 + \int_0^t (1 + s)ds = 1 + t + \tfrac{t^2}{2}, \\ &\vdots \\ f_n(t) &= \sum_{k=0}^{n-1} \frac{t^k}{k!}. \end{aligned}$$
It ie easy to see that $[T^n(0)](t) = f_n(t)$. Thus our n^{th} estimate gives us the Maclaurin expansion of e^t truncated at term $n - 1$. From this it is clear that $[T^n(0)](t) \to e^t$ uniformly as $n \to \infty$ which we expect since we know that (DE) has unique solution $f(t) = e^t$.

Problem F12.1. Let $\{b_n\}_{n=1}^\infty$ be a sequence of real numbers and let $M > 0$ be such that
$$\left| \sum_{n=1}^N b_n \right| \le M$$

for all $N \in \mathbb{N}$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive real numbers which decreases to 0 as $n \to \infty$. Show that the series $\sum_{n=1}^\infty a_n b_n$ converges.

Solution. Let $B_n = \sum_{k=1}^n b_k$ and $s_n = \sum_{k=1}^n a_k b_k, n \in \mathbb{N}$. We know that B_n is a bounded sequence and we need to prove that s_n has a limit as $n \to \infty$. Consider, for $m, n \in \mathbb{N}$, $n > m$,

$$|s_n - s_m| = \left| \sum_{k=m+1}^n a_k b_k \right|.$$

Using summation by parts, we see

$$\sum_{k=m+1}^n a_k b_k = (B_n a_{n+1} - B_m a_{m+1}) - \sum_{k=m+1}^n B_k (a_{k+1} - a_k).$$

Then

$$\begin{aligned} |s_n - s_m| &= \left| (B_n a_{n+1} - B_m a_{m+1}) - \sum_{k=m+1}^n B_k (a_{k+1} - a_k) \right| \\ &\le |B_n| \, |a_{n+1}| + |B_m| \, |a_{m+1}| + \left| \sum_{k=m+1}^n B_k (a_{k+1} - a_k) \right| \\ &\le M(a_{n+1} + a_{m+1}) + \sum_{k=m+1}^n M(a_k - a_{k+1}), \end{aligned}$$

where we have used the fact that (a_n) is a positive decreasing sequence. Then the sum telescopes so

$$|s_n - s_m| \le M(a_{n+1} + a_{m+1}) + M(a_{m+1} - a_{n+1}) = 2M a_{m+1}.$$

Since the sequence (a_n) goes to zero as $n \to \infty$, this shows that (s_n) is a Cauchy sequence and hence converges. Thus $\sum_{n=1}^\infty a_n b_n$ converges.

Problem F12.3. Let $\{f_n\}$ be a sequence of non-negative continuous functions on a *compact* metric space X. Assume that $f_n(x) \ge f_{n+1}(x)$ for all n and x so that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for every $x \in X$. Show that f is continuous if and only if $f_n \to f$ uniformly in X.

Solution. $\underline{\Longleftarrow.}$ Assume that $f_n \to f$ uniformly. Let $\varepsilon > 0$. Then, by uniform convergence, there is $N \ge 0$, such that for any $x \in X$,

$$n \ge N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon/3.$$

Further, for $x, y \in X$, and fixed $n \ge N$, by continuity of f_n, there is $\delta > 0$, such that

$$d(x,y) < \delta \quad \Longrightarrow \quad |f_n(x) - f_n(y)| < \varepsilon/3.$$

By repeated use of the triangle inequality,

$$\begin{aligned}|f(x)-f(y)| &\le |f(x)-f_n(x)| + |f_n(x)-f(y)| \\ &\le |f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \quad \text{whenever } d(x,y) < \delta.\end{aligned}$$

Thus f is continuous.

$\underline{\Longrightarrow .}$ Assume f is continuous and that $f_n \to f$ pointwise. Let $\varepsilon > 0$. Define $A_n = \{x \in X : |f_n(x) - f(x)| < \varepsilon\}$. Since f, f_n are continuous, A_n is open since it is the pullback of the open set $[0,\varepsilon)$ under the continuous function $|f_n - f|$ (we consider $[0,\varepsilon)$ open because it is open in the relative topology on $[0,\infty)$ which is the largest potential co-domain of $|f - f_n|$; alternatively, we could replace it with $(-\infty,\varepsilon)$ if we consider the range of $|f - f_n|$ to be $\mathbb{R}$). Since $f_n \to f$ pointwise, eventually $|f_n(x) - f(x)| < \varepsilon$ for every x. Hence (A_n) form an open cover of X. Since X is compact, there is a finite subcover:

$$X = \bigcup_{n=1}^{N} A_n.$$

By construction, we have $A_1 \subset A_2 \subset A_3 \subset \cdots$. Thus $X = A_N$. But this implies that $f_n \to f$ uniformly, because for all $x \in X$, $|f_N(x) - f(x)| < \varepsilon$ (and the same will hold for any $n > N$).

Problem F12.4. A subset K of a metric space (X,d) is called *nowhere dense* is K has empty interior. Prove the Baire Category Theorem that if (X,d) is a complete metric space, then X is not a countable union of closed nowhere dense sets.

Solution. First, we need to get the correct definition. A subset K of a metric space (X,d) is *nowhere dense* if the *closure* of K has empty interior. The *closure* part is key because, e.g., the rationals should not be considered nowhere dense. [Note: in the case of closed K, we have that K equals its closure so the correct definition reduces to the definition given.]

First, it is clear that X is not nowhere dense since the interior of the closure of X is X. Assume to the contrary that $X = \cup_n K_n$ where each K_n is closed and nowhere dense. Since X is not nowhere dense, $X \neq K_1$ and since K_1 is closed, $X - K_1$ is open. Take $x_1 \in X - K_1$. By openness, there is $\delta_1 > 0$ such that $B(x_1,\delta_1) \subset X - K_1$; wlog take $\delta_1 < 1/2$ (we can do this by simply reducing δ_1 if necessary). Since K_2 is nowhere dense, it cannot contain $B(x_1, \delta_1/2)$. Take $x_2 \in (X - K_2) \cap B(x_1, \delta_1/2)$ (which is an open set). Then there is $\delta_2 < \delta_1/2 < 1/4$ such that $B(x_2,\delta_2) \subset (X - K_2) \cap B(x_1,\delta_1/2)$. Continue this procedure inductively to construct a sequence (x_n) in X such that $x_n \notin K_1, \ldots, K_n$ and $d(x_n, x_{n-1}) \le \frac{1}{2^{n-1}}$. Then for $n > m$, $d(x_n, x_m) \le d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) < \sum_{j=m}^{\infty} \frac{1}{2^m}$. Taking m large enough shows that this is a Cauchy sequence. Since X is complete, the sequence converges to some $x \in X$. However, for any $m \in \mathbb{N}$, the sequence (x_n) eventualy resides in a ball of distance at least $1/2^{m-1}$ away from $X - K_m$. Thus $x \notin K_m$ for all $m \in \mathbb{N}$. Hence $x \in X - \cup_n K_n$ which is a contradiction to our assumption that $X = \cup_m K_m$. Thus X is not the union of countably many closed, nowhere dense sets.

Problem S12.1. Let Ω denote the set of all closed subsets of $[0,1]$ and let $\rho : \Omega \times \Omega \to [0,1]$ be defined by

$$\rho(A,B) \coloneqq \max\left\{\sup_{x\in A}\inf_{y\in B}|x-y|\,, \sup_{y\in B}\inf_{x\in A}|x-y|\right\},$$

for $A,B \in \Omega$. Prove that (Ω,ρ) is a metric space.

Solution. Symmetry is obvious, as if the fact that $\rho(A,A) = 0$. We simply have to prove that ρ is positive on unequal sets and that the triangle inequality is satisfied. Suppose that $A, B \in \Omega, A \neq B$. Then (without loss of generality) we can take $x \in A$ such that $x \notin B$. Then, since B is closed, there is $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap B = \varnothing$. This means that for any $y \in B$, $|x-y| > \varepsilon$. Then

$$\inf_{y\in B}|x-y| \geq \varepsilon > 0$$

which leads to

$$\rho(A,B) \geq \sup_{x\in A}\inf_{y\in B}|x-y| > 0.$$

Next, let $A, B, C \in \Omega$. Then for all $a \in A, b \in B, c \in C$,

$$|a-b| \leq |a-c| + |c-b|\,.$$

Taking the infimum over all $b \in B$ gives

$$\inf_{b\in B}|a-b| \leq |a-c| + \inf_{b\in B}|c-b|\,.$$

But $\inf_{b\in B}|c-b| \leq \sup_{c\in C}\inf_{b\in B}|c-b|$, so

$$\inf_{b\in B}|a-b| \leq |a-c| + \sup_{c\in C}\inf_{b\in B}|c-b|\,.$$

Next taking the infimum over all $c \in C$ and then the supremum over all $a \in A$ (in that order), gives

$$\sup_{a\in A}\inf_{b\in B}|a-b| \leq \sup_{a\in A}\inf_{c\in C}|a-c| + \sup_{c\in C}\inf_{b\in B}|c-b| \leq \rho(A,C) + \rho(B,C).$$

A very similar sequence of operations gives $\sup_{b\in B}\inf_{a\in A}|a-b| \leq \rho(A,C) + \rho(B,C)$ so we can conclude that $\rho(A,B) \leq \rho(A,C) + \rho(B,C)$, which completes the proof.

Problem S12.2. Recall that $f : [a,b] \to \mathbb{R}$ is convex if for all $x, y \in [a,b]$,

$$f(\alpha x + (1-a)y) \leq \alpha f(x) + (1-\alpha)f(y), \;\; \alpha \in [0,1].$$

Let $f_n : [a,b] \to \mathbb{R}$ be convex functions and suppose that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in [a,b]$. Prove that if $f(x)$ is continuous on $[a,b]$, then $f_n \to f$ uniformly.

Solution. Fix $\varepsilon > 0$. Since f is continuous on a compact set, it is uniformly continuous. Thus we can take a uniform partition $a = a_0 < a_1 < \cdots < a_N = b$ fine enough that for all $x \in [a,b]$,

$$|f(x) - f(a_i)|\,, |f(x) - f(a_{i+1})| \le \varepsilon$$

where $[a_i, a_{i+1}]$ is the subinterval containing x. Thus

$$f(x) \ge f(a_i) - \varepsilon \quad \text{and} \quad f(x) \ge f(a_{i+1}) - \varepsilon \quad \implies \quad f(x) \ge \max\{f(a_i), f(a_{i+1})\} - \varepsilon.$$

Now choose M large enough that for $n \ge M$, we have $|f_n(a_i) - f(a_i)| < \varepsilon$ for all a_i. Then for any $x \in [a,b]$, using the convexity and the above inequality, we see that for $n \ge M$,

$$f_n(x) \le \max\{f_n(a_i), f_n(a_{i+1})\} \le \max\{f(a_i), f(a_{i+1})\} + \varepsilon \le f(x) + 2\varepsilon.$$

This gives the necessary uniform upper bound. I could not figure out how to achieve the corresponding lower bound.

Problem S12.3. Prove the Bolzano-Weierstrass Theorem in the following form: Every sequence $(a_n)_{n\in\mathbb{N}}$ of numbers in $[0,1]$ has a convergent subsequence.

