The Jordan Canonical Form

Recall: diagonalizable linear operators have a particularly simple description
Such operators have diagonal matrix representation, equivalently there is an ordered basis of e.vect's.
Problem: most operators are not diagonalizable!
Aim: obtain nice matrix representations for more general operators.
We will obtain a general answer for linear operators whose char. poly. splits (if $F=\mathbb{C}$, this applies to all operators!)

Def Let $T \in \mathcal{L}(V)$, V is a v.s. with $\dim(V) < \infty$.
Assume that β is a basis for V s.t.

$$[T]_\beta = \begin{pmatrix} \boxed{A_1} & & & O \\ & \boxed{A_2} & & \\ & & \ddots & \\ O & & & \boxed{A_k} \end{pmatrix}, \text{ where each } A_i \text{ is a square matrix of the form } (\lambda) \text{ or } \begin{pmatrix} \lambda & 1 & & O \\ & \lambda & 1 & \\ O & & \lambda & 1 \\ & & & \ddots \lambda \end{pmatrix} \text{ for}$$

some e.val. λ of T.
Such a matrix A_i is called a Jordan block corresponding to λ. So $[T]_\beta = A_1 \oplus \dots \oplus A_k$.
And the matrix $[T]_\beta$ is called a Jordan canonical form of T.
And such basis β is a Jordan Canonical basis for T.

Rem Observe that each Jordan block A_i is "almost" a diagonal matrix, and
$[T]_\beta$ is diagonal $\Leftrightarrow$ each A_i is of the form (λ).

Ex1 Supp. $T \in \mathcal{L}(\mathbb{C}^8)$ and $\beta = \{v_1, \dots, v_8\}$ is an ordered basis for $\mathbb{C}^8$ s.t.

$$J = [T]_\beta = \begin{pmatrix} 2 & 1 & 0 & & & & & \\ 0 & 2 & 1 & & & & O & \\ 0 & 0 & 2 & & & & & \\ & & & 2 & & & & \\ & & & & 3 & 1 & & \\ & & & & 0 & 3 & & \\ & O & & & & & 0 & 1 \\ & & & & & & 0 & 0 \end{pmatrix}$$

is a Jordan canonical form of T.
Then the char. poly. of T is
$\det(J - tI) = \det\begin{pmatrix} 2-t & 1 & 0 \\ 0 & 2-t & 1 \\ 0 & 0 & 2-t \end{pmatrix} \cdot \det(2-t) \cdot \det\begin{pmatrix} 3-t & 1 \\ 0 & 3-t \end{pmatrix} \cdot$

$\cdot \det\begin{pmatrix} -t & 1 \\ 0 & -t \end{pmatrix} = (t-2)^4 (t-3)^2 t^2.$

So the multiplicity of each e.val. is the number of times it appears on the diagonal of J.

And v_1, v_4, v_5, v_7 are the only vectors in β that are eigenvectors of T — they correspond to the columns of J with no "1" above the diagonal.

We will show that every operator whose char. poly. splits admits a Jordan canonical form.

Ex2 However, the Jordan canonical form of T is not determined by the char. poly. of T!
Let $T' \in \mathcal{L}(\mathbb{C}^8)$ be s.t. $[T']_\beta = J'$, where β is an ordered basis from Ex 1, and

$$J' = \begin{pmatrix} 2 & & & & & & & \\ & 2 & & & & & O & \\ & & 2 & & & & & \\ & & & 2 & & & & \\ & & & & 3 & & & \\ & O & & & & 3 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

Then the char. poly of T' is also $(2-t)^4(3-t)^2 t^2$, but $J \neq J'$ are two different canonical forms.

Notice that from J we have:
$T(v_2) = v_1 + 2v_2$, so $(T-2I)(v_2) = v_1$.
Similarly, $(T-2I)(v_3) = v_2$.
v_1, v_4 — e.vect's of T with e.val. $\lambda = 2$, so $(T-2I)^3(v_i) = 0$ for $i=1,2,3,4$.

Similarly, $(T-3I)^2(v_i) = 0$ for $i = 5,6$.
$(T-0I)^2(v_i) = 0$ for $i = 7,8$.

What happens in general?

Prop. If v lies in a Jordan canonical basis for $T \in \mathcal{L}(V)$ and its column in $[T]_\beta$ has diagonal entry λ, then $(T-\lambda I)^p(v)=0$ for sufficiently large $p \in \mathbb{N}$.
Eigenvectors satisfy this with $p=1$.
Proof H/W 8.

