MATH 115B (CHERNIKOV), SPRING 2019

PROBLEM SET 3

DUE FRIDAY, APRIL 26

Problem 1. Let T be a linear operator on V, $\dim(V) < \infty$.

(1) Let W be a T-invariant subspace of V. Prove that W is $g(T)$-invariant for any polynomial $g(t)$.
(2) Let $v \in V$ be a non-zero vector, and let W be the T-cyclic subspace of V generated by v. For any $w \in V$, prove that $w \in W$ if and only if there exists a polynomial $g(t)$ such that $w = g(T)(v)$.
(3) Prove that the polynomial $g(t)$ in (2) can always be chosen so that its degree is less than or equal to $\dim(W)$.

Problem 2. Let A be an $n \times n$ matrix. Prove that $\dim\left\{\operatorname{Span}\left(\left\{I_n, A, A^2, \ldots\right\}\right)\right\} \leq n$.

Problem 3. Let V be a finite-dimensional vector space with a basis β, and let $\beta_1, \ldots, \beta_k$ be a partition of β (that is, $\beta_1, \ldots, \beta_k$ are subsets of β such that $\beta = \beta_1 \cup \ldots \cup \beta_k$ and $\beta_i \cap \beta_j = \emptyset$ if $i \neq j$). Prove that

$$V = \operatorname{Span}(\beta_1) \oplus \ldots \oplus \operatorname{Span}(\beta_k).$$

Problem 4. Prove Theorem 5.25:

Let T be a linear operator on a finite-dimensional vector space V, and let $W_1, \ldots, W_k$ be T-invariant subspaces of V such that $V = W_1 \oplus \ldots \oplus W_k$. For each i, let β_i be an ordered basis for W_i, and let $\beta = \beta_1 \cup \ldots \cup \beta_k$. Let $A = [T]_\beta$ and $B_i = [T_{W_i}]_{\beta_i}$ for $i = 1, \ldots, k$. Then $A = B_1 \oplus \ldots \oplus B_k$.

(Hint: by induction on k, starting with $k = 2$ as in the proof of Theorem 5.24.)

Problem 5. Let T be a linear operator on a finite-dimensional vector space V. Prove that T is diagonalizable if and only if V is the direct sum of *one-dimensional* T-invariant subspaces.

Problem 6. Let T be a linear operator on a finite-dimensional vector space V, let $W_1, \ldots, W_k$ be T-invariant subspaces of V such that $V = W_1 \oplus \ldots \oplus W_k$. Prove that

$$\det(T) = \det(T_{W_1}) \cdot \ldots \cdot \det(T_{W_k}).$$

Problem 7. Let T be a linear operator on a finite-dimensional vector space V, let $W_1, \ldots, W_k$ be T-invariant subspaces of V such that $V = W_1 \oplus \ldots \oplus W_k$. Prove that T is diagonalizable if and only if T_{W_i} is diagonalizable for all $i, 1 \leq i \leq k$.

Problem 8. Let $n \in \mathbb{N}$ and let

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{pmatrix}.$$

Find the characteristic polynomial of A.

(Hint: first show that A has rank 2 and that $\operatorname{Span}\left(\left\{(1,1,\ldots,1),(1,2,\ldots,n)\right\}\right)$ is L_A-invariant).