Solution. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$. Divide the interval into $[0,1/2]$ and $[1/2,1]$ At least one of these subintervals must contain infinitely many of (a_n). Let $(a_{1,n})$ be a subsequence that is entirely in one of the half intervals. Divide that interval into two subintervals which cover the set and are of length $1/4$. Then there is an infinite subsequence of $(a_{1,n})$ which lies entirely in one interval of length $1/4$. Repeat this process indefinitely. Then we have inductively created sequences $(a_{k,n})_{n=1}^\infty$ such that each $(a_{k,n})_{n=1}^\infty$ is a subsequence of $(a_{k-1,n})_{n=1}^\infty$ and each sequence $(a_{k,n})_{n=1}^\infty$ is contained in an interval of length $1/2^k$. Define $x_n = a_{n,n}, n \in \mathbb{N}$. Then $(x_n)_{n=1}^\infty$ is a subsequence of $(a_n)_{n=1}^\infty$ and for each $k \in \mathbb{N}$, $(x_n)_{n=k}^\infty$ is a subsequence of $(a_{k,n})_{n=1}^\infty$. Let $\varepsilon > 0$. Choosing $k \in \mathbb{N}$ so that $1/2^{k-1} < \varepsilon$, we see that $m, n \ge k$ implies

$$|x_m - x_n| \le \tfrac{1}{2^{k-1}} < \varepsilon.$$

Thus (x_n) is a Cauchy sequence and thus converges since $[0,1]$ is complete (which is because it is a closed subset of a complete space). Thus every sequence (a_n) in $[0,1]$ has a convergent subsequence.

Problem S12.4. For a sequence (a_n) of *non-negative* numbers, let $s_n := \sum_{i=1}^n a_i$. Suppose that (s_n) tends to a number $s \in \mathbb{R}$ in the Cesaro sense:

$$\lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n} = s.$$

Show that $\sum_{n=1}^\infty a_n$ exists and is equal to s.

Solution. To show the infinite sum exists, we use a proof by contradiction. Assuming the sum diverges, since all terms are non-negative, the sum must diverge to $+\infty$. Then for any $L > 0$, there is $N \in \mathbb{N}$, such that $s_n = \sum_{i=1}^n a_i > L$ for all $n > N$. Choosing $n > N$, we see

$$\frac{s_1 + s_2 + \cdots + s_n}{n} = \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n} > \frac{(n-N)L}{n}.$$

Letting $L \to \infty$ implies that

$$\frac{s_1 + s_2 + \cdots + s_n}{n}$$

can be made arbitrarily large so that (s_n) cannot converge in the Cesaro sense which contradicts our assumption. We conclude that $\sum_{n=1}^{\infty} a_n$ exists.

To prove that $\sum_{n=1}^{\infty} a_n = s$, we need to prove that $s_n \to s$. Suppose that $s_n \to s'$ for some $s' \in \mathbb{R}$. Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ so that $|s_n - s'| < \varepsilon/2$, $n > N$. Then $|s_1 - s'| + \cdots |s_N - s'|$ is some finite number. Choose $N^* > N$ so that

$$\frac{|s_1 - s'| + \cdots |s_N - s'|}{N^*} < \varepsilon/2.$$

Then for $n > N^*$, we have

$$\begin{aligned}\left|\frac{s_1 + \cdots + s_n}{n} - s'\right| &= \left|\frac{(s_1 - s') + \cdots (s_n - s')}{n}\right| \\ &\leq \left|\frac{(s_1 - s') + \cdots (s_N - s')}{n}\right| + \left|\frac{(s_{N+1} - s') + \cdots (s_n - s')}{n}\right| \\ &\leq \frac{|s_1 - s'| + \cdots |s_N - s'|}{n} + \frac{|s_{N+1} - s'| + \cdots |s_n - s'|}{n} \\ &< \frac{|s_1 - s'| + \cdots |s_N - s'|}{N^*} + \frac{(n-N)}{n} \cdot \frac{\varepsilon}{2} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{s_1 + \cdots + s_n}{n} = s'.$$

Since limits in $\mathbb{R}$ are unique, we must have $s' = s$ and so $\lim_{n\to\infty} s_n = s$ as desired.

Problem S12.5. Prove there is a unique continuous function $y : [0,1] \to \mathbb{R}$ such that

$$y(x) = e^x + \frac{y(x^2)}{2}, \quad x \in [0,1].$$

Solution. Define the operator T on $C[0,1]$ by

$$(Ty)(x) = e^x + \frac{y(x^2)}{2}, \quad x \in [0,1].$$

Then clearly, Ty is continuous whenever y is continuous so $T : C[0,1] \to C[0,1]$. Consider, for any $y, z \in C[0,1]$,

$$|(Ty)(x) - (Tz)(x)| = \left|\frac{z(x^2)}{2} - \frac{y(x^2)}{2}\right| = \frac{1}{2}\left|z(x^2) - y(x^2)\right| \leq \frac{1}{2}\,||z - y||_{\infty}, \quad x \in [0,1].$$

Thus taking the supremum over all x, we have

$$||Ty - Tz||_{\infty} \leq \frac{1}{2}\,||y - z||_{\infty}.$$

Hence T is a contraction mapping and since $C[0,1]$ is complete with respect to the sup norm, there is a unique $y \in C[0,1]$ such that $Ty = y$. This y is the unique continuous function satisfying the desired equation.

Problem S12.6. Let γ be a smooth curve in $\mathbb{R}^2 - \{(0,0)\}$ which begins and ends at $(1,0)$ and winds once around the origin counterclockwise. Compute the integral

$$I(\gamma) = \oint_\gamma \frac{y\,dx - x\,dy}{x^2+y^2}.$$

Solution. We can parameterize the curve in any way we want since the problem doesn't specify. The most natural parameterization is

$$x = \cos\theta, \quad y = \sin\theta, \quad \theta \in [0, 2\pi].$$

Then

$$\begin{aligned} I(\gamma) &= \int_0^{2\pi} \frac{\sin\theta(-\sin\theta) - \cos\theta\cos\theta}{\cos^2\theta + \sin^2\theta} d\theta \\ &= -\int_0^{2\pi} d\theta = -2\pi. \end{aligned}$$

Problem F13.1. For a sequence $\{a_n\}$ of positive real numbers, define $P_n = \prod_{j=1}^n (1+a_j)$. Prove that $\lim_{n\to\infty} P_n$ exists if and only if $\lim_{n\to\infty} \sum_{j=1}^n a_j$ exists.

Solution. Since all terms are positive P_n and $S_n = \sum_{j=1}^n a_j$ are increasing sequences. If we can show they are bounded above then they converge.

Suppose that S_n converges. Then S_n is a bounded sequence; let $M = \sup_n S_n$. For all $x \ge 0$, we have $1 + x \le e^x$ (this is easily verified by looking at the Taylor Expansion of e^x). Then

$$P_n = \prod_{j=1}^n (1+a_j) \le \prod_{j=1}^n e^{a_j} = \exp\left(\sum_{j=1}^n a_j\right) = e^{S_n} \le e^M.$$

Thus P_n converges.

Suppose P_n converges. We see that

$$P_n = \prod_{j=1}^n (1+a_j) = 1 + \left(\sum_{j=1}^n a_j\right) + \left(\sum_{i\neq j} a_i a_j\right) + \cdots + \prod_{j=1}^n a_j = S_n + (\text{positive terms}).$$

Thus $S_n \le P_n$, so S_n converges since P_n converges.

Problem F13.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function.

(a) Prove that $\{x \in \mathbb{R} : f \text{ is not continuous at } x\}$ is countable.

(b) Let $S \subset \mathbb{R}$ be a countable set. Prove there is a nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ such that f is discontinuous at x if and only if $x \in S$.

Solution.

(a) Let $\Omega = \{x \in \mathbb{R} : f \text{ is not continuous at } x\}$. Then for $y \in \Omega$, we must have

$$\lim_{x\to y^-} f(x) \neq \lim_{x\to y^+} f(x),$$

because otherwse f would be continuous at y. In particular, since f is nondecreasing,

$$\lim_{x\to y^-} f(x) < \lim_{x\to y^+} f(x).$$

Since the rationals are dense in the reals, for each $y \in \Omega$, we can find $q(y) \in \mathbb{Q}$ so that

$$\lim_{x\to y^-} f(x) < q(y) < \lim_{x\to y^+} f(x).$$

Consider, for $y \neq z \in \Omega$ (wlog we take $y < z$), we have

$$\lim_{x\to y^-} f(x) < q(y) < \lim_{x\to y^+} f(x) \leq \lim_{x\to z^-} f(x) < q(z) < \lim_{x\to z^+} f(x),$$

and thus $q(y) \neq q(z)$. This means that Ω injects into $\mathbb{Q}$ and thus must be countable.

(b) Let $S \subset \mathbb{R}$ be countable. Then we can say $S = \{s_n : n \in \mathbb{N}\}$. For any x define $A_x \subset \mathbb{N}$ such that $A_x = \{n \in \mathbb{N} : s_n < x\}$. Then

$$f(x) = \sum_{n\in A_x} \frac{1}{2^n}$$

has the desired property. Indeed for $x < y$, if there is any $n \in \mathbb{N}$ such that $x < s_n < y$, then clearly $f(y)$ is greater than $f(x)$ by at least 2^{-n}; if there is no such n, then $f(x) = f(y)$. Thus f is nondecreasing. Further, for any $n \in \mathbb{N}$,

$$\lim_{x\to s_n^-} f(x) < \lim_{x\to s_n^+} f(x) = \lim_{x\to s_n^-} f(x) + \frac{1}{2^n}$$

so f is discontinuous at all points in S. Also, for $y \notin S$, take $\delta > 0$ small enough so that $s_1, \ldots, s_N$ are not in $(y - \delta, y + \delta)$ [this is possible for any N since none of these values can equal y]. Then for $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq \sum_{n=N}^{\infty} \frac{1}{2^n}.$$

Choosing N large enough, this bound becomes arbitrarily small. Thus f is continuous at y when $y \notin S$.

Problem F13.3. Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a continuous injective function. By definition, the length of the range $\gamma([0, 1])$ is

$$L(\gamma) = \sup\left\{\sum_{j=0}^{n-1} ||\gamma(t_{j+1}) - \gamma(t_j)|| : 0 \leq t_0 < t_1 < \cdots < t_n \leq 1, n < \infty\right\}.$$

(a) Suppose that f is continuous and non-decreasing on $[0,1]$ and let $\gamma(t) = (t, f(t))$ (so that the range of γ is the graph of f). Prove

$$L(\gamma) \le 1 + f(1) - f(0).$$

(b) Show there exists a continuous nondecreasing $f(t)$ on $[0,1]$ such that $f(0) = 0, f(1) = 1$ and $L(\gamma) = 2$ when $\gamma(t) = (t, f(t))$.

Solution.

(a) Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be an arbitrary partition of $[0,1]$ Then

$$\begin{aligned} L &\le \sum_{j=0}^{n-1} ||\gamma(t_{j+1}) - \gamma(t_j)|| \\ &= \sum_{j=0}^{n-1} ||(t_{j+1}, f(t_{j+1})) - (t_j, f(t_j))|| \\ &= \sum_{j=0}^{n-1} ||(t_{j+1}, f(t_{j+1})) - (t_{j+1}, f(t_j)) + (t_{j+1}, f(t_j)) - (t_j, f(t_j))|| \\ &\le \sum_{j=0}^{n-1} ||(t_{j+1}, f(t_{j+1})) - (t_{j+1}, f(t_j))|| + ||(t_{j+1}, f(t_j)) - (t_j, f(t_j))|| \\ &= \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)| + |t_{j+1} - t_j| \\ &= f(t_n) - f(t_0) + t_n - t_0 = 1 + f(1) - f(0). \end{aligned}$$

Note we use $|f(t_{j+1}) - f(t_j)| = f(t_{j+1}) - f(t_j)$ which is valid since f is non-decreasing.