This motivates the following definition.

Def Let $T \in \mathcal{L}(V)$ and $\lambda \in F$.
A vector $x \neq 0$ in V is a generalized e.vect. of T corresp. to λ if $(T-\lambda I)^p(x)=0$ for some positive integer p.

Rem If $x \in V$ is a gen. e.vect. of T corresp. to λ, and p is the <u>smallest</u> positive integer s.t. $(T-\lambda I)^p(x)=0$, then $(T-\lambda I)^{p-1}(x)$ is an e.vect. of T corresp. to λ. So λ is an e.val. of T!

Ex 3 In Ex 1, each vector in β is a gen. e.vect. of T.
v_1, v_2, v_3 corresp. to the scalar 2, v_5 and v_6 to 3, v_7 and v_8 to 0.

Def Let $T \in \mathcal{L}(V)$, let λ be an e.val. of T.
The generalized eigenspace of T corresponding to λ is
$K_\lambda = \{x \in V : (T-\lambda I)^p(x)=0 \text{ for some positive integer } p\}$.
So K_λ consists of 0 and all gen. e.vect's of T corresp. to λ.

Thm 7.1 Let $T \in \mathcal{L}(V)$, let $\lambda \in F$ be an e.val. of T. Then:
a) K_λ is a T-inv. subspace of V containing E_λ.
b) For any $\mu \neq \lambda$ in F, the restriction of $T-\mu I$ to K_λ is one-to-one.

Proof a) K_λ is a subspace of V.
Clearly $0 \in K_\lambda$.
Supp. $x, y \in K_\lambda$. Then by def. $\exists\, p, q \in \mathbb{N}_{>0}$ s.t.
$(T-\lambda I)^p(x) = (T-\lambda I)^q(y) = 0$. So
$(T-\lambda I)^{p+q}(x+y) = (T-\lambda I)^{p+q}(x) + (T-\lambda I)^{p+q}(y)$ (as T lin $\Rightarrow (T-\lambda I)^{p+q}$ is lin.)
$= (T-\lambda I)^q(0) + (T-\lambda I)^p(0)$ (by the choice of p and q)
$= 0$ (as $(T-\lambda I)^q, (T-\lambda I)^p$ are both linear).
$\Rightarrow x+y \in K_\lambda$.
Similarly, K_λ is closed under scalar multiplication.

K_λ is T-inv
Let $x \in K_\lambda$.
Let $p \in \mathbb{N}_{>0}$ be such that $(T-\lambda I)^p(x)=0$. Then
$(T-\lambda I)^p T(x) = T(T-\lambda I)^p(x)$ (as T commutes with $(T-\lambda I)^p$ using linearity)
$= T(0) = 0$.
$\Rightarrow T(x) \in K_\lambda$.

$E_\lambda \subseteq K_\lambda$
If $x \in E_\lambda$, then $(T-\lambda I)^p(x)=0$ with $p=1$ by def. of e.spaces.

b) We show that $N\left((T-\mu I)_{K_\lambda}\right) = \{0\}$.
Assume $x \in K_\lambda$ and $(T-\mu I)(x)=0$.
Towards contradiction, suppose $x \neq 0$.
Let p be the smallest integer >0 s.t. $(T-\lambda I)^p(x)=0$.
Let $y = (T-\lambda I)^{p-1}(x)$. Then
$(T-\lambda I)(y) = (T-\lambda I)^p(x) = 0$, so $y \in E_\lambda$.

Furthermore,

$$(T-\mu I)(y) = (T-\mu I)(T-\lambda I)^{p-1}(x) \underset{\substack{\uparrow \\ \text{linear}}}{=} (T-\lambda I)^{p-1}\underbrace{(T-\mu I)(x)}_{=0} = 0$$

So $y \in E_\mu$.

But $E_\lambda \cap E_\mu = \{0\}$ as $\lambda \neq \mu$, so $y=0$ – contradicting minimality of p.

So $x=0 \Rightarrow (T-\mu I)_{K_\lambda}$ is one-to-one.

Thm 7.2 Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$, and char. poly of T splits.
Supp. λ is an e.val. of T with multiplicity m. Then:

a) $\dim(K_\lambda) \leq m$.

b) $K_\lambda = N((T-\lambda I)^m)$.

