MATH 115B (CHERNIKOV), SPRING 2019
PROBLEM SET 9
DUE FRIDAY, JUNE 7

Problem 1. Show that the characteristic polynomial of a Jordan canonical form splits.

Problem 2. Let T be a linear operator on a finite-dimensional vector space V whose characteristic polynomial splits.

(1) Prove Theorem 7.5(b).
(2) Suppose that β is a Jordan canonical basis for T, and let λ be an eigenvalue of T. Let $\beta' = \beta \cap K_\lambda$. Prove that β' is a basis for K_λ.

Problem 3. Let $\gamma_1, \ldots, \gamma_p$ be cycles of generalized eigenvectors of a linear operator T corresponding to an eigenvalue λ. Prove that if the initial eigenvectors are distinct, then the cycles are disjoint.

Problem 4. Let T be a linear operator on a finite-dimensional vector space V whose characteristic polynomial splits. Prove that then V is the direct sum of the generalized eigenspaces of T.

Problem 5. For each linear operator T, find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T.

(1) T is the linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = 2f(x) - f'(x)$.
(2) V is the real vector space of functions spanned by the set of real valued functions $\left\{1, t, t^2, e^t, te^t\right\}$, and T is the linear operator on V defined by $T(f) = f'$.
(3) T is the linear operator on $M_{2\times 2}(\mathbb{R})$ defined by $T(A) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot A$ for all $A \in M_{2\times 2}(\mathbb{R})$.
(4) $T(A) = 2A + A^t$ for all $A \in M_{2\times 2}(\mathbb{R})$.

Problem 6. Let T be a linear operator on a vector space V, and let γ be a cycle of generalized eigenvectors that corresponds to the eigenvalue λ. Prove that $\mathrm{Span}(\gamma)$ is a T-invariant subspace of V.

Problem 7. Use Theorem 7.4 to prove that the vectors $v_1, \ldots, v_k$ in the statement of Theorem 7.3 are unique.

Problem 8. Let T be a linear operator on a finite-dimensional vector space whose characteristic polynomial splits. Let λ be an eigenvalue of T.

(1) Suppose that γ is a basis for K_λ consisting of the union of q disjoint cycles of generalized eigenvectors. Prove that $q \leq \dim(E_\lambda)$.
(2) Let β be a Jordan canonical basis for T, and suppose that $J = [T]_\beta$ has q Jordan blocks with λ in the diagonal position. Prove that $q \leq \dim(E_\lambda)$.