(b) This question seems unreasonably difficult for the basic exam but here we go.

Let $f_0(x) = x, x \in [0,1]$ and iteratively define

$$f_{n+1}(x) = \begin{cases} f_n(3x)/2, & 0 \le x \le 1/3, \\ 1/2, & 1/3 \le x \le 2/3, \\ 1/2 + f_n(3x-2)/2, & 2/3 \le x \le 1. \end{cases}$$

We argue that these are continuous functions. If f_n is continuous, then f_{n+1} is continuous on $[0, 1/3)$ and $(2/3, 1]$. It is also clear that f_{n+1} is continuous on $(1/3, 2/3)$. By induction, we see that $f_n(0) = 0, f_n(1) = 1$ for all $n \in \mathbb{N}$ and thus $f_{n+1}(1/3) = f_n(1)/2 = 1/2$ and $f_{n+1}(2/3) = 1/2 + f_n(0)/2 = 1/2$ so f_{n+1} is continuous at $1/3, 2/3$. Thus f_{n+1} is continuous whenever f_n is continuous. Thus, since f_0 is continuous, we have f_n continuous for all n. Next, we see that if $x \in (1/3, 2/3)$, then $f_n(x) = f_{n+1}(x)$, $n \ge 1$. Next, we see

$$\begin{aligned} \sup_{0 \le x \le 1/3} |f_{n+1}(x) - f_n(x)| &= \sup_{0 \le x \le 1/3} \left| \tfrac{1}{2} f_n(3x) - \tfrac{1}{2} f_{n-1}(3x) \right| \\ &= \tfrac{1}{2} \sup_{0 \le x \le 1} |f_n(x) - f_{n-1}(x)| \end{aligned}$$

and similarly

$$\sup_{2/3\le x\le 1} |f_{n+1}(x) - f_n(x)| = \sup_{2/3\le x\le 1} \left|\tfrac{1}{2} - \tfrac{1}{2} f_n(3x-2) - \tfrac{1}{2} + \tfrac{1}{2} f_{n-1}(3x-1)\right|$$
$$= \tfrac{1}{2} \sup_{0\le x\le 1} |f_n(x) - f_{n-1}(x)|.$$

Thus we have

$$\sup_{0\le x\le 1} |f_{n+1}(x) - f_n(x)| = \tfrac{1}{2} \sup_{0\le x\le 1} |f_n(x) - f_{n-1}(x)|.$$

Hence by induction,

$$\sup_{0\le x\le 1} |f_{n+1}(x) - f_n(x)| = \tfrac{1}{2^n} \sup_{0\le x\le 1} |f_1(x) - f_0(x)|.$$

This shows that f_n is uniformly Cauchy and thus converges uniformly to some function f. Since f is the uniform limit of continuous functions, it is continuous.

By constuction, f is constant on the intervals $[1/3, 2/3], [1/9, 2/9], [7/9, 8/9], [1/27, 2/27]$, etc. The sum of the lengths of the intervals is

$$\tfrac{1}{3} + 2 \cdot \tfrac{1}{9} + 4 \cdot \tfrac{1}{27} + \cdots = \tfrac{1}{3} \sum_{k=0}^{\infty} \left(\tfrac{2}{3}\right)^k = \tfrac{1}{3} \left(\tfrac{1}{1-1/3}\right) = 1.$$

However, $f(0) = 0$ and $f(1) = 1$, thus on the set where f is non-constant, the arc length must be at least 1 since the vertical change is 1. This means that the total arc length is greater than or equal to 2. Using (a), we see that the arc length must be 2.

[Note: The function f we constructed is called the Cantor step function and has a few remarkable properties. It is constant on the complement of the Cantor set. As a consequence, it is differentiable almost everywhere and has derivative zero almost everywhere. However, it still increases from 0 to 1. It is uniformly continuous (since it is continuous on a compact set) but it is not absolutely continuous since it cannot be written as the derivative of its (Lebesgue) integral. Finally, the function $g(x) = x + f(x), x \in [0, 1]$ maps the Cantor set (a set of measure zero) homeomorphically to a set of measure 1. This is a bizarre result because, while we know continuous functions can stretch and enlarge volume, it seems counterintuitive that they can "create" volume.]

Problem F13.5. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \quad \text{for all} \quad x, y \in \mathbb{R}^d, 0 \le t \le 1.$$

Assume that f is continuously differentiable and

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \ge 0, \;\; x, y \in \mathbb{R}^d.$$

Prove that f is convex.

Solution. For simplicity, for fixed $x, y \in \mathbb{R}^d$, define $F, G : [0,1] \to \mathbb{R}$ by

$$F(t) = f(tx + (1-t)y), \quad G(t) = tf(x) + (1-t)f(y), \quad t \in [0,1].$$

We need to prove that $F(t) \le G(t)$ or rather $G(t) - F(t) \ge 0$, for all $t \in [0,1]$. We see

$$G'(t) - F'(t) = f(x) - f(y) - \nabla f(tx + (1-t)y) \cdot (x-y).$$

For all $r > s$, we have

$$(G'(r) - F'(r)) - (G'(s) - F'(s)) = [\nabla f(sx + (1-s)y) - \nabla f(rx + (1-r)y)] \cdot (x-y).$$

But $sx + (1-s)y - (rx + (1-r)y) = (s-r)(x-y)$, and $s - r < 0$ so by our assumption

$$(s-r)((G'(r) - F'(r)) - (G'(s) - F'(s))) \ge 0$$

so $(G'(r) - F'(r)) - (G'(s) - F'(s)) \le 0$ or

$$G'(r) - F'(r) \le G'(s) - F'(s), \quad r, s \in [0,1], \;\; r > s.$$

That is, the derivative of $G - F$ is decreasing.

It is clear that $G(1) - F(1) = 0 = G(0) - F(0)$. Now assume that there is $t \in (0,1)$ such that $G(t) - F(t) < 0$. Then by the mean value theorem, there are $c \in (0,t), d \in (t,1)$ such that

$$0 < G(t) - F(t) - (G(0) - F(0)) = (G'(c) - F'(c))t$$

and

$$0 < G(t) - F(t) - (G(1) - F(1)) = (G'(d) - F'(d))(t-1).$$

Then

$$G'(d) - F'(d) < 0 < G'(c) - F'(c)$$

which contradicts that the derivative of $G - F$ is decreasing. Hence there is no such t, so $G(t) \ge F(t)$ as required.

Problem F13.6. Let X be a metric space and let (x_n) be a sequence in X such that if (y_n) is a subsequence of (x_n) then there is a subsequence (z_n) of (y_n) converging to $x \in X$. Prove that (x_n) converges to $x \in X$.

Solution. Suppose that $x_n \not\to x$. Then there is $\varepsilon > 0$ such that for any $k \in \mathbb{N}$ there is $n \ge k$ such that $d(x_n, x) \ge \varepsilon$. For each $k \in \mathbb{N}$, let $n_k \ge k$ be a natural number such that $d(x_n, x) \ge \varepsilon$. Then (x_{n_k}) is certainly a subsequence of (x_n) but any member of (x_{n_k}) is at least ε away from x. Hence no subsequence of (x_{n_k}) converges to x which is a contradiction. Thus $x \to x$.

Problem S13.1.

(a) Define what it means for a function on $[0,1]$ to be Riemann integrable.

(b) Show that every monotone increasing function on $[0,1]$ is Riemann integrable.

(c) Use part (b) to construct a Riemann integrable function on $[0, 1]$ which has infinitely many discontinuities.

Solution.

(a) Let $f : [0, 1] \to \mathbb{R}$. Then by definition, f is Riemann integrable if and only if

$$\overline{\int_0^1} f(x)dx = \underline{\int_0^1} f(x)dx,$$

where

$$\overline{\int_0^1} f(x)dx = \inf\left\{\int_0^1 g(x)dx : g \text{ is a piecewise constant majorizing function of } f \text{ on } [0,1]\right\},$$

and

$$\underline{\int_0^1} f(x)dx = \sup\left\{\int_0^1 h(x)dx : h \text{ is a piecewise constant minorizing function of } f \text{ on } [0,1]\right\}.$$

(b) Left $\varepsilon > 0$, and $n \in \mathbb{N}$ such that $\frac{f(1)-f(0)}{n} < \varepsilon$. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be the partition of $[0, 1]$; i.e., $x_i = i/n, i = 0, \ldots, n$. Define

$$g(x) = f(x_{i+1}), \quad x \in [0,1], x_i \le x < x_{i+1}$$

where $i = 0, \ldots, n-1$ and by convention $g(1) = f(1)$. Then g is piecewise constant and majorizes f. Likewise, let

$$h(x) = f(x_i), \quad x \in [0,1], x_i \le x < x_{i+1},$$

for $i = 0, \ldots, n-1$, and $h(1) = f(x_{n-1})$. Then h is piecewise constant and minorizes f. Further

$$\begin{aligned}
\overline{\int_0^1} f(x)dx - \underline{\int_0^1} f(x)dx &\le \int_0^1 g(x)dx - \int_0^1 h(x)dx \\
&= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (g(x) - h(x))dx \\
&= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} (f(x_{i+1}) - f(x_i))dx \\
&= \sum_{i=1}^{n-1} (x_{i+1} - x_i)(f(x_{i+1}) - f(x_i)) \\
&= \frac{1}{n} \sum_{i=1}^{n-1} (f(x_{i+1}) - f(x_i)) \\
&= \frac{1}{n}(f(1) - f(0)) < \varepsilon.
\end{aligned}$$

Since ε was arbitrary, this implies that $\overline{\int_0^1} f(x)dx = \underline{\int_0^1} f(x)dx$ so f is Riemann integrable.

(c) Define

$$f(x) = \begin{cases} 0, & x = 0, \\ \frac{1}{n+1}, & x \in \left[\frac{1}{n+1}, \frac{1}{n}\right), n = 1, 2, 3, \ldots, \\ 1, & x = 1. \end{cases}$$

Then f is nondecreasing, and hence Riemann integrable, but f is discontinuous at $x = \frac{1}{n}$ for all $n = 1, 2, 3, \ldots$.

Problem S13.2. The approximation from Simpson's Rule for $\int_a^b f(x)dx$ is given by

$$S(f) = \left[\frac{2}{3} f\left(\frac{a+b}{2}\right) + \frac{1}{3}\left(\frac{f(a)+f(b)}{2}\right)\right](b-a).$$

Suppose that f has continuous derivatives up to order 3. Show that

$$\left| \int_a^b f(x)dx - S(f) \right| \leq C(b-a)^4 \sup_{a \leq x \leq b} \left| f^{(3)}(x) \right|$$

where C is a constant which does not depend on f.