Proof

a) Let $W = K_\lambda$.
Let $h(t)$ be the char. poly of T_W.
By Thm 5.21, $h(t)$ divides the char. poly. of T, so $h(t)$ splits.
By Thm 7.1(b), λ is the only e.val. of T_W.
Hence $h(t) = (-1)^d (t-\lambda)^d$, where $d = \dim(W)$, and $d \leq m$.

b) Clearly $N((T-\lambda I)^m) \subseteq K_\lambda$ by def. of K_λ.
Let W and $h(t)$ be as in (a).
Then $h(T_W) = T_0$ by the Cayley-Hamilton Thm.
So $(T-\lambda I)^d(x) = 0$ for all $x \in W$.
Since $d \leq m$, we have $K_\lambda \subseteq N((T-\lambda I)^m)$.

Thm 7.3 Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$ and the char. poly. of T splits.
Let $\lambda_1, \dots, \lambda_k$ be the distinct e.val's of T.
Then for every $x \in V$ there exist vectors $v_i \in K_{\lambda_i}$, $1 \leq i \leq k$, s.t.
$x = v_1 + \dots + v_k$.

Proof.
By induction on the number k of distinct e.val's of T.

$k=1$ Let m be the multiplicity of λ_1.
Then $(\lambda_1 - t)^m$ is the char. poly. of T, hence
$(\lambda_1 I - T)^m = T_0$ – by the Cayley-Hamilton thm.
Thus $V = K_{\lambda_1}$, and the result follows.

$k>1$, and assume the result holds for $<k$ distinct e.val's.
Supp. T has k distinct e.vals.
Let m be the multiplicity of λ_k, let $f(t)$ be the char. poly. of T.
Then $f(t) = (t-\lambda_k)^m g(t)$ for some poly. $g(t)$ not divisible by $(t-\lambda_k)$.
Let $W = R((T-\lambda_k I)^m)$.
Then W is T-inv. ($x \in W \Leftrightarrow \exists y \in V$ s.t. $(T-\lambda_k I)^m(y) = x \Rightarrow T(x) = T(T-\lambda_k I)^m(y) = (T-\lambda_k I)^m(\overset{V}{\overset{\in}{T(y)}})$, so $T(x) \in R((T-\lambda_k I)^m) = W$)

Claim 1 $(T-\lambda_k I)^m$ maps K_{λ_i} onto itself for $i<k$.
Indeed, supp. $i<k$.
K_{λ_i} is T-invariant $\Rightarrow K_{\lambda_i}$ is $(T-\lambda_k I)^m$-invariant, in other words $(T-\lambda_k I)^m$ maps K_{λ_i} into itself.
And $\lambda_k \neq \lambda_i$, so $(T-\lambda_k I)_{K_{\lambda_i}}$ is one-to-one by Thm 7.1(b), hence onto.
But then $(T-\lambda_k I)^m_{K_{\lambda_i}}$ is also onto.

Claim 1 implies that for $i<k$, K_{λ_i} is contained in W. (as $K_{\lambda_i} = (T-\lambda_k I)^m (K_{\lambda_i}) \subseteq R((T-\lambda_k I)^m)$).
Hence λ_i is an e.val. of T_W for $i<k$.

Claim 2 λ_k is not an e.val. of T_W.
Supp. $T(v) = \lambda_k v$ for some $v \in W$.
$v \in W \iff \exists y \in V$ s.t. $v = (T-\lambda_k I)^m (y)$. Hence
$0 = (T-\lambda_k I)(v) = (T-\lambda_k I)^{m+1}(y)$.
$\Rightarrow y \in K_{\lambda_k}$.
By Thm 7.2(b) $\Rightarrow$ $v = (T-\lambda_k I)^m (y) = 0$, so v is not an e.val. of T_W.

Since every e.val. of T_W is also an e.val. of T, we conclude that the distinct e.val's of T_W are $\lambda_1, \dots, \lambda_{k-1}$.