Solution. We construct a second-degree polynomial p that agrees with f at a, b and $\frac{a+b}{2}$. Put $c = \frac{a+b}{2}$. The correct polynomial is

$$p(x) = f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)} + f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)}.$$

Put $g(x) = f(x) - p(x)$. Next, choose $d \in [a, b]$, $d \neq a, d \neq b, d \neq c$. Then

$$q(x) = g(d)\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}$$

is a third degree polynomial which agrees with g at 4 points. Hence by higher order Rolle's theorem, there is $y \in (a, b)$ such that $g'''(y) - q'''(y) = 0$. But $g''' = f'''$ so

$$f'''(y) = q'''(y) = \frac{6g(d)}{(d-a)(d-b)(d-c)}$$

and thus

$$|g(d)| = \frac{|f'''(x)|\,|d-a|\,|d-b|\,|d-c|}{6} \leq \frac{(b-a)^3}{6} \sup_{a \leq t \leq b} |f'''(t)|.$$

Since d was arbitrary, we conclude that

$$|g(x)| = |f(x) - p(x)| \leq \frac{(b-a)^3}{6} \sup_{a \leq t \leq b} |f'''(t)|, \quad \text{for all } x \in [a, b].$$

Finally, let $h = c - a = b - c$, then

$$p(x) = -\frac{f(c)}{h^2}(x-a)(x-b) + \frac{f(b)}{2h^2}(x-a)(x-c) + \frac{f(a)}{2h^2}(x-b)(x-c).$$

So

$$\begin{aligned}
\int_a^b p(x)dx &= \int_{-h}^h p(x+c)dx \\
&= \frac{1}{h^2}\int_{-h}^h \frac{1}{2}\left(f(b)x(x+h)+f(a)x(x-h)\right) - f(c)(x-h)(x+h)dx \\
&= \frac{1}{h^2}\left(\frac{1}{2}\cdot\frac{2h^3(f(b)+f(a))}{3} + \frac{4h^3 f(c)}{3}\right) \\
&= \frac{h}{3}(f(b)+f(a)) + 4f(c)) \\
&= (b-a)\left[\frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{3}\left(\frac{f(a)+f(b)}{2}\right)\right] = S(f).
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\int_a^b f(x)dx - S(f)\right| &= \left|\int_a^b f(x) - p(x)dx\right| \\
&\le \int_a^b |f(x)-p(x)|\,dx \\
&\le \frac{(b-a)^3}{6}\sup_{a\le t\le b}|f'''(t)|\int_a^b dx \\
&= \frac{(b-a)^4}{6}\sup_{a\le t\le b}|f'''(t)|,
\end{aligned}$$

so $C = 1/6$.

Problem S13.3. Prove that a metric space is sequentially compact if and only if it is complete and totally bounded.

Solution. Let X be a metric space which is complete and totally bounded. Let (x_n) be a sequence in X. Since we will be inductively choosing more and more subsequences, it is convenient to say $(x_n) = (x_{1,n})$. Since X is totally bounded, it can be covered with finitely many balls of radius $1/2$. One of these balls must contain infinitely many of $(x_{1,n})$. Let $(x_{2,n})$ be a subsequence of $(x_{1,n})$ which is contained in a ball of radius $1/2$. Cover this ball with finitely many balls of radius $1/3$. Then infinitely many of $(x_{2,n})$ must be contained in a single ball. Let $(x_{3,n})$ be a subsequence of $(x_{2,n})$ which is completely contained in a ball of radius $1/3$. Continuing this process we get sequences $(x_{k,n})$ each contained in a ball of radius $1/k$ and each a subsequence of $(x_{k-1,n})$. Let (y_n) be defined $y_n = x_{n,n}$ for all $n \in \mathbb{N}$. Then (y_n) is a subsequence of (x_n) and by construction,

$$d(y_n, y_m) < \max\left\{\tfrac{2}{n}, \tfrac{2}{m}\right\}.$$

From this we see that (y_n) is a Cauchy sequence and is hence convergent since X is complete. Thus (x_n) has a convergent subsequence and so X is sequentially compact.

Now suppose that X is sequentially compact. Let (x_n) be a Cauchy sequence in X. Then there is a subsequence (x_{n_k}) which converges to some $x \in X$. Let $\varepsilon > 0$, then there is $N \in \mathbb{N}$ such that $d(x_n, x_{n_k}) < \varepsilon/2$ and $d(x_{n_k}, x) < \varepsilon/2$ when $n, k > N$. For such n, k, we have

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$$

so (x_n) converges to x as well. Thus Cauchy sequences in X converge and so X is complete.

Again, suppose X is sequentially compact. Let $\varepsilon > 0$. For contradiction, assume that X cannot be covered by finitely many balls of radius ε. Let $x_1 \in X$. Then by assumption $B(x_1, \varepsilon)$ does not cover X, so there is x_2 outside $B(x_1, \varepsilon)$. Again, by assumption, $B(x_1, \varepsilon), B(x_2, \varepsilon)$ do not cover X so there is x_3 outside of both $B(x_1, \varepsilon), B(x_2, \varepsilon)$. Repeating this process inductively, we construct a sequence (x_n) such that $d(x_n, x_m) \geq \varepsilon$ for all $m \neq n$. This sequence has no Cauchy subsequence and hence no convergent subsequence. This is a contradiction since we assumed our space is sequentially compact. Hence X can be covered by finitely many balls of radius ε and so X is totally bounded.

Problem S13.4. Denote by h_n the n^{th} harmonic number:

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Prove that there is a limit

$$\gamma = \lim_{n \to \infty} (h_n - \ln n).$$

Solution. Let $x_n = h_n - \ln n$. Define $f_n : [1, \infty) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1/j, & j \leq x < j+1, j = 1, 2, \ldots n-1, \\ 0, & x > n. \end{cases}$$

Then each f_n is piecewise constant an majorizes $f(x) = 1/x$ for $x \in [1, n]$. Then

$$h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \int_1^n f_n(x) dx \geq \int_1^n \frac{1}{x} dx = \ln n.$$

Thus $x_n = h_n - \ln n \geq 0$. Also

$$x_{n+1} - x_n = h_{n+1} - h_n + \ln n - \ln(n+1) = \frac{1}{n+1} + \ln n - \ln(n+1).$$

By the mean value theorem, there is $c \in (n, n+1)$ such that

$$\ln(n) - \ln(n+1) = \frac{1}{c}(n - (n+1)) = -\frac{1}{c}.$$

Then

$$x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{c} \leq 0.$$

Thus x_n is a decreasing sequence which is bounded below so it converges to its infimum.

Problem S13.5. Define polynomials $U_n(x), n = 0, 1, 2, \ldots$ as follows:

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, 4, \ldots.$$

(a) Prove that $U_n(\cos\theta) = \dfrac{\sin((n+1)\theta)}{\sin\theta}$ for all n.

(b) Prove that the polynomials satisfy

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}dx = \frac{\pi}{2}\delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker delta (i.e., $\delta_{m,n} = 1$ if $m = n$ and 0 otherwise).

Solution.

(a) The claim is most easily proven by induction. Note $U_0(\cos\theta) = 1 = \frac{\sin((0+1)\theta)}{\sin\theta}$ so the claim clearly holds for $n = 0$. Also

$$U_1(\cos\theta) = 2\cos\theta = \frac{2\cos\theta\sin\theta}{\sin\theta} = \frac{\sin(2\theta)}{\sin\theta} = \frac{\sin((1+1))\theta}{\sin\theta}.$$

Hence the claim also holds for $n = 1$.

Set $n \geq 1$ and assume the claim holds for all $k \leq n$. Then

$$\begin{aligned}
U_{n+1}(\cos\theta) &= 2\cos\theta U_n(\cos\theta) - U_{n-1}(\cos\theta) \\
&= 2\cos\theta\frac{\sin((n+1)\theta)}{\sin\theta} - \frac{\sin(n\theta)}{\sin\theta} \\
&= \frac{2\cos\theta\sin((n+1)\theta) - \sin((n+1)\theta - \theta)}{\sin\theta} \\
&= \frac{2\cos\theta\sin((n+1)\theta) - (\cos\theta\sin((n+1)\theta) - \cos((n+1)\theta)\sin\theta)}{\sin\theta} \\
&= \frac{\cos\theta\sin((n+1)\theta) + \cos((n+1)\theta)\sin\theta)}{\sin\theta} \\
&= \frac{\sin((n+1)\theta + \theta)}{\sin\theta} = \frac{\sin((n+2)\theta)}{\sin\theta}.
\end{aligned}$$

This completes the induction so the claim holds for all n.

(b) Use the substitution $x = \cos\theta$. Then

$$\begin{aligned}
\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}dx &= \int_{\pi}^{0} U_m(\cos\theta)U_n(\cos\theta)\sqrt{1-\cos^2\theta}(-\sin\theta)d\theta \\
&= \int_0^{\pi} \frac{\sin((m+1)\theta)}{\sin\theta}\frac{\sin((n+1)\theta)}{\sin\theta}\sin^2\theta d\theta \\
&= \int_0^{\pi} \sin((m+1)\theta)\sin((n+1)\theta)d\theta.
\end{aligned}$$

From here, the using the identity $\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ and performing the integration gives the answer.

Problem S13.11. Define the Fibonacci sequence F_n by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n = 2, 3, 4, \ldots$.

(a) Show that the limit as $n \to \infty$ of $\frac{F_n}{F_{n-1}}$ exists and find its value.

(b) Prove that $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1, n \geq 1$.

Solution.

(a) To find the limit, we simply solve the recurrence relation and take the limit explicitly. Assume $F_n = C \cdot x^n$ for some real number x. Then from the recurrence relation, we see

$$Cx^n = Cx^{n-1} + Cx^{n-2} \implies x^2 - x - 1 = 0.$$

Solving gives

$$x_\pm = \frac{1 \pm \sqrt{5}}{2}.$$

In particular, $|x_-| < 1$ and

$$F_n = C_1 x_+^n + C_2 x_-^n.$$

We could solve for C_1, C_2 using F_0, F_1, but we will see it doesn't matter what they are. We notice

$$\frac{F_n}{F_{n-1}} = \frac{C_1 x_+^n + C_2 x_-^n}{C_1 x_+^{n-1} + C_2 x_-^{n-1}}$$

and since $\lim\limits_{n\to\infty} x_-^n = 0$, we have

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{C_1 x_+^n}{C_1 x_+^{n-1}} = x_+ = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

(b) We prove the claim by induction, noticing that $F_3F_1 - F_2^2 = 2 \cdot 1 - 1^2 = 1$. Assume for some $n \geq 1$, we have $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$. Then

$$\begin{aligned}
F_{2n+3}F_{2n+1} - F_{2n+2}^2 &= (F_{2n+2} + F_{2n+1})F_{2n+1} - F_{2n+2}^2 \\
&= F_{2n+2}F_{2n+1} + F_{2n+1}^2 - F_{2n+2}^2 \\
&= F_{2n+2}F_{2n+1} + F_{2n+1}^2 - (F_{2n+1} + F_{2n})^2 \\
&= F_{2n+2}F_{2n+1} - 2F_{2n+1}F_{2n} - F_{2n}^2 \\
&= F_{2n+1}(F_{2n+2} - 2F_{2n}) - F_{2n}^2 \\
&= F_{2n+1}(F_{2n+1} - F_{2n}) - F_{2n}^2 \\
&= F_{2n+1}F_{2n-1} - F_{2n}^2 = 1 \quad \text{by our inductive hypothesis.}
\end{aligned}$$

Thus the claim is proven for all $n \geq 1$.

Problem F14.1. Show that the function

$$H(x, y) = x^2 + y^2 + |x - y|^{-1}$$

achieves its global minimum somewhere in the set $\{(x, y) \in \mathbb{R}^2 : x \neq y\}$.

Solution. By symmetry, it is enough to check $x > y$. In that region,

$$\frac{\partial H}{\partial x} = 2x - \frac{1}{(x-y)^2},$$
$$\frac{\partial H}{\partial y} = 2y + \frac{1}{(x-y)^2}.$$

Setting these both equal to zero, we see

$$2x(x-y)^2 = 1, \tag{3}$$
$$-2y(x-y)^2 = 1. \tag{4}$$

Adding (1),(2) gives

$$(x-y)^3 = 1 \quad \implies \quad x = y + 1.$$

Plugging this into (2) yields $y = -1/2$ and so $x = 1/2$. This point must be a minimum since there are no local/global maxima for this function (by the very nature of the square and absolute value functions). By symmetry, the function has global minima at $(x, y) = (1/2, -1/2)$ and $(x, y) = (-1/2, 1/2)$.

Problem F14.2. Let A, B be closed subsets of $\mathbb{R}^n$ such that $A \cup B$ and $A \cap B$ are connected. Prove that A is connected.

Solution. Suppose to the contrary that A is disconnected. Then there are open, disjoint, nonempty C, D such that $A = C \cup D$. By shrinking each of C, D and taking the closures, it is actually alright to take C, D to be closed. Then since $A \cap B$ is connected, it must be contained in either C or$D$.Without loss of generality, say $A \cap B \subset C$. Then $A \cup B = D \cup (B \cup C)$. However, since C, D are disjoint, and since $A \cap B, D$ are disjoint, this shows that $A \cup B$ is not connected; a contradiction. Thus A is connected.

Problem F14.3. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of monotonically increasing continuous functions. Assume that f_n converge pointwise to a continuous function $f : [0, 1] \to \mathbb{R}$. Show that the convergence is uniform.

Solution. Let $\varepsilon > 0$.

Since each of f_n is increasing, by passing to the limit, we see that f is increasing. Also, since f is continuous and $[0, 1]$ is compact, f is uniformly continuous. Take $\delta > 0$ such that for all $x, y \in [0, 1]$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/5$.

Choose $N \in \mathbb{N}$ such that $1/N < \delta$ and let $0 = x_0 < x_1 < \ldots < x_N = 1$ be the uniform partition of $[0, 1]$; i.e., $x_i = i/N$, $i = 0, 1, \ldots, N$.

Since $f_n \to f$ pointwise, for each $i = 0, 1, \ldots, N$ there is $M_i \in \mathbb{N}$ such that $n \geq M_i$ implies that $|f_n(x_i) - f(x_i)| < \varepsilon/5$ for each i. Take $M = \max\{M_0, M_1, \ldots, M_N\}$ (so that M is a uniform constant that guarantees convergence at the grid points).

For any $x \in [0, 1]$, there is $i = 0, 1 \ldots, N - 1$ such that $x \in [x_i, x_{i+1}]$. Then for $n \geq M$, we have

$$\begin{aligned}|f_n(x) - f(x)| &= |f_n(x) - f_n(x_i) + f_n(x_i) - f(x_i) + f(x_i) - f(x)| \\ &\le |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &< f_n(x_{i+1}) - f_n(x_i) + \varepsilon/5 + \varepsilon/5 \\ &< f(x_{i+1}) + \varepsilon/5 - (f(x_i) - \varepsilon/5) + 2\varepsilon/5 \\ &< f(x_{i+1}) - f(x_i) + 4\varepsilon/5 < \varepsilon.\end{aligned}$$

Notice that M does not depend on x, thus the convergence is uniform.

Problem F14.4. Let $f_n : [-2, 2] \to [0, 1]$ be a sequence of convex functions. Show that there is a subsequence which converges uniformly on $[-1, 1]$.

Solution. From the range of f, we see $|f_n(x)| \le 1$ so the sequence is uniformly bounded. Also, since each f_n is convex, we know that

$$\frac{f_n(s) - f_n(t)}{s - t}$$

is a non-decreasing function of s and t. Then for $s, t \in [-1, 1]$, we have

$$\frac{f_n(s) - f_n(t)}{s - t} \le \frac{f_n(2) - f_n(t)}{2 - t} \le \frac{f_n(2) - f_n(1)}{2 - 1} \le 1.$$

Also,

$$\frac{f_n(s) - f_n(t)}{s - t} \ge \frac{f_n(-1) - f_n(t)}{-1 - t} \ge \frac{f_n(-1) - f_n(-2)}{-1 - (-2)} \ge -1.$$

Thus

$$\left|\frac{f_n(s) - f_n(t)}{s - t}\right| \le 1 \quad \Longrightarrow \quad |f_n(s) - f_n(t)| \le |s - t|\,.$$

Thus each f_n is Lipschitz with the same Lipschitz constant. Hence the sequence is equicontinuous. Thus by the Arzela-Ascoli Theorem, there is a subsequence that converges uniformly.

Problem F14.5. Consider the sequence

$$a_1 = \sqrt{2} \quad \text{and} \quad a_{n+1} = \sqrt{2 + a_n}, \quad n \ge 1.$$

Prove that the sequence converges and find the limit.

Solution. We prove convergence by the monotone convergence theorem. It is clear that all terms in the sequence are non-negative. Consider, $a_1 \le 2$. Also, if $a_n \le 2$ then

$$2 + a_n \le 2 \quad \Longrightarrow \quad \sqrt{2 + a_n} \le 2 \quad \Longrightarrow \quad a_{n+1} \le 2.$$

Thus by induction, the sequence is bounded above by 2. Also $a_1 = \sqrt{2} \le \sqrt{2 + \sqrt{2}} = a_2$ and if $a_n \le a_{n+1}$, we have

$$2 + a_n \le 2 + a_{n+1} \quad \Longrightarrow \quad \sqrt{2 + a_n} \le \sqrt{2 + a_{n+1}} \quad \Longrightarrow \quad a_{n+1} \le a_{n+2}.$$

Thus by induction the sequence is increasing. Thus the sequence converges.

Let $\lim_{n\to\infty} a_n = a$. Then by continuity, a must satisfy

$$a = \sqrt{2+a} \quad \Longrightarrow \quad a^2 - a - 2 = 0 \quad \Longrightarrow \quad a = -1 \text{ or } a = 2.$$

But all a_n are nonnegative, so we cannot have $a = -1$. Thus $\lim_{n\to\infty} a_n = 2$.

Problem F14.6. Let $f : [0,1] \to \mathbb{R}$ be a C^1 function. Prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \left| f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right) \right| = \int_0^1 |f'(t)|\, dt.$$

Solution. Since f' is continuous so is $|f'|$ and since [0,1] is compact, $|f'|$ is uniformly continuous. Let $\varepsilon > 0$. then there is $\delta > 0$ such that $|x-y| < \delta \implies ||f'(x)| - |f'(y)|| < \varepsilon$. Choose $N \in \mathbb{N}$ such that $1/N < \delta$. For $n > N$, let $0 = x_0 < x_1 < \cdots < x_n = 1$ be the uniform partition of $[0,1]$. By the mean value theorem, in each interval (x_i, x_{i+1}), $i = 0, 1, \ldots, n-1$, there is y_i such that

$$|f(x_{i+1}) - f(x_i)| = |f'(y_i)|\, |x_{i+1} - x_i| = \tfrac{1}{n} |f'(y_i)|\,.$$

Putting this all together:

$$\begin{aligned}
\left| \sum_{k=0}^{n-1} \left| f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right) \right| - \int_0^1 |f'(x)|\, dx \right| &= \left| \sum_{k=0}^{n-1} \left(\frac{1}{n} |f'(y_i)| - \int_{x_k}^{x_{k+1}} |f'(x)|\, dx \right) \right| \\
&= \left| \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} [|f'(y_i)| - |f'(x)|] dx \right| \\
&\le \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} ||f'(y_i)| - |f'(x)||\, dx \\
&< \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \varepsilon dx = \sum_{k=0}^{n-1} \frac{\varepsilon}{n} = \varepsilon.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \left| f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right) \right| = \int_0^1 |f'(t)|\, dt$$

as desired.

Problem S14.7. Find a doubly infinite sequence $\{a_{n,m},\ n, m \in \mathbb{Z}\}$ such that for all m,

$$\sum_{n\in\mathbb{Z}} a_{n,m} = 0$$

and for all n,

$$\sum_{m\in\mathbb{Z}} a_{n,m} = 0$$

with all of these series converging absolutely but such that

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|a_{n,m}| = \infty.$$

Solution. Define

$$a_{n,m} = \begin{cases} 1, & m = n, \\ -1, & m = n+1, \quad 0, \quad m \neq n, n+1. \end{cases}$$

Then $a_{n,m}$ is an "infinite matrix" with 1 along the diagonal and -1 on the first superdiagonal. Fixing m and summing down any "column," we will get

$$\sum_{n\in\mathbb{Z}} a_{n,m} = a_{m,m-1} + a_{m,m} = -1 + 1 = 0 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |a_{n,m}| = |a_{m-1,m}| + |a_{m,m}| = 2.$$

Similarly, fixing n and summing across any "row," we have

$$\sum_{m\in\mathbb{Z}} a_{n,m} = a_{n,n} + a_{n,n+1} = 1 + (-1) = 0 \quad \text{and} \quad \sum_{m\in\mathbb{Z}} |a_{n,m}| = |a_{n,n}| + |a_{n,n+1}| = 2.$$

Thus all these sums are zero and all of them converge absolutely. However,

$$\sum_{n\in\mathbb{Z}}\sum_{m\in Z}|a_{n,m}| = \infty$$

because we are adding 1 infinitely many times.

Problem S14.8.

(a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

is infinitely differentiable.

(b) Find a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^n$, $\varphi(x) = 0$ if $|x| > 1$ and

$$\int_{\mathbb{R}^n} \varphi(x)dx = 1.$$

Solution.

(a) It is clear that f is infinitely differentiable on each of its pieces. We simply need to to prove that it is infinitely differentiable at 0. We see that, $f^{(n)}(t) = 0$ for all $t < 0$, $n \in \mathbb{N}$. Thus $\lim_{t\to 0^-} f^{(n)}(t) = 0$. Thus the goal is to show that $\lim_{t\to 0^+} f^{(n)}(t) = 0$ for all $n \in \mathbb{N}$. For $t > 0$,

$$f'(t) = \frac{1}{t^2}e^{-\frac{1}{t}}, \quad f''(t) = \left(\frac{1}{t^4} - \frac{2}{t^3}\right)e^{-\frac{1}{t}}, \ldots.$$

In general, $f^{(n)}(t) = p_n\left(\frac{1}{t}\right)e^{-\frac{1}{t}}$, for $t > 0$, where p_n is a polynomial of degree $2n$. Thus if we can prove that $p\left(\frac{1}{t}\right)e^{-\frac{1}{t}} \to 0$ as $t \to 0^+$ for all polynomials p, then we will be done. This is easy by a simple change of variables. Put $s = \frac{1}{t}$. This is a bijective map for $t > 0$ and $t \to 0^+ \implies s \to +\infty$. Then

$$\lim_{t\to 0^+} p\left(\tfrac{1}{t}\right)e^{-\frac{1}{t}} = \lim_{s\to+\infty} p(s)e^{-s} = 0, \quad \text{by L'Hôpital's rule.}$$

Thus f is infinitely differentiable.