Now let $x \in V$.
Then $(T-\lambda_k I)^m (x) \in W$ by def. of W.
Let $K'_{\lambda_i} \subseteq W$ denote the gen. e. space of T_W corresp. to λ_i, for $1 \le i \le k-1$.
Since T_W has $k-1$ distinct e.val's $\lambda_1, \dots, \lambda_{k-1}$, the induction hypothesis applies and gives:
$\exists$ $w_i \in K'_{\lambda_i}$, $1 \le i \le k-1$ s.t.
$(T-\lambda_k I)^m (x) = w_1 + \dots + w_{k-1}$.
Since $K'_{\lambda_i} \subseteq K_{\lambda_i}$ for $i<k$ and $(T-\lambda_k I)^m$ maps K_{λ_i} onto itself (by Claim 1) for $i<k$,
there exist vectors $v_i \in K_{\lambda_i}$ s.t.
$(T-\lambda_k I)^m (v_i) = w_i$ for $i<k$. Thus:
$(T-\lambda_k I)^m (x) = (T-\lambda_k I)^m (v_1) + \dots + (T-\lambda_k I)^m (v_{k-1})$.
By linearity $\Rightarrow$ $(T-\lambda_k I)^m (x - (v_1 + \dots + v_{k-1})) = 0$ $\overset{\text{def.}}{\iff} v_k = x - (v_1 + \dots + v_{k-1}) \in K_{\lambda_k}$.

Thus $x = v_1 + \dots + v_k$ and $v_i \in K_{\lambda_i}$ for all $1 \le i \le k$.

Now we can generalize Thm 5.9 from diagz. operators to all operators with splitting char. poly., replacing e. spaces by gen. e. spaces.

Thm 7.4. Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$ and char. poly. of T splits.
Let $\lambda_1, \dots, \lambda_k$ be the distinct e.val's of T, with corresp. multiplicities $m_1, \dots, m_k$.
For $1 \le i \le k$, let β_i be an ordered basis for K_{λ_i}. Then:
a) $\beta_i \cap \beta_j = \emptyset$ for $i \ne j$.
b) $\beta = \beta_1 \cup \dots \cup \beta_k$ is an ordered basis for V.
c) $\dim(K_{\lambda_i}) = m_i$ for all i.

Proof.
a) Supp. $x \in \beta_i \cap \beta_j \subseteq K_{\lambda_i} \cap K_{\lambda_j}$ for $i \ne j$. In particular $x \ne 0$.
By Thm 7.1(b), $T - \lambda_i I$ is one-to-one on K_{λ_j}, therefore
$(T-\lambda_i I)^p (x) \ne 0$ for any $p \in \mathbb{N}_{>0}$. (as $x \notin N(T-\lambda_i I)$).
But this contradicts $x \in K_{\lambda_i}$!
b) Let $x \in V$. By Thm 7.3, for $1 \le i \le k$ there exist $v_i \in K_{\lambda_i}$ s.t.
$x = v_1 + \dots + v_k$.
As each v_i is a lin. comb. of vectors in β_i $\Rightarrow$ x is a lin. comb. of vectors in β. $\Rightarrow$ $V = \mathrm{Span}(\beta)$.
Let $q = |\beta|$. Then $\dim(V) \le q$.
For each i, let $d_i = \dim(K_{\lambda_i})$. By Thm 7.2(a):
$q = \sum_{i=1}^{k} d_i \le \sum_{i=1}^{k} m_i = \dim(V)$. Hence $q = \dim(V)$, so β is a basis for V.

c) By (b) we have $\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i$.
But $d_i \le m_i$ by Thm 7.2(a).
So $d_i = m_i$ for all i.

Cor Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$ and char. poly. of T splits.
Then T is diagz $\Leftrightarrow$ $E_\lambda = K_\lambda$ for every e.val. λ of T.

Proof
By Thm 7.4 and 5.9, T is diagz $\Leftrightarrow$ $\dim(E_\lambda) = \dim(K_\lambda)$ for each e.val. λ of T.
But $E_\lambda \le K_\lambda$, so $E_\lambda = K_\lambda \Leftrightarrow \dim(E_\lambda) = \dim(K_\lambda)$.

Our next aim is to develop a general method for finding a Jordan canonical basis for an operator.
In view of Thm 7.4, we should first learn to select suitable bases for gen. e.spaces.

Def. Let $T \in \mathcal{L}(V)$, let x be a gen. e.vect. of T corresp. to the e.val. λ.
Supp. that $p \in \mathbb{N}_{>0}$ is the smallest s.t. $(T-\lambda I)^p(x) = 0$. Then the ordered set
$\{(T-\lambda I)^{p-1}(x), (T-\lambda I)^{p-2}(x), \dots, (T-\lambda I)(x), x\}$
is called a cycle of generalized e.vect.'s of T corresponding to λ.
The vectors $(T-\lambda I)^{p-1}(x)$ and x are called the initial vector and the end vector of the cycle, respectively. The length of the cycle is p.