(b) Define $\psi : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

Then ψ is nonnegative, zero outside the unit ball and infinitly differentiable by the same reasoning as in (a). By continuity, ψ has a finite integral on the unit ball in $\mathbb{R}^n$ and thus on all of $\mathbb{R}^n$ since it is zero outside the unit ball. Define $\varphi : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi(x) = \frac{\psi(x)}{\int_{\mathbb{R}^n}\psi(y)dy}, \quad x \in \mathbb{R}^n.$$

Then φ has all the desired properties.

Problem S14.9. Find a function that minimizes $\int_0^1 |f'(x)|^2\,dx$ among all $f \in C^1[0,1]$ such that $f(0) = 0$, $f(1) = 1$. Is the minimizing function unique?

Solution. By the Cauchy- Schwarz inequality

$$\int_0^1 f'(x)dx \leq \left(\int_0^1 1^2 dx\right)\left(\int_0^1 f'(x)^2 dx\right) = \int_0^1 |f'(x)|^2\,dx$$

with equality if and only if 1 and $f'(x)$ are linearly dependent. However, using our conditions $f(0) = 0$ and $f(1) = 1$, we can evaluate the right hand side to see

$$1 \leq \int_0^1 |f'(x)|^2\,dx.$$

Thus we have an upper bound for all such quantities. We can achieve equality by ensuring that 1 and $f'(x)$ are linearly dependent; that is $f'(x) = \alpha$, a constant. Then $f(x) = \alpha x + \beta$ for some constant β. But using the conditions $f(0) = 0, f(1) = 1$ again, we see that $f(x) = x$. Checking, we see that the solution does indeed meet the lower bound and it is unique since the Cauchy-Schwarz inequality gives necessary and sufficient conditions for equality.

Problem S14.10. Let $\mathcal{F}$ be a set of continuous real valued functions on $[0,1]$. Assume that

(i) $\mathcal{F}$ is uniformly bounded; i.e., there is $M < \infty$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$, $x \in [0,1]$.

(ii) $\mathcal{F}$ is equicontinuous; i.e., for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in [0,1]$,

$$|x-y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Prove that every sequence in $\mathcal{F}$ has a convergent subsequence.

Solution. This is the forward direction of the Arzela-Ascoli Theorem.

Let $\varepsilon > 0$. Take $\delta > 0$ be such that $|x-y| < \delta \implies |f(x)-f(y)| < \varepsilon/3$ for all $x, y \in [0,1]$, $f \in \mathcal{F}$.

Let (f_n) be a sequence in $\mathcal{F}$. Since we will be inductively creating more and more sequences, it is convenient to write $(f_n) = (f_{0,n})$. Let (x_k) be a denumeration of the rationals in $[0,1]$. By uniform boundedness, $(f_{0,n}(x_1))$ is a bounded sequence in $\mathbb{R}$. Thus by Bolzano-Weierstrass, there is a convergent subsequence $(f_{1,n}(x_1))$. Again, $(f_{1,n}(x_2))$ is bounded so there is a convergent subsequence $(f_{2,n}(x_2))$. Since this is a subsequence, we still have convergence of $(f_{2,n}(x_1))$ as well.

Continuing this by induction, we create subsequences $(f_{m,n})$ such that $(f_{m,n})$ is a subsequence of $(f_{m-1,n})$ for all m and $(f_{m,n}(x_j))$ converges for all $j \le m$. Taking $g_n = f_{n,n}$ for all $n \in \mathbb{N}$, we see that (g_n) is a subsequence of (f_n) and $(g_n(x))$ converges for all rational $x \in [0,1]$ by construction. We prove that (g_n) is uniformly Cauchy and thus converges uniformly.

The open intervals $(x_k - \delta, x_k + \delta)$ create an open cover of $[0,1]$ since the rationals are dense. By compactness, there is a finite subcover: $(x_{k_1} - \delta, x_{k_1} + \delta), \ldots, (x_{k_N} - \delta, x_{k_N} + \delta)$. Since (g_n) converges at each rational (and is thus Cauchy at each rational), there are $K_1, \ldots, K_j \in \mathbb{N}$ such that $n, m \ge K_j$ implies

$$|g_n(x_k) - g_m(x_k)| < \varepsilon/3,$$

for each $j = 1, \ldots, N$. Take $K = \max_j K_j$.

Let $x \in [0,1]$. Then $x \in (x_{k_j} - \delta, x_{k_j} + \delta)$ for some $j = 1, \ldots, N$. Then

$$\begin{aligned} |g_n(x) - g_m(x)| &= \left| g_n(x) - g_n(x_{k_j}) + g_n(x_{k_j}) - g_m(x_{k_j}) + g_m(x_{k_j}) - g_m(x) \right| \\ &\le \underbrace{\left| g_n(x) - g_n(x_{k_j}) \right|}_{<\varepsilon/3 \text{ by equicontinuity}} + \underbrace{\left| g_n(x_{k_j}) - g_m(x_{k_j}) \right|}_{<\varepsilon/3 \text{ by convergence at } x_{k_j}} + \underbrace{\left| g_m(x_{k_j}) - g_m(x) \right|}_{<\varepsilon/3 \text{ by equicontinuity}} \\ &< \varepsilon. \end{aligned}$$

Thus (g_n) is uniformly Cauchy and so (f_n) has a uniformly convergent subsequence.

Problem S14.11. Let $\mathcal{F}$ be a set of continuous real values functions on $[0,1]$. Assume that every sequence in F has a uniformly convergent subsequence. Prove that

(i) $\mathcal{F}$ is uniformly bounded; i.e., there is $M < \infty$ such that $|f(x)| \le M$ for all $f \in \mathcal{F}$, $x \in [0,1]$.

(ii) $\mathcal{F}$ is equicontinuous; i.e., for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in [0,1]$,

$$|x-y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Solution. This is the backwards direction of the Arzela-Ascoli Theorem. We prove both (i) and (ii) by contradiction. We denote

$$||f||_\infty = \sup_{0 \le x \le 1} |f(x)|, \quad f \in \mathcal{F}.$$

(i) Assume that $\mathcal{F}$ is not uniformly bounded. Then for each $n \in \mathbb{N}$ there is a function $f_n \in \mathcal{F}$ such that $||f_n||_\infty > n$. By assumption, this sequence (f_n) must have a subsequence (g_n) converging uniformly to some f [it may not be the case that $f \in \mathcal{F}$]. Then f must be continuous as a uniform limit of continuous function. Then

$$n \le ||g_n||_\infty \le ||g_n - f||_\infty + ||f||_\infty.$$

Taking the limit as $n \to \infty$, we see that $||g_n - f||_\infty \to 0$ by uniform convergence and so $||f||_\infty$ is unbounded. But this is impossible since f is a continuous function on a compact set. This contradiction implies that $\mathcal{F}$ is uniformly bounded.

(ii) Assume that $\mathcal{F}$ is not equicontinuous. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in [0,1]$, $f \in \mathcal{F}$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \ge \varepsilon$. Choosing $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$, we get sequences $(x_n), (y_n)$ in $[0,1]$ and (f_n) in $\mathcal{F}$ such that

$$|x_n - y_n| < \frac{1}{n} \qquad \text{but} \qquad |f_n(x_n) - f_n(y_n)| \ge \varepsilon.$$

This sequence (f_n) must have a subsequence (g_n) converging uniformly to some f [again, not necessarily $f \in \mathbb{F}$ but f will be continuous since each g_n is continuous]. Since f is continuous, there is N_1 sufficiently large so that $|x - y| < 1/N_1 \implies |f(x) - f(y)| < \varepsilon/4$. Further there is N_2 such that $n \ge N_2$ implies that $|g_n(x) - f(x)| < \varepsilon/4$ for all $x \in [0,1]$ (by uniform convergence). Then for $n \ge \max\{N_1, N_2\}$,

$$\begin{aligned} \varepsilon \le |g_n(x_n) - g_n(y_n)| &= |g_n(x_n) - f(x_n) + f(x_n) - f(y_n) + f(y_n) - g_n(y)| \\ &= |g_n(x_n) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - g_n(y_n)| \\ &< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4. \end{aligned}$$

This is a contradiction. Thus $\mathcal{F}$ must be equicontinuous.

Problem S14.12. Assume $[0,1] = \bigcup_{n=1}^{\infty} I_n$ where $I_n = [a_n, b_n] \neq \varnothing$ and

$$I_n \cap I_m = \varnothing$$

when $n \neq m$.

(a) Let $E = \{a_n : n \ge 1\} \cup \{b_n : n \ge 1\}$. Prove that E is closed.

(b) Prove no such family of intervals $\{I_n\}$ exists.

Solution.

(a) Let $x \in [0,1]$. Then since the given intervals are disjoint, there is a unique $n \in \mathbb{N}$ such that $x \in [a_n, b_n]$. Then $x \notin E$ implies $x \in (a_n, b_n)$. Conversely, if $x \in (a_n, b_n)$ for some $n \in \mathbb{N}$ then $x \notin E$. Hence

$$[0,1] - E = \bigcup_{n=1}^{\infty} (a_n, b_n)$$

and so $[0,1] - E$ is open since it is a union of open sets. Thus E is closed.

(b) No idea.

Problem S15.1. Let $f : [0,\infty) \to [0,\infty)$ be continuous with $f(0) = 0$. Show that if

$$f(t) \leq 1 + \frac{1}{10} f(t)^2, \quad \text{for all } t \geq 0$$

then f is bounded on $[0,\infty)$.

Solution. Assume there is $t \in [0,\infty)$ such that $f(t) \geq 2$. Then by continuity, there must be $s \in [0,\infty)$ such that $f(s) = 2$. But then

$$2 = f(s) \leq 1 + \frac{1}{10} f(s)^2 \leq 1 + \frac{4}{10} = 1.4;$$

a contradiction. Hence $f(t) < 2$ for all $t \in [0,\infty)$. By assumption f is a non-negative function so this implies f is bounded.

Problem S15.2. Let $f : [0,1] \to \mathbb{R}$, $\alpha \in (0,1)$. We say that f is α-Holder continuous (and write $f \in C^\alpha[0,1]$) if

$$||f||_{C^\alpha} = \sup_{x\in[0,1]} |f(x)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

This defines a norm on $C^\alpha[0,1]$ (no need to prove this).

Prove that any bounded sequence in $C^{1/2}[0,1]$ admits a convergent subsequence in $C^{1/3}[0,1]$.

Solution. Let (f_n) be a bounded sequence in $C^{1/2}[0,1]$; say M is the bound. Then in particular, for all $x, y \in [0,1]$, $|x-y| \leq 1$ so $|x-y|^{1/2} \leq |x-y|^{1/3}$. Then $\frac{1}{|x-y|^{1/3}} \leq \frac{1}{|x-y|^{1/2}}$ so

$$\begin{aligned}
||f_n||_{C^{1/3}} &= \sup_{x\in[0,1]} |f_n(x)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f_n(x) - f_n(y)|}{|x-y|^{1/3}} \\
&\leq \sup_{x\in[0,1]} |f_n(x)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f_n(x) - f_n(y)|}{|x-y|^{1/2}} = ||f_n||_{C^{1/2}} \leq M.
\end{aligned}$$

Thus (f_n) is uniformly bounded in $C^{1/3}[0,1]$. Further, it is clear from $||f_n||_{C^{1/3}} \leq M$ that

$$|f_n(x) - f_n(y)| \leq M\,|x-y|^{1/3}.$$

Let $\varepsilon > 0$. Then putting $\delta = (\varepsilon/M)^3$, we see that

$$|x - y| < \delta \quad \Longrightarrow \quad |f_n(x) - f_n(y)| < M((\varepsilon/M)^3)^{1/3} = \varepsilon$$

for all $n \in \mathbb{N}$ and $x, y \in [0,1]$. Thus (f_n) is equicontinuous. Hence by Arzela-Ascoli, there is a subsequence of (f_n) which converges uniformly in $C^{1/3}[0,1]$.