Rem 1) The initial vector is the only e.vect. of T in the cycle.
2) If x is an e.vect. of T corresp. to λ, then the set $\{x\}$ is a cycle of gen. e.vect's of T corresp. to λ of length 1.

Thm 7.5 Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$ and the char. poly. of T splits.
Supp. β is a basis for V s.t. β is a disjoint union of cycles of gen. e.vect.'s of T. Then:
a) For each cycle γ of gen. e.vect's contained in β, $W = \mathrm{Span}(\gamma)$ is T-inv, and $[T_W]_\gamma$ is a Jordan block.
b) β is a Jordan canonical basis for V.

Proof
a) Supp. that γ corresponds to λ, γ has length p, and x is the end vector of γ.
Then $\gamma = \{v_1, \dots, v_p\}$, where
$v_i = (T-\lambda I)^{p-i}(x)$ for $i<p$ and $v_p = x$.
So $(T-\lambda I)(v_1) = (T-\lambda I)^p(x) = 0$,
hence $T(v_1) = \lambda v_1$. For $i>1$,
$(T-\lambda I)(v_i) = (T-\lambda I)^{p-(i-1)}(x) = v_{i-1}$, so $\overbrace{T(v_i) = v_{i-1} + \lambda v_i}^{*} \in W$.
Therefore T maps W into itself, and by $(*)$ $[T_W]_\gamma = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ is a Jordan block.

b) Repeating the argument of (a) for each cycle in β (see H/W 9).

In view of this result, to find a Jordan canonical basis, we must show that under appropriate conditions there exist bases that are disjoint unions of cycles of gen. e.vect.'s.
Since the char. poly. of a Jordan canonical form splits (H/W 9), this is a necessary condition.
Our aim is to show that it is also sufficient.
But we need some preparatory results first.

Thm 7.6. Let $T \in \mathcal{L}(V)$, let λ be an e.val. of T.
Supp. $\gamma_1, \dots, \gamma_q$ are cycles of gen. e.vect.'s of T corresp. to λ s.t. the <u>initial</u> vectors of the γ_i's are distinct and form a lin. indep. set.
Then the γ_i's are disjoint, and $\gamma = \bigcup_{i=1}^{q} \gamma_i$ is lin. indep.

Proof

γ_i's are disjoint - see H/W 9.

γ is lin. indep.

By induction on the number of vectors in γ.

If $|\gamma| < 2$ - clear.

Supp. $|\gamma| = n \geq 2$, and the result holds whenever $|\gamma| < n$.

Let $W = \mathrm{Span}(\gamma)$.

Then W is $(T-\lambda I)$-inv (by def. of cycle), and $\dim(W) \leq n$.

Let $U = (T-\lambda I)_W$.

For each i, let γ_i' be the cycle obtained from γ_i by deleting the end vector.

If $|\gamma_i| = 1$, then $\gamma_i' = \emptyset$.

$\left.\begin{array}{l}\text{If } \gamma_i' \neq \emptyset \text{, then each vector of } \gamma_i' \text{ is the image under } U \text{ of a vector in } \gamma_i. \\ \text{Conversely, every non-zero image under } U \text{ of a vector of } \gamma_i \text{ is contained in } \gamma_i'.\end{array}\right\} (*)$

Let $\gamma' = \bigcup_i \gamma_i'$. Then by $(*)$ γ' generates $R(U)$.

Also $|\gamma'| = n - q$, and the initial vector of γ_i' is the initial vector of γ_i, for all i.

Thus, by induction hypothesis, γ' is lin. indep.

So γ' is a basis for $R(U)$, hence $\dim(R(U)) = n - q$.

Since the q initial vectors of the γ_i's form a lin. indep. set (by assumption) and lie in $N(U)$, we have $\dim(N(U)) \geq q$.

Combining and using the rank-nullity thm:

$n \geq \dim(W) = \dim(R(U)) + \dim(N(U)) \geq (n-q) + q = n.$

So $\dim(W) = n$.

As $W = \mathrm{Span}(\gamma)$ and $|\gamma| = n$, it follows that γ is a basis for W, hence lin. indep.

Cor Every cycle of gen. e.vect.'s of a lin. op. is lin. indep.
(if the initial vector $\neq 0$).

Thm 7.7 Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$ and λ an e.val. of T.
Then K_λ has an ordered basis consisting of a union of disjoint cycles of gen. e.vect.'s corresp. to λ.

Proof.

By induction on $n = \dim(K_\lambda)$.

$n=1$ - clear.

Assume $\dim(K_\lambda) = n > 1$, and the result is valid whenever $\dim(K_\lambda) < n$.

Let $U = (T-\lambda I)_{K_\lambda}$.

Then $R(U)$ is a subspace of K_λ and $R(U) \neq K_\lambda$ (Why? Assume $R(U) = K_\lambda$, and take any $x_0 \neq 0$ in K_λ. Then $\exists x_1, x_2, \dots$ in K_λ s.t. $x_0 = U(x_1) = U^2(x_2) = \dots$, and $x_i \neq 0$ by linearity of U, so $x_{m+1} \notin N(U^m)$, where m = multiplicity of λ - contradicting Thm 7.2.)

So $\dim(R(U)) < \dim(K_\lambda) = n$.

And $R(U)$ is the space of gen. e.vect.'s corresp. to λ for the restriction of T to $R(U)$.

Then by induction hypothesis: $\exists$ disjoint cycles $\gamma_1, \dots, \gamma_q$ of gen. e.vect.'s of this restriction, and hence of T itself, corresp. to λ for which
$\gamma = \bigcup_{i=1}^{q} \gamma_i$ is a basis for $R(U)$.

For $1 \le i \le q$, the end vector of γ_i is the image under U of a vector $v_i \in K_\lambda$, so we can extend each γ_i to a larger cycle $\tilde{\gamma}_i = \gamma_i \cup \{v_i\}$ of gen. e.vect.'s of T corresp. to λ.
For $1 \le i \le q$, let w_i be the initial vect. of $\tilde{\gamma}_i$ (and hence of γ_i as well).
Since $\{w_1, \dots, w_q\}$ is a lin. indep. subset of E_λ (see Remark after the def. of cycles), this set can be extended to a basis $\{w_1, \dots, w_q, u_1, \dots, u_s\}$ for E_λ.
Then $\tilde{\gamma}_1, \dots, \tilde{\gamma}_q, \{u_1\}, \dots, \{u_s\}$ are disjoint cycles of gen. e.vect.'s of T corresp. to λ, s.t. their initial vectors are lin. indep.
Hence $\tilde{\gamma} = \tilde{\gamma}_1 \cup \dots \cup \tilde{\gamma}_q \cup \{u_1\} \cup \dots \cup \{u_s\}$ is a lin.-indep. subset of K_λ by Thm 7.6.

We show that $\tilde{\gamma}$ is a basis for K_λ.
Supp. γ consists of $r = \text{rank}(U)$ vectors.
Then $\tilde{\gamma}$ consists of $r+q+s$ vectors
As $\{w_1, \dots, w_q, u_1, \dots, u_s\}$ is a basis for $E_\lambda \overset{\text{by def.}}{=} N(U)$, it follows that $\text{nullity}(U) = q+s$. So $\dim(K_\lambda) = \text{rank}(U) + \text{nullity}(U) = r+q+s$.
So $\tilde{\gamma}$ is a lin.-indep. subset of K_λ with $|\tilde{\gamma}| = K_\lambda \Rightarrow \tilde{\gamma}$ is a basis for K_λ.

We are ready to obtain the promised result.

Cor 1 Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$ and char. poly. of T splits.
Then T has a Jordan canonical form.
Proof.
Let $\lambda_1, \dots, \lambda_k$ be the distinct e.val.'s of T.
By Thm 7.7, for each i there is an ordered basis β_i consisting of a disjoint union of cycles of gen. e.vect.'s of T corresp. to λ_i.
Let $\beta = \beta_1 \cup \dots \cup \beta_k$.
By Thm 7.4 (b), β is an ordered basis for V.
And by Thm 7.5(b) β is a Jordan canonical basis for V.

We have an analog of this result for matrices.
Def Let $A \in M_{n \times n}(F)$ be s.t. the char. poly. of A (and hence of L_A) splits.
Then the **Jordan canonical form** of A is defined to be the Jordan canonical form of L_A.

We have immediately from Cor 1:
Cor 2 Let $A \in M_{n \times n}(F)$ have a splitting char. poly. Then A has a Jordan canonical form J, and A is similar to J.