Problem S15.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Suppose that for every $x \in \mathbb{R}$,

$$\lim_{n\to\infty} n\left[f\left(x + \tfrac{1}{n}\right) - f(x)\right] = 0.$$

Prove that f is differentiable.

Solution. Since f is Lipschitz, it is differentiable Lebesgue almost everywhere and

$$f(x) = f(0) + \int_0^x f'(t)d\lambda(t)$$

for all $x \in \mathbb{R}$ where λ is the Lebesgue measure on $\mathbb{R}$ (actually absolute continuity is enough to guarantee this; Lipschitz continuity is a stronger condition). Wherever the derivative is defined, the given assumption implies that the derivative is zero by the sequential criterion theorem. Thus $f'(t)$ exists and is zero for Lebesgue almost every $t \in \mathbb{R}$. But then

$$\int_0^x f'(t)d\lambda(t) = 0, \qquad \text{for all } x \in \mathbb{R}.$$

Hence $f(x) = f(0)$ for all $x \in \mathbb{R}$. Thus f is constant which certainly implies that it is differentiable.

[Note: I'm not sure this was the intended solution since the concepts of absolute continuity and the Lebesgue measure aren't covered in boot camp but I can't find an elementary solution to this problem which uses only the definition of the derivative and related concepts.]

Problem S15.4. Let $f : [0,1] \to \mathbb{R}$ be a function satisfying the intermediate value property; that is, whenever $0 \le a < b \le 1$ and y lies between $f(a)$ and $f(b)$, there is $x \in (a,b)$ such that $f(x) = y$. Assume that for any $y \in \mathbb{R}$, the preimage $f^{-1}(\{y\})$ is closed. Prove that f is continuous.

Solution. Suppose f is discontinuous at some point $x \in \mathbb{R}$. Then there is an $\varepsilon > 0$ such that for any $\delta > 0$ there is x_0 such that $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \ge \varepsilon$. Specifically, for each $n \in \mathbb{N}$, there is $x_n \in \mathbb{R}$ such that $|x - x_n| < \frac{1}{n}$ but

$$|f(x_n) - f(x_n)| \ge \varepsilon.$$

Each $f(x_n)$ is either greater than or less than $f(x)$. Thus there are either infinitely many n such that $f(x_n) \ge f(x) + \varepsilon$ or infinitely many n such that $f(x_n) \le f(x) - \varepsilon$. Without loss of generality, we can assume there are infinitely many n such that $f(x_n) \ge f(x) + \varepsilon$. We consider this subsequence of (x_n) but for notational convenience, we still call it (x_n).

Take $c \in (f(x), f(x_n))$ for each n. Note, c need not depend on n because $|f(x) - f_n(x)| \geq \varepsilon$ for all n. Then by the intermediate value property, there is $y_n \in (x, x_n)$ such that $f(y_n) = c$. Clearly the sequence y_n converges to x because for each n, we have $|x - y_n| < |x - x_n| < \frac{1}{n}$. But each $y_n \in f^{-1}(\{c\})$ which is closed by assumption and so $x \in f^{-1}(\{c\})$ which implies $f(x) = c$. This is a contradiction because we assumed that $f(x) < c$.

Hence f is continuous at all $x \in \mathbb{R}$.

Problem S15.5. Let $f : [1, \infty) \to [0, \infty)$ be a bounded decreasing function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_1^{n+1} f(x)dx - \sum_{k=1}^{n} f(k)$$

tends to a finite limit as $n \to \infty$.

Solution. Write

$$x_n = \int_1^{n+1} f(x)dx - \sum_{k=1}^{n} f(k), \quad n \in \mathbb{N}.$$

We prove that x_n is a Cauchy sequence and hence converges.

Consider, for $n > m$, we have

$$\begin{aligned} x_n - x_m &= \int_1^{n+1} f(x)dx - \sum_{k=1}^{n} f(k) - \int_1^{m+1} f(x)dx + \sum_{k=1}^{m} f(k) \\ &= \int_{m+1}^{n+1} f(x) - \sum_{k=m+1}^{n} f(k) \\ &= \sum_{k=m+1}^{n} \left(\int_k^{k+1} f(x)dx - f(k) \right). \end{aligned}$$

Since f is decreasing, we have $f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k)$ for all k. Then

$$-f(m+1) \leq f(n+1) - f(m+1) = \sum_{k=m+1}^{n} (f(k+1) - f(k)) \leq x_n - x_m \leq \sum_{k=m+1}^{n} (f(k) - f(k)) = 0.$$

Let $\varepsilon > 0$. Since $f(x) \to 0$ as $x \to \infty$, there is $N \in \mathbb{N}$ such that $x \geq N \implies f(x) < \varepsilon$. Taking $n, m \geq N$, we see $|x_n - x_m| < \varepsilon$. Thus (x_n) is Cauchy and hence convergent.

Problem S15.6. Prove that the integral equation

$$f(x) = e^{x^2} + \tfrac{1}{2} \int_0^1 \cos(y) f(y) dy$$

admits a unique continuous solution $f : [0, 1] \to \mathbb{R}$.

Solution. Define the operator T on $C[0,1]$ by

$$(Tf)(x) = e^{x^2} + \tfrac{1}{2}\int_0^1 \cos(y)f(y)dy, \quad x \in [0,1], \quad f \in C[0,1].$$

Since $C[0,1]$ is complete with respect to the supremum norm, it will suffice to prove that $T : C[0,1] \to C[0,1]$ and that T is a contraction; then the Banach Fixed Point Theorem assures the existence of a unique fixed point for T which will be a unique solution to the integral equation in $C[0,1]$.

If $f \in C[0,1]$, then $\int_0^1 \cos(y)f(y)dy$ converges, so it is clear that Tf is a well-defined and continuous function since it is a constant added to the continuous function e^{x^2}. Thus $T : C[0,1] \to C[0,1]$.

Take $f, g \in C[0,1]$. Then for $x \in [0,1]$,

$$\begin{aligned} |(Tf)(x) - (Tg)(x)| &= \left| e^{x^2} + \tfrac{1}{2}\int_0^1 \cos(y)f(y)dy - e^{x^2} - \tfrac{1}{2}\int_0^1 \cos(y)g(y)dy \right| \\ &\le \tfrac{1}{2}\int_0^1 \cos(y)\,|f(y) - g(y)|\,dy \\ &\le \tfrac{1}{2}\int_0^1 |f(y) - g(y)|\,dy \\ &\le \tfrac{1}{2}\,||f - g||_\infty \,. \end{aligned}$$

Since this holds for all $x \in [0,1]$, we see

$$||Tf - Tg||_\infty \le \tfrac{1}{2}\,||f - g||_\infty$$

so T is a contraction. Thus there is a unique function f such that $Tf = f$. This function is the unique solution to the integral equation.

Note: it is likely that the integral equation was supposed to read

$$f(x) = e^{x^2} + \tfrac{1}{2}\int_0^x \cos(y)f(y)dy.$$

In this case, the property still holds but we need a bit more work to prove continuity of T.

Let $\varepsilon > 0, f \in C[0,1]$. Then since e^{x^2} is continuous (and thus uniformly continuous on $[0,1]$), there is $\delta > 0$ such that for all $x, y \in [0,1]$,

$$|x - y| < \delta \quad \Longrightarrow \quad \left| e^{x^2} - e^{y^2} \right| < \tfrac{\varepsilon}{2}.$$

Also, since f is continuous on a compact set, it is bounded by some $M \in \mathbb{R}$. We can take $\delta < \varepsilon/2M$ by making it smaller if necessary. Then for $|x - y| < \delta$ (wlog $x > y$), we see

$$\begin{aligned} |(Tf)(x) - (Tf)(y)| &= \left| e^{x^2} - e^{y^2} + \int_y^x \cos(z)f(z)dz \right| \\ &\le \left| e^{x^2} - e^{y^2} \right| + \int_y^x |f(z)|\,dz \\ &< \varepsilon/2 + M(x - y) < \varepsilon/2 + M(\varepsilon/2M) = \varepsilon. \end{aligned}$$

Thus Tf is continuous. The proof that T is a contraction follows exactly as above.

Problem F15.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers such that

$$a_{n+m} \le a_n + a_m, \quad n, m \ge 1.$$

Prove that $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists.

Solution. Using the "subadditivity," it is not difficult to see that for any $n, m \ge 1$, we have

$$a_{nm} = a_{n+\cdots+n} \le m a_n.$$

Clearly the set $\{a_1/1, a_2/2, \ldots\}$ is bounded below (by zero) and so it has a finite infimum L. We show that $\lim\limits_{n\to\infty} \frac{a_n}{n} = L$.

Let $\varepsilon > 0$. Since L is the infimum of the sequence, there is $k \in \mathbb{N}$ such that $a_k/k \le L+\varepsilon/2$. There is also $N \in \mathbb{N}$ such that $N \ge k$ and

$$\frac{\max\{a_1, \ldots, a_{k-1}\}}{N} \le \varepsilon/2.$$

Then for $n \ge N$, write $n = dk + r$ for some $d, r \in \mathbb{N}$, $r \le k$. Then

$$L \le \frac{a_n}{n} = \frac{a_{dk+r}}{dk+r} \le \frac{a_{dk} + a_r}{dk+r} = \frac{a_{dk}}{dk+r} + \frac{a_r}{n} \le \frac{da_k}{dk} + \frac{\max\{a_1, \ldots, a_{k-1}\}}{N} \le \frac{a_k}{k} + \varepsilon/2 \le L + \varepsilon.$$

Thus by definition, $\lim\limits_{n\to\infty} \frac{a_n}{n} = L$.

Problem F15.2. Let $a, b \in \mathbb{R}$, $a < b$. Show that if $g, h : [a, b] \to \mathbb{R}$ are continuous with $h \ge 0$, then there is $c \in [a, b]$ such that

$$\int_a^b g(x)h(x)dx = g(c) \int_a^b h(x)dx.$$

Solution. If $h \equiv 0$ and/or g is constant, the conclusion is trivial and actually holds for all $c \in [a, b]$.

Assume $h \not\equiv 0$ and g is not constant. Then a standard result tells us that $\int_a^b h(x)dx > 0$. We seek to prove that there is $c \in [a, b]$ such that $g(c) = \frac{\int_a^b g(x)h(x)dx}{\int_a^b h(x)dx}$.

Assume no such c exists. If there are values $c_+, c_- \in [a, b]$ such that

$$g(c_+) > \frac{\int_a^b g(x)h(x)dx}{\int_a^b h(x)dx} \quad \text{and} \quad g(c_-) < \frac{\int_a^b g(x)h(x)dx}{\int_a^b h(x)dx}$$

then we have violated the intermediate value theorem and we are done. Otherwise one of c_+, c_- doesn't exists, and then

$$g(c) \quad \text{is always greater than or always less than} \quad \frac{\int_a^b g(x)h(x)dx}{\int_a^b h(x)dx}.$$

Assume wlog that the latter is true. Then

$$\int_a^b (g(x) - g(c))h(x)dx > 0$$

for all $c \in [a, b]$.

Since $[a, b]$ is compact and g is continuous, there is $z \in [a, b]$ such that $g(z) \geq g(x)$ for all $x \in [a, b]$. Then $g(x) - g(z) \leq 0$, for all $x \in [a, b]$. Then $(g(x) - g(z))h(x) \leq 0$ for all $x \in [a, b]$. But this implies that

$$\int_a^b (g(x) - g(z))h(x)dx \leq 0$$

which contradicts the above statement. This implies the existence of some such c.

Problem F15.3. Let $\{f_n\}$ be a sequence of continuous functions $f_n : [-1, 1] \to [0, 1]$ such that for every $x \in [-1, 1]$

(a) the sequence of numbers $\{f_n(x)\}$ is non-increasing, and

(b) $\lim_{n\to\infty} f_n(x) = 0$.

Define

$$g_n(x) = \sum_{m=1}^{n} (-1)^m f_m(x).$$

Prove that g_n converges pointwise to some function g on $[-1, 1]$ and that g is continuous.

Solution. The series

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \sum_{m=1}^{n} (-1)^m f_m(x)$$

converges for each $x \in [-1, 1]$ by the alternating series test. Define

$$g(x) = \lim_{n\to\infty} \sum_{m=1}^{n} (-1)^m f_m(x), \quad x \in [-1, 1].$$

It remains to prove that g is continuous.

We first show that $f_n \to 0$ uniformly (this is actually called Dini's Theorem and proven a bit more generally in **F12.3**). Indeed, let $\varepsilon > 0$ and define $A_n = \{x \in [-1, 1] : f_n(x) < \varepsilon\}$. Each A_n is open since it is the pullback of an open set under a continuous function. Since $f_n(x) \to 0$ for all $x \in [-1, 1]$, we see that

$$[-1, 1] = \bigcup_{n=1}^{\infty} A_n.$$

Then since $[-1, 1]$ is compact there is a finite subcover $A_{n_1}, \ldots, A_{n_k}$. Then

$$[-1, 1] = \bigcup_{\ell=1}^{k} A_{n_\ell}.$$

However, by construction $A_n \subset A_m$ when $n \le m$. Thus $[-1,1] = A_{n_k}$. Then for all $n \ge n_k$, we have $f_n(x) < \varepsilon$ for all $x \in [-1,1]$. Thus $f_n \to 0$ uniformly.

Take $n, m \in \mathbb{N}$ (wlog $n \ge m$). Then

$$|g_n(x) - g_m(x)| = |f_m(x) - f_{m+1}(x) \pm \cdots \pm f_n(x)|.$$

However, since $f_n(x)$ is non-increasing for each x, we see that $f_{k+1}(x) - f_k(x) \le 0$ for all x. Then

$$|g_n(x) - g_m(x)| \le f_m(x).$$

Since $f_m \to 0$ uniformly, this shows that g_n is uniformly Cauchy and hence converges uniformly. Clearly each g_n is continuous since it is a finite sum of continuous functions. So g is the uniform limit of continuous functions and is thus continuous.

Problem F15.4. Let $f_n : [0,\infty) \to \mathbb{R}$ be functions defined recursively by $f_0(x) = 0,\ x \in [0,\infty)$ and

$$f_{n+1}(x) = e^{-2x} + \int_0^x f_n(t)e^{-2t}dt, \quad n \ge 1, \quad x \in [0,\infty).$$

Show that $f(x) = \lim_{n\to\infty} f_n(x)$ exists and explicitly find f.

Solution. Let $V = \{g \in C[0,\infty) : ||f||_\infty \le 2\}$. Assume that g_n is a sequence in V which converges to some function $g \in C[0,\infty)$. Then by the continuity of the norm, we see

$$||g||_\infty = \left|\left| \lim_{n\to\infty} g_n \right|\right|_\infty = \lim_{n\to\infty} ||g_n||_\infty \le 2.$$

Thus V is a closed subspace of a complete space and is thus complete.

For $f \in V$, define

$$(Tf)(x) = e^{-2x} + \int_0^x f(t)e^{-2t}dt, \quad x \in [0,\infty).$$

Then $f_0 = T^0 f_0, f_1 = Tf_0, f_2 = T^2 f_0, \ldots$. Thus it will suffice to show that $\lim_{n\to\infty} T^n f_0$ exists.

It is clear that T is a linear operator; we must prove that $T : V \to V$ and that T is a contraction. Then by the Banach Fixed Point theorem, T has a unique fixed point f. Moreover, to find the fixed point, we can begin from an arbitrary member of V and iterate with T. That is $\lim_{n\to\infty} f_n = f$ where $Tf = f$.

Let $f \in V$. Then for $x \in [0,\infty)$,

$$\begin{aligned} |(Tf)(x)| &= \left| e^{-2x} + \int_0^x f(t)e^{-2t}dx \right| \\ &\le \left|e^{-2x}\right| + \int_0^x |f(t)|\, e^{-2t}dt \\ &\le 1 + 2\int_0^x e^{-2t}dt \\ &\le 1 + 2\int_0^\infty e^{-2t}dt = 2. \end{aligned}$$

Thus $||Tf||_\infty \le 2$ when $||f||_\infty \le 2$ and so $T : V \to V$.

Let $f, g \in V$.

Then

$$\begin{aligned} |(Tf)(x) - (Tg)(x)| &= \left| e^{-2x} + \int_0^x f(t)e^{-2t}dt - e^{-2x} - \int_0^x g(t)e^{-2t}dt \right| \\ &= \left| \int_0^x (f(t) - g(t))e^{-2t}dt \right| \\ &\le \int_0^x |f(t) - g(t)|\, e^{-2t}dt \\ &\le ||f - g||_\infty \int_0^x e^{-2t}dt \\ &\le ||f - g||_\infty \int_0^\infty e^{-2t}dt = \tfrac{1}{2}\, ||f - g||_\infty \,. \end{aligned}$$

Thus

$$||Tf - Tg||_\infty \le \tfrac{1}{2}\, ||f - g||_\infty$$

and so T is a contraction.

Thus $f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} T^n f_0$ exists. Further, $f(x)$ is continuous and

$$Tf = f \quad \implies \quad f(x) = e^{-2x} + \int_0^x f(t)e^{-2t}dt, \quad x \in [0, \infty).$$

However, if f is continuous, then $\int_0^x f(t)e^{-2t}dt$ is differentiable, so f is differentiable. Differentiating the above equation, we see f satisfies the differential equation

$$\begin{aligned} f'(x) - e^{-2x}f(x) &= -2e^{-2x} \\ f(0) &= 1. \end{aligned}$$

Using an integrating factor, we see

$$\frac{d}{dx}\left(f(x)\exp\left(\tfrac{1}{2}e^{-2x}\right)\right) = -2e^{-2x}\exp\left(\tfrac{1}{2}e^{-2x}\right)$$

which implies that

$$f(x)\exp\left(\tfrac{1}{2}e^{-2x}\right) - e^{1/2} = -2\int_0^x e^{-2t}\exp\left(\tfrac{1}{2}e^{-2t}\right)dt = 2\left(\exp\left(\tfrac{1}{2}e^{-2t}\right) - e^{1/2}\right).$$

Then

$$f(x) = \exp\left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2x}\right) + 2\left(1 - \exp\left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2x}\right)\right) = 2 - \exp\left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2x}\right).$$

Problem F15.5. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Suppose $F(0, 0, 0) = 0$ and that no component of ∇F is 0 at $(0, 0, 0)$. Show that if $x = x(y, z), y = y(x, z), z = z(x, y)$ define the surface $F(x, y, z) = 0$ in a neighborhood of $(0, 0, 0)$, then

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1.$$

Solution. Alternately using the implicit value theorem in each coordinate, we get differentiable functions $x(y,z), y(x,z), z(x,y)$ such that

$$\begin{aligned} F(x(y,z),y,z) &= 0, \\ F(x,y(x,z),z) &= 0, \\ F(x,y,z(x,y)) &= 0, \end{aligned}$$

in some neighborhood of $(0,0,0)$. Differentiate the first equation with respect to y, the second with respect to z and the third with respect to x to get

$$\begin{aligned} \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} &= 0, \\ \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial F}{\partial z} &= 0, \\ \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial x} &= 0. \end{aligned}$$

Rearranging and using that no component ∇F is 0 in a neighborhood of $(0,0,0)$ [which, incidentally, we also needed to call on the implicit function theorem] yields the result.

Problem F15.6. Let $X = \mathbb{R} - \{0\}$. Find a metric ρ on X such that

(i) (X,ρ) is a complete metric space, and

(ii) for any sequence (x_n) in X and $x \in X$,

$$|x_n - x| \to 0 \quad \text{if and only if} \quad \rho(x_n,x) \to 0 \qquad (\text{as } n \to \infty).$$

Solution. We immediately see that any that any sequence in $\mathbb{R}$ which goes to zero in absolute value cannot be a Cauchy sequence with respect to ρ. A first thought may be

$$\rho(x,y) = \frac{|x-y|}{|xy|} = \left|\frac{1}{x} - \frac{1}{y}\right|, \quad x,y \in X$$

since this will blow up for sequences that go to zero. However, we also need to require that any sequence that does not converge in absolute value is not a Cauchy sequence in ρ and this doesn't hold for ρ defined above because that choice of ρ makes the natural numbers a Cauchy sequence. To make sure this doesn't happen, we modify ρ a bit by setting

$$\rho(x,y) = \max\left\{|x-y|, \left|\frac{1}{x} - \frac{1}{y}\right|\right\}, \quad x,y \in X.$$

If (x_n) is a sequence in X and $x \in X$, then certainly $\rho(x_n,x) \to 0$ implies that $|x_n - x| \to 0$. Conversely, if $x_n \to x \neq 0$, we know that $\frac{1}{x_n} \to \frac{1}{x}$ and so $|x_n - x| \to 0$ implies that $\rho(x_n,x) \to 0$. Finally, if (x_n) is a Cauchy sequence in (X,ρ) then it is also a Cauchy sequence in $\mathbb{R}$ with the absolute value so there is limit $x \in \mathbb{R}$. We simply need to prove that

$x \neq 0$ (i.e., $x \in X$) to conclude that X is complete. Since (x_n) is Cauchy in (X, ρ), there is $N \in \mathbb{N}$, so that $m, n \geq N$ give

$$\frac{|x_n - x_m|}{|x_n x_m|} < 1 \qquad \Longrightarrow \qquad |x_n - x_m| < |x_n x_m| \, .$$

If $x = 0$ [and thus $x_n \to 0$], then there is $M \in \mathbb{N}$ so that $m \geq M$ gives $|x_m| < 1$ and so for $n, m \geq \max\{M, N\}$ we have

$$|x_n| > |x_n - x_m| \, .$$

Plugging in $m = M$ and using the reverse triangle inequality gives that

$$|x_n| \geq |x_M| - |x_n| \quad \Longrightarrow \quad |x_n| \geq \tfrac{1}{2} |x_M| > 0, \ \ n \geq \max\{M, N\}.$$

But this directly contradicts that $x_n \to 0$ since it gives a positive lower bound for sufficiently large n. Hence we cannot have $x = 0$ and we conclude that X is